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ON ANYON SUPERCONDUCTIVITY

YI-HONG CHEN

FRANK WILCZEK*

EDWARD WITTEN†
School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

and

BERTRAND I. HALPERIN‡
Physics Department, Harvard University, Cambridge, MA 02138, USA

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We investigate the statistical mechanics of a gas of fractional statistics particles in $2+1$ dimensions. In the case of statistics very close to Fermi statistics (statistical parameter $\theta = \pi(1 - 1/n)$, for large n), the effect of the statistics is a weak attraction. Building upon earlier RPA calculation of Fetter, Hanna, and Laughlin for the case $n = 2$, we argue that for large n perturbation theory is reliable and exhibits superfluidity (or superconductivity after coupling to electromagnetism). We attempt to describe the order parameter for this superconducting phase in terms of "spontaneous breaking of commutativity of translations" as opposed to the usual pairing order parameters. The vortices of the superconducting anyon gas are charged, and superconducting order parameters of the usual type vanish. We investigate the characteristic P and T violating phenomenology.

1. Introduction

Since the early days of quantum mechanics it has been appreciated that the behavior of assemblies of identical particles is influenced not only by conventional "forces" but also by the particle statistics. Indeed, the ideal Bose and Fermi gases are the points of departure for most studies of condensed matter at low temperature. It has been extremely useful to have these simple paradigms; for example such ubiquitous concepts as the Fermi surface and Bose condensation were abstracted from their study.

While Bose and Fermi statistics are the only logical possibilities in three spatial dimensions (and the whole notion of quantum statistics degenerates in one

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spatial dimension), in two dimensions the situation is more interesting. In two spatial dimensions, the possibilities for quantum statistics are not limited to bosons and fermions, but rather allow continuous interpolation between these extremes. The quantum statistics is defined by the phase of the amplitude associated with slow motion of distance particles around one another. If the phase $e^{i\theta}$ on interchanging the particles is $+1$ the particles are bosons, if it is -1 the particles are fermions; but other values of the phase are allowed, and give us generically *anyons*.

It is a very attractive problem, to figure out the behavior of these new quantum ideal gases. The high temperature, low density behavior was addressed several years ago in a paper by Arovas, Schrieffer, Wilczek, and Zee.[1] They calculated, in particular, the value of the second virial coefficient. A simple answer was found, that interpolates continuously between bosons and fermions. While this result was significant as a check of the consistency of the whole circle of ideas, and as an exercise for sharpening technique, it hardly addressed the central questions regarding the new quantum ideal gases. The most important effects of quantum statistics, of course, occur only at low temperatures or high density. The existence of a cusp in the virial coefficient at Bose statistics was one of several indications that the behavior of anyon gases at low temperatures would be interesting and probably far from smooth. However, it has proved quite difficult to extend the calculations starting from the high-temperature end, and since the problem seemed both esoteric and inaccessible it was largely abandoned.

Recently, however, there has been a sharp increase in the interest in this problem — for reasons we shall review shortly — and important progress, especially through the work of Laughlin,[2,4] Kalmeyer and Laughlin,[3] and Fetter, Hanna, and Laughlin[5] on high-temperature superconductivity.

In this paper we report further progress in understanding the behavior of the anyon gases with statistics parametrized by

$$\theta = \pi(1 - 1/n), \tag{1.1}$$

at zero temperature. Here n is a positive integer. $n = 1$ corresponds to bosons, while we approach fermions at large n. In accord with Laughlin and coworkers, we shall argue that these gases generically form superfluids, and become superconductors if the anyons are electrically charged. The mechanism of superfluidity seems rather different from conventional pairing, and seems to lie outside the usual Nambu-Goldstone-Higgs framework. Our conclusions are based both on detailed calculations in a controlled approximation, and on qualitative symmetry arguments we expect to be quite robust. We will also derive an effective Lagrangian, that summarizes the electromagnetic response of the charged anyon gas. This Lagrangian, which to a first approximation is of the usual London or Landau-Ginzburg form, also contains small but characteristic interactions violating the discrete symmetries P and T. These terms lead to novel effects, whose occurrence (or not) should enable us to determine whether the anyon gas is realized in concrete physical systems.

At this point it would be disingenuous not to remark that much of the stimulus for the recent upsurge in interest in the anyon gas are some theoretical speculations that quasiparticles in CuO planes, which presumably are the key actors in high temperature superconductivity, are in fact anyons. These speculations were motivated by analysis of excitations around certain types of ordered states (chiral spin liquids) that have been proposed for the electronic ground state in the planes. Needless to say, the fact that superconductivity is an automatic by-product makes these ideas considerably more compelling.

For simplicity, most of the discussion of this paper will be given for the case in which there is a single type of anyon. The discussion can be readily generalized to a set of two or more types of anyons, possessing identical charge and mass, but distinguished by an isospin index τ. Although some of the quantitative formulas will be modified, the qualitative results will be generally similar. For reasons which will be discussed elsewhere we believe that in models of relevance to high-temperature superconductivity there will always be an even number of anyon species.

Before we embark on the analysis, it seems appropriate to establish the context with a brief quasi-historical account of the development of the circle of ideas we are dealing with.

Many of the basic principles involved in fractional quantum statistics were clearly stated and illustrated in a remarkable paper by Leinaas and Myrheim.[6] Unfortunately this paper received little notice at the time, and did not enter the general consciousness, presumably because it was felt to be a purely academic exercise without a broader context. The continuous modern development of the ideas began as part of the recent interest in peculiar, and in particular fractional, quantum numbers more generally.

In fact, it was argued long ago in prescient work by Skyrme[7] that in a nonlinear sigma model of pions, particles with the quantum numbers of nucleons can emerge in the form of solitons. What was surprising about this is that spin and isospin one-half can emerge in a theory in which the elementary fields have integer spin and isospin. Later, Finkelstein and Rubinstein[8] clarified the topological considerations responsible for Skyrme's phenomenon, and showed by a topological argument that solitons of half integral spin in fact obey Fermi statistics, as one would expect on general grounds. (This work probably also represented the first study of what would now be called a θ angle in quantum field theory.) In a somewhat analogous fashion, magnetic monopoles in 3+1 dimensions can be fermions even in a theory in which the elementary fields are all bosons,[9] and can carry fractional[10] and even irrational[11] electric charge. Also, Skyrme's spontaneous generation of half integral spin turns out to have an analog[12] for the case of more than two "flavors" of strong interactions, provided one takes account of the global effects of the Wess-Zumino coupling.[13] Closely related phenomena occur in condensed matter systems[14] and in a wide variety of quantum field theories.[15,16]

Of course in three spatial dimensions the non-trivial commutation relations of the angular momentum algebra ensure that the spin of any particle, regardless of microscopic origins, must be an integer or half-integer. Thus, the above cited results generating half-integer spin from integer spin in $3+1$ dimensions are in a sense the best possible. In two dimensions the situation is different. The rotation group has a single generator which in principle can have any real eigenvalue. For instance, particles orbiting around gauge theory strings, or even around ordinary magnetic flux tubes, can readily be seen to carry fractional angular momentum.[17] Once this is realized, it is then natural to ask (as Finkelstein and Rubinstein had done in connection with Skyrme's work in $3+1$ dimensions) what happens to the spin-statistics connection in these circumstances. This was investigated in a series of papers[18,19,1,20] at first largely rediscovering (in ignorance) the results of Leinaas and Myrheim, but soon going beyond them in various ways, particularly in suggesting how objects of fractional statistics could actually be realized in the physical world. (For an account of early controversies surrounding these ideas, and their resolution, see Goldhaber and MacKenzie.[21])

One early application of the idea of fractional statistics[19] was to the $2+1$-dimensional S^2 σ model, used to model the low-energy excitations of planar ferromagnets and antiferromagnets. It was shown that the classical σ model does not determine a unique quantum theory. The quantum theory allows inclusion of a new interaction, represented by the so-called Hopf term, which is invisible classically. The coefficient of the Hopf term is an angle θ, closely related to the θ introduced in connection with fractional statistics. Indeed, in the σ model the coefficient of the Hopf term determines the spin and statistics of certain collective excitations, the baby Skyrmions. Roughly speaking, the Hopf term plays a role for these excitations somewhat similar to the role played by the Wess-Zumino interaction in connection with $3+1$ dimensional Skyrmions.

Soon afterward the most important realization of fractional statistics so far established arose from a most unexpected quarter, in studies of the behavior of semiconductor heterojunctions held at millikelvin temperature in a strong external magnetic field. The fractional quantized Hall effect (FQHE), discovered in this context, established the existence of a rich new state, or actually series of states, of matter. The theory of these states was developed mainly by Laughlin,[22] with important contributions from Haldane[23] and from Halperin.[24] At the foundation of the theory is the idea that the new states are best described as incompressible quantum liquids, around which the low-energy excitations are localized quasiparticles with unusual quantum numbers, including notably fractional statistics. Using this idea, Halperin was able to predict the values of the allowed fractions in the FQHE hierarchy in a simple and convincing, as well as observationally successful, way. Arovas, Schrieffer, and Wilczek, using the Berry phase technique, showed directly[25] that the quasiparticles had the properties assumed by Halperin. (For an account of early objections to these ideas, and their resolution, see Laughlin.[26]) They also suggested that since the statistical "interaction", together of course with ordinary electromagnetism, is the dominant interaction of the quasiparticles at long distances, it should be possible to write an effective Lagrangian for the long-wavelength behavior of the quasiparticle gas, using just these interactions. The formal implementation of this idea was carried through in the abovementioned paper by Arovas, Schrieffer, Wilczek, and Zee.[1] An important element of that paper, which has played a key role in the further development of the subject, is the introduction of a local implementation of fractional quantum statistics, through the Chern-Simons interaction.

It is also quite likely that fractional statistics excitations exist for liquid ^{3}He films in the *A* phase.[27]

The application of this circle of ideas to superconductivity is by no means as certain or well-developed even as it is in the contexts mentioned above. It is surely premature to be writing even the most informal of histories here. Still, it may be useful to orient ourselves with respect to some of the relevant recent literature on high-temperature superconductivity.

Immediately upon the experimental discovery of the new superconductors, Anderson[28] stressed their essentially two-dimensional character, the importance of strong magnetic ordering, and the possible existence of excitations with exotic quantum numbers. A relatively concrete proposal embodying one form of Anderson's vision was put forward by Kivelson, Rokshar, and Sethna.[29] They showed that division of valence bonds on a square lattice occupied by approximately one valence electron per site into localized dimers, as suggested by the phase "resonating valence bond", could plausibly support excitations — specifically, defects in the pair-bonding of electrons, trapping a single unpaired site — which are charged, spinless bosons. The initial thought was that Bose condensation of such charged excitations was the mechanism of superconductivity. A closely related proposal was made by Dzyaloshinskii, Polyakov, and Wiegmann.[30] Their starting point was a σ-model description of the spin ordering in the CuO layers. They proposed that one employ the Hopf term, as we mentioned above, with $\theta = \pi$. (The paper contains the remark, without elaboration, that only $\theta = 0$ or $\theta = \pi$ are consistent with unitarity. This is mistaken.) The effect of this term is to make the baby Skyrmions of the pure spin model obey Fermi statistics. The idea then is that the charge carriers plausibly induce or bind to these baby Skyrmions, making the composite a boson. Although the microscopic basis of this picture was never clear, and in fact the whole scenario now appears rather dubious, this paper caught the imagination of many physicists. Altogether, these early papers focused considerable attention on the possibility of exotic quantum numbers and statistical transmutation in two dimensions.

Unfortunately, the most immediate natural consequence of all these suggestions is that, since one has direct Bose condensation instead of pairing, the flux quantum should be h/e. Experimentally, it appears to be $h/2e$, at least in the regimes where it has been studied so far. Various modifications of the ideas have been proposed,[29] but it is difficult to know what conclusions to trust when such a seemingly straightforward one must be abandoned. Also, with the loss of the compellingly simple concept of Bose condensation as a mechanism of superconductivity, the motivation for the suggestion of exotic quantum numbers becomes much less clear.

An essentially new set of ideas was added by Laughlin and collaborators, in Refs. 2, 3 5. Kalmeyer and Laughlin made an approximate mapping of certain frustrated spin models onto Bose gases with short range repulsive interactions and subject to a strong external magnetic field. The latter situation is completely analogous to that in the quantized Hall effect, and one can therefore take battle-tested knowledge of the ground state and low-lying excitations in the Hall system over into the spin models. Given the previous discussion of the FQHE, it should not seem shocking that the quasiparticles are then found to obey fractional statistics. Wen, Wilczek, and Zee[31] have given a more abstract treatment of the problem, not relying on the details of a specific wave function, indicating what sort of spin ordering is essential to obtain fractional statistics quasiparticles. We follow them in referring to this class of ordered systems as chiral spin liquids.

Once one has a chiral spin liquid, it is plausible that charged particles doped into the system induce or bind to the fractional statistics quasiparticles, thus themselves acquiring fractional statistics. In several papers, Laughlin and his collaborators have argued that fractional statistics in and of itself leads to superconductivity. The present paper sharpens and extends these arguments.

An important feature of most models incorporating anyons is that they violate the discrete symmetries P and T. This is quite natural for the FQHE, which takes place in an external magnetic field. It occurs spontaneously in ^{3}He-A. It would also have to occur spontaneously in high-temperature superconductors, if anyon models are to describe them. It is, of course, characteristic of chiral spin liquids. That such symmetry breaking could occur, and can have important experimental consequences, was first emphasized by March-Russell and Wilczek,[33] and considerably elaborated recently by these two together with Halperin.[34] Some of the issues have also been discussed recently by Wen and Zee[35] and by Anderson.[36] The considerations of this paper suggest some additional possibilities, and allow us to begin to discuss them quantitatively.

Calculations of the energy of the undoped spin systems using variational wave functions of the Kalymeyer-Laughlin type have not yielded particularly good energies for simple model Hamiltonians, such as Heisenberg antiferromagnets with any combination of couplings to a few near neighbours. Moreover, for the undoped parent compounds of the actual copper-oxide superconductors (e.g., La_2CuO_4) there is compelling evidence that the planes of copper spins are well described by a nearest-neighbour Heisenberg model on a square lattice, with a ground state that has conventional antiferromagnetic order.[37–39] It is known, however, that the addition of a relatively small concentration of holes is sufficient to destroy the antiferromagnetic order. It is certainly possible that the holes also induce an effective multispin-interaction which favors a chiral spin state for the remaining copper spins. If this is the case, then it is reasonable to approach the superconducting state by *starting* with a model Hamiltonian where the spins form a chiral spin liquid even in the absence of free charges. Laughlin has shown that there exists in fact a model Hamiltonian (with long-range four-spin interactions, and with explicitly broken time-reversal and chiral symmetries) for which the quantum-Hall-effect wave function is the exact ground state.[32] There is little reason to doubt that there exists also a class of Hamiltonians which only have finite range interactions, and are invariant under P and T, for which the ground state is a chiral spin liquid.

Shraiman and Siggia[40] have argued that a very dilute concentration of holes in a copper-oxygen plane may lead to a ground state with a spiral spin structure, assuming that one can ignore the effects of the compensating charged impurities, which must be present and would tend to localize the holes in an actual system at low concentrations. A spiral spin structure, in general, would have a chiral character, as well as a broken translational invariance. It is then plausible that above a certain critical concentration of holes, the broken translational symmetry will be destroyed by fluctuations, but the chiral character will persist.

Finally let us note that while the work reported here was proceeding, Hosotani carried out some calculations of the properties of the anyon gas using a somewhat different approximation scheme. Where they overlap, our conclusions agree. Also, Wen and Zee[41] have attempted to study some questions related to those studied in this paper, by perturbing from bosons. Also, interesting numerical studies of small systems of anyons subject to an external magnetic field have been reported recently.[42]

2. The Hamiltonian

In this section we derive a non-local Hamiltonian formulation of the anyon interaction, starting from a formulation in terms of a Chern-Simons Lagrangian. The Chern-Simons formulation is local, but contains redundant variables. The point of the exercise is that each description has its virtues. The Chern-Simons form clearly exhibits the full symmetry and global nature of the interaction. The Hamiltonian form, on the other hand, has the great advantage that its variables represent true physical degrees of freedom. It is therefore better suited to approximations and explicit calculations.

The Lagrangian for an ideal gas of fractional statistics particles is

$$L = \sum_{\alpha} \left\{ \frac{m}{2} \dot{x}_\alpha^2 + e[-a_0(x_\alpha) + \dot{x}_\alpha \cdot a(x_\alpha)] \right\} + \frac{\mu}{2} \int d^2x \varepsilon^{\rho\sigma\tau} a_\rho \partial_\sigma a_\tau \,. \qquad (2.1)$$

Here the x_a are particle coordinates and a is a vector field. The coupling of the particles to the gauge field is standard, but the gauge field action is unusual. Instead of a conventional kinetic energy for the gauge field, one has the final term in (1). This term, the so-called Chern-Simons term, is special to 2+1 dimensions. The action is gauge invariant, despite the explicit appearance of undifferentiated vector potentials. This is because these vector potential always appears contracted with conserved currents — either the conventional particle current, or the unusual "current" $\varepsilon^{\rho\sigma\tau} f_{\sigma\tau}$ which is automatically conserved because of the Bianchi identity.

Varying with respect to a, we find the field equations

$$ej^{\rho} = \frac{\mu}{2}\varepsilon^{\rho\sigma\tau} f_{\sigma\tau}, \tag{2.2}$$

where j is the standard point-particle current and f the standard field strength. These equations indicate that the gauge invariant content of the vector field a is entirely determined by the particle current. In other words, a has no independent dynamics. To avoid confusion with the true electromagnetic potentials and fields, it is convenient to refer to these a fields, whose only purpose in life is to be integrated out and implement fractional statistics, as "fictitious" fields. It follows from the field equation that the field strength f is confined to the particle worldlines, and determined locally by the current of these lines. Thus there are no classical Lorentz forces among the particles.

Integrating the 0 component of the field equation, we find the fundamental relation

$$eN = \mu\Phi, \tag{2.3}$$

where N is the particle number and Φ the fictitious flux. This indicates that the effect of the Chern-Simons term is to associate with each particle fictitious flux e/μ. Of course, the particles also carry fictitious charge e. Thus as they wind around one another, they acquire phase through the Aharonov-Böhm effect. The consequence of all this is that the sole result of adding the fictitious fields is to alter quantum-mechanical amplitude for trajectories where the particles wind around one another, or are interchanged, by a phase proportional to the amount of winding. In other words, the quantum statistics has been altered. A simple calculation shows this alteration of statistics is parametrized by

$$\Delta\theta = \frac{e^2}{2\mu} \tag{2.4}$$

in terms of the angle θ mentioned before.

Turning to the Hamiltonian formulation, we find again that the system has a unique underlying simplicity. Writing out the Lagrangian in more extended form:

$$L = \sum_\alpha \frac{m}{2}\dot{x}_\alpha^2 + \int d^2x\, a_0(-ej_0 + \mu\varepsilon_{ij}\partial_i a_j) + e\sum_\alpha \dot{x}_\alpha \cdot a + \frac{\mu}{2}\int d^2x\, \varepsilon_{ij} a_i \dot{a}_j\,. \tag{2.5}$$

we see that apart from the first term, the rest are either linear in a_0 or linear in time derivatives. Since the time derivative of a_0 never appears, varying with respect to it simply yields the constraint

$$ej_0 = \mu\varepsilon_{ij}\partial_i a_j \equiv \mu b\,. \tag{2.6}$$

Also, when we pass from the Lagrangian to the Hamiltonian terms linear in time derivatives cancel. Thus the Hamiltonian is *numerically* equal to the free-particle Hamiltonian — the net effect of all the extra terms is to enforce an unusual relationship between the canonical momentum and the velocity. The classical equations of motion are just those of non-interacting free particles; the non-trivial dynamics arises entirely from the altered quantum commutation relations.

Since a is a redundant variable we can eliminate it. To do this conveniently, we impose the gauge condition

$$\partial_i a_i = 0. \tag{2.7}$$

Then we can solve the constraint (6) to find

$$a_i(x) = \frac{e}{2\pi\mu}\int d^2y \frac{\varepsilon_{ij}(x-y)_j}{|x-y|^2}\rho(y) = \frac{e}{2\pi\mu}\sum_\alpha \varepsilon_{ij}\frac{(x-x_\alpha)_j}{|x-x_\alpha|^2}\,. \tag{2.8}$$

The final result is that the Hamiltonian is simply

$$H = \frac{1}{2m}\sum_\alpha (p_\alpha - ea(x_\alpha))^2\,, \tag{2.9}$$

with a given as a function of x according to (8).

The Hamiltonian (9) forms the starting point for most of the further considerations in this paper. It was also the starting point adopted by Fetter, Hanna, and Laughlin.[5] As far as we know it has not previously been explicitly derived in full generality from the Chern-Simons Lagrangian, though the result was stated in Ref. 1 and a proof has been sketched before.[34]

To conclude this section we add a few remarks that are not strictly essential to the logical development, but address some points that might be puzzling.

If one were given the Hamiltonian (9) without any explanation of its origins, it might be hard to believe that this Hamiltonian does not lead to classical forces among the particles. Indeed, H looks like the Hamiltonian for a charged particle interacting with an electromagnetic field, in a gauge where $a_0 = 0$. Since the vector potential depends on the particle positions, it varies in time, and one might therefore expect there to be electric fields depending on the relative positions of the particles, and therefore forces among them. Of couse we know from the preceding discussion that it is not so: what gives? Another puzzle is this: how does our H, lacking as it does the standard scalar potential piece, manage to give gauge-invariant results?

The resolution of these puzzles is really quite simple. The resemblance between our H and the standard Hamiltonian for an assembly of particles interacting with an external gauge field is in one crucial respect misleading. That is, our a is given as an explicit *non-local* function of the particle positions. This means, in particular, $a(x_\alpha)$ depends not only on the position of particle α, but on the position of all the other particles as well. Thus when we derive the Hamiltonian equations of motion, there will be additional terms that do not appear in the usual equations for particles interacting with an external gauge field. Keeping this in mind, a straightforward analysis of the equations of motion derived from the Hamiltonian H resolves both our puzzles at the same time. It is found that the additional terms serve exactly to reconstitute the full fictitious electric field, including specifically the gradient of the scalar potential a_0, as determined from (2) in the gauge (7), in the Lorentz force equation. And the full fictitious electric field, as we discussed before, does not depend on the positions or velocities of distant particles, and does not generate classical interparticle forces.

At the risk of being pedantic, we wish to emphasize explicitly one implication of the preceding discussion. No approximation has been made in deriving H. Especially — despite apparent instantaneous interaction terms — retardation effects have not been neglected.

3. Approach to the Problem

The statistical mechanics of an ideal gas of anyons has a very different flavor from that of the more familiar quantum ideal gases of bosons and fermions.

In the case of bosons or fermions, one can construct the eigenstates of the many particle Hamiltonian directly from the eigenstates of the single-particle Hamiltonian, simply by taking tensor products. The sole effect of the statistics, in these two cases, is that one restricts to the subspace of many-body wave functions either symmetric or antisymmetric under permutations, respectively. The reason why this familiar, simple procedure fully incorporates the quantum statistics, is ultimately that the rule for assigning amplitudes to trajectories beginning at $x_1, x_2, \ldots$ and ending at $x_{P_1}, x_{P_2}, \ldots$ depends only on the sign of the permutation P. Thus symmetry or antisymmetry in these coordinates is a condition stable in time. Also, we can obtain all trajectories with the proper weighting from trajectories along which the particles do not change their identity, if we allow all permutations of identity, with the appropriate sign factors, in the initial state. (Indeed, we have just the same trajectories, but with P^{-1} acting on the initial configuration instead of P on the final one.)

For generic anyons, the situation is different. The amplitude assigned to a trajectory depends not only upon the permutation suffered by the particles as they follow the trajectory, but also on other aspects of the trajectories by which they wind around one another. Mathematically, while the Hilbert space of a system of identical bosons and fermions gives a representation of the permutation group, the Hilbert space of a system of identical anyons gives a representation of the "braid group", in which one distinguishes topologically inequivalent trajectories leading to the same permutations of the particles. Incidentally, in $2+1$ dimensional many-body physics it is possible in principle to have a system even more exotic than "ordinary" fractional statistics, in which trajectories that involve braidings of identical particles are represented by non-commuting matrices, not just by abelian phases. (It is far from straightforward to construct representations of the N particle braid group $\mathscr{B}_N$ that are compatible with all the physical requirements of locality and cluster decomposition, but the Jones representations of the braid group[43] satisfy all of the physical conditions, and in fact have a realization in local quantum field theory via a non-abelian Chern-Simons theory.[44]) Leaving aside these more exotic possibilities, which may or may not eventually play a role in condensed matter physics, our interest here is with the anyon gas in which particle trajectories are represented by phases. In fact, the phase associated with a given trajectory is the product of the statistical parameter and the linking number of the trajectory.[18,1]

Once the permutation group is replaced by the braid group, the simple construction passing from the solution of one-particle problems to the solution of many-particle problems, familiar for free bosons and free fermions, does not work any more. It seems most unlikely that there is any comparably simple substitute. For this reason, even an ideal gas of anyons must be regarded as an interacting system.

Since an exact solution seems out of reach, it seems a good strategy to attempt to begin to understand anyon gases by perturbing around the familiar cases of free bosons or fermions, taking advantage of the tools developed over many years for the study of interacting systems of identical particles.

There is an extremely naive argument, which suggests that in general — excluding fermions — an anyon gas will be superfluid (or, for electromagnetically charged anyons, superconducting) at zero temperature. It goes as follows. Fermions with arbitrarily weak attractive forces are known to form superfluids at zero temperature. But there is a real sense in which anyons in general can be considered as fermions with an additional attractive interaction. Indeed, the most important effect of quantum statistics at short distances is that it determines the allowed values of kinetic angular momentum, and thus the strength of the centrifugal barrier. For bosons the allowed values are even integers; for fermions they are odd integers, and for general θ they are $\theta/\pi +$ *even* **integer.** Thus the minimum allowed absolute value is generically smaller than it is for fermions; and so generic anyons can be regarded as fermions with an additional attractive interaction. Although it will become evident in the following that this argument is really much too naive, clearly it points us in the direction of suspecting superfluidity in the **anyon** gas at zero temperature.

With this suspicion, it might seem logical to try to perturb around Bose statistics. After all, the ideal Bose gas exhibits the phenomenon we are after — superfluidity — already in the zeroth approximation. (It is sometimes said that the ideal Bose gas requires a repulsive interaction to become superfluid. We think it is more accurate to say that the ideal Bose gas is a **superfluid** with zero critical velocity, and poised on the brink of instability — a weak attraction will cause it to cease to have a sensible thermodynamic limit.) On further reflection, however, several difficulties with this approach become apparent. The most important one is the following. Consider the gas with statistical parameter

$$\theta = \frac{\pi}{n}. \tag{3.1}$$

Now if we imagine that superfluidity is characterized by an effective condensation into bosons — generalizing ordinary Bose condensation or Cooper pairing — then we must ask: how many of these anyons does it take, to form a boson? If we take one m-tuple around another, we find the accumulated phase $\pi m^2/n$. Thus the condition is

$$\frac{m^2}{n} \equiv 0 \pmod 2. \tag{3.2}$$

Clearly, the minimum required number grows with n, roughly as the square root. It is not easy to see how to obtain this behavior smoothly, starting from condensation of single particles in the Bose gas.

Anyons near θ near zero are similar to a system of bosons with a weak repulsion of statistical origin (representing the centrifugal barrier that is present at $\theta \neq 0$) and in a background magnetic field (representing the interaction of one particle with the average statistical background of the others; this interpretation will be clearer in Sec. 4). Now, bosons with a weak repulsion undergo bose condensation and become superfluid. In the presence of a magnetic field, bose condensation still occurs but not in a translationally invariant fashion; one should expect to form some sort of vortex lattice.

Our approach instead will be to work near Fermi statistics:

$$\theta = \pi\left(1 - \frac{1}{n}\right). \tag{3.3}$$

Then as n gets large the expectation that condensation requires more and more particles appears rather as a virtue than as an embarrassment — it allows us to lose superfluidity in the limit of fermions.

One reason that we think it is natural to work near θ = a is the following. In order to establish that the statistical attraction (relative to fermions) of a departure from θ = A gives rise to superfluidity, it seems to us that the key case is to show that even a weak statistical attraction among a system of otherwise free fermions leads to superfluidity. Once it is established that a weak statistical attraction gives superfluidity, it is natural to expect the same for the strong statistical attraction that arises at the case $(\theta = \pi/2)$ that is believed to be of most interest. Once the effects of a weak statistical attraction are understood qualitatively, it is reasonable to hope that the effects of a strong statistical attraction are similar qualitatively. Our basic strategy is thus to attempt to understand the statistical mechanism for superfluidity starting from the regime of θ near π where this mechanism is operating weakly and can be studied in a controlled way.

Both the qualitative arguments of the next section and the detailed calculations which follow are based on an approximation procedure suggested by Arovas et al.' and employed to great effect by Laughlin* and by Fetter, Hanna, and Laughlin.' We now describe this procedure, and identify a limit in which it is expected to be valid.

Above, we have seen that in a precise sense the statistical interaction can be implemented by attaching fictitious charge and flux to fermions. It is, however, very awkward to deal with the resulting long-range interactions directly. In other problems involving long-range interactions, it is sometimes valid to replace the effect of many distant particles by a mean field or collective variable, with the deviations from the mean represented by residual weak or short-range interactions. Could something like that occur in our problem?

We will argue that in fact very plausibly it does. To get started, let us consider the self-consistency of the approach. Suppose, then, that we do replace the total effect of the distant particles by their average. In our context, this means we are replacing the many singular flux tubes by a smooth magnetic field with the same flux density. For $\theta = \pi(1 - 1/n)$, the resulting magnetic field is related to the average particle density $\bar{\rho}$ by

$$b = \frac{2\pi}{en}\bar{\rho}. \tag{3.4}$$

In such a magnetic field, the particles move along cyclotron orbits with radius

$$r = \frac{mv}{eb}. \tag{3.5}$$

Taking for the velocity the velocity at the nominal Fermi surface, we substitute

$$v = \frac{\sqrt{4\pi\bar{\rho}}}{m} \tag{3.6}$$

and find that a typical cyclotron orbit contains

$$\bar{\rho}\pi r^2 = n^2 \tag{3.7}$$

particles on the average. If the number of particles inside the typical significant orbit is much greater than 1, we should expect that it is indeed valid as a first approximation to replace the field generated by the particles by its average value. since fluctuations will be small compared to the total.

While this argument can and should be sharpened, it seems clear that in the limit of large n it is at least self-consistent as a first approximation to replace our anyon gas by a gas of fermions carrying fictitious charge and propagating in a fictitious magnetic field tied to their density according to (3.4).

4. A Qualitaive Picture

Several of the most important qualitative features of the anyon gas can be understood readily from the simple starting point defined in the previous section.

There, the anyon gas was replaced to a first approximation by fermions propagating in a uniform background fictitious magnetic field given by $b = 2\pi\bar{\rho}/n$. In the fictitious background field b, the energy eigenstates of the fermions form Landau bands, each with degeneracy

$$\rho_\ell = \frac{eb}{2\pi} = \frac{\bar{\rho}}{n} \tag{4.1}$$

per unit area, with energy eigenvalues

$$\varepsilon_\ell = \left(\ell + \frac{1}{2}\right)\frac{eb}{m} = \left(l + \frac{1}{2}\right)\omega_c, \tag{4.2}$$

where $l = 0, 1, 2, \ldots$. When the statistical parameter is $\theta = \pi(1 - 1/n)$, the density is just such as to fill n Landau levels exactly. (In the next approximation we will find a massless particle that will give the fermions a logarithmically divergent self-energy, which we ignore for the present.)

The fact that the bands are exactly filled suggests that the ground state will have a particularly favorable energy at these values of the statistical parameter. Exactly filling the top band ought to be analogous to completing a shell in atomic or nuclear physics, or filling an ordinary band in a solid. If this is true, the ground state should exhibit a certain rigidity, and exhibit an energy gap.

To test and quantify these expectations, let us consider the effect of adding a small *real* magnetic field B to the fictitious one b. The situation is asymmetric with respect to the sign of the real field relative to the fictitious field, and we must consider the two cases where the fields add or cancel separately.

If the real field is in the same direction as the fictitious one, the density of states per Landau level will be somewhat greater, and we will not quite completely fill n levels anymore. Let us denote the fractional filling of the highest level by $1-x$. Then from the conservation of particle number we derive

$$(b+B)(n-x) = bn; \; (b+B)x = Bn. \tag{4.3}$$

For the total energy we have then

$$E = \frac{e(b+B)}{2\pi}\frac{e(b+B)}{m}\left\{\sum_{\ell=0}^{n-1}\left(\ell+\frac{1}{2}\right)-\left(n-\frac{1}{2}\right)x\right\} = \frac{n^2e^2}{4\pi m} \times\left\{b^2+\frac{1}{n}bB-\left(1-\frac{1}{n}\right)B^2\right\}. \tag{4.4}$$

Thus the energy relative to the ground state is positive, and grows linearly with B for small B.

If the real field is in the opposite direction from the fictitious one, the density of states per Landau level will be smaller, and we will have to promote some particles to the $(n+1)$ level. Denoting the fractional filling of this level by x, we have from particle conservation

$$(b-B)(n+x) = bn; \; (b-B)x = Bn, \tag{4.5}$$

and for the energy

$$E = \frac{e(b-B)}{2\pi}\frac{e(b-B)}{m}\left\{\sum_{\ell=0}^{n-1}\left(\ell+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)x\right\} = \frac{n^2e^2}{4\pi m} \times\left\{b^2+\frac{1}{n}bB-\left(1+\frac{1}{n}\right)B^2\right\}. \tag{4.6}$$

Thus in this case too the energy relative to the ground state is positive, and grows linearly with B for small B. Despite the asymmetry of the situation, the coefficients of the terms linear in B are equal in the two cases. The quadratic terms differ.

These arguments though simple are quite significant. They suggest that the anyon gas, at the statistics considered, will strive to exclude external magnetic fields. This is the germ of the Meissner effect, a hallmark of superconductivity. At the same time they suggest the existence of an energy gap in the charged particle spectrum. Indeed, the energy to create a separated particle-hole pair should be just the energy to excite a fermion into the lowest empty Landau band, viz.

$$E_{\text{pair}} = \frac{eb}{m} = \frac{2\pi\bar{\rho}}{mn}. \tag{4.7}$$

Considered more closely, these arguments also suggest a close connection between vortices and fermion excitations that seems to be something new in the theory of superconductivity. This connection is characteristic of anyon superconductivity, and will play a key role below both in its deeper theory and in its phenomenology. The point is this: since the fictitious field is uniquely tied to the particle density, and is appropriate to n Landau levels being exactly filled, to accommodate any additional real magnetic field we will necessarily have to excite particles across the gap. (Or to create holes, a process which we have seen is also characterized by a gap.) Conversely, if the particles do not fill the Landau levels exactly, there must be a real magnetic field present to account for the mismatch. Anticipating that the filled Landau level state, and its possible adiabatic modulations, is the superfluid component, we are led to conclude that *in anyon superconductivity, charged quasiparticles and vortices do not constitute two separate sorts of elementary excitations — they are one and the same.*

We can also infer the value of the flux quantum, from this identification. Adding a single fundamental unit $2\pi/e$ of real flux increases the number of available states by one per Landau level. Thus, for n filled Landau levels, the act of piercing the material by a unit flux tube creates n holes. Clearly this is not the most elementary excitation. The most elementary excitation is to produce just one hole. Thus the elementary fluxoid is $1/n$ of the fundamental unit, or $2\pi/ne$.

Although these simple arguments have taken us a long way, there remains a central feature of superfluidity that is not at all obvious, or even true, in the simple approximation described thus far. This feature is the existence of a sharp Nambu-Goldstone mode, or concretely an excitation with the dispersion relation $\omega^2 \propto k^2$ at low frequency and small wave vector. It does exist. It was discovered in a remarkable calculation by Fetter, Hanna, and Laughlin.[5] They calculated the effect of adding back the residual interactions, and found that these interactions produced the necessary pole in the current-current correlation function. In physical terms, this means that there are particle-hole bound states at zero energy. In the following two sections we shall review and generalize these calculations.

Unfortunately these calculations do not by themselves make it clear *why* the massless mode exists. Aside from being emotionally disturbing, it is not objectively satisfactory to lack such understanding. Without it, one may be left uncertain whether this central qualitative feature of the anyon gas is robust, or an artifact of the approximations employed in the calculation. Similarly, one may be left uncertain whether small changes in the model Hamiltonian itself — which after all, is highly idealized — might change this feature. Fortunately, the existence of the massless mode can also be demonstrated simply, and it can be understood qualitatively using arguments closely related to those in the present section. This is the subject of Sec. 7.

To conclude the present section we would like to make some brief remarks concerning the anyon gas at other values of the statistical parameter, when $\theta \neq \pi(1 - 1/n)$.

If the top Landau level were not completely filled, then the second of our calculations above (leading to Eq. (4.6)) would be valid for either sign of the field. The energy is then analytic in B, and the presence of a linear term is indicative of the fact that the ground state of the anyon system possesses an orbital ferromagnetic moment in this case. (We also find that there is an orbital magnetic moment when $\theta = \pi(1 - 1/n)$ but the analysis is considerably more complicated.[34])

For more general rational values of θ/π, it is possible that the anyons in the highest Landau level will form a correlated many-body state, similar to the states of the fractional quantized Hall effect.[45] In this case there is again an energy gap for vortex excitations, and we expect again to find a superfluid ground state.

For most of our discussion, up to and including the previous sentence, we have assumed that the ground state is homogeneous. (An exception was when we discussed the expected ground state for fractional statistics near bosons.) This is almost surely true for the values $\theta = \pi(1 - 1/n)$ which are our main concern. However, it is almost surely *not* true in general. For example, let us consider again statistics very close to, but not equal to, one of our favored values, say $n = n_0$. Then clearly instead of expanding around $n = \infty$ — fermions — we should expand around $n = n_0$. The particles will then have a small residual interaction. More important, the particle density will then not quite fit the density appropriate to the fictitious magnetic field. It seems very likely that the best way to accommodate this situation is to allow an occasional normal particle — or equivalently, an occasional vortex — rather than to disrupt the superfluid state globally. Thus, operationally, one would separate the anyons into two classes — the first, with fractional density n_0/n to be treated as an anyon gas with $\theta = \pi(1 - 1/n_0)$ and the remainder to be treated as vortices or antivortices in that background. Readers familiar with the fractional quantized Hall effect[45] will recognize a strong resemblance to the situation that occurs there, when the density is close to but not quite equal to one of the favored rational filling fractions.

These considerations are by no means rigorous or complete, but they do serve to suggest that the physics of the anyon gas at general values of θ is likely to be quite rich and to depend quite strongly on "number-theoretic" properties of θ.

5. The RPA Calculation

In this section we discuss the mechanics of calculations in the random phase approximation. The method follows closely that of Fetter, Hanna and Laughlin; we have merely added a few observations and elaborated several points left implicit in their very concise presentation.

To begin with, as we discussed in Sec. 2, the Hamiltonian of the anyon gas is (changing notation slightly to agree with Ref. 5)

$$H = \sum_{\alpha} \frac{1}{2m} |p + a(r_\alpha)|^2, \tag{5.1}$$

where r_α is a two-dimensional vector specifying the position of particle α and

$$a(r_\alpha) = \frac{1}{n} \sum_{\beta \neq \alpha} \frac{\hat{z} \times r_{\alpha\beta}}{|r_{\alpha\beta}|^2} \tag{5.2}$$

with $r_{\alpha\beta} = r_\alpha - r_\beta$. Here the particles are to be regarded (in the absence of interactions) as fermions; the interaction then makes them anyons with statistical parameter $\theta = \pi(1 - 1/n)$.

It will be convenient to use second quantized notation, in which

$$H = \int d^2 r \Psi^\dagger(r) \frac{1}{2m} |\hat{p} + a(r)|^2 \Psi(r). \tag{5.3}$$

Here Ψ is a spinless fermion field, and

$$a(r) = \frac{1}{n} \int d^2 r' \, \frac{\hat{z} \times (r - r')}{|r - r'|^2} \Psi^\dagger(r') \Psi(r'). \tag{5.4}$$

The Hamiltonian describes a system of spinless fermions interacting through long range gauge potentials.

Actually these expressions are somewhat formal, in that if the density is constant the integral for a will diverge. For this reason, and also to implement the ideas of Sec. 3, it is useful to separate a into a background part and a fluctuating part. This is analogous to the familiar use of normal ordering or subtractions in defining the vacuum quantum numbers in quantum field theory. It should be considered as part of *defining* the theory. We shall have to check whether the theory so defined retains the properties — and in particular, the symmetries —

we expected of the naive model. Alternatively, one could in principle formulate the theory in a finite geometry, say on a torus.

If we ignore fluctuations and substitute the average density $\bar{\rho}$ for the density operator in $a(r)$, we expect that the system should reduce to spinless fermions propagating in a constant fictitious magnetic field. Thus we are led to define

$$a(r) = \bar{a}(r) + \frac{1}{n}\int d^2r' \frac{\hat{z}\times(r-r')}{|r-r'|^2}(\Psi^\dagger\Psi(r) - \bar{\rho}), \tag{5.5}$$

where

$$\bar{a}(r) = \frac{1}{2} b\hat{z}\times r, \tag{5.6}$$

$$b = \frac{2\pi\bar{\rho}}{n}. \tag{5.7}$$

This definition of a replaces (5.4). However, the formula (5.6) for $\bar{a}(r)$ requires some explanation. The mean vector potential $\bar{a}$ should naturally be defined by the same integral

$$\bar{a}(r) = \frac{1}{n}\int d^2r' \frac{\hat{z}\times(r-r')}{|r-r'|^2}\bar{\rho} \tag{5.8}$$

as (5.4), with the true charge density $\Psi^\dagger\Psi$ replaced by the mean density $\bar{\rho}$. The only problem with this is that the integral in (5.8) is not unambiguously convergent if $\bar{\rho}$ is strictly constant. To interpret this integral, note that for arbitrary $\bar{\rho}$ such that the integral in (5.8) is well-defined, that integral computes an abelian gauge field $\bar{a}$ such that $b = 2\pi\bar{\rho}/n$, where $b = \partial_1\bar{a}_2 - \partial_2\bar{a}_1$, and moreover such that $\nabla\cdot\bar{a} = 0$, and such that $\bar{a}$ vanishes at ∞. $\bar{a}$ is uniquely determined by those conditions, and the integral in (5.8) has exactly the kernel required to produce the field $\bar{a}$ obeying those conditions. For the limiting case in which the support of $\bar{\rho}$ extends over all of space, the integral in (5.8) is ambiguous (not absolutely convergent), and it is impossible to obey all of the conditions that would hold if $\bar{\rho}$ had compact support (to give the right b, $\bar{a}$ cannot vanish at ∞). We interpret the integral in (5.8) as giving an average $\bar{a}$ field that gives the right b and obeys the gauge condition and has a behavior at ∞ that is as good as possible. The proposed form in (5.6) obeys these desiderata, but is not quite unique since without changing b or violating the gauge condition or worsening the behavior at ∞, one could add a constant to $\bar{a}$. This ambiguous integration constant is actually closely related to the physics that we will eventually find. Modulo an integration constant, the answer in (5.6) is certainly what one would get by doing the integral in (5.4) for some almost constant $\bar{\rho}$ of compact support, and then taking the limit as the support of $\bar{\rho}$ extends over all space. The value that one would get for the integration constant would depend on exactly how one took the limit.

For later use, we define current operators

$$J_i(r) \equiv \Psi^\dagger(r)\frac{1}{m}(p_i + a_i(r))\Psi(r) \tag{5.9}$$

$$\hat{j}_i(r) \equiv \Psi^\dagger(r)\frac{1}{m}(p_i + \bar{a}_i)\Psi(r)\,. \tag{5.10}$$

Since on the one hand it is a soluble problem, and on the other we have argued it contains much of the important physics, we will treat the system of otherwise free fermions propagating in the average field as the reference problem, and regard the rest of the Hamiltonian as a perturbation. The interaction Hamiltonian for this perturbation scheme is

$$\begin{aligned} H_I = H - H_0 &= \int d^2r \frac{1}{2m}\Psi^\dagger(r)\{2(p+\bar{a})(a-\bar{a}) + (a-\bar{a})^2\}\Psi(r) \\ &= H_1 + H_2\,, \end{aligned} \tag{5.11}$$

$$\begin{aligned} H_1 &= \frac{1}{nm}\int\int d^2r d^2r' \Psi^\dagger(r)(p_i + \bar{a}_i)\Psi(r)\frac{\varepsilon_{ij}(r-r')_j}{|r-r'|^2}\{\Psi^\dagger(r')\Psi(r') - \bar{\rho}\} \\ &= \frac{1}{n}\int\int d^2r\, d^2r'\, \hat{j}_i(r)\frac{\varepsilon_{ij}(r-r')_j}{|r-r'|^2}(\rho-\bar{\rho})(r'), \end{aligned} \tag{5.12}$$

$$\begin{aligned} H_2 &= \frac{1}{2mn^2}\int\int\int d^2r\, d^2r'\, d^2r''\Psi^\dagger(r)\Psi(r)\left\{\frac{\varepsilon_{ij}(r-r')_j}{|r-r'|^2}(\Psi^\dagger(r')\Psi(r') - \bar{\rho})\right\} \\ &\quad\times\left\{\frac{\varepsilon_{ik}(r-r'')_k}{|r-r''|^2}(\Psi^\dagger(r'')\Psi(r'') - \bar{\rho})\right\} \\ &= \frac{1}{2mn^2}\int\int\int d^2r d^2r' d^2r'' \rho(r)\frac{(r-r')\cdot(r-r'')}{|r-r'|^2|r-r''|^2}(\rho(r') - \bar{\rho})\,(\rho(r'') - \bar{\rho})\,. \end{aligned} \tag{5.13}$$

In the same spirit let us reorganize H_2 into two pieces, using $\rho(r) = \bar{\rho} + (\rho(r) - \bar{\rho})$.

The first half of the resulting expression is expected to dominate for large n, when fluctuations in density are relatively small. Its meaning becomes transparent upon doing the integral:

$$\int d^2r \frac{(r-r')\cdot(r-r'')}{|r-r'|^2\cdot|r-r''|^2} = -2\pi \ln|r'-r''| . \tag{5.14}$$

It represents an effective Coulomb interaction! The interaction is repulsive between like-signed particles, attractive between oppositely-signed particles.

The existence of such an interaction is important in two respects. First, it generates an effective long range *repulsion* between two particles, or two holes. Given the identification of these excitations with vortices, this is responsible for the anyon superconductor being type II.

Second, it generates an effective long range *attraction* between particles and holes. This is responsible for the formation of the zero-mass bound state.

The nature of the interaction can be given an interesting interpretation. Imagine that a massless gauge field has developed dynamically, such that our particles couple to this field. Then there would be a logarithmic interaction of precisely the calculated form. Later we shall see that the premises in this interpretation do actually hold.

If we simply drop the other half of H_2, we are left with two-body interactions only, and can make great progress. Note that the discarded term, besides being intrinsically small, is manifestly translation, rotation, and (even if we couple in electromagnetism) gauge invariant. The remaining interactions can be written

$$H_I = \frac{1}{2}\int d^2r \int d^2r' \hat{j}_\mu(r) V_{\mu\nu}(r,r')\hat{j}_\nu(r'), \tag{5.15}$$

where the spatial part of $\hat{j}_\mu$ has been defined before, and

$$\hat{j}_0(r) = \rho(r) - \bar{\rho} . \tag{5.16}$$

There is no distinction between $\hat{j}_0$ and $\hat{J}_0$. V takes a simple form in Fourier space. To exhibit this, we take a momentum vector q with component only in the t and x directions, and we order the coordinates as (t, x, y). Then one has

$$V_{\mu\nu}(q) \equiv \int d^2r' \, V_{\mu\nu}(r,r')e^{iq(r-r')}$$

$$= \frac{1}{n}\begin{pmatrix} \frac{\bar{\rho}(2\pi)^2}{mnq^2} & 0 & \frac{i2\pi}{q} \\ 0 & 0 & 0 \\ \frac{-i2\pi}{q} & 0 & 0 \end{pmatrix}. \tag{5.17}$$

The appearance of the Coulomb interaction suggests the importance of summing bubble graphs, as in the standard treatment of the electron gas. Since the interaction Hamiltonian can be written in terms of $\hat{j}$, the correlation function of $\hat{j}$ obeys a simple geometric equation, in this approximation. Thus defining

$$D_{\mu\nu}(1, 2) = -i\langle T(\hat{j}_\mu(1)\hat{j}_\nu(2)\rangle \tag{5.18}$$

as the matrix of time-ordered expectation values in the true ground state, and $D^0_{\mu\nu}$ as the corresponding object in the non-interacting ground state, we have in this approximation

$$D = D_0 + D_0 VD. \tag{5.19}$$

The product is to be regarded as convolution in real space or simple multiplication in Fourier space.

Solving this equation, we find

$$D = (1 - D^0 V)^{-1} D^0 . \tag{5.20}$$

Another perspective on the bubble-graph approximation, that is actually superior from a logical point of view, is to regard it as simply a *perturbative* evaluation of the inverse propagator D^{-1}. The previous equation, in the form

$$D^{-1} = D^{0^{-1}}(1 - D^0 V), \tag{5.21}$$

is then simply lowest-order perturbation theory. Why is it more appropriate to perturb in the inverse propagator than in the propagator itself? That is a standard story that we shall not belabor here; the key point is that the inverse propagator, unlike the propagator itself, is regular at small frequency and wave vector, so whereas for the propagator itself we find immediately that the limits $\omega, q \to 0$ and $n \to \infty$ do not commute, there is every reason to expect the perturbative evaluation of the inverse propagator to become accurate as $n \to \infty$.

The calculation of $D^0_{\mu\nu}$ is straightforward though rather arduous; it is presented in Appendix A. The result may be parametrized in the form

$$D^0 = \frac{n}{2\pi\omega_c}\begin{pmatrix} q^2\Sigma_0 & q\omega\Sigma_0 & -iq\omega_c\Sigma_1 \\ q\omega\Sigma_0 & (\omega^2\Sigma_0 - \omega_c^2\Sigma_3) & -i\omega\omega_c\Sigma_1 \\ iq\omega_c\Sigma_1 & i\omega\omega_c\Sigma_1 & \omega_c^2\Sigma_2 \end{pmatrix}. \tag{5.22}$$

In writing this result we have specialized to the case $q_y = 0$; this involves no real loss of generality.

D is not quite the object we want. The electromagnetic response is rather given in terms of the true current-current correlation function

$$\Lambda_{\mu\nu}(1, 2) = -i\langle|T[\hat{J}_\mu(1)\hat{J}_\nu(2)]|\rangle \tag{5.23}$$

where (1) denotes the dependence on r_1, t_1, and $|\rangle$ denotes the exact ground state. Fortunately, Λ and D are closely related. Consider, for example, the 10 entry. We have

$$(\Lambda - D)_{10} = \frac{-i}{m}\langle|T[\Psi^\dagger\Psi(1)(a-\bar{a})(1),\quad \Psi^\dagger\Psi(2)-\bar{\rho}]|\rangle$$

$$= \frac{-i}{mn}\langle|T[\rho(1)\int d^2r_3\frac{\hat{z}\times(r_3-r_1)}{|r_3-r_1|^2}\hat{j}_0(r_3, t_1)\hat{j}_0(2)|\rangle . \tag{5.24}$$

In the now familiar manner, we separate ρ into an average and a fluctuating part:

$$(\Lambda - D)_{10} = \frac{-i}{mn}\bar{\rho}\langle|T\left[\int d^2r_3\frac{\hat{z}\times(r_3-r_1)}{r_3-r_1{}^2}\hat{j}_0(r_3, t_1)\hat{j}_0(2)\right]|\rangle$$

$$\frac{-i}{mn}\langle|T\left[\hat{j}_0(1)\int d^2r_3\frac{\hat{z}\times(r_3-r_1)}{|r_3-r_1|^2}\hat{j}_0(r_3, t_1)\hat{j}_0(2)\right]|\rangle . \tag{5.25}$$

The contribution to $\Lambda - D$ involving the average can be simply expressed in terms of D itself; the contribution from fluctuations is small in the $n \to \infty$ limit and we drop it. Passing to Fourier space, we arrive at

$$(\Lambda - D)_{10} = +i\frac{2\pi\bar{\rho}}{mn}\frac{1}{q}D_{00} . \tag{5.26}$$

A generalization of this argument leads easily to

$$\Lambda \simeq (1+\bar{\rho}u^\dagger)D(1+\bar{\rho}u) ,$$

$$u = \frac{2\pi}{mnq}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.27}$$

Finally, the true electromagnetic response includes not only the current-current correlation (which essentially represents the iteration of the first order term in the true electromagnetic potential A) but also a contact term, from the direct appearance of A^2 in the Lagrangian, which is quadratic in momentum. Thus the final expression for the response function, defined according to

$$4\pi \langle J_\mu(q, \omega) \rangle = -K_{\mu\nu}(q, \omega) A_\nu^{\text{ext}}(q, \omega) \tag{5.28}$$

is

$$K_{\mu\nu}(q, \omega) = \frac{e^2}{m}\bar{\rho}\delta_{\mu\nu}(1 - \delta_{\mu 0}) + e^2\Lambda_{\mu\nu}(q, \omega)\,. \tag{5.29}$$

Collecting the various formulae, we find (setting, for reasons discussed in Appendix A, $\Sigma_3 = 1$)

$$K = \frac{e^2 n}{2\pi \det}\begin{pmatrix} \frac{q^2}{\omega_c}\Sigma_0 & q\frac{\omega}{\omega_c}\Sigma_0 & iq\Xi \\ q\frac{\omega}{\omega_c}\Sigma_0 & \frac{\omega^2}{\omega_c}\Sigma_0 & i\omega\Xi \\ -iq\Xi & -i\omega\Xi & \omega_c(\Xi - \Sigma_1 + \Sigma_2 + \det) \end{pmatrix} \tag{5.30}$$

where

$$\Xi = -\Sigma_1 - \Sigma_1^2 + \Sigma_0\Sigma_2 + \Sigma_0 \tag{5.31}$$

and

$$\det = 1 - \Sigma_0 + 2\Sigma_1 + \Sigma_1^2 - \Sigma_0\Sigma_2\,. \tag{5.32}$$

In arriving at this expression, we have made approximations at three stages: in the perturbative evaluation of the inverse propagator D^{-1}, in formulating the interaction Hamiltonian, and in passing from D to Λ. We have discussed the first of these above, now let us address the other two. Both these approximations were of the same general form: in an expression involving the correlation of the density at one point with density fluctuations at two other points, we replaced the density with its average. In concluding this section, we wish to remark that this approximation can be justified in the large n limit. Indeed, the triple correlations of density fluctuations satisfy a simple Dyson equation. Although we will not present the details here, a straightforward analysis based on this equation shows that the terms dropped involve a higher power of the interaction than the terms kept, and thus a higher power of $1/n$. Clearly, these remarks also point the way to a practical method of calculating to higher order.

6. Results of the RPA Calculation

We now evaluate the electromagnetic response $K_{\mu\nu}(q, \omega)$ for small q and ω explicitly. From Appendix A we derive in this limit

$$\Sigma_0 \approx -1 - \left(\frac{\omega}{\omega_c}\right)^2 + \frac{3n}{8}\left(\frac{q}{\lambda}\right)^2 ,$$

$$\Sigma_1 \approx -1 - \left(\frac{\omega}{\omega_c}\right)^2 + \frac{3n}{4}\left(\frac{q}{\lambda}\right)^2 , \tag{6.1}$$

$$\Sigma_2 \approx -1 - \left(\frac{\omega}{\omega_c}\right)^2 + n\left(\frac{q}{\lambda}\right)^2 .$$

It is noteworthy that to this order only transitions between the two top filled Landau levels and the two bottom empty ones contribute.

There is evidently a pole in the response function, at

$$\left(\frac{\omega}{\omega_c}\right)^2 \approx n\left(\frac{q}{\lambda}\right)^2 . \tag{6.2}$$

The physical significance of K becomes more transparent if we reformulate it in terms of an effective Lagrangian. We have found that we can reproduce the response function at low frequency and small wave vector using an effective model which contains a massless scalar field interacting with the electromagnetic gauge field, of the form

$$L = \frac{1}{2}(\dot{\phi} - CA_0)^2 - \frac{v^2}{2}(\partial_i\phi - CA_i)^2$$
$$+ a\varepsilon_{ij}\partial_i A_j(\dot{\phi} - CA_0) + b\varepsilon_{ij}(\partial_0 A_i - \partial_i A_0)(\partial_j\phi - CA_j) . \tag{6.3}$$

Notice that this Lagrangian is invariant under the gauge transformation:

$$\begin{aligned} \phi &\to \phi + Cf \\ A_\mu &\to A_\mu + \partial_\mu f . \end{aligned} \tag{6.4}$$

This model exhibits the Higgs mechanism in its pristine form (due to Stuckelberg): ϕ, which in the absence of electromagnetism represents a scalar degree of freedom — essentially a sound wave, with v^2 equal to the speed of sound squared — loses its independent significance when thus coupled to electromagnetism. Indeed, it can be set to zero by a gauge transformation.

The first two terms in L are familiar in the theory of superconductivity. They generate the ordinary London equations. The next two terms are higher order in gradients, and thus formally subdominant. However, we have kept them because they display a qualitatively new feature. Whereas the first two terms are automatically invariant under parity and time reversal, the next two are not. They are of course fully rotationally and gauge invariant, but violate both P and T, in such a way that PT is conserved. In a word, they obey the symmetries of our underlying microscopic model — the anyon gas — and we have every right to expect that they should occur. The fact that these terms are in a real sense small is both entertaining and significant. It is entertaining, in that it is a rather unexpected analogue of a familiar situation in high-energy physics. There, it is an important result that in QCD, parity violation and time-reversal violation cannot occur through low-dimension (renormalizable) interactions. It is this fact that makes it comprehensible that parity and the time-reversal violation are hard to observe, even though neither is fundamentally a good symmetry. Similarly here, it is very significant that parity and time-reversal symmetry are in some sense automatically hidden in anyon superconductivity. This makes the phenomenology more challenging to work out and the experiments to meaningfully test the symmetries necessarily subtle.

If we put $\phi = 0$ inside the Lagrangian, we see that these new terms are closely related to gauge theory Chern-Simons terms. It is amusing that upon dropping the requirement of relativistic invariance we find there are two possible Chern-Simons like terms. To a first approximation the charge density and electric current associated with ϕ are

$$\begin{aligned} \rho &= -C(\dot{\phi} - CA_0) \\ j_i &= Cv^2(\partial_i\phi - CA_i) \end{aligned} \tag{6.5}$$

as follows from varying the Lagrangian with respect to A_0, A_i respectively and dropping the terms proportional to a and b. Using these approximate expressions, we can write the new terms in a more transparent form:

$$a\varepsilon_{ij}\partial_i A_j(\dot{\phi} - CA_0) = -\frac{a}{C}\rho B\,, \tag{6.6}$$

$$b\varepsilon_{ij}(\partial_0 A_i - \partial_i A_0)(\partial_j \phi - CA_j) = -\frac{b}{v^2 C}\, j \times E\,. \tag{6.7}$$

Thus we see that a correlates electric charge density with magnetic fields, and b correlates current with perpendicular electric field in a manner reminiscent of the Hall effect. Also, we see that a change $\delta\rho$ in the density is generally accompanied by a change in the magnetic moment density, proportional to a.

The numerical evaluation is carried out by comparing the photon two-point function calculated from L with the response function K. A few details of the calculation are presented in Appendix B. Our results, valid in the limit $n \to \infty$, are:

$$\begin{aligned} v^2 &= \frac{2\pi\bar{\rho}\hbar^2}{m^2}\,, \\ C &= e\sqrt{\frac{m}{2\pi\hbar}}\,, \\ a &= ne\sqrt{\frac{\hbar}{32\pi m}}\,, \\ b &= 0 \end{aligned} \tag{6.8}$$

where proper units have been restored.

The values of v^2 and C are just such as to reproduce the standard formula for the London penetration depth. The vanishing of b can be understood on physical grounds. We will discuss this, as well as some phenomenological implications of L, in Sec. 8.

One can also obtain the coefficient a by another type of analysis, which we believe to be exact, and whose details will be given elsewhere.[34] The correct formula for a differs from that in (6.8) by an additional factor $(1 - n^{-2})$. Note that this gives $a = 0$ for the case of bosons ($n = 1$) as we expect for this situation, where P and T are actually good symmetries.

To conclude this section we would like to comment on the relation of the effective Lagrangian discussed above to a more complete effective Lagrangian, and how the latter might be calculated. These comments illustrate certain points but do not incorporate the special features of the order parameter discussed in Sec. 7; thus the equations that follow should be interpreted metaphorically.

In the Landau-Ginzburg generalization of the London framework one considers that the density as well as the phase of the superfluid condensate can vary. In this generalization, we would have instead of the Lagrangian considered above, a Lagrangian of the form

$$\begin{aligned} L_{\text{L.G.}} = & \frac{1}{2}|\dot{\Phi} - iqA_0\Phi|^2 - \frac{v^2}{2}|\partial_i\Phi - iqA_i\Phi|^2 \\ & - \frac{ia}{2V}\varepsilon_{ij}\partial_i A_j\{\Phi^\dagger(\dot{\Phi} - iqA_0\Phi) - \text{c.c.}\} \\ & - \frac{ib}{2V}\varepsilon_{ij}(\partial_0 A_i - \partial_i A_0)\{\Phi^\dagger(\partial_j\Phi - iqA_j\Phi) - \text{c.c.}\} \\ & + m^2|\Phi|^2 - \lambda|\Phi|^4 . \end{aligned} \tag{6.9}$$

Our previous considerations on the unit of quantized flux suggest $q = ne$. This form goes over into the London Lagrangian if we specialize the complex scalar Φ to the form

$$\Phi = Ve^{i\phi/V} \tag{6.10}$$

where $V = \sqrt{m^2/2\lambda}$ is the vacuum expectation value of Φ, and work to lowest order in gradients. Notice that the mass term m and the self-interaction λ lose their significance in this limit.

We determined the coefficients of the London Lagrangian by matching to the electromagnetic response at low frequency and small wave vector. One could in principle determine the coefficients of the Landau-Ginzburg Lagrangian, or an appropriate modification of it, within the framework of the calculations reported above, by matching to the response at higher frequency and larger wave vector. It should be remarked, however, that the unique feature of the statistical interaction — its long-range nature — does not guarantee, or even make it reasonable to expect, that it is a good guide with respect to short-distance or small-time behavior. Other interactions of a more prosaic sort will surely come into play. Therefore the idealization involved in treating the quasiparticles in any real material as an ideal gas of anyons generally becomes more severe as we move away from the London regime, except for certain qualitative questions of a global character.

We might also step back one more step, and try to build into an effective Lagrangian the fact that the P and T violation, which we have been treating as if it were fundamental, must actually have its origins in spontaneous symmetry breaking. A simple possibility is the following. Let η be a *real* scalar field, meant to parametrize the degree of chiral spin liquid order. Then let

$$L = \tilde{L}_{\text{L.G.}} + L_\eta \tag{6.11}$$

where

$$L_{\text{L.G.}} = \frac{1}{2}|\dot{\Phi} - iqA_0\Phi|^2 - \frac{v^2}{2}|\partial_i\Phi - iqA_i\Phi|^2 + i\tilde{a}\eta\varepsilon_{ij}\partial_i A_j\{\Phi^\dagger(\dot{\Phi} - iqA_0\Phi) - \text{c.c.}\}$$

$$+ i\tilde{b}\eta\varepsilon_{ij}(\partial_0 A_i - \partial_i A_0)\{\Phi^\dagger(\partial_j\Phi - iqA_j\Phi) - \text{c.c.}\} + (\kappa\eta^2 - m_0^2)|\Phi|^2 - \lambda|\Phi|^4\,, \tag{6.12}$$

is a modified version of the Landau-Ginzburg Lagrangian considered before, and

$$L_\eta = \frac{1}{2}\dot{\eta}^2 - \frac{\gamma}{2}(\partial_i\eta)^2 + M^2\eta^2 - \Lambda\eta^4\,. \tag{6.13}$$

The Lagrangian is invariant under P and T if η is defined to be P and T odd. Now if η acquires a vacuum expectation value, ($\langle\eta\rangle = \pm\sqrt{M^2/2\Lambda}$), then clearly the modified Landau-Ginzburg Lagrangian takes the same form as the original Landau-Ginzburg Lagrangian. The signs of the coefficients of the P and T violating terms a and b will depend on the sign of the vacuum expectation value of η. Notice that if m_0^2 is positive but

$$\kappa\langle\eta\rangle^2 - m_0^2 > 0 \tag{6.14}$$

then chiral spin order will drive the onset of superconductivity. At the level of the Landau-Ginzburg Lagrangian discussed here the two transitions are in principle quite distinct, however.

Another direction in which the effective Lagrangian can be extended usefully is to take into account the coupling of the superfluid to normal electrons, or vortices. This will be discussed extensively in the following section.

7. The Order Parameter

One of the mysterious features of the RPA treatment of the anyon gas in Ref. 5, and its further elaboration in the present paper, is that the calculation proceeds without exhibiting the superconducting order parameter. One finds a massless pole in the two point function of the electromagnetic current, but the computation that reveals the existence of this pole does not also exhibit a local order parameter analogous to the charge-violating local order parameters familiar in the theory of conventional superconductors. In contrast, in conventional treatments of ordinary superconductors, it would be practically impossible to compute the interesting physical observables without at the same time exhibiting the key order parameter.

We may restate this puzzle in terms of the mechanics of the calculation. In general, we would expect that in constructing a broken symmetry ground state we would have to make some arbitrary choice among a set of energetically degenerate possibilities. Thus for instance in a ferromagnet we would have to choose a definite direction for the magnetization; in a BCS superconductor we would have to choose a phase for the condensate, and so forth. However, in the RPA calculation presented above it is not at all obvious where such a choice has been made. Indeed, if there were a conventional condensate it would necessarily, for large n, be very complicated, for reasons we mentioned in Sec. 3. For it to influence a calculation, the calculation would need to involve high-order correlation functions somewhere along the way. But the computation we actually performed involved only simple correlation functions, with intermediate states.

This mystery of the order parameter is a familiar story in some of the other $2+1$ dimensional systems in which fractional statistics play a role. In particular, there has never been a fully satisfactory description of the relevant order parameter in the fractionally quantized Hall effect — a description, that is, of what is the general class of things of which the celebrated Laughlin wave function is an example. We will unfortunately not be able in this paper to shed much light on the fractional Hall effect, but we hope to clarify the nature of the order parameter in the case of the superfluid anyon gas.

In a way, it is encouraging that the order parameter of the superfluid anyon gas should be rather elusive and somewhat novel. The reasoning that begins with two-dimensional spin models, proceeds (for example, via the mean field theory of Ref. 31) to fractional statistics, and then attempts to derive superconductivity from properties of the anyon gas, is long and indirect. It would be less than satisfying if the output of all this were to be merely a strongly coupled version of BCS theory. The anyon gas as a mechanism for superconductivity is far more interesting if it leads to a new universality class (but see the remarks at the end of Secs. 7.1 and 7.6.)

Of course, spontaneous P and T violation is essential in this circle of ideas, and is absent in usual superconductors. However, there is no problem in having P and T violation coexist with the ordinary superconducting order parameter. The BCS theory could perfectly well be elaborated to describe a system with both spontaneous breakdown of P and T and spontaneous violation of charge conservation. Such a situation actually arises in the conventional description of the A phase of liquid ^{3}He. But we will argue that in the case of the anyon gas, superconductivity does not merely coexist with spontaneous P and T violation; P and T violation are built into the correct description of the order parameter responsible for superconductivity.

7.1. *Sum Rule Argument*

The necessity for the existence of a zero-energy boson-like mode at long wavelengths can actually be demonstrated by a direct argument,[46] which makes no reference to an order parameter or a broken symmetry. Let us define a spectral weight,

$$W(k, \omega) = \sum_l |\langle l|\rho_k|0\rangle|^2 \omega^{-1} \delta(\omega - E_l + E_0), \tag{7.1}$$

where ρ_k is the density operator at wave vector k, $|0\rangle$ is the ground state of the system, and the sum is over all excited states $|l\rangle$, while E_0 and E_l are the respective energy eigenvalues. For a system of non-relativistic particles of mass m, with forces that are velocity-independent, there is a well-known sum rule:

$$\int_0^\infty W(k, \omega)\omega^2 d\omega = \frac{\bar{\rho} k^2}{m}. \tag{7.2}$$

This sum-rule, which is obtained by evaluating the quantity $\langle 0|[[\rho_{-k}, H], \rho_k]|0\rangle$, is easily derived for the anyon system using a representation given below, where the wavefunction is multivalued and the kinetic energy has just the free-particle form. At the same time, we know that

$$\int_0^\infty W(k, \omega) d\omega = \Lambda_{00}(k), \tag{7.3}$$

where $\Lambda_{00}(k)$ is the density response function defined by (5.23) evaluated at $\omega = 0$. The $k \to 0$ limit of this function is the compressibility, which is finite for our system since the ground state energy is an analytic function of $\bar{\rho}$. (For non-interacting anyons, the energy per particle is simply proportional to $\bar{\rho}$.) It follows that the root-mean square value of the energy in the spectral density at wave vector k is given, in the limit $k \to 0$, by,

$$\bar{\omega}_k = v_0 k, \qquad v_0 = (\bar{\rho}/m\Lambda_{00}(0))^{1/2}. \tag{7.4}$$

Now there are two possibilities. The spectral density may be exhausted by a single mode, in which case its frequency must be precisely equal to $v_0 k$. (This is what happens in superfluid ^{4}He, or in a neutral fermion superconductor, such as ^{3}He.) Alternatively, there may be a spread of energies entering the spectral weight at wave vector k. In this case there will be some excitations with energies greater than v_0k, while others must have energy less than v_0k. This is the case in a normal Fermi liquid, where there are particle-hole excitations throughout the interval $0 < \omega < v_F k$, where v_F is the Fermi velocity. For anyons, there is no continuum of particle-hole excitations at low energies, so we are not surprised to find that there is an isolated boson mode, with energy $\omega = v_0 k$.

The very generality of the sum rule argument means that it provides only a limited amount of insight about the properties of a particular system. For more insight, one might try to find a conventional order parameter for the system. Specifically, we would like to find an operator $\Psi(\mathbf{r})$ which reduces the charge in the vicinity of the point $\mathbf{r}$ by n units, and which has the property that for large separations $|\mathbf{r}-\mathbf{r}'|$, the correlation function $\langle 0|\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r})|0\rangle$ approaches a finite constant, or at worst falls off as a power of $|\mathbf{r}-\mathbf{r}'|$. Operators that satisfy these requirements can possibly be constructed in direct analogy with the order parameters employed recently to describe the quantized Hall effect.[47-49] These operators are highly non-local, however, at least when they are expressed in terms of anyon operators in the fermion representation used above. In fact, we shall argue below that there can be no superfluid order parameter of the conventional type for this system that is local in this representation.

The fact that an operator $\Psi(\mathbf{r})$ is non-local in terms of the anyon operators does not necessarily imply that it is non-local in the underlying electron operators, when applied to a solid state system on a lattice. To investigate this question ultimately we must refer to the specific microscopic model from which the anyons were derived. We shall discuss elsewhere some insight into this issue that can be derived from general symmetry properties.[34] Preliminary results of this analysis suggest that the symmetry of the order parameter $\Psi(\mathbf{r})$ for a system containing two kinds of anyons with half-Fermi statistics ($\theta = \pi/2$) is compatible with the symmetry of a Cooper pair of electrons in a spin-singlet state.

7.2. *Translation Invariance of the Underlying System*

It is instructive to begin by considering some elementary facts about the spin models that can be considered to underlie the anyon gas. In these models, one has a system of quantum spins arranged on a two-dimensional lattice L. The total Hamiltonian H is a sum over lattice sites $a \in L$ of a local Hamiltonian density $\mathscr{H}_a$:

$$H = \sum_{a \in L} \mathscr{H}_a . \tag{7.5}$$

The density $\mathscr{H}_a$ is constructed from the spins at the site a and their close neighbors. The construction of the $\mathscr{H}_a$ is translation invariant. This means that if

$\mathbf{e}_1$ and $\mathbf{e}_2$ are elementary lattice vectors, then the operators T_1 and T_2 that translate the spins one step in the $\mathbf{e}_1$ or $\mathbf{e}_2$ directions commute with the Hamiltonian:

$$[T_1, H] = [T_2, H] = 0. \tag{7.6}$$

In addition, of course, they commute with one another,

$$[T_1, T_2] = 0. \tag{7.7}$$

Continuous Translation Symmetry

Although the spin models (like most condensed matter systems) possess only discrete translation invariance, the anyon gas which is conjectured to give an approximate description of a system of electrons interacting with a suitable spin model is a system with continuous translational symmetries. The translation generators of the anyon gas are the momentum operators. The anyon gas can be described in a variety of mathematical formalisms. Each formalism leads to a different description of the momentum operators P_i and the Hamiltonian H. In any formalism, these obey the fundamental microscopic relations

$$[H, P_i] = 0, \tag{7.8}$$

and

$$[P_i, P_j] = 0. \tag{7.9}$$

At the risk of belaboring the obvious, we will review the definition of appropriate operators H and P_i obeying (7.8) and (7.9) in several possible formalisms.

To begin with, one can treat the anyons as a gas of N particles with position operators $\mathbf{x}_\alpha$, $\alpha = 1, \ldots, N$ and a wave function $\psi(x_1, \ldots, x_N)$ that is multivalued and changes by a factor of $\exp(2\pi i/n)$ when one particle loops around another. In this formalism, H and P_i are defined by the familiar free particle formulas

$$H = \sum_\alpha \left(-\frac{1}{2m} \frac{d^2}{d(x_\alpha{}^i)^2} \right) \tag{7.10}$$

and

$$P_i = \sum_\alpha -i \frac{d}{dx_\alpha{}^i}. \tag{7.11}$$

Clearly, (7.8) and (7.9) are obeyed.

Alternatively, if one wishes to work with ordinary single-valued wave functions, then the replacement

$$\psi(x_1, \ldots, x_N) \to \prod_{\alpha<\beta} (z_\alpha - z_\beta)^{1/n} \psi'(x_1, \ldots, x_N) \tag{7.12}$$

(where $z_\alpha = x_\alpha^{\ 1} + ix_\alpha^{\ 2}$) permits us to replace ψ with a single-valued wave function ψ'. As a result, one gets

$$H = \sum_\alpha \left(-\frac{1}{2m} \frac{D^2}{D(x_\alpha^{\ i})^2} \right) \tag{7.13}$$

and

$$P_i = \sum_\alpha -i \frac{D}{Dx_\alpha^{\ i}}. \tag{7.14}$$

Here the covariant derivatives are defined by

$$\frac{D}{Dx_\alpha^i} = \frac{d}{dx_\alpha^{\ i}} + ia_{\alpha i}, \tag{7.15}$$

with the effective vector potential seen by particle α being

$$a_{\alpha i} = \frac{1}{n} \sum_{\beta \neq \alpha} \frac{\varepsilon_{ij}(x_{\alpha j} - x_{\beta j})}{|x_\alpha - x_\beta|^2}. \tag{7.16}$$

Obviously, (7.8) and (7.9) are still obeyed, since we have merely made the redefinition (7.12).

Finally, one can derive the anyon gas in a second quantized formalism from the Chern-Simons Lagrangian

$$\mathscr{L} = \frac{n}{4\pi} \int dtd^2x\,(\varepsilon^{ijk} a_i \partial_j a_k) + \int dtd^2x \left(\psi^* i \frac{D}{Dt} \psi - \frac{1}{2m} D_k \psi^* D_k \psi \right). \tag{7.17}$$

Here ψ is a second quantized "electron" field. It is known[1] that the system obtained by quantizing (7.17) is a system of particles (conserved in number) that interact only via the statistical interaction of the anyon gas. The conserved particle number is

$$Q = \int d^2x J_0 = \int d^2x \psi^* \psi. \tag{7.18}$$

Conservation of the particle number follows from the current conservation law

$$\partial_0 J^0 + \partial_i J^i = 0, \tag{7.19}$$

where

$$J_i = -\frac{i}{2m}(\psi^* D_i \psi - (D_i \psi^*)\psi). \tag{7.20}$$

The Hamiltonian and momentum operators derived from (7.17) are

$$H = \int d^2x \, T_{00}, \tag{7.21}$$

and

$$P_i = \int d^2x \, T_{0i}, \tag{7.22}$$

where the energy density is

$$T_{00} = \frac{1}{2m} D_i \psi^* D_i \psi, \tag{7.23}$$

and the momentum density is

$$T_{0i} = \frac{-i}{2}(\psi^* D_i \psi - (D_i \psi^*)\psi). \tag{7.24}$$

The equivalence with the particle description of the anyon gas ensures that the P_i commute with each other and with H. This can be directly verified in the second quantized description using the commutation relations

$$\{\psi^*(x), \psi(y)\} = \delta(x-y), \qquad [A_i(x), A_j(y)] = -\frac{2\pi i}{n}\varepsilon_{ij}\delta(x-y) \tag{7.25}$$

and the Gauss law constraint

$$f_{ij}(x) = \varepsilon_{ij}\frac{2\pi}{n}\psi^*\psi(x), \tag{7.26}$$

where $f_{ij} = \partial_i a_j - \partial_j a_i$.

Before leaving this subject let us note the amusing fact (visible in the above formulas) that the particle current and the momentum density of the anyon gas are equal:

$$T_{0i} = mJ_i\,. \tag{7.27}$$

This reflects the fact that at the microscopic level, the system is invariant under Galilean transformation and all particles have a common charge to mass ratio. We will later have use for this fact.

In summary, the anyon gas, in any mathematical formalism, has at the microscopic level a Hamiltonian H and momentum operators P_i that obey the basic relations (7.8) and (7.9). The following discussion will focus on trying to understand how those properties are realized macroscopically. Then, since in realistic superconductors the continuous translation symmetries are broken down to discrete translation symmetries by the presence of a lattice, we will consider the more realistic case of (7.6) and (7.7) with discrete translational symmetries only.

7.3. *Macroscopic Realization of Translation Invariance*

The question now arises of how the symmetries we have just surveyed are realized macroscopically, at the level of the physical excitations of the system.

It is a familiar story in condensed matter and particle physics that a symmetry of the microphysics is not necessarily manifested as a symmetry of the macroscopic physics. An underlying symmetry that does not leave invariant the vacuum state is "spontaneously broken". Spontaneous breaking of a continuous symmetry leads to the existence of a massless mode which in particle physics is called a Goldstone boson. Spontaneous symmetry breaking is the key to most modern understanding of superfluids, and has offered such a fruitful perspective for understanding superfluids that one tends to assume that it has universal applicability.

We would now like to claim, however, that the key concept for understanding the superfluidity of the anyon gas is not really spontaneous breaking of a symmetry but what might be called spontaneous violation of a fact. The fact that is spontaneously violated is the fact that the momentum generators commute. While microscopically

$$[P_i, P_j] = 0, \tag{7.28}$$

macroscopically, at the level of quasi-particles, one obtains

$$[P_i, P_j] = if \cdot \varepsilon_{ij} Q, \tag{7.29}$$

where Q is the particle number and f is a constant which we would like to interpret as the fundamental order parameter of the anyon gas. From $f \neq 0$, we will deduce the existence of a massless mode. This will be our explanation of the mode first uncovered in Ref. 5, the mode that is responsible for the superfluidity of the anyon gas.

Axiomatically, this mode can be interpreted as a Goldstone boson, since it appears as a pole in the two point function of the electromagnetic current, as was already seen in Ref. 5. In this interpretation, the existence of this mode is rather mysterious, since it seems (and it will be argued later) that there is no local order parameter that would naturally explain the existence of a Goldstone boson. We believe that the crucial massless mode does have a natural explanation as a consequence of spontaneous violation of the fact that P_i and P_j commute. Its role as a Goldstone boson (appearing as a pole in the two point function of the electromagnetic current) can then be deduced as a corollary.

7.4. *Plane Waves and Landau Levels*

It is easy to see why (7.29) is true. If the translation generators P_i are conserved and commute, it must be possible to take the quasi-particle excitations to be momentum eigenstates. This is what is most definitely not possible in the perturbative calculations that we have been pursuing. The charged quasi-particles in those calculations are not in plane wave states but in Landau levels. It is precisely because the quasi-particle states are not plane waves that the perturbative computations are difficult.

That the quasi-particle states are not plane waves could be well understood, of course, if translation invariance were spontaneously broken — if the P_i did not annihilate the vacuum. This is not the case here, however. It is because of the interaction with a non-zero expectation value of the fictitious magnetic field $f = \frac{1}{2}\varepsilon^{ij}(\partial_i a_j - \partial_j a_i)$ that the charged quasiparticles are not plane waves. Because we take f to be translation invariant, this background is translation invariant, and conservation of the P_i is not spontaneously broken. However, in a magnetic field, the translation generators do not commute, so the nonzero expectation value of f results in a spontaneous violation of the commutation relation $[P_i, P_j] = 0$.

We can make this somewhat more precise. Consider, first of all, a single particle moving in a constant magnetic field. The one particle Hamiltonian is

$$H = -\frac{1}{2m}\sum_k D^{(0)2}_k, \tag{7.30}$$

where the covariant derivatives $D^{(0)}_i$ obey

$$[D^{(0)}{}_i, D^{(0)}{}_j] = i\varepsilon_{ij} f. \tag{7.31}$$

The superscript "(0)" is meant to indicate that we are considering the interaction with a fixed vector potential; the gauge field is not dynamical. It is important to realize that the translation generators are not simply the covariant derivatives $D^{(0)}{}_i$; these do not commute with the Hamiltonian. Rather, the conserved translation operators are

$$P_i = -iD^{(0)}{}_i + f\varepsilon_{ij}x^j; \tag{7.32}$$

these are easily seen to commute with H. They do not commute with each other, however, but obey

$$[P_i, P_j] = if\varepsilon_{ij}. \tag{7.33}$$

To express the same thing in a second quantized language, recall first that in studying the anyon gas, one finds, in lowest order in $1/n$, an expectation value of the fictitious magnetic field f, and the following "obvious" elementary excitations: quasiparticles that can occupy all Landau levels but the first n, and quasiholes that can fill any state in the first n Landau levels. What must be explained is why one finds in addition one more type of elementary excitation, namely the massless boson. The "obvious" elementary excitations can be represented by an effective fermion field χ with a Lagrangian

$$\mathscr{L} = \int dt d^2x \left(\chi^* i \frac{D^{(0)}}{Dt} \chi - \frac{1}{2m} D_k^{(0)} \chi^* D_k^{(0)} \chi \right). \tag{7.34}$$

The gauge field is no longer dynamical; and instead of the elementary fermion field ψ, we use a quasiparticle field χ to emphasize that (7.34) is meant to be not a microscopic Lagrangian but (a piece of) a phenomenological Lagrangian in which as much as possible of the relevant physics is visible at tree level.

Can (7.34) be the whole of such a phenomenological Lagrangian? To investigate this, we examine the realization of translation invariance. The Hamiltonian derived from (7.34) is

$$H = \frac{1}{2m} \int d^2x D_k^{(0)} \chi^* D_k^{(0)} \chi. \tag{7.35}$$

It may not be immediately obvious what the translation generators can be, but by virtue of the single particle result (7.32) one can see that the operators that generate translations and commute with H are

$$P^{(\chi)}{}_i = \int d^2x\,(\chi^*(-iD_i^{(0)})\chi + f\varepsilon_{ij}x^j\chi^*\chi)\,. \tag{7.36}$$

The quasiparticles and quasiholes appearing in (7.35) cannot be the whole story because the translation generators in (7.36) do not commute; they obey a relation

$$[P_i^{(\chi)}, P_j^{(\chi)}] = if\varepsilon_{ij}Q \tag{7.37}$$

where

$$Q = \int d^2x\,\chi^*\chi \tag{7.38}$$

is the conserved charge operator. (7.37) is the second-quantized version of the single particle result (7.33) (being a single particle result, (7.33) effectively corresponds to the sector $Q = 1$).

Now we can see that the quasiparticles and quasiholes that are visible in lowest order in $1/n$ cannot be the whole story. At a microscopic level the translation generators of the anyon gas commute, as we emphasized in the last subsection. But the translation generators of the phenomenological model (7.34) do not commute. Something must be done to correct this discrepancy between the microphysics and the putative macroscopic realization in (7.34).

There is another, closely related reason that (7.34) cannot be the whole story. In the underlying microscopic anyon gas, the translation generators P_i are the integrals of intrinsically defined local densities T_{0i}, for which a formula was given in (7.24). In the macroscopic model (7.34) this is not true. The translation generators can be written, as in (7.36), as the integrals of local densities, but because of the "x^j" in (7.36), the definition of these local densities does not depend only on the intrinsic local physics, but also depends on the arbitrary choice of an origin of coordinates.

This second version of the problem, though it may sound more abstract, is in a way a more powerful formulation, since this version of the difficulty is relevant to the sector of $Q = 0$ as well as to the charged sectors.

7.5. *Restoring Commutativity of the Translation Generators*

We will now see that if, in addition to the quasiparticles and quasiholes described in (7.34), we assume the existence of an additional spin zero massless boson, the above-cited problems can be repaired. This massless boson is analogous to a Goldstone boson, since its role is to correct for a discrepancy between the microscopic properties of a system and the macroscopic realization.

However, while a Goldstone boson is tied to the violation of a symmetry, the massless boson present in this problem is tied to the violation of a fact — the fact that $[P_i, P_j] = 0$.

Of course, we cannot *prove* on grounds such as these that a spin zero massless boson must exist. There would be other logical possibilities, notably the possibility that the approximation leading to the excitations that appear in (7.34) is wrong even for large n. The best that we can say is that *if* one postulates the existence of the excitations in (7.34), *then* this creates problems that can be cured by the additional existence of a massless boson with certain properties.

The obvious way to represent a spinless massless boson by a quantum field is to consider a scalar field ϕ with Lagrangian

$$L_\phi = \int dt\, d^2x \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{v^2}{2}(\partial_i\phi)^2\right). \tag{7.39}$$

Here v is the velocity of propagation of the massless boson. While the description (7.39) of a massless boson is possible in any dimension, in 2+1 dimensions there is another possibility. One can represent a massless spinless boson by an abelian gauge field b_i with field strength $h_{ij} = \partial_i b_j - \partial_j b_i$. The Lagrangian for b_i is

$$L_b = \frac{1}{2}\int dt\, d^2x \left(\sum_i h_{0i}{}^2 - v^2 h_{12}^2\right). \tag{7.40}$$

The equivalence between the two descriptions is made by the change of variables

$$\partial_0\phi = h_{12}, \qquad v^2\partial_i\phi = \varepsilon_{ij}h_{0j}. \tag{7.41}$$

For our present purposes, it is most convenient to first consider the description in terms of b_i. The conventional translation generator of an abelian gauge field is

$$P_i = \int d^2x\, T^{(b)}{}_{0i}, \tag{7.42}$$

where the conventional form of the momentum density is

$$T^{(b)}_{0i} = -\sum_j h_{0j}h_{ij}. \tag{7.43}$$

This leads to the standard result $[P_i, P_j] = 0$. But if one adds an additional term to the momentum density, taking

$$T^{(b)}_{0i} = -\sum_j h_{0j}h_{ij} + \varepsilon_{ij} f h_{0j}, \tag{7.44}$$

then a short computation, using the canonical commutation relations derived from (7.44), gives the result

$$[P^{(b)}{}_i, P^{(b)}{}_j] = if\varepsilon_{ij} \int d^2x\, (\partial_k h_{0k}) = if\varepsilon_{ij} \oint dl\, n^k h_{0k}, \tag{7.45}$$

where the integral is over a large circle at infinity, and n^k is the normal vector to this circle.

Evidently, therefore, if we combine the χ and b systems, and form the total momentum operators $P_i = P^{(\chi)}{}_i + P^{(b)}{}_i$, then

$$[P_i, P_j] = if\varepsilon_{ij}\left(Q - \oint dl\, n^k h_{0k}\right). \tag{7.46}$$

Thus, all is well if we restrict ourselves to the subspace of Hilbert space for which

$$Q - \oint dl\, n^k h_{0k} = 0. \tag{7.47}$$

If we take the free Lagrangians (7.34) and (7.40) literally, then the Gauss' law constraint (the equation of motion obtained by varying with respect to b_0) would give $\partial_k h_{0k} = 0$ and therefore $\oint dl\, n^k h_{0k} = 0$, in contradiction to the desired result (7.47). To obtain (7.47) (and at the same time give this condition a physical interpretation), we must modify the free Lagrangians by adding a suitable term coupling b to χ. This term means that χ is a charged field interacting with the dynamical gauge field b. The requisite Lagrangian is

$$L = \frac{1}{2}\int dt d^2x\, ({h_{0i}}^2 - v^2 h_{12}^2) + \int dt d^2x \left(\chi^* i\frac{D}{Dt}\chi - \frac{1}{2m} D_k\chi^* D_k\chi\right), \tag{7.48}$$

where now $D_i = \partial_i + ib_i$, and b_i is a dynamical gauge field. It is now easy to see what is the role of b_i in restoring the commutativity of the translations. Indeed, (7.48) is a perfectly normal Lagrangian with commuting translation generators. By expanding around a constant expectation value f of the "magnetic field" $\partial_1 b_2 - \partial_2 b_1$, one will find the χ excitations to be Landau orbits with apparent violation of commutativity of translations. But this phenomenon is "spontaneous", simply reflecting the non-zero expectation value of h. Because the Lagrangian (7.48) has "normal" translation invariance, the translation operators will commute regardless of what background one expands around, when one includes all contributions to these operators.

The way that commutativity of translations is realized in Eq. (7.46) is very similar to the way that relations expressing an underlying symmetry are usually realized in systems with spontaneously broken symmetry. The translation generators commute — but only if one takes into account surface terms involving massless particles. The local measurement of the motion of the χ quasiparticles sees broken commutativity of the translations.

Finally, we can now readily resolve the other difficulty noted in connection with Eq. (7.36), which was that the quasiparticle momentum density could not be written as the integral of a local density that could be defined in a natural way. The offending term can be rewritten by using the Gauss law constraint and integration by parts:

$$\int d^2x f\varepsilon_{ij}x^j\chi^*\chi = \int d^2x f\varepsilon_{ij}x^j\partial_k h_{0k}$$

$$= -f\int d^2x\varepsilon_{ik}h_{0k}. \tag{7.49}$$

Thus, the offending term in (7.36) is actually equivalent, using Gauss's law, to the integral of a local density which is naturally defined, since it does not contain any explicit factors of "x^i".

Let us now briefly consider the formulation in which b_i is replaced by an equivalent scalar boson ϕ. To restore translation invariance, one must include in the stress tensor $T^{(\phi)}$ a non-minimal term similar to the one required in $T^{(b)}$, or specifically

$$T^{(\phi)}_{0i} = -\partial_0\phi\partial_i\phi + f\partial_i\phi. \tag{7.50}$$

The surface integrals transform as follows:

$$\oint dl n^k h_{0k} = v^2\oint dl^i\partial_i\phi. \tag{7.51}$$

The operator on the right-hand side of (7.51) is usually called the vortex number Φ,

$$\Phi = v^2 \oint dl^i \partial_i \phi \tag{7.52}$$

There is a subtlety here. In any state in which $\Phi \neq 0$, the scalar field ϕ must have a singularity somewhere in space. Thus, the description in terms of a scalar field ϕ is only adequate at infinity; usually, of course, the situation is repaired by interpreting ϕ as the argument of a complex-valued scalar field that may have zeros.

In terms of ϕ, the fundamental relation (7.47) says that

$$Q = \Phi. \tag{7.53}$$

In other words, the charged excitations must have vorticity. We first encountered this result in another way in Sec. 4, where it appeared in the opposite fashion: there it was more natural to say that we had learned that the vortices must carry charge.

Since we know that the eigenvalues of Q are arbitrary integers, we know the allowed values of vorticity; and we can say, in particular, that ϕ should be regarded as a periodic field with periodicity equal to $1/2\pi v^2$.

Incidentally, although this point possibly should be addressed with greater care, it would appear that the fluxons (or χ quasiparticles) obeying (7.53) are ordinary fermions with $\theta = \pi$, at least if n is large enough so that the discussion is valid. In fact, the gauge field b_i in the effective Lagrangian (7.48) has no Chern-Simons term (otherwise it would get a mass and the surface term needed to make sense of the situation would vanish). This being so, the statistics of the χ particles is unshifted from its free field value of $\theta = \pi$.

7.6. *Superfluidity Without Charge Violation*

We have now learned that the system under study must — given the existence of the χ quasiparticles — have a massless boson ϕ. What is more, in view of the term $T^{(\phi)}{}_{0i} \sim f\partial_i\phi$ in (7.50), this boson must appear as a pole in the two point function of the momentum density T_{0i}. We would now like to know, however, whether the system is superfluid in the usual sense, or in other words whether this boson appears as a pole in the two point function of the current density J_i.

For the simple anyon gas, this question can be answered quickly. Because of the microscopic formula $T_{0i} = mJ_i$ (Eq. (7.27)) a massless boson that appears as a pole in T_{0i} must also appear as a pole in J_i.

Of course, a realistic two-dimensional CuO plane will not be described precisely by the simple anyon gas. At best the latter is an approximation of some kind. However, once we know that the idealized anyon gas has a current density obeying

$$mJ_i \sim f\partial_i\phi, \tag{7.54}$$

so that ϕ appears as a pole in the two point function of J_i, this situation cannot be ruined if one makes a generic small perturbation in the Hamiltonian. At most such a generic small perturbation would change the value of the coefficient f in (7.54).

This argument for why there must be a massless pole in the two point function of the current should seem peculiar. Usually such a result is deduced as a corollary of the existence of spontaneous breakdown of conservation of electric charge. The latter in turn is related to the existence of a non-vanishing expectation value for some charge-bearing local observable. In the present discussion, we have instead understood superfluidity as a consequence of the fact that the commutativity of translations is spontaneously modified to a phenomenological law (ignoring massless boson contributions)

$$[P_i, P_j] = if\varepsilon_{ij}Q. \tag{7.55}$$

We would like to regard the parameter f that appears here as the fundamental order parameter characterizing the superfluid anyon gas. Note that (as the ε_{ij} symbol in (7.55) is P violating and the complex number i is odd under T), P and T violation are prerequisites for the ability even to define this order parameter. The order parameter is odd under P and T but conserves PT — as it must, since the anyon gas from this order parameter is derived is PT invariant.

One might wonder whether in fact there is a local order parameter of the usual kind — an expectation value of a charged observable — which we have merely overlooked. There is, however, a strong argument that this is not the case. This follows from the fundamental relation

$$Q = \Phi. \tag{7.56}$$

(It must be understood that Q here is the quasiparticle number, not the total electric charge which includes a supercurrent contribution.) In a two-dimensional superconductor of any kind, the number Φ of vortices is absolutely conserved (since it can be measured as a surface integral at infinity, which cannot change, or since after coupling to dynamical electromagnetism it can be identified with the physical magnetic flux). Therefore, (7.56) means that in the two-dimensional superconductors under discussion, Q must also be conserved.

To say this differently, in a two-dimensional BCS superconductor, one begins microscopically with a law of conservation of Q; this conservation law is then lost (spontaneously broken). One also usually generates spontaneously a conservation law — conservation of vortex number. The total number of conservation laws is unchanged, but the conservation law lost is different from the conservation law gained.

In the case at hand, the fundamental relation $Q = \Phi$ means that the law of conservation of Q is not broken; it is just reinterpreted macroscopically as the law of conservation of Φ. Thus, it must be that a local order parameter of the usual kind (which would trigger non-conservation of Q) does not exist.

For applications to realistic superconductors, we must consider the interaction of the anyon gas with a crystal lattice. We then need not realize arbitrary translations, but only those consistent with the symmetry of the lattice. The essential feature of the anyon gases that concern us, namely that when one anyon winds around another the phase of the amplitude changes by $\pi(1 - 1/n)$, then need not be implemented by a $U(1)$ gauge field — a Z_n gauge field should suffice. The anyons will have unit charge with respect to this gauge field.

The analysis then proceeds in complete analogy to the continuum case, leading to the spontaneous violation of the commutation relations for quasiparticle translations, and their dynamical restoration through vortex coupling to a massless field. Notice that both charge and vorticity are both defined as integers modulo n, so that the fundamental relation $Q = \Phi$, central for the whole mechanism, makes sense.

However, there is a crucial difference between the lattice and continuum models, arising from the fact that for a Z_n gauge theory charge and vorticity are defined only modulo n. It is, that in the lattice model the product of n anyon fields is a fictitious gauge singlet. It could therefore conceivably acquire a vacuum expectation value, without breaking the crucial fictitious gauge invariance. This would be an order parameter of the more familiar kind for a superconductor — an expectation value of a field of charge n.

Also, such a product could well couple to the product of an equal number of copies of (the complex conjugate of) the elementary electron field, in a fully real and fictitious gauge invariant manner. Such a coupling would allow Josephson coupling between anyon and usual superconductors

7.7. *Consequences of the Relation between Current and Momentum*

The simple anyon gas is a Galilean invariant system in which all particles have the same charge to mass ratio. This fact is conveniently expressed in the relation

$$mJ_i = T_{0i} \tag{7.57}$$

that we have discussed earlier. This relation, together with current conservation, has very strong implications for the low energy interactions of the massless bosons. It is these implications that we wish to consider in this subsection. The properties that we will deduce are exact properties of the low energy physics obtained from the anyon gas as long as there are no massless particles other than the ϕ field that we will consider. These properties will not hold exactly in realistic superconductors, where there is not a simple relation between J_i and T_{0i}.

In treating this problem, we will use the formulation in which the massless boson is represented by a scalar field ϕ. To begin with, at the level of precision that we have considered so far, the ϕ field is described by a quadratic Lagrangian,

$$\mathscr{L}_\phi = \int dt d^2x \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{v^2}{2}(\partial_i\phi)^2\right), \tag{7.58}$$

which leads to the equation of motion

$$\frac{\partial^2}{\partial t^2}\phi - v^2 \frac{\partial^2}{\partial(x^i)^2}\phi = 0. \tag{7.59}$$

The current and momentum density are

$$mJ_i^{(\phi)} = T_{0i}^{(\phi)} = f\partial_i\phi + \partial_0\phi\partial_i\phi. \tag{7.60}$$

Now we wish to ask whether there is a charge density $J_0^{(\phi)}$ such that the current conservation law

$$\partial_0 J_0 + \partial_i J_i = 0 \tag{7.61}$$

is obeyed. For a first orientation to the problem, ignore the term in (7.60) that is quadratic in ϕ. One may readily see that in this approximation, the charge density that works is

$$mJ_0 = \frac{f}{v^2}\partial_0\phi. \tag{7.62}$$

In verifying (7.61), one uses the equation of motion (7.59).

Now we wish to include the term of order $(\nabla\phi)^2$ in J_i. We have to assume that there might be a term of order $(\nabla\phi)^2$ in J_0 as well, so now

$$mJ_0 = -\frac{f}{v^2}\partial_0\phi + Y, \tag{7.63}$$

where Y is to be quadratic in ϕ. In addition, we have to assume that there might be a term of order $(\nabla\phi)^2$ in the equation of motion, so this will now be

$$\frac{\partial^2}{\partial t^2}\phi - v^2 \frac{\partial^2}{\partial(x^k)^2}\phi + Z = 0, \tag{7.64}$$

where again we assume that Z is of order $(\nabla\phi)^2$.

Current conservation now gives the statement

$$0 = \frac{f}{v^2} Z + \partial_0 Y + \partial_i(\partial_0\phi\partial_i\phi). \tag{7.65}$$

Though (7.65) by itself does not uniquely determine Y and Z, these are uniquely determined when one requires that the correction Z to the free equation of motion must be derivable from a Lagrangian. One finds that the Lagrangian must be corrected to

$$\mathcal{L}_\phi = \int dt d^2x \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{v^2}{2}(\partial_i\phi)^2 - \frac{v^2}{2f}\partial_0\phi \cdot (\partial_k\phi)^2\right), \tag{7.66}$$

and the charge density must be

$$mJ_0 = -\frac{f}{v^2}\partial_0\phi + \frac{1}{2}(\partial_i\phi)^2. \tag{7.67}$$

This is not the end of the story, because the term that we have added to the Lagrangian results in an addition to the momentum density. (The extra term in the Lagrangian results in a modification of the canonical commutation relations, and as a result the objects $\int d^2xT^{(\phi)}{}_{0i}$, with $T^{(\phi)}{}_{0i}$ as defined in (7.60), no longer generate translations.) One can now take the momentum density to be

$$T^{(\phi)}_{0i} = f\partial_i\phi + \partial_0\phi\partial_i\phi - \frac{v^2}{2f}\partial_i\phi \cdot (\partial_k\phi)^2. \tag{7.68}$$

However, the current J_i derived from (7.66) is unchanged from (7.60). Therefore, (7.66) does not lead to the desired equality of current and momentum density. To save the day, it is necessary to add a term of order $(\nabla\phi)^4$ to the Lagrangian, which now becomes

$$\mathcal{L}_\phi = \int dt d^2x \left(\frac{1}{2}(\partial_0\phi)^2 - \frac{v^2}{2}(\partial_i\phi)^2 - \frac{v^2}{2f}\partial_0\phi \cdot (\partial_k\phi)^2 \right.$$
$$\left. + \frac{1}{8}\left(\frac{v^2}{f}\right)^2 (\partial_i\phi)^2(\partial_k\phi)^2\right). \tag{7.69}$$

It is now possible to assume that J_i is given by $1/m$ times the formula in (7.68); there is no modification of J_0. This latest addition to the Lagrangian does not require any further modification of T_{0i} (since it does not bring about a change in the canonical commutation relations), and therefore (7.69) is compatible with the underlying fact that all particles have the same charge to mass ratio.

A more systematic treatment of the consequences of $T_{0i} = mJ_i$ will be given separately elsewhere.[50] It can be used to explain the vanishing of the b term in the anyon model discussed above, and to derive some interesting, though from a modern perspective rather peculiar-looking, relations proposed by London.[51] London's original motivation for his proposals was a pre-BCS, hydrodynamic picture of superconductivity. Let us emphasize again that we do not expect these relations (or $b = 0$) to be exact in real materials, though they can be exact consequences of highly non-trivial models.

8. Phenomenology

In this section we will discuss three distinctive phenomenological effects associated with anyon superconductivity. Our emphasis will be on effects that can be motivated directly within the framework of the models discussed above. Other aspects of possible P and T violating phenomenology are discussed in Ref. 34.

The basis for our discussion will be the effective Lagrangian (6.3). We will mainly consider the most naive extension to three dimensions, in which ϕ is taken to be independent of the direction $\hat{z}$ perpendicular to the plane. In the Maxwell Lagrangian the interplanar spacing s then appears as a multiplicative factor:

$$L_{\mathrm{Maxwell}} = \frac{s}{8\pi} \int d^2x\, dt (E^2 - c^2 B^2). \tag{8.1}$$

Implicit in this framework is the assumption that the couplings a and b are constant; and in particular that they do not change sign from plane to plane. Even if the basic idea of anyon superconductivity does apply to the actual high-T_c materials, and even if the energetics favors alignment of the sense of P and T violation in neighboring planes, it is unlikely that such alignment can persist through a bulk sample. For this reason among others, we cannot attempt to give a complete or quantitative discussion of possible experiments at this time. However, we can indicate what appear to be some promising directions, and to point out some surprising qualitative aspects of the suggested phenomenology.

1) *Charge inhomogeneities around vortices*

As we have emphasized repeatedly, in anyon superconductivity the charged excitations are intimately related to vortices; indeed, in a strong sense they are identical. In the absence of screening the vortices would be electrically charged. Unfortunately in bulk the electric screening length is

$$\ell = \frac{1}{C}\sqrt{\frac{\hbar s}{4\pi}} = \left(\frac{s\hbar}{2\alpha mc}\right)^{1/2} \tag{8.2}$$

and if we take as typical values $s = 20$ Å and m the electron mass, this is only 3 Å. Of couse this value, extracted from our simple London Lagrangian, is not really applicable in the neighborhood of a vortex, but it should indicate the correct order of magnitude. It might be possible to observe even such small-scale inhomogeneities by scanning tunnelling microscopy. The screening is of course less effective for thin films.

We expect transverse voltage to accompany current gradients generally, as we shall discuss in some detail immediately below. In this sense, the charge inhomogeneity associated with a vortex should extend at least over the region where there are significant current gradients, i.e. over a coherence length.

2) *Zero-field Hall effect*

We have just argued that there is a charge inhomogeneity associated with vortices; naturally this implies a potential difference between the center of the vortex and infinity. Now a vortex is in some sense a small circulating current, and we can imagine straightening it out. This leads us to suspect that there will in general be an electric field transverse to current flow: a sort of Hall effect, but persisting in zero external field. The existence of such an effect would of course be direct evidence for P and T violation.

The simplest case to analyze is the flow of small currents in a semi-infinite bulk sample. For sufficiently small currents will be in the Meissner regime, with no vortices. Then the London Lagrangian is adequate, and we find the Maxwell-London equations:

$$-CA_0 + v^2 C\partial_i A_i + (a-b)\varepsilon_{ij}(\partial_i A_j) = 0, \tag{8.3}$$

$$\frac{1}{4\pi s}\partial_i(\partial_0 A_i - \partial_i A_0) = -C^2 A_0 + (a+b)C\varepsilon_{ij}\partial_i A_j, \tag{8.4}$$

$$\frac{1}{4\pi s}\left(c^2\partial_\ell(\partial_\ell A_i - \partial_\ell A_\ell) - \partial_0(\partial_0 A_i - \partial_i A_0)\right)$$

$$= v^2 C^2 A_i + Ca\varepsilon_{i\ell}\partial_\ell A_0 - Cb\varepsilon_{i\ell}(\partial_0 A_\ell - \partial_\ell A_0) - Cb\varepsilon_{ij}\partial_0 A_j. \tag{8.5}$$

To get oriented, let us first consider the situation with $a = b = 0$. Trying the *ansatz*

$$A_0 = f(x);\quad \rho = C^2 f, \tag{8.6}$$

$$A_y = g(x);\quad j_y = -C^2 v^2 g, \tag{8.7}$$

(other components zero) appropriate to a sample with $x \geq 0$ we find two solutions:

$$\begin{aligned}
&(\mathrm{I})\ f(x) = e^{-x/\ell_1},\quad g = 0;\quad \ell_1^2 = \frac{s}{4\pi C^2}\\
&(\mathrm{II})\ f = 0,\quad g(x) = e^{-x/\ell_2};\quad \ell_2^2 = \frac{c^2}{v^2}\frac{s}{4\pi C^2}.
\end{aligned} \tag{8.8}$$

The first corresponds to expulsion of an electric field; it contains an inhomogeneous charge density but no current. The second corresponds to expulsion of a magnetic field; it contains a current but no variation in charge density. Clearly the a and b terms will mix these modes. Regarding a and b as small, we can solve for the asymptotic charge distribution using the zeroth order "magnetic expulsion" solution; thus starting from (II) and perturbing (I) we find:

$$\begin{aligned}
&A_y = e^{-x/\ell_2} \Rightarrow j_y = C^2 v^2 e^{-x/\ell_2}\\
&\delta A_0 = V e^{-x/\ell_2}\\
&V \cong \frac{a+b}{C\ell_2}.
\end{aligned} \tag{8.9}$$

The relation between the potential drop and total current in the asymptotic region is given by:

$$\frac{\delta A_0}{j} = \frac{a+b}{C\ell_2}\frac{1}{C^2 v^2}. \tag{8.10}$$

Charge accumulates at the surface, to compensate the charge accumulated in the asymptotic region. However, it is not difficult to see that the dominant contribution to the voltage drop comes from the asymptotic region. Taking a and b from the microscopic theory and the numerical values as before, we find for the ratio of potential to current:

$$\frac{V}{\int j} = \frac{a+b}{C\ell_2^2}\frac{1}{C^2 v^2} = \frac{4\pi(a+b)}{sc^2 C}. \tag{8.11}$$

In this equation the current is the current *per layer*; for many layers the current adds while the voltage stays the same. For $n = 2$ and the parameter values mentioned above we have:

$$\frac{V}{\int j} = \frac{\hbar \pi n}{smc^2} = .03 \text{ volt/amp.} \tag{8.12}$$

The electrostatic potential V can in principle be measured by transporting a test charge through empty space from just outside one side of the sample, where current is flowing, to just outside another side of the same sample, in a region where there is no surface current. (Here and below we shall no longer insist on a semi-infinite sample; rather we imagine it large but finite in the $\hat{x}$ direction.) A more subtle issue is whether it can be measured using an ordinary voltmeter with contact to the sample which measure the *electrochemical* potential. Can the electrochemical potential across the superconductor be non-zero?

One might well doubt the possibility, on the following grounds. An ordinary voltmeter requires a non-zero current flow through it, and dissipates a small but finite amount of energy in its operation. On the other hand, at small current densities, the current-carrying state in a superconductor can, for most purposes, be considered an equilibrium state. A small persistent current flowing around a loop, for example, is really a metastable state. Thus if we could use it to drive a voltmeter, we would violate the principles of thermodynamics. The electrochemical potential in this situation must in fact be constant, so the voltmeter will measure no voltage.

A different situation can occur if the current exceeds a critical value, so that dissipation can occur by nucleation and flow of vortices. In this case there can be a Hall voltage in a superconductor measurable with a voltmeter, which can occur in principle without an external magnetic field, in the case of an anyon superconductor.

There is a general inequality between voltage and current, in the anyon case, that follows from this line of reasoning. The force on the normal electron must always be such as to keep it moving in the right direction; this implies (for the core region)

$$\Phi j \geq eE, \tag{8.13}$$

with $\Phi = 2\pi/ne$, and its integrated form

$$\frac{2\pi}{ne^2} J \geq V. \tag{8.14}$$

This inequality is comfortably satisfied in our case. It is saturated by the Hall current for n filled Landau levels.

The problem of the Hall voltage in a conventional superconductor subject to an external magnetic field has been studied by various authors. Experiments[52] in that case confirmed that there exists a Hall electric field, which was observed using an a.c. technique with a capacitative pickup; whereas no Hall voltage can be measured by a voltmeter through ohmic contacts except in the regime where there is dissipation associated with the motion of magnetic vortex lines.

It is interesting to consider how the current carrying state behaves as we approach fermions, that is $n \to \infty$. At first the situation seems quite disturbing, because (for constant density and mass) the penetration depth remains fixed at the London value. Superconductivity this robust is too much of a good thing. However, we should realize that the gap (4.7) to create vortices shrinks with n. Thus the domain of non-dissipative, vortex-free superconductivity becomes vanishingly small — the amount of current that can be carried, or magnetic field expelled, shrinks to zero.

3) *Reflection of polarized light*

A very interesting possible manifestation of P and T violation, pointed out by Wen and Zee,[35] is that the direction of polarization of linearly polarized light is subject to rotation by reflection at normal incidence. It is not difficult to see that the a term, despite its being P and T violating, does not lead to this effect. Indeed this term couples the charge density to the perpendicular magnetic field, and does not affect the propagation of fields tangential to the plane. An effect of the type proposed by Wen and Zee would indeed arise from the b term. However, as we have seen, the b term does not arise in the simplest anyon model, for reasons we alluded to in Sec. 7. A more direct argument is the following. In a translation invariant system such that the charged particles all have the same charge to mass ratio, a spatially constant electric field couples to the center of mass coordinate. Since the equation of motion of the center of mass is not affected by the interactions, the response must be that of free particles; in particular, it cannot violate P and T. Now the b term, if present, would lead to transverse current flow in response to a uniform electric field. Since the anyon models we have studied satisfy the premises of this argrument, they cannot generate a non-zero b term.

Now of course in real materials the charged particles do not all have the same charge to mass ratio, and translation invariance is spontaneously broken. Thus the optical rotation effect should exist, but it will be suppressed.

There *will* be characteristic, unsuppressed polarization effects at non-normal incidence, due to the a term. The necessary computations are rather cumbersome and will not be attempted here.

The underlying reason for these optical effects will be readily appreciated by those readers familiar with the field theory literature on Chern-Simons terms. In fact these sorts of terms were first studied not in connection with fractional statistics, but rather as a means of giving mass to gauge bosons.[53] The effect of a

Chern-Simons term, when combined with the ordinary Higgs mechanism, is essentially to give unequal masses to the two circular polarization states of the photon. Below threshold, they will be exponentially attenuated with distance, but at different rates. This way of regarding the situation suggests that the most sensitive way to search for the effects of interest is actually in transmission rather than in reflection.

9 Concluding Remarks

In this paper we have investigated the ground state of the anyon gas for statistical parameters of the form $\theta = \pi(1 - 1/n)$. Using an analysis which is valid at least in the case of large n, we confirm the existence of a superconducting ground state with a low frequency sound mode at long wavelengths, and quasiparticle excitations that are identified as charged vortices. We have introduced an effective Lagrangian for the superconductor, which contains anomalous terms that reflect the lack of P and T symmtry, and we have calculated the values of the associated coefficients in the large n limit. We have noted the existence of a violation of the commutativity of the generators of translations of the quasiparticle excitations, which is a key to understanding this system. We have also noted some phenomena which are consequences of the violations of P and T symmetry in the model, whose observation would establish the model's relevance to actual high-temperature superconductors.

As a theory of high-temperature superconductivity, the anyon model is clearly incomplete. Most pressing, of course, are the need to establish the connection to microscopic models of interacting electrons more convincingly, and to understand how this new mechanism of two-dimensional superconductivity can be extended into the third dimension.

Despite such major gaps in the theory, it is certainly suggestive that the new high-temperature superconductivity arises in a variety of highly anisotropic materials sharing a common two-dimensional structure. If these materials are as two-dimensional and their behavior as qualitatively new as they seem to be, it is tempting to think that in them Nature has realized anyon superconductivity.

Appendix A

Here we present some details of the calculation of the unperturbed correlation function $D^0_{\mu\nu}(q, \omega)$. We shall work in the second quantized scheme. We define the inverse magnetic length and cyclotron frequency for a particle of mass m in the presence of a statistical field b:

$$\lambda = \sqrt{\frac{m\omega_c}{\hbar}} \qquad \text{where} \qquad \omega_c = \frac{\hbar}{m} b \,. \tag{A.1}$$

By definition, at zero temperature

$$D^0_{\mu\nu}(r_1, t_1; r_2, t_2) = -i\langle|T(\Psi^\dagger(1)j_\mu\Psi(1)\Psi^\dagger(2)j_\nu\Psi(2))|\rangle_0 \tag{A.2}$$

where j_μ is as defined in (5.10) and Ψ is the electron field operator

$$\Psi(r, t) = \sum_{l,k} \varphi_{lk}(r)c_{lk}(t) \tag{A.3}$$

with $\varphi_{lk}(r)$ being the Landau wave function. We have found it convenient to work in the asymmetric gauge

$$a_x = -\hbar\, by, \qquad a_y = 0 \tag{A.4}$$

$$\varphi_{lk}(r) = e^{ikx}\left(\frac{\lambda}{\sqrt{\pi}2^l l!}\right)^{1/2} \exp\left[-\frac{\lambda^2}{2}\left(y+\frac{k}{b}\right)^2\right] H_l\left(\lambda\left(y+\frac{k}{b}\right)\right) \tag{A.5}$$

where H_l is the Hermite polynomial. The expectation value is taken with respect to the state with n completely filled Landau levels.

We can now express $D^0_{\mu\nu}(1, 2)$ in terms of Landau wave functions

$$\begin{aligned}
D^0_{\mu\nu}(1, 2) = {} & -i \sum_{a,b,c,d} \varphi^*_a(1) j_\mu \varphi_b(1)\varphi^*_c(2) j_{\nu d}(2) \\
& = \langle 0| \prod_{l,k_1} a_{lk_1}\, T\left(a^\dagger_a(1)a_b(1)\left(\sum_{m,k_2} |m,k_2\rangle\langle m,k_2|\right)a^\dagger_c(2)a_d(2)\right) \\
& \quad \times \prod_{l,k_1} a^\dagger_{lk_1}|0\rangle \\
& = -i\sum_{l,k_1}\sum_{m,k_2}\Big(e^{i(t_2-t_1)(\omega_m-\omega_l)}\varphi^*_{lk_1}(1)j_\mu\varphi_{mk_2}(1)\varphi^*_{mk_2}(2)j_\nu\varphi_{lk_1}(2)\Theta(t_1-t_2) \\
& \quad + e^{-i(\omega_m-\omega_l)(t_2-t_1)}\varphi^*_{mk_2}(1)j_\mu\varphi_{lk_1}\Big)(1)\varphi^*_{lk_1}(2)j_\nu\varphi_{mk_2}(2)\Theta(t_2-t_1)\,.
\end{aligned} \tag{A.6}$$

We will calculate the Fourier transform of the correlation function:

$$D^0_{\mu\nu}(\mathbf{q}, \omega) = 2\pi\int dr_1^2 dr_2^2 dt_1 dt_2 e^{i(q_x(x_2-x_1)+q_y(y_2-y_1))}e^{-i\omega(t_2-t_1)}D^0_{\mu\nu}(1,2) \tag{A.7}$$

without loss of generality, we choose the momentum transfer to be in the x-direction

$$q_x = q, \quad q_y = 0. \tag{A.8}$$

We now proceed to evaluate each component of $D^0_{\mu\nu}(q,\omega)$ in turn. First,

$$\begin{aligned} D^0_{00}(q,\omega) = {} & \sum_{l=0}^{n-1} \sum_{m=n}^{\infty} c_{lm} \int \left(\prod_{i=1,2} dx_i dy_i dk_i \right) e^{iq(x_2 - x_1)} e^{i(k_1 - k_2)(x_2 - x_1)} \\ & \times \left(\frac{1}{\omega - (\omega_m - \omega_l) + i\varepsilon} - \frac{1}{\omega + (\omega_m - \omega_l) - i\varepsilon} \right) \\ & \times \prod_{i=1,2} \exp\left[-\frac{\lambda^2}{2} \left(y_i + \frac{k_1}{b} \right)^2 \right] H_l\left(\lambda \left(y_i + \frac{k_1}{b} \right) \right) \\ & \times \prod_{i=1,2} \exp\left[-\frac{\lambda^2}{2} \left(y_i + \frac{k_2}{b} \right)^2 \right] H_m\left(\lambda \left(y_i + \frac{k_2}{b} \right) \right) \end{aligned} \tag{A.9}$$

where

$$C_{lm} = \frac{1}{2\pi^2} \frac{\lambda^4}{2^m m! 2^l l!}. \tag{A.10}$$

Let

$$u_i \equiv \left(y_i - \frac{k_1 + k_2}{2b} \right), \qquad i = 1, 2, \qquad v \equiv \frac{\lambda(k_1 - k_2)}{2b} \tag{A.11}$$

and change integration variables:

$$\int dk_1 dk_2 = 2\left(\frac{b}{\lambda}\right) \int dv d\left(\frac{k_1 + k_2}{2}\right) \tag{A.12}$$

$k_1 + k_2$ is essentially the y coordinate of the center of mass; the integrand will be essentially independent of it. The integration over v is very simple as well, as the integration over x_2, x_1 results in a delta function:

$$\delta(k_1 - k_2 + q) = \left(\frac{\lambda}{2b}\right) \delta\left(v + \frac{\lambda q}{2b}\right). \tag{A.13}$$

Now using

$$\exp\left[-\frac{\lambda^2}{2}\left(y_i+\frac{k_1}{b}\right)^2\right]\exp\left[-\frac{\lambda^2}{2}\left(y_i+\frac{k_2}{b}\right)^2\right] = e^{-u_i^2}e^{-v^2}$$

$$H_l\left(\lambda\left(y_i+\frac{k_1}{b}\right)\right) = H_l(u_i+v)$$

$$H_m\left(\lambda\left(y_i+\frac{k_2}{b}\right)\right) = H_m(u_i-v)$$

we find

$$D^0_{00}(q,\omega) = \sum_{l=0}^{n-1}\sum_{m=n}^{\infty} c_{lm}\left(\frac{1}{\lambda^2}\right)\int du_1 du_2 dv \exp[-u_1^2-u_2^2-2v^2]$$

$$\times\left(\frac{\delta(v+\lambda q/2b)}{\omega-(\omega_m-\omega_l)+i\varepsilon}-\frac{\delta(-v+\lambda q/2b)}{\omega+(\omega_m-\omega_l)-i\varepsilon}\right)\cdot$$

$$H_l(u_1-v)H_l(u_2-v)H_m(u_1+v)H_m(u_2+v) \tag{A.14}$$

Using the identity

$$\int_{-\infty}^{\infty} dx \exp[-x^2]H_l(x+y)H_m(x+z) = 2^m\sqrt{\pi}\,l!\,z^{m-l}L_l^{m-l}(-2yz) \tag{A.15}$$

where $m \geq l$, we get:

$$D^0_{00}(q,\omega) = \sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{\lambda^2}{2\pi^2 2^m m! 2^l l!}\int dv\left(\frac{\delta\left(v+\frac{\lambda q}{2b}\right)}{\omega-(\omega_m-\omega_l)+i\varepsilon}-\frac{\delta\left(-v+\frac{\Delta\lambda q}{2b}\right)}{\omega+(\omega_m-\omega_l)-i\varepsilon}\right)$$

$$\times \exp[-2v^2]2^{2m}\pi(l!)^2 v^{2(m-1)}L_l^{m-l}(-2v^2)$$

$$= \frac{1}{2\pi}\lambda^2\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{l!}{m!}\frac{2\omega_c(m-l)}{\omega^2-\omega_c^2(m-l)^2}\exp[-X]X^{m-l}[L_l^{m-1}(X)]^2$$

$$= \frac{1}{2\pi\omega_c}q^2\Sigma_0 \tag{A.16}$$

where

$$X \equiv \frac{\lambda^2 q^2}{2b} = \frac{q^2}{2\lambda^2} \tag{A.17}$$

and

$$\Sigma_0 \equiv \sum_{l=0}^{n-1} \sum_{m=n}^{\infty} \frac{l!}{m!} \frac{m-l}{\left(\frac{\omega}{\omega_c}\right)^2 - (m-l)^2} \exp[-X] X^{m-l-1} [L_l^{m-l}(X)]^2 . \tag{A.18}$$

It is difficult to simplify further; fortunately this equation lends itself readily to expansion in ω and q.

Now we pass to the other components. The entries involving j_X depend on the quantity

$$\varphi_{lk_1}^*(1) j_x \varphi_{mk_2}(1) = \frac{\hbar}{m} \varphi_{lk_1}^*(1) \left(\frac{k_1 + k_2}{2} - by_1\right) \varphi_{mk_2}(1) \tag{A.19}$$

where

$$\frac{\hbar}{m} = \frac{\omega_c}{\lambda^2} . \tag{A.20}$$

After shifting the origin of the y, only the gauge potential part contributes. Hence

$$\begin{aligned} D_{x0}^0(q, \omega) = {} & \frac{\hbar}{m} \sum_{l=0}^{n-1} \sum_{m=n}^{\infty} c_{lm} \int \left(\prod_{i=1,2} dx_i dy_i dk_i\right) e^{iq(x_2 - x_1)} e^{i(k_1 - k_2)(x_2 - x_1)} \\ & \times \left(\frac{1}{\omega - (\omega_m - \omega_l) + i\varepsilon} - \frac{1}{\omega + (\omega_m - \omega_l) - i\varepsilon}\right)(-by_1) \\ & \times \prod_{i=1,2} \exp\left[-\frac{\lambda^2}{2}\left(y_i + \frac{k_1}{b}\right)^2\right] H_l\left(\lambda\left(y_i + \frac{k_1}{b}\right)\right) \\ & \times \prod_{i=1,2} \exp\left[-\frac{\lambda^2}{2}\left(y_i + \frac{k_2}{b}\right)^2\right] H_m\left(\lambda\left(y_i + \frac{k_2}{b}\right)\right) \end{aligned} \tag{A.21}$$

$$= \frac{\hbar}{m} \sum_{l=0}^{n-1} \sum_{m=n}^{\infty} c_{lm} \left(\frac{1}{\lambda^2}\right) \int du_1 du_2 dv$$

$$\times \left(\frac{\delta(v+\lambda q/2b)}{\omega-(\omega_m-\omega_l)+i\varepsilon}-\frac{\delta(-v+\lambda q/2b)}{\omega+(\omega_m-\omega_l)-i\varepsilon}\right)$$

$$\times \left(\frac{-b}{\lambda}\right)(u_1-v+v)\exp[-u_1^2-u_2^2-2v^2]$$

$$\times H_l(u_1-v)H_l(u_2-v)H_m(u_1+v)H_m(u_2+v)\,.$$

Using the recursion relationship:

$$xH_l(x)=\frac{1}{2}H_{l+1}(x)+lH_{l-1}(x) \tag{A.22}$$

we have

$$D^0_{x0}(q,\omega)=\frac{\hbar}{m}\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}c_{lm}\left(\frac{1}{\lambda^2}\right)\int du_1du_2dv\left(\frac{-b}{\lambda}\right)\exp[-u_1^2-u_2^2-2v^2]$$

$$\times \left(\frac{\delta(v+\lambda q/2b)}{\omega-(\omega_m-\omega_l)+i\varepsilon}-\frac{\delta(-v+\lambda q/2b)}{\omega+(\omega_m-\omega_l)-i\varepsilon}\right)H_l(u_2-v)$$

$$\times \left(\frac{1}{2}H_{l+1}(u_1-v)+lH_{l-1}(u_1-v)+vH_l(u_1-v)\right)H_m(u_1+v)$$

$$\times H_m(u_2+v)$$

$$=\frac{1}{2\pi}\left(\frac{\omega_c b^2}{\lambda^2 q}\right)\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{l!}{m!}\left(\frac{1}{\omega-(\omega_m-\omega_l)+i\varepsilon}+\frac{1}{\omega+(\omega_m-\omega_l)-i\varepsilon}\right)$$

$$\times \exp[-X]X^{m-l}L_l^{m-l}(X)\Big((l+1)L_{l-1}^{m-l-1}(X)+X(L_{l-1}^{m-l+1}(X)(1-\delta_{l,0})$$

$$+L_l^{m-l}(X))\Big)\,. \tag{A.23}$$

Now using the recursion relationship

$$xL_l^{a+1}(x)=(l+a+1)L_l^a(x)-(l+1)L_{l+1}^a(x) \tag{A.24}$$

we have

$$(l+1)L_{l+1}^{m-l-1}(X)+X\big(L_{l-1}^{m-l+1}(X)(1-\delta_{l,0})+L_l^{m-1}(X)\big)=(m-l)L_l^{m-l}(X)$$

and finally

$$D_{x0}^{0}(q,\omega)=\frac{1}{2\pi}\left(\frac{\omega_c b^2}{\lambda^2 q}\right)\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{l!}{m!}\frac{2\omega(m-l)}{\omega^2-\omega_c^2(m-l)^2}$$

$$\exp[-X]X^{m-l}[L_l^{m-l}(X)]^2$$

$$=\frac{1}{2\pi\omega_c}q\omega\Sigma_0\,. \tag{A.25}$$

A similar calculation gives

$$D_{xx}^{0}(q,\omega)=\frac{1}{2\pi}\frac{\hbar}{m}\left(\frac{b^2}{\lambda^2 q}\right)^2\omega_c\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{l!}{m!}\frac{2\omega_c(m-l)}{\omega^2-\omega_c^2(m-l)^2}$$

$$\exp[-X]X^{m-l}[(m-l)L_l^{m-l}(X)]^2$$

$$=\frac{1}{2\pi}\frac{\hbar}{m}\left(\frac{b}{\lambda}\right)^2\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{l!}{m!}(m-l)$$

$$\left(\frac{\omega^2}{\omega^2-\omega_c^2(m-l)^2}-1\right)\exp[-X]X^{m-l-1}[L_l^{m-l}(X)]^2$$

$$=\frac{1}{2\pi\omega_c}(\omega^2\Sigma_0-\omega_c^2\Sigma_3) \tag{A.26}$$

where

$$\Sigma_3-n\equiv\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\exp[-X]X^{m-l-1}\frac{l!}{m!}(m-l)[L_l^{m-l}(X)]^2$$

$$\approx\frac{n(n-1)(n-2)}{6}X^2+o(X^3)\,. \tag{A.27}$$

Notice that a possible term proportional to q^2 vanishes.

Σ_3-n, taken beyond the first term, is something of an embarrassment. Current conservation, applied to the response function K, should force it (given the other formulas in Sec. 5) to terminate after the first term. We suspect that the method of including the electromagnetic contact interaction used there is not quite right, and when done correctly it will cancel the offending part of Σ_3. We hope to remedy this defect soon. In any case, since the trouble starts at order q^4, none of our conclusions are affected.

Now we calculate $D^0_{y0}(q,\omega)$. The current in the y-direction is: ($a_y = 0$)

$$\begin{aligned}\varphi^*_{lk_1}(1) j_y \varphi_{mk_2}(1) = {} & \frac{\hbar}{m}\frac{\lambda}{2i} e^{ix_1(k_1-k_2)} \exp\left[-\frac{\lambda^2}{2}\left(y_1+\frac{k_1}{b}\right)^2\right] \exp\left[-\frac{\lambda^2}{2}\left(y_1+\frac{k_2}{b}\right)^2\right] \\ & \left(\left(lH_{l-1}\left(\lambda\left(y_1+\frac{k_1}{b}\right)\right) - \frac{1}{2}H_{l+1}\left(\lambda\left(y_1+\frac{k_1}{b}\right)\right)\right)\right. \\ & \times H_m\left(\lambda\left(y_1+\frac{k_2}{b}\right)\right) \\ & -\left(mH_{m-1}\left(\lambda\left(y_1+\frac{k_2}{b}\right)\right) - \frac{1}{2}H_{m+1}\left(\lambda\left(y_1+\frac{k_2}{b}\right)\right)\right) \\ & \left.\times H_l\left(\lambda\left(y_1+\frac{k_1}{b}\right)\right)\right). \end{aligned} \tag{A.28}$$

Therefore

$$\begin{aligned} D^0_{y0}(q,\omega) = {} & -i\frac{1}{2\pi}\frac{\hbar}{m}\lambda^2\frac{b}{q}\sum_{l=0}^{n-1}\sum_{m=n}^{\infty}\frac{l!}{m!}\frac{2(m-l)\omega_c}{\omega^2-\omega_c^2(m-l)^2} \\ & \exp[-X]X^{m-l}L_l^{m-l}(X)L'(l,m,X) \\ \equiv {} & -i\frac{1}{2\pi\omega_c}\omega_c q\,\Sigma_1 \end{aligned} \tag{A.29}$$

where

$$\begin{aligned} L'(l,m,X) \equiv {} & \frac{1}{2}(XL_{l-1}^{m-l+1}(X)(1-\delta_{l,0}) - (l+1)L_{l+1}^{m-l-1}(X) \\ & - mL_l^{m-l-1}(X) + XL_l^{m-l+1}(X)). \end{aligned} \tag{A.30}$$

By the same method of calculation, we find

$$D^0_{yy}(q,\omega) = \frac{1}{2\pi}\left(\frac{\hbar}{m}\right)^2 \lambda^2 \left(\frac{b}{q}\right)^2 \sum_{l=0}^{n-1}\sum_{m=n}^{\infty} \frac{l!}{m!}\frac{2(m-l)\omega_c}{\omega^2-\omega_c^2(m-l)^2}$$

$$\times \exp[-X]X^{m-l}[L'(l,m,X)]^2$$

$$\equiv \frac{1}{2\pi\omega_c}\omega_c^2\Sigma_2$$

$$D^0_{xy}(q,\omega) = -i\frac{1}{2\pi}\left(\frac{\hbar}{m}\right)^2 \lambda^2 \left(\frac{b}{q}\right)^2 \sum_{l=0}^{n-1}\sum_{m=n}^{\infty} \frac{l!}{m!}\frac{2(m-l)\omega}{\omega^2-\omega_c^2(m-l)^2}$$

$$\times \exp[-X]X^{m-l}L_l^{m-l}(X)L'(l,m,X)$$

$$= -i\frac{1}{2\pi\omega_c}\omega_c\omega\,\Sigma_1\,. \tag{A.31}$$

We now calculate Σ_i in the small q, ω limit. To first order in q^2, ω^2

$$\Sigma_0 \approx \sum_{m-l=1,2}(1-X)X^{m-l-1}\frac{l!}{m!}\frac{-1}{m-l}\left(1-\left(\frac{\omega}{\omega_c}\right)^2\right)$$

$$\times\left(\frac{m!}{l!(m-l)!}-\frac{m!}{(l-1)!(m-l+1)!}X\right)^2$$

$$\approx n\left(-1-\left(\frac{\omega}{\omega_c}\right)^2+\frac{3}{4}nX\right). \tag{A.32}$$

Notice that by demanding low powers of X we are restricted to a very limited range of m and l, thus $m-l=1$ implies $m=n$, $l=n-1$; $m-l=2$ implies $m=n$, $l=n-2$ or $m=n+1$, $l=n-1$. So the results for the low frequency, long wavelength limit used in the text are sensitive only to the top two occupied Landau levels, and the two empty ones right above the Fermi level.

Similar calculations give

$$\Sigma_1 \approx n\left(-1-\left(\frac{\omega}{\omega_c}\right)^2+\frac{3}{2}nX\right),$$

$$\Sigma_2 \approx n\left(-1-\left(\frac{\omega}{\omega_c}\right)^2+2nX\right). \tag{A.33}$$

For convenience, let us define

$$\tilde{\Sigma}_i \equiv \frac{\Sigma_i}{n} \tag{A.34}$$

and drop the tilde. Putting everything together, we have then

$$D^0 = \frac{n}{2\pi\omega_c}\begin{pmatrix} q^2\Sigma_0 & q\omega\Sigma_0 & -iq\omega_c\Sigma_1 \\ q\omega\Sigma_0 & (\omega^2\Sigma_0 - \omega_c^2\Sigma_3) & -i\omega\omega_c\Sigma_1 \\ iq\omega_c\Sigma_1 & i\omega\omega_c\Sigma_1 & \omega_c^2\Sigma_2 \end{pmatrix}. \tag{A.35}$$

With the interaction matrix derived in Sec. 5:

$$V = \frac{2\pi}{n}\begin{pmatrix} \frac{\omega_c}{q^2} & 0 & \frac{i}{q} \\ 0 & 0 & 0 \\ -\frac{i}{q} & 0 & 0 \end{pmatrix} \tag{A.36}$$

and the correction matrix

$$u = \frac{2\pi}{nq}\frac{\hbar}{m}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.37}$$

we find

$$K = e^2\Lambda_{\mu\nu} + e^2\bar{\rho}\frac{\hbar}{m}\delta_{\mu\nu}(1-\delta_{\mu 0})$$

$$= \frac{e^2 n}{2\pi \det}\begin{pmatrix} \frac{q^2}{\omega_c}\Sigma_0 & q\frac{\omega}{\omega_c}\Sigma_0 & iq\Xi \\ q\frac{\omega}{\omega_c}\Sigma_0 & \frac{\omega_2}{\omega_c}\left(\Sigma_0 - \det(\Sigma_3 - 1)\frac{\omega_c^2}{\omega^2}\right) & i\omega\Xi \\ -iq\Xi & -i\omega\Xi & \omega_c(\Xi - \Sigma_1 + \Sigma_2 + \det) \end{pmatrix} \tag{A.38}$$

where

$$\begin{aligned}\Xi &= -\Sigma_1 - \Sigma_1^2 + \Sigma_0\Sigma_2 + \Sigma_0 \\ &\approx -\frac{n}{4}\left(\frac{q}{\lambda}\right)^2 \end{aligned} \tag{A.39}$$

$$\begin{aligned}\det &= 1 - \Sigma_0 + 2\Sigma_1 + \Sigma_1^2 - \Sigma_0\Sigma_2 \\ &\approx -\left(\frac{\omega}{\omega_c}\right)^2 + n\left(\frac{q}{\lambda}\right)^2 . \end{aligned} \tag{A.40}$$

Appendix B

We show here how we determine the effective Lagrangian by matching the two point function calculated from the Lagrangian to the linear response function $K_{\mu\nu}$, which is calculated in Appendix A.

Consider the effective Lagrangian

$$\begin{aligned}L = \frac{1}{2}(\dot{\phi} - CA_0)^2 - \frac{v^2}{2}(\partial_i\phi - CA_i)^2 \\ + a\varepsilon_{ij}\partial_i A_j(\dot{\phi} - CA_0) + b\epsilon_{ij}(\partial_0 A_i - \partial_i A_0)(\partial_j\phi - CA_j) . \end{aligned} \tag{B.1}$$

The sound velocity v and the other coefficients a, b, C are to be determined.

In the calculation of $K_{\mu\nu}$ we have chosen the momentum transfer to be in the x direction only, i.e. $q_y = 0$. Here we shall consider the same situation.

For this choice, the Feynman rules for the effective Lagrangian are (Fig. 1)

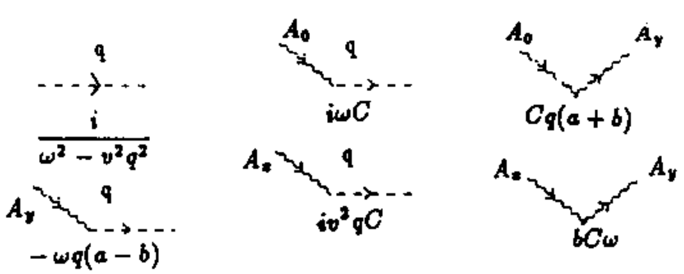

Fig. 1

where the broken line denotes the scalar field ϕ, and the wiggled line denotes the gauge field A_μ.

We now calculate the two point functions from the effective Lagrangian, to the lowest order Feynman diagrams. The two point function that couples to A_x, A_y comes from two diagrams (Fig. 2):

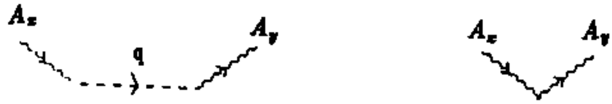

Fig. 2

They yield:

$$\frac{iC\omega(av^2q^2 - b\omega^2)}{\omega^2 - v^2q^2}. \tag{B.2}$$

In the small q, ω limit we have

$$\det \approx -\left(\frac{\omega}{\omega_c}\right)^2 + n\left(\frac{q}{\lambda}\right)^2. \tag{B.3}$$

This determines the position of the pole, and thus identifies

$$v^2 = n\frac{\omega_c^2}{\lambda^2} = 2\pi\bar{\rho}\left(\frac{\hbar}{m}\right)^2. \tag{B.4}$$

We shall assume this identification in the following formulas. We have:

$$K_{xy} = \frac{e^2}{2\pi}\frac{n^2}{4}\frac{\omega_c^2}{\lambda^2}\frac{q^2\omega i}{\omega^2 - v^2q^2}. \tag{B.5}$$

Equating (B.2) and (B.5) we have

$$b = 0, \tag{B.6}$$

$$Ca = \frac{e^2}{8\pi}n. \tag{B.7}$$

Similarly, the two point function that couples to A_0, A_y is (Fig. 3)

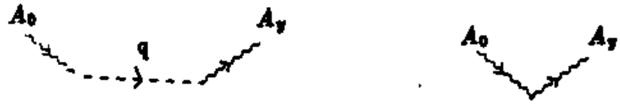

Fig. 3

$$\frac{iCav^2q^3}{\omega^2 - v^2q^2} \tag{B.8}$$

and

$$K_{0y} = \frac{e^2}{2\pi}\frac{n^2}{4}\frac{\omega_c^2}{\lambda^2}\frac{q^3 i}{\omega^2 - v^2q^2}\,. \tag{B.9}$$

The matching here simply confirms the result from the K_{xy} matching.

For K_{0x}, the corresponding two point function comes from only one diagram (Fig. 4)

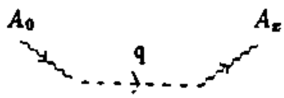

Fig. 4

$$\frac{v^2\omega q C^2}{\omega^2 - v^2q^2}\,. \tag{B.10}$$

Matching to

$$K_{0x} = \frac{e^2}{2\pi}n\omega_c\frac{q\omega}{\omega^2 - v^2q^2} \tag{B.11}$$

we find

$$C^2 = \frac{e^2}{2\pi}\frac{m}{\hbar} \tag{B.12}$$

if we choose $C > 0$, then we have

$$C = e\sqrt{\frac{m}{2\pi\hbar}} \tag{B.13}$$

and

$$a = en\sqrt{\frac{\hbar}{32\pi m}}\,. \tag{B.14}$$

Our effective Lagrangian is thus fully determined.

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