Quantum Supersymmetric Bianchi IX Cosmology and its Hidden Kac-Moody Structure

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Hidden Kac-Moody symmetries (E_{10} or AE_n) in (super)gravity

Damour, Henneaux 01; Damour, Henneaux, Julia, Nicolai 01; Damour, Henneaux, Nicolai 02; Damour, Kleinschmidt, Nicolai 06; de Buyl, Henneaux, Paulot 06; Kleinschmidt, Nicolai 06; ...

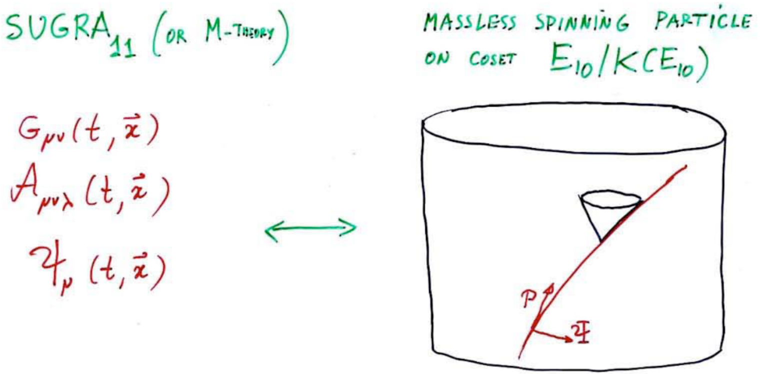
[first conjectured by Julia 82; related conjectures Ganor 99, 04; West (E_{11}) 01; DHN work stemmed from a study of Belinski-Khalatnikov-Lifshitz chaotic billiard dynamics]

$\longrightarrow$ Gravity/Coset conjecture (DHN02)

'Duality' between $D = 11$ supergravity (or, hopefully, M-theory) and the (quantum) dynamics of a massless spinning particle on $E_{10}/K(E_{10})$

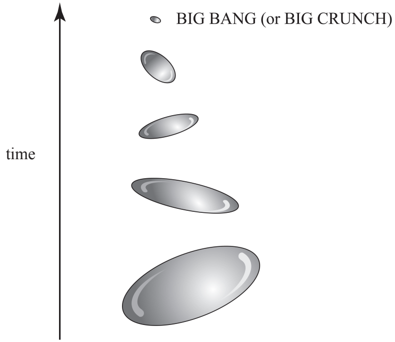
Gravity/Coset Correspondence

E_{10} : Damour, Henneaux, Nicolai $'02$;
related: Ganor $'99$ $'04$; E_{11} : West $'01$



A concrete case study (Damour, Spindel 2013, 2014)

• Quantum supersymmetric Bianchi IX, i.e. quantum dynamics of a supersymmetric triaxially squashed three-sphere



Susy Quantum Cosmology: Obregon et al $\geq$ 1990; D'Eath, Hawking, Obregon, 1993, D'Eath $\geq$ 1993, Csordas, Graham 1995, Moniz $\geq$ 94, . . .

A concrete case study (Damour, Spindel 2013, 2014)

Technically: Reduction to one, time-like, dimension of the action of $D = 4$ simple supergravity for an $SU(2)$-homogeneous (Bianchi IX) cosmological model

$$
\begin{aligned}
g_{\mu\nu}\, dx^\mu\, dx^\nu &= -N^2(t) dt^2 \\
&+ g_{ab}(t)(\tau^a(x) + N^a(t) dt)(\tau^b(x) + N^b(t) dt)\,,
\end{aligned}
$$

τ^a: left-invariant one-forms on $SU(2) \approx S_3$: $d\tau^a = \frac{1}{2}\, \varepsilon_{abc}\, \tau^b \wedge \tau^c$

Dynamical degrees of freedom

- Gauss-decomposition of the metric:

$$g_{bc} = \sum_{\widehat{a}=1}^{3} e^{-2\beta^a} \, S^{\widehat{a}}{}_b(\varphi_1, \varphi_2, \varphi_3) \, S^{\widehat{a}}{}_c(\varphi_1, \varphi_2, \varphi_3)$$

six metric dof:

$$\begin{aligned} \beta^a &= (\beta^1(t), \beta^2(t), \beta^3(t)) \\ &= \text{cologarithms of the squashing parameters } a, b, c \text{ of 3-sphere} \end{aligned}$$

$$a = e^{-\beta^1}, \qquad b = e^{-\beta^2}, \qquad c = e^{-\beta^3}$$

and three Euler angles:

$$\varphi_a = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$$

parametrizing the orthogonal matrix $S^{\hat{a}}_b$

Dynamical degrees of freedom

• Gravitino components in specific gauge-fixed orthonormal frame $\theta^{\widehat{\alpha}}$ canonically associated to the Gauss-decomposition

$$\theta^{\widehat{0}} = N(t)dt\,,\ \theta^{\widehat{a}} = \sum_b e^{-\beta^a(t)} S^{\widehat{a}}{}_b(\varphi_c(t))(\tau^b(x) + N^b(t)dt)$$

• redefinitions of the gravitino field:

$$\Psi^A_{\hat{\alpha}}(t) := g^{1/4}\,\psi^A_{\hat{\alpha}} \quad \text{and} \quad \Phi^a_A := \Sigma_B\,\gamma^{\hat{a}}_{AB}\,\Psi^B_{\hat{a}} \quad \text{(no summation on } \hat{a}\text{)}$$

• 3×4 gravitino components Φ^a_A, $a = 1, 2, 3$; $A = 1, 2, 3, 4$.

Supersymmetric action (first order form)

$$S = \int dt \left[\pi_a \, \dot{\beta}^a + p_{\varphi^a} \, \dot{\varphi}^a + \frac{i}{2} \, G_{ab} \, \Phi_A^a \, \dot{\Phi}_A^b + \bar{\Psi}'^A_{\hat{0}} \, \mathcal{S}_A - \tilde{N} H - N^a H_a \right]$$

G_{ab}: Lorentzian-signature quadratic form:

$$G_{ab} \, d\beta^a \, d\beta^b \equiv \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a \right)^2$$

G_{ab} defines the kinetic terms of the gravitino, as well as those of the β^a's:

$$\frac{1}{2} \, G_{ab} \, \dot{\beta}^a \, \dot{\beta}^b$$

Lagrange multipliers $\longrightarrow$ Constraints $\mathcal{S}_A \approx 0$, $H \approx 0$, $H_a \approx 0$

Quantization

• Bosonic dof:

$$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \; ; \quad \widehat{p}_{\varphi_a} = -i \frac{\partial}{\partial \varphi_a}$$

• Fermionic dof:

$$\widehat{\Phi}_A^a \, \widehat{\Phi}_B^b + \widehat{\Phi}_B^b \, \widehat{\Phi}_A^a = G^{ab} \, \delta_{AB}$$

This is the Clifford algebra $\mathrm{Spin}\,(8^+, 4^-)$

• The wave function of the universe $\Psi_\sigma(\beta^a, \varphi_a)$ is a 64-dimensional spinor of $\mathrm{Spin}\,(8, 4)$ and the gravitino operators Φ_A^a are 64×64 "gamma matrices" acting on Ψ_σ, $\sigma = 1, \ldots, 64$

• Similar to Ramond string $\psi_0^\mu \sim \gamma^\mu \rightarrow$ spinorial amplitude

Dirac Quantization of the Constraints

$$\widehat{\mathcal{S}}_A \Psi = 0\,, \quad \widehat{H}\Psi = 0\,, \quad \widehat{H}_a \Psi = 0$$

Diffeomorphism constraint $\Leftrightarrow \widehat{p}_{\varphi_a} \Psi = -i \frac{\partial}{\partial \varphi_a} \Psi = 0$: "*s* wave" w.r.t. the Euler angles

$\longrightarrow$ Wave function $\Psi(\beta^a)$ submitted to constraints

$$\widehat{\mathcal{S}}_A(\widehat{\pi}, \beta, \widehat{\Phi})\, \Psi(\beta) = 0\,, \quad \widehat{H}(\widehat{\pi}, \beta, \widehat{\Phi})\, \Psi(\beta) = 0$$

$\widehat{\pi}_a = -i \frac{\partial}{\partial \beta^a} \Rightarrow 4 \times 64 + 64$ PDE's for the 64 functions $\Psi_\sigma(\beta^1, \beta^2, \beta^3)$

Heavily overdetermined system of PDE's

Explicit form of the SUSY constraints

$(\gamma^5 \equiv \gamma^{\hat{0}\hat{1}\hat{2}\hat{3}}$, $\beta_{12} \equiv \beta^1 - \beta^2$, $\widehat{\Phi}^{12} \equiv \widehat{\Phi}^1 - \widehat{\Phi}^2)$

$$\begin{aligned}
\widehat{\mathcal{S}}_A &= -\frac{1}{2}\sum_a \widehat{\pi}_a \, \Phi_A^a + \frac{1}{2}\sum_a e^{-2\beta^a}(\gamma^5 \, \Phi^a)_A \\
&- \frac{1}{8} \coth \beta_{12} (\widehat{S}_{12}(\gamma^{12} \, \widehat{\Phi}^{12})_A + (\gamma^{12} \, \widehat{\Phi}^{12})_A \, \widehat{S}_{12}) \\
&+ \text{cyclic}_{(123)} + \frac{1}{2}(\widehat{\mathcal{S}}_A^{\text{cubic}} + \widehat{\mathcal{S}}_A^{\text{cubic}\,\dagger})
\end{aligned}$$

$$\begin{aligned}
\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\bar{\widehat{\Phi}}^3 \, \gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1 + \widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}^1 \, \gamma^{\hat{0}\hat{1}\hat{2}} \, \widehat{\Phi}^1) \\
&+ (\bar{\widehat{\Phi}}^2 \, \gamma^{\hat{0}\hat{1}\hat{2}} \, \widehat{\Phi}^2) - (\bar{\widehat{\Phi}}^1 \, \gamma^{\hat{0}\hat{1}\hat{2}} \, \widehat{\Phi}^2)] \, ,
\end{aligned}$$

$$\begin{aligned}
\widehat{\mathcal{S}}_A^{\text{cubic}} &= \frac{1}{4}\sum_a (\bar{\widehat{\Psi}}_0 , \gamma^{\hat{0}} \, \widehat{\Psi}_{\hat{a}}) \, \gamma^{\hat{0}} \, \widehat{\Psi}_{\hat{a}}^A - \frac{1}{8}\sum_{a,b} (\bar{\widehat{\Psi}}_{\hat{a}} \, \gamma^{\hat{0}} \, \widehat{\Psi}_{\hat{b}}) \, \gamma^{\hat{a}} \, \widehat{\Psi}_{\hat{b}}^A \\
&+ \frac{1}{8}\sum_{a,b} (\bar{\widehat{\Psi}}_0 , \gamma^{\hat{a}} \, \Psi_{\hat{b}})(\gamma^{\hat{a}} \, \Psi_{\hat{b}}^A + \gamma^{\hat{b}} \, \Psi_{\hat{a}}^A) \, ,
\end{aligned}$$

(Open) Superalgebra satisfied by the $\widehat{\mathcal{S}}_A$'s and $\widehat{H}$

$$\widehat{\mathcal{S}}_A \, \widehat{\mathcal{S}}_B + \widehat{\mathcal{S}}_B \, \widehat{\mathcal{S}}_A = 4\, i \sum_C \widehat{L}_{AB}^{C}(\beta) \, \widehat{\mathcal{S}}_C + \frac{1}{2} \, \widehat{H} \delta_{AB}$$

$$[\widehat{\mathcal{S}}_A, \widehat{H}] = \widehat{M}_A^B \, \widehat{\mathcal{S}}_B + \widehat{N}_A \widehat{H}$$

Dynkin Diagrams (= Cartan Matrix) of E_{10} and AE_3

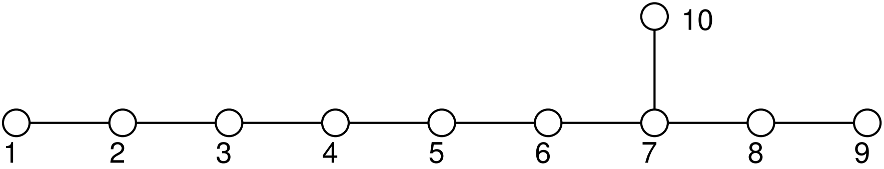
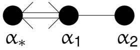


Figure: *Dynkin diagram of E_{10} with numbering of nodes.*

Cartan matrix of AE_3: $(A_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$



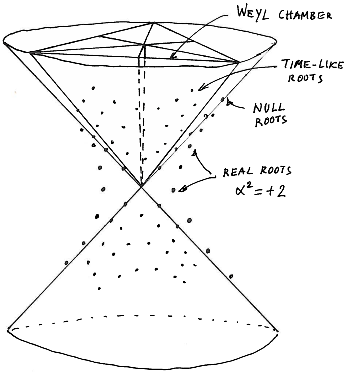
Dynkin diagram AE_3

Root diagram of $AE_3 = A_1^{++}$

3-dimensional Lorentzian-signature space: metric in Cartan sub-algebra

$$G_{ab}\, d\beta^a\, d\beta^b = \sum_a (d\beta^a)^2 - \left(\sum_a d\beta^a\right)^2$$

(directly linked to Einstein action: $K_{ij}^2 - (K_i^i)^2$)



Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$2\,\widehat{H} = G^{ab}(\widehat{\pi}_a + i\,A_a)(\widehat{\pi}_b + i\,A_b) + \widehat{\mu}^2 + \widehat{W}(\beta)\,,$$

$G_{ab} \leftrightarrow$ metric in Cartan subalgebra of AE_3

$$\widehat{W}(\beta) = W_g^{\rm bos}(\beta) + \widehat{W}_g^{\rm spin}(\beta) + \widehat{W}_{\rm sym}^{\rm spin}(\beta)\,.$$

$$W_g^{\rm bos}(\beta) = \frac{1}{2}\,e^{-4\beta^1} - e^{-2(\beta^2+\beta^3)} + {\rm cyclic}_{123}$$

Linear forms $\alpha_{ab}^g(\beta) = \beta^a + \beta^b \Leftrightarrow$ six level-1 roots of AE_3

Kac-Moody Structures Hidden in the Quantum Hamiltonian

$$\widehat{W}_g^{\rm spin}(\beta, \widehat{\Phi}) = e^{-\alpha_{11}^g(\beta)} \widehat{J}_{11}(\widehat{\Phi}) + e^{-\alpha_{22}^g(\beta)} \widehat{J}_{22}(\widehat{\Phi}) + e^{-\alpha_{33}^g(\beta)} \widehat{J}_{33}(\widehat{\Phi}) \,.$$

$$\widehat{W}_{\rm sym}^{\rm spin}(\beta) = \frac{1}{2} \, \frac{(\widehat{S}_{12}(\widehat{\Phi}))^2 - 1}{\sinh^2 \alpha_{12}^{\rm sym}(\beta)} + {\rm cyclic}_{123} \,,$$

Linear forms $\alpha_{12}^{\rm sym}(\beta) = \beta^1 - \beta^2$, $\alpha_{23}^{\rm sym}(\beta) = \beta^2 - \beta^3$, $\alpha_{31}^{\rm sym}(\beta) = \beta^3 - \beta^1$
$\Leftrightarrow$ three level-0 roots of AE_3

Spin dependent (Clifford) Operators coupled to AE_3 roots

$$
\begin{aligned}
\widehat{S}_{12}(\widehat{\Phi}) &= \frac{1}{2}[(\bar{\widehat{\Phi}}^3\,\gamma^{\hat{0}\hat{1}\hat{2}}(\widehat{\Phi}^1+\widehat{\Phi}^2)) + (\bar{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^1) \\
&+ (\bar{\widehat{\Phi}}^2\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2) - (\bar{\widehat{\Phi}}^1\,\gamma^{\hat{0}\hat{1}\hat{2}}\,\widehat{\Phi}^2)],
\end{aligned}
$$

$$
\widehat{J}_{11}(\widehat{\Phi}) = \frac{1}{2}[\;\bar{\widehat{\Phi}}^1\gamma^{\hat{1}\hat{2}\hat{3}}(4\widehat{\Phi}^1+\widehat{\Phi}^2+\widehat{\Phi}^3) + \;\bar{\widehat{\Phi}}^2\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^3].
$$

- $\widehat{S}_{12}$, $\widehat{S}_{23}$, $\widehat{S}_{31}$, $\widehat{J}_{11}$, $\widehat{J}_{22}$, $\widehat{J}_{33}$ generate (via commutators) a 64-dimensional representation of the (infinite-dimensional) "maximally compact" sub-algebra $K(AE_3) \subset AE_3$. [The fixed set of the (linear) Chevalley involution, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which is generated by $x_i = e_i - f_i$.]

Maximally compact sub-algebra of a Kac-Moody alg.

Generalization of $so(n) \subset gl(n)$:

$gl(n)$: arbitrary matrices: m_{ij}

$so(n)$: antisymmetric matrices: $a_{ij} = -a_{ij}$

projection: $gl(n) \to so(n)$: $m_{ij} \to \frac{1}{2}(m_{ij} - m_{ji})$

Kac-Moody alg. generated by h_i, e_i, f_i:

"Maximally compact subalgebra" generated by $x_i = e_i - f_i$:

i.e. fixed set of the (linear) Chevalley involution $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$, $\omega(h_i) = -h_i$, which corresponds to $a \to -a^T$

The "squared-mass" Quartic Operator $\widehat{\mu}^2$ in $\widehat{H}$

In the middle of the Weyl chamber (far from all the hyperplanes $\alpha_i(\beta) = 0$):

$$2\,\widehat{H} \simeq \widehat{\pi}^2 + \widehat{\mu}^2$$

where $\widehat{\mu}^2 \sim \sum \widehat{\Phi}^4$ gathers many complicated quartic-in-fermions terms (including $\sum \widehat{S}_{ab}^2$ and the infamous ψ^4 terms of supergravity).

Remarkable Kac-Moody-related facts:

• $\widehat{\mu}^2 \in$ *Center* of the algebra generated by the $K(AE_3)$ generators $\widehat{S}_{ab}$, $\widehat{J}_{ab}$

• $\widehat{\mu}^2$ is $\sim$ the square of a very simple operator $\in$ Center

$$\widehat{\mu}^2 = \frac{1}{2} - \frac{7}{8}\,\widehat{C}_F^2$$

where $\widehat{C}_F := \frac{1}{2}\,G_{ab}\,\widehat{\bar{\Phi}^a}\,\gamma^{\hat{1}\hat{2}\hat{3}}\,\widehat{\Phi}^b$.

Solutions of SUSY constraints

Overdetermined system of 4×64 Dirac-like equations

$$\widehat{\mathcal{S}}_A \Psi = \left(\frac{i}{2} \Phi_A^a \frac{\partial}{\partial \beta^a} + \ldots \right) \Psi = 0$$

Space of solutions is a mixture of "discrete states" and "continuous states", depending on fermion number $N_F = C_F - 3$. $\exists$ solutions for both even and odd N_F.

$\exists$ continuous states (parametrized by initial data comprising arbitrary *functions*) at $C_F = -1, 0, +1$.

Our results complete inconclusive studies started long ago: D'Eath 93, D'Eath-Hawking-Obregon 93, Csordas-Graham 95, Obregon 98, ...

Fermionic number operator

Fermionic number operator $\widehat{N}_F = \widehat{C}_F + 3$, with $\widehat{C}_F := \frac{1}{2}\, G_{ab}\, \widehat{\bar{\Phi}^a}\, \gamma^{\hat{1}\hat{2}\hat{3}}\, \widehat{\Phi}^b$.

Eigenvalues $C_F = -3, -2, -1, 0, 1, 2, 3$ or $0 \leq N_F \leq 6$

Fermionic "annihilation operators" $b^a_\pm$ and "creation operators" $\tilde{b} \equiv b^\dagger$

$$b^a_+ = \widehat{\Phi}^a_1 + i\, \widehat{\Phi}^a_2\,, \qquad b^a_- = \widehat{\Phi}^a_3 - i\, \widehat{\Phi}^a_4$$
$$\tilde{b}^a_+ = \widehat{\Phi}^a_1 - i\, \widehat{\Phi}^a_2\,, \qquad \tilde{b}^a_- = \widehat{\Phi}^a_3 + i\, \widehat{\Phi}^a_4$$

$$\left\{ b^a_\epsilon, \tilde{b}^b_\sigma \right\} = 2\, G^{ab} \delta_{\epsilon\, \sigma}\,,$$
$$\left\{ b^a_\epsilon, b^b_\sigma \right\} = \left\{ \tilde{b}^a_\epsilon, \tilde{b}^b_\sigma \right\} = 0\,.$$

with $\epsilon, \sigma = \pm$

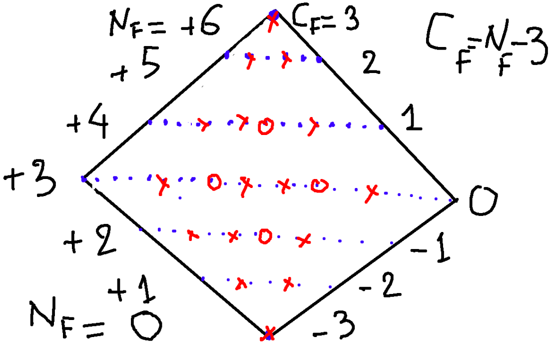
"Vacuum state" (empty) $b^a_\epsilon\, |0\rangle_- = 0 : N_F = 0$

[Filled state: $\tilde{b}^a_\epsilon\, |0\rangle_+ = 0 : \widehat{N}_F = 6$]

$$\widehat{N}_F = G_{ab}\, \tilde{b}^a_+\, b^b_+ + G_{ab}\, \tilde{b}^a_-\, \tilde{b}^b_-$$

Spin diamond

Spin $(8,4) =$ 64-dim space spanned by the spinor wavefunction Ψ_σ can be sliced by a fermionic number operator $\widehat{N}_F = \widehat{C}_F + 3$. Among these, the solutions (discrete = crosses or continuous = circle) of the susy constraints are marked in red.



Solution space of quantum susy Bianchi IX: $N_F = 0$

Level $N_F = 0$: $\exists$ unique "ground state" $|f\rangle = f(\beta)\,|0\rangle_-$ with

$$f = f_0\,[(y-x)(z-x)(z-y)]^{3/8}\,\frac{e^{-\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}}{(x\,y\,z)^{5/4}}$$

where $x := e^{2\beta_1} = \frac{1}{a^2}$, $y := e^{2\beta_2} = \frac{1}{b^2}$, $z := e^{2\beta_3} = \frac{1}{c^2}$

$|f\rangle$ differs from previously discussed "ground state" $\propto \exp\left(-\frac{1}{2}(a^2+b^2+c^2)\right)$ (Moncrief-Ryan, D'Eath, ...) by factors $\sim (x-y)^{3/8}\ldots$ vanishing on the symmetry walls. ["centrifugal barriers"]

Solution space of quantum susy Bianchi IX: $N_F = 1$

Previous approaches (D'Eath, Graham, ...) concluded to the absence of states at odd fermionic levels.

Level $N_F = 1$: $\exists$ two dimensional solution space:

$$c_+ \, f_a \, \tilde{b}_+^a \, |0\rangle_- + c_- \, f_a \, \tilde{b}_-^a \, |0\rangle_-$$

$$f_a = \{x(y-z), y(z-x), z(x-y)\} \, \frac{e^{-\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}}{(x\,y\,z)^{5/4}((x-y)(y-z)(z-x))^{3/8}}$$

Solution space: $N_F = 2$

$$\sum_{\substack{\sigma,\sigma'=\pm \\ k,k'=1,\dots,3}} f^{\sigma\sigma'}_{kk'}\,\tilde{b}^{k}_{\sigma}\,\tilde{b}^{k'}_{\sigma'}\,|0\rangle_{-}\,, \qquad f^{\sigma\sigma'}_{kk'} = -f^{\sigma'\sigma}_{k'k}$$

$\exists$ mixture of discrete (finite-dimensional) and continuous (infinite-dimensional; free functions) space of solutions

Example of two-dimensional subspace of discrete solutions

$$\{f^{\epsilon\epsilon}_{12}, f^{\epsilon\epsilon}_{23}, f^{\epsilon\epsilon}_{31}\} = f^{\epsilon\epsilon}\left\{x(y-z)-yz+\frac{xyz}{2},\ y(z-x)-zx+\frac{xyz}{2},\ z(x-y)-xy+\frac{xyz}{2}\right\}$$

with the prefactor functions $f^{\epsilon\epsilon}$ given by

$$f^{\epsilon\epsilon} = e^{-\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}(xyz)^{-3/4}\,[(x-y)(y-z)(x-z)]^{-1/8}$$
$$\left[C_1(x-z)^{-1/2} + \epsilon\, C_2(y-z)^{-1/2}\right]$$

Solution space: $N_F = 2$ (continued)

Example of infinite-dimensional space of solutions

$$k_{(ab)}\, \tilde{b}_+^a\, \tilde{b}_-^b\, |0\rangle_-$$

with a symmetric $k_{ab} = f^{+-}_{(ab)}$, with 6 components, that satisfies Maxwell-type equations

$$\frac{i}{2}\, \partial^k\, k_{kp} + \varphi^k\, k_{kp} - 2\, \rho^{kl}{}_p k_{kl} = 0$$

$$\frac{i}{2}\, \partial_{[p}\, k_{q]r} - \varphi_{[p}\, k_{q]r} + \mu_{pq}{}^k k_{kr} + 2\, \rho_{[p|r|}{}^k k_{q]k} = 0$$

similar to $\delta k \sim 0$, $dk \sim 0$. The compatibility equations (linked to $d^2 = 0 = \delta^2$) are satisfied because of Bianchi-like identities. Then one can separate the problem into:

(i) five constraint equations (containing only "spatial" ∂_{β^x} derivatives)

(ii) six evolution equations (containing "time" derivative ∂_{β^0} with $\beta^0 \equiv \beta^1 + \beta^2 + \beta^3$)

$N_F = 2, 3, 4, 5, 6$

Finally, the general solution is parametrized by giving, as free data, two arbitrary functions of two variables (β^x, β^y). These free data allow one to compute the Cauchy data for the six k_{ab}, which can then be evolved by the six evolution equations: $\partial_0 \, k_{ab} = \dots$

At level $N_F = 3$, there is a similar mixture between discrete solutions and continuous ones.

Moreover, when $4 \leq N_F \leq 6$, there is a duality mapping solutions for N_F into solutions for $N'_F = N_F - 6$. E.g. $\exists$ unique state at $N_F = 6$:

$$[(y-x)(z-x)(z-y)]^{3/8} \, \frac{e^{+\frac{1}{2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}}{(x\,y\,z)^{5/4}} \, |0\rangle_+$$

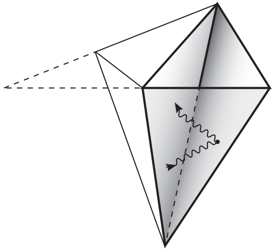
with $\tilde{b}^a_\epsilon \, |0\rangle_+ = 0$ (filled state)

Quantum Supersymmetric Billiard

The spinorial wave function of the universe $\Psi(\beta^a)$ propagates within the (various) Weyl chamber(s) and "reflects" on the walls (= simple roots of AE_3). In the small-wavelength limit, the "reflection operators" define a spinorial extension of the Weyl group of AE_3 (Damour Hillmann 09) defined within some subspaces of $\mathrm{Spin}(8,4)$

$$\widehat{\mathcal{R}}_{\alpha_i} = \exp\left(-i\,\frac{\pi}{2}\,\widehat{\varepsilon}_{\alpha_i}\,\widehat{J}_{\alpha_i}\right)$$

with $\widehat{J}_{\alpha_i} = \{\widehat{S}_{23}, \widehat{S}_{31}, \widehat{J}_{11}\}$ and $\widehat{\varepsilon}^2_{\alpha_i} = \mathrm{Id}$



Fermions and their dominance near the singularity

Crucial issue of boundary condition near a big bang or big crunch or black hole singularity: DeWitt 67, Vilenkin 82 ..., Hartle-Hawking 83 ..., ..., Horowitz-Maldacena 03

Finding in Bianchi IX SUGRA: the WDW square-mass term $\widehat{\mu}^2$ is *negative* (i.e. tachyonic) in most of the Hilbert space (44 among 64).

$$\mu^2 = \left(-\frac{59}{8}\Big|_0^1, -3\Big|_1^6, -\frac{3}{8}\Big|_2^{15}, +\frac{1}{2}\Big|_3^{20}, -\frac{3}{8}\Big|_4^{15}, -3\Big|_5^6, -\frac{59}{8}\Big|_6^1 \right) \qquad (1)$$

This is a quantum effect quartic in fermions:

$$\rho_4 \sim \psi^4 \sim \mu^2 \, (\mathcal{V}_3)^{-2} = \mu^2 \, (a\, b\, c)^{-2} = \mu^2 \bar{a}^{-6}$$

which dominates the other contributions near a small volume singularity $abc \to 0$.

Bouncing Universes and Quantum Boundary Conditions at a Spacelike Singularity

When μ^2 is negative, such a "stiff", negative $\rho_4 = p_4$ classically leads to a quantum avoidance of a singularity, i.e. a bounce of the universe. Quantum mechanically, the general solution of the WDW equation (in "hyperbolic polar coordinates" $\beta^a = \rho\gamma^a$)

$$\left(\frac{1}{\rho}\partial_\rho^2\,\rho - \frac{1}{\rho^2}\Delta_\gamma + \hat{\mu}^2\right)\Psi'(\rho,\gamma^a) = 0$$

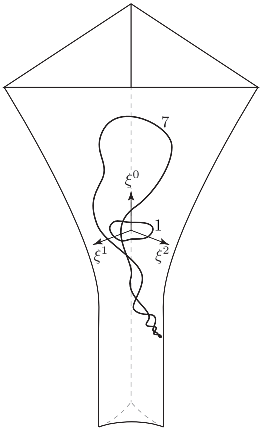
behaves, after a quantum-billiard mode-expansion $\Psi'(\rho,\gamma^a) = \sum_n R_n(\rho)\, Y_n(\gamma^a)$, as

$$\rho\, R_n(\rho) \equiv u_n(\rho) \approx a_n e^{-|\mu|\rho} + b_n e^{+|\mu|\rho}\ , \text{ as } \rho \to +\infty$$

This suggests to impose the boundary condition $\Psi' \sim e^{-|\mu|\rho} \to 0$ at the singularity, which is, for a black hole crunch, a type of "final-state boundary condition.

Classical Bottle Effect

Classical confinement between $\mu^2 < 0$ for small volumes, and the usual closed-universe recollapse (Lin-Wald) for large volumes $\Rightarrow$ periodic, cyclically bouncing, solutions (Christiansen-Rugh-Rugh 95).

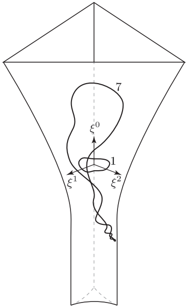


Quantum Bottle Effect ?

? $\exists$ a set of discrete quantum states, corresponding (à la Selberg-Gutwiller) to the classical periodic solutions ? These would be excited avatars of the $N_F = 0$ "ground state"

$$\Psi_0 = (abc)\left[(b^2 - a^2)(c^2 - b^2)(c^2 - a^2)\right]^{3/8} e^{-\frac{1}{2}(a^2+b^2+c^2)} | 0 \rangle_-$$

and define a kind of quantum storage ring of near-singularity states (ready for tunnelling, via inflation, toward large universes).



Conclusions

• The case study of the quantum dynamics of a triaxially squashed 3-sphere (Bianchi IX model) in (simple, $D = 4$) supergravity confirms the hidden presence of hyperbolic Kac-Moody structures in supergravity. [Here, AE_3 and $K(AE_3)$]

• The wave function of the universe $\Psi(\beta^1, \beta^2, \beta^3)$ is a 64-dimensional spinor of $\mathrm{Spin}(8, 4)$ which satisfies Dirac-like, and Klein-Gordon-like, wave equations describing the propagation of a "quantum spinning particle" reflecting off spin-dependent potential walls which are built from quantum operators $\widehat{S}_{12}, \widehat{S}_{23}, \widehat{S}_{31}$, $\widehat{J}_{11}, \widehat{J}_{22}, \widehat{J}_{33}$ that generate a 64-dim representation of $K(AE_3)$. The squared-mass term $\widehat{\mu}^2$ in the KG equation belongs to the center of this algebra.

• The space of solutions is a mixture of "discrete states" and "continuous states" (depending on fermion number $N_F = C_F + 3$). $\widehat{\mu}^2$ is mostly negative and can lead to a quantum avoidance of the singularity, and to a bottle effect near the singularity.