Lecture notes on modular tensor categories and braid group representations

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Please do not share these notes outside of our class yet, as they are under construction and the material is to appear in a chapter of my dissertation.

Link to Week 1 lecture notes: `http://web.math.ucsb.edu/~shokrian/notes_227C`

Contents

1 Motivations for modular tensor category theory

1.1 Review: the bulk-boundary correspondence

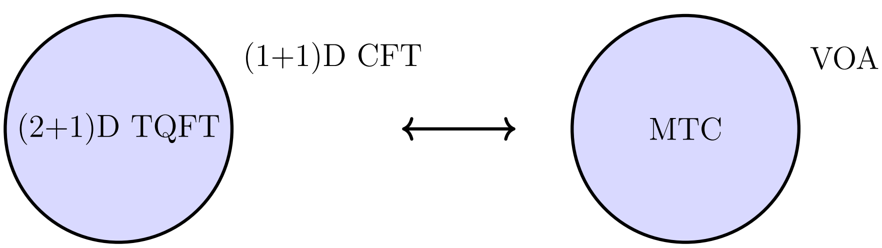


Figure 1: The bulk-boundary correspondence between topological quantum field theory in a “bulk” region of 2-dimensional space and the 1-dimensional conformal field theory describing physics on the boundary of the region.

Conjecture 1 *Every modular tensor category is the representation category of some vertex operator algebra.*

One goal of this course is to study a closely related problem.

Question 1 *Does every vector-valued modular form come from a chiral conformal field theory?*

1.2 MTCs as models for anyons in (2+1)D topological phases of matter

Another way to get at the relationship between TQFT and MTCs is to understand topological phases of matter.

These are materials which behave according to the laws of topological quantum field theory in the low energy limit. When quantum systems of physical bosons (or fermions) are confined to 2 spatial dimensions, (for example by applying a strong magnetic field $\vec{B}$ to a sea of electrons), the ideal, zero-temperature physics is described by a (2+1)D unitary (spin) TQFT. The collective quantum system is then said to be in a bosonic (or fermionic) (2+1)D topological phase of matter (TPM) and may support emergent quasiparticles called anyons and a phenomenon known as ground state degeneracy.

1.2.1 Gapped (2+1)D TPM and quantum error correction

When the next excited states above the degenerate ground states are separated by a finite amount of energy ΔE, the system is said to be *gapped*. Otherwise, there exist states with arbitrarily small energy above the ground state, and it is *gapless*.

One thing that makes these topological phases of matter *topological* is that they can support gaps which are large, in the sense that the separation between ground states and excited states decays exponentially slowly in the "size" of the system. For example, the distance between sites on a lattice, or on the length scale of a topological nanowire.

We say that the ground states are *topologically protected* from the excited states.

The physical theory of these anyons in the TPM is then given by a modular tensor category - objects are collections of anyons and morphisms are anyon processes.

We will see that the structure of a unitary modular tensor category (UMTC) is precisely what is required to do topological quantum mechanics on the Hilbert spaces of states of collections of anyons.

Further, a skeletal UMTC will be precisely what is required to do topological quantum mechanics on Hilbert spaces *with bases*.

There are several essentially equivalent ways to think about modular tensor categories. Abstractly, an MTC will be a special kind of *braided fusion category*.

2 Overview of MTC theory

We have several ways to think about an anyon model: as a category, as some complex numbers satisfying certain equations, or as rules for drawing pictures. We will work with these different formulations and the tools each affords interchangeably.

2.1 Braided fusion categories

Definition 1 *A modular tensor category is a nondegenerate ribbon fusion category.*

To unpack this definition, we need to understand all of the additional structure that a MTC has on top of being a monoidal category.

Definition 2 *A tensor category is a locally finite, $\mathbb{C}$-linear, rigid monoidal category with simple tensor unit.*

Definition 3 *A fusion category is a finite semisimple $\mathbb{C}$-linear category.*

Definition 4 *A ribbon fusion category is a spherical braided fusion category.*

Two MTCs are equivalent if they are related by a braided auto-equivalence functor.

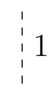
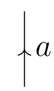
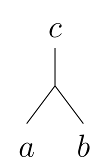
2.2 6j braided fusion systems

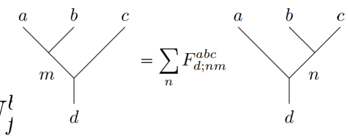
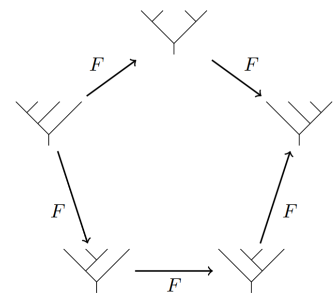
In a way that categorifies the fact that every finite-dimensional vector space is isomorphic to a finite-dimensional vector space with basis, every braided fusion category (BFC) is equivalent to a *skeletal* braided fusion category.

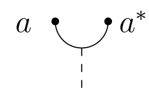
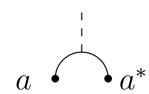
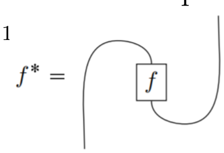
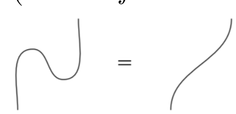
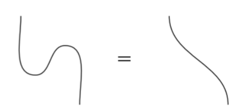
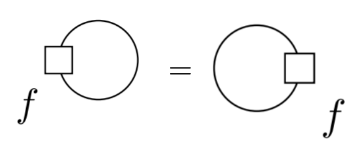
A skeletal BFC category can be astracted as a finite set of symbols $\{N_c^{ab}, R_c^{ab}, [F_d^{abc}]_{ef}\}$ satisfying a finite set of equations, with a notion of when two solutions to these equations are equivalent. These are *6j-braided fusion systems*.

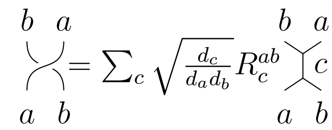
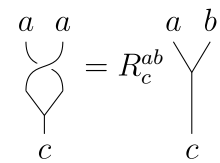
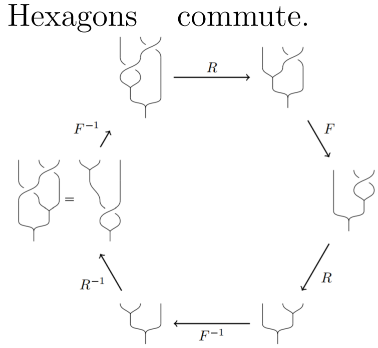
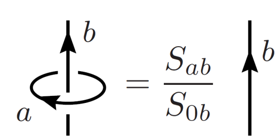
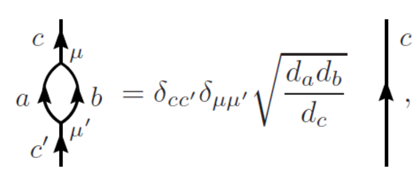
In the unitary case, writing down a specific set of solutions $\{N_c^{ab}, R_c^{ab}, [F_d^{abc}]_{ef}\}$ entails "fixing a gauge". Transforming the data to an equivalent set of data is referred to as "using gauge freedom".

3 Modular tensor category dictionary

Anyon model	Unitary modular tensor category (UMTC)	6j-braided fusion system	Graphical calculus
Algebraic theory of anyons in a (2+1)D TPM	Non-degenerate ribbon fusion category $$(\mathcal{C}, \oplus, \otimes, \mathbb{1}, \alpha, c, \phi)$$	Complex numbers $$\{N_c^{ab}, R_c^{ab}, [F_d^{abc}]_{nm}\}$$ satisfying certain equations	Admissibly labeled trivalent graphs satisfying local relations
Anyon types	Isomorphism classes of simple objects $\mathrm{Irr}(\mathcal{C})$	Label set $\mathcal{L}$	$\bullet\, a$
Vacuum	Monoidal unit $\mathbb{1}$ is simple	$1 \in \mathcal{L}$	
Time evolution	$$\mathrm{id}_a \in \mathrm{Hom}(a, a)$$ $$\mathrm{Hom}(a, b) = 0 \text{ if } a \ncong b$$	$$N_a^{a1} = N_a^{1a} = 1$$ for all $a \in \mathcal{L}$	
Quantum superposition	$\mathcal{C}$ is an abelian category with $\mathbb{C}$-linear bifunctor $$\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$ with bilinear composition so that $\mathrm{Hom}(X, Y)$ is a $\mathbb{C}$-vector space		Formal $\mathbb{C}$-linear addition of diagrams
Fusion	Bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ so that $\mathrm{Hom}(X, Y)$ is a $\mathbb{C}$-vector space	Fusion coefficients N_c^{ab} $$N_c^{ab} = \dim\left(\mathrm{Hom}(c, a \otimes b)\right)$$	Admissibly labeled trivalent graph 

Anyon model	Unitary modular tensor category (UMTC)	6j-braided fusion system	Graphical calculus
Change of basis on state space	Monoidal structure $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ $$\alpha_{W,X,Y\otimes Z} \circ \alpha_{W\otimes X,Y,Z} \quad (1)$$ $$= (\mathrm{id}_W \otimes \alpha_{X,Y,Z}) \circ \alpha_{W,X\otimes Y,Z} \circ (\alpha_{W,X,Y} \otimes \mathrm{id}_Z) \quad (2)$$	Fusion associative on level of isomorphism classes $\sum_e N_e^{ab} N_d^{ec} = \sum_f N_d^{af} N_f^{b}$ $[F_d^{abc}]_{nm}$ satisfying	 
	Triangle axiom	$F_d^{abc} = I$ when any of $a, b,$ or c are 1.	

Anyon model	Unitary modular tensor category (UMTC)	6j-braided fusion system	Graphical calculus
Generation	$\mathrm{coev}_a : \mathbb{1} \to a \otimes a^*$ $\mathrm{coev}'_a : \mathbb{1} \to a^* \otimes a$	For all $a \in \mathcal{L}$, there exists a unique $a^* \in \mathcal{L}$ such that $N_1^{aa^*} = N_1^{a^*a} = 1$	
Annihilation	$\mathrm{ev}_a : a \otimes a^* \to \mathbb{1}$ $\mathrm{ev}'_a : a^* \otimes a \to \mathbb{1}$		
Time reversal	Rigidity: the evaluation and co-evaluation morphisms satisfy $(\mathrm{id}_X \otimes \mathrm{ev}_X) \circ \alpha_{X,X^*,X} \circ (\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X$ $(\mathrm{ev}_X \otimes \mathrm{id}_{X^*}) \circ \alpha^{-1}(X^*, X, X^*) \circ (\mathrm{id}_{X^*} \otimes \mathrm{coev}_X)$ and similarly for coev′ and ev′.	For any $a \in \mathcal{L}$, $\left([F^{a^*aa^*}_{a^*}]^{-1}\right)_{11} = [F^{aa^*a}_{a}]_{11}$	Dual morphisms  Rigidity axiom (Take f = id)   "lines can be straighted"
Hilbert space of states	Spherical pivotal structure on $\mathcal{C}$ Isomorphisms $\phi_X : X \to X^{**}$ satisfying $\phi_{X\otimes Y} = \phi_X \otimes \phi_Y$ and $f^{**} = f$ for morphisms $f : X \to Y$ $\mathrm{Tr}^R(f) = \mathrm{ev} \circ (\phi_X \otimes \mathrm{id}_{X^*}) \circ (f \otimes \mathrm{id}_{X^*}) \circ \mathrm{coev}_X$ and similarly for Tr^L. Spherical if $\mathrm{Tr}^L(f) = \mathrm{Tr}^R(f)$	For each $a \in \mathcal{L}$, there exist a root of unity t_a called the pivotal coefficient so that $t_1 = 1$ $t_{a^*} = t_a^{-1}$ $t_a^{-1} t_b^{-1} t_c = [F_1^{abc}]_{a^*,c}[F_1^{bc^*a}]_{a^*a}[F_1^{c^*ab}]_{b^*b}$ The pivotal coefficients are spherical if $t_a \in \{\pm 1\}$.	Diagrammatic left and right traces equal  "Diagrams live on a sphere"

Anyon model	Unitary modular tensor category (UMTC)	6j-braided fusion system	Graphical calculus
Anyonic quantum systems evolve under exchange	Braiding morphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$	R-symbols R^{ab}_c	Crossings can be resolved.   and similarly for R^{-1}.
Exchange compatible with fusion	Braiding given by family of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$	R^{ab}_c satisfies hexagon equations $R^{ac}_m [F^{acb}_d]_{nm} R^{bc}_n \quad (3)$ $= \sum_l [F^{cab}_d]_{lm} R^{lc}_d [F^{abc}_d]_{nl} \quad (4)$ and $(R^{ca}_m)^{-1} [F^{acb}_d]_{nm} (R^{cb}_n)^{-1} \quad (5)$ $= \sum_l [F^{cab}_d]_{lm} (R^{cl}_d)^{-1} [F^{abc}_d]_{nl} \quad (6)$	Hexagons commute.  and similarly for R^{-1}.
Can put TPM on surface Σ_g	Non-degenerate ribbon fusion category $\mathcal{C}$ (Modular tensor category)	S-matrix is non-singular	
Unitarity	Conjugation on Hom spaces: for every $f \in \mathrm{Hom}(X,Y)$, there exists $\bar{f} \in \mathrm{Hom}(Y,X)$ which is conjugate linear, $\bar{\bar{f}} = f,\ f \mathbin{\bar{\otimes}} g = \bar{f} \otimes \bar{g},\ f \mathbin{\bar{\circ}} g = \bar{g} \circ \bar{f}$ $\overline{\mathrm{coev}_X} = \mathrm{ev}'_X, \overline{\mathrm{ev}_X} = \mathrm{coev}'_X, \overline{c_{X,Y}} = c^{-1}_{X,Y}, \overline{\theta_X} = \theta^{-1}_X$ $\mathrm{Tr}(f \circ \overline{f}) \geq 0$ for all f	$R^{ab}_c, [F^{abc}_d]$ unitary with respect to conjugate transposition of matrices, S-matrix unitary, $d_a \geq 1$ for all $a \in \mathcal{L}$	Conjugation is horizontal reflection of diagrams 

4 Examples

Example 1 (Semion topological order)

Anyons	$\mathcal{L} = \{1, s\}$
Fusion	$s \otimes s = 1$
R-symbols	$R_1^{ss} = i$
F-symbols	$F_s^{sss} = -1$
Quantum dimensions	$d_s = 1$
	$\mathcal{D} = \sqrt{2}$
Twists	$\theta_s = i$
S-matrix	$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Example 2 (Fibonacci topological order)

Anyons	$\mathcal{L} = \{1, \tau\}$
Fusion	$\tau \otimes \tau = 1 \oplus \tau$
R-symbols	$R_1^{\tau\tau} = e^{-4\pi i/5}$
	$R_\tau^{\tau\tau} = e^{3\pi i/5}$
F-symbols	$F_\tau^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}$
Quantum dimensions	$d_\tau = \phi$
	$\mathcal{D} = \sqrt{2 + \phi}$
Twists	$\theta_\tau = e^{4\pi i/5}$
S-matrix	$S = \frac{1}{\sqrt{2+\phi}} \begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix}$

5 Invariants of UMTCs/(Symbols invariant under skeletalization)

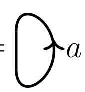
Next we introduce several important invariants of a UMTC $\mathcal{C}$. These quantities are preserved by braided-tensor autoequivalence functors of $\mathcal{C}$. In particular, given a skeletal UMTC, such quantities will be independent of the specific set of solutions $\{N_c^{ab}, R_c^{ab}, [F_d^{abc}]_{nm}\}$ to the consistency equations.

In this section we will denote the rank of the UMTC by $rank(\mathcal{C}) = |\mathcal{L}| = n$.

5.1 Quantum dimensions

The traces of the identity morphisms on simple objects id_a give invariants called *quantum dimensions* d_a.

$$\mathrm{Tr}(\mathrm{id}_a) =$$



$$= d_a.$$

The quantum dimensions satisfy $d_1 = 1$ and $d_a = d_{a^*}$ for all $a \in \mathcal{L}$.

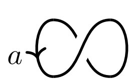
A related invariant of UMTCs is the *global quantum dimension* $\mathcal{D}$, which is the positive square root

$$\mathcal{D} = \sqrt{\sum_a d_a^2}.$$

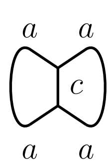
5.2 Twists

The trace of the braiding isomorphism of a simple object with itself is an invariant.

$$\mathrm{Tr}(c_{aa}) =$$



$$= \sum_c R_c^{aa}$$



$$= \sum_c \sqrt{\frac{d_c}{d_a^2}} R_c^{aa} \sqrt{d_a d_a d_c} = \sum_c d_c R_c^{aa}$$

Dividing by the quantum dimension of the anyon a gives the invariant θ_a, called the *topological twist* of a.

$$\theta_a = \frac{1}{d_a}$$

$$= \sum_c \frac{d_c}{d_a} R_c^{aa}$$

The twists can be organized into the diagonal T-matrix

$$T_{ab} = \theta_a \delta_{ab}$$

It is a theorem due to Vafa that each θ_i is a root of unity. Put $\theta_i = e^{2\pi i/r_i}$. Then it follows that

$$|T| = \mathrm{lcm}(r_1, r_2, \ldots, r_n).$$

5.2.1 The ribbon property

The R-symbols and twists satisfy the equation

$$\sum_c R_c^{ab} R_c^{ba} = \frac{\theta_c}{\theta_a \theta_b}$$

.

5.3 The central charge

Define

$$p_\pm = \sum_i \theta_i^{\pm 1} d_i^2.$$

When $\mathcal{C}$ is an MTC,

$$p_+/\mathcal{D} = e^{2\pi i c/8}$$

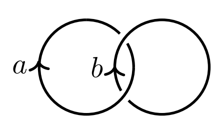
for some $c \in \mathbb{Q}$, known as the *central charge* of $\mathcal{C}$. The *topological central charge* c_{top} of $\mathcal{B}$ is

$$c_{top} = c \mod 8.$$

5.4 S-matrix

Another important invariant is given by the trace of the double braiding:

$$S_{ab} = \mathrm{Tr}(c_{b,a^*} \circ c_{a^*,b}) = \frac{1}{\mathcal{D}}$$



Exercise 1 *Use the graphical calculus and the ribbon property to show that*

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c N_c^{a^* b} \frac{\theta_c}{\theta_a \theta_b} d_c.$$

6 The modular data and modular representation

Taken together, the set of matrices $\{S, T\}$ is called the modular data of a modular tensor category.

6.1 The modular representation

Let C be the *charge conjugation* matrix

$$C_{ab} = \begin{cases} 1 & b = a^* \\ 0 & b \neq a^* \end{cases}.$$

Observe that $C^2 = I$, since the dual of an anyon a^* is a.

Theorem 1 *[2] The matrices S and T satisfy the equations*

$$(ST)^3 = \Theta C \tag{7}$$
$$S^2 = C \tag{8}$$
$$C^2 = I_n \tag{9}$$

where

$$\Theta = \frac{1}{\mathcal{D}} \sum_{a \in \mathcal{C}} d_a^2 \theta_a = e^{2\pi i c/8}.$$

Recall c is the central charge of $\mathcal{C}$.

It follows that the map $\rho : PSL(2, \mathbb{Z}) \to U(n)$ that sends

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mapsto T$$

is a linear representation of $PSL(2,\mathbb{Z})$, and hence a projective representation of $SL(2,\mathbb{Z})$. Thus every UMTC gives a projective representation of the mapping class group of the torus $SL(2,\mathbb{Z})$. More generally, the non-degeneracy of the S-matrix means that we get projective representations of all mapping class groups.

Example 3 *Let $\mathcal{C}$ be the semion theory with $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. (See Zhenghan's lecture notes from April 3rd with the representation corresponding to A= (2).) Defining $\eta = e^{2\pi i c/8}$ and sending $T \mapsto \tilde{T} = \eta^{-1/3}T$, one has a linear representation of $SL(2,\mathbb{Z})$.*

Theorem 2 *[2] The image of the modular representation coming from a UMTC $\mathcal{C}$ has finite image.*

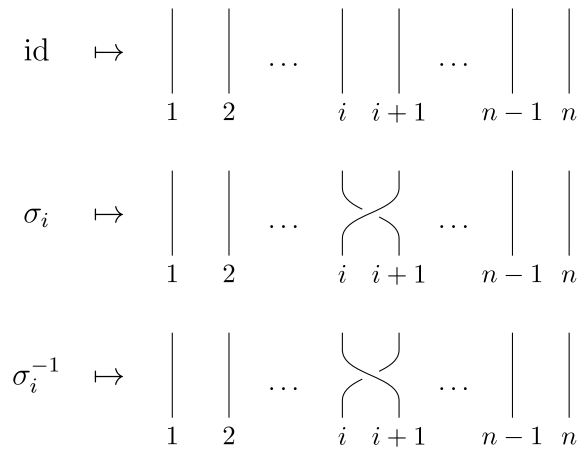
7 Representations of $\mathcal{B}_3$ from UMTCs

7.1 The n-strand braid group

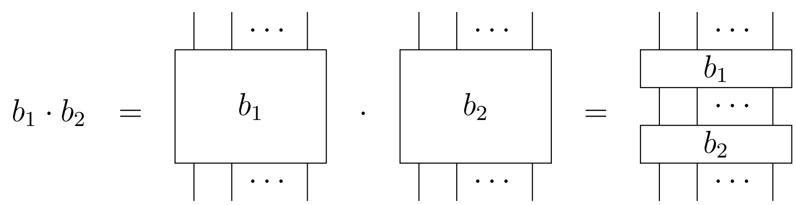
The n-strand braid group $\mathcal{B}_n$ has presentation

$$\mathcal{B}_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{r} \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } i = 1,2,\ldots,n-1 \end{array} \right\rangle.$$

The n braid group can be understood diagrammatically by identifying its elements as elementary braid diagrams on n-strands. We use the following conventions.

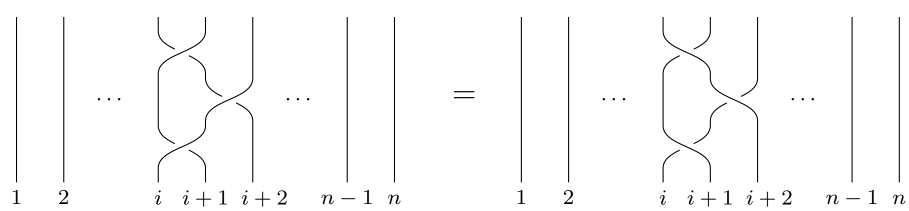


Multiplication is given by stacking of diagrams, and we take the convention that $b_1 \cdot b_2$ is b_1 stacked on top of b_2.



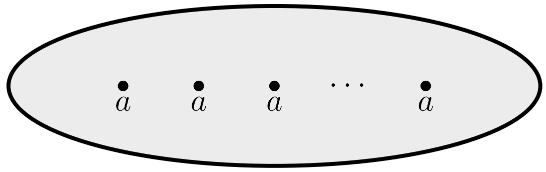
The far commutativity and braid relations then correspond to braid-isotopy and the Reidemeister III move.

picture of far commutativity coming soon



7.2 Action of $\mathcal{B}_n$ on $\mathrm{Hom}(i, a^{\otimes n})$

Recall from the two previous lectures the Hilbert space of states associated to a collection of n anyons of type a with total charge i, which we denote here by $V_i^{a^{\otimes n}}$.



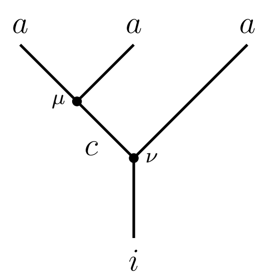
$$\mathrm{Hom}(i, a \otimes a \otimes \cdots \otimes a) \cong V_i^{a^{\otimes n}}.$$

The n-strand braid group acts on this vector space, and an explicit matrix representation can be found by stacking braid diagrams on a chosen fusion tree basis and resolving using the R- and F-moves of the graphical calculus. See the next section for a concrete example.

8 The action of $\mathcal{B}_3$ on state spaces of three anyons

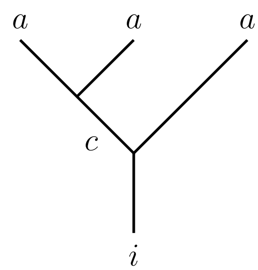
In this course we will primarily be interested in representations of the 3-strand braid group $\mathcal{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. Observe that the far commutativity relation does not apply because none of the strands are ever more than one strand apart.

$\mathrm{Hom}(i, a \otimes a \otimes a) \cong \mathbb{C}\left[\left\{\right.\right.$



$\left.\left. \Bigg| \; N_c^{aa} \cdot N_i^{ca} \neq 0, 1 \leq \mu \leq N_c^{aa}, 1 \leq \nu \leq N_i^{ca} \right\}\right]$

For multiplicity-free theories, this simplifies to

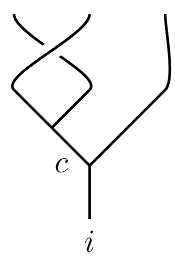
$$\mathrm{Hom}(i, a \otimes a \otimes a) \cong \mathbb{C}\left[\left\{\right.\right.$$


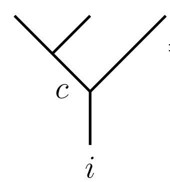
$$\left.\left.\middle|\; N_c^{aa} \cdot N_i^{ca} \neq 0 \right\}\right]$$

Below we use the notation $|a, a, a; i; c\rangle$ to indicate the fusion tree basis vector of $V_i^{a^{\otimes 3}}$ with internal edge labeled by $c \in \mathcal{L}$.

To understand the group action of $\mathcal{B}_3$ on $V^{a_i^{\otimes 3}}$, it suffices to understand how the generators σ_1 and σ_2 act.

With the left-associated basis, the braid that results from stacking σ_1 can be resolved with a single R-move.

$$\sigma_1 \cdot |a, a, a; i; c\rangle \quad =$$


$$= R_c^{aa}$$


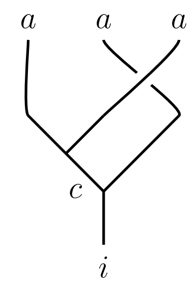
$$= R_c^{ab}|a, a, a; i; c\rangle$$

It follows that the matrix representation of σ_1 is diagonal in the left associated basis:

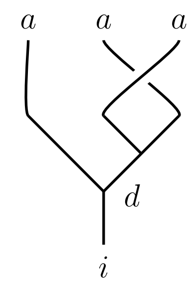
$$\rho(\sigma_1) = \mathrm{diag}(R_{c_1}^{aa}, R_{c_2}^{aa}, \ldots R_{c_d}^{aa})$$

As for the second generator, we have

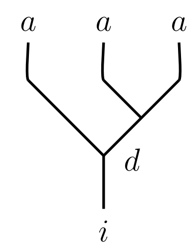
$$\sigma_2 \cdot |a,a,a;i;c\rangle \quad =$$



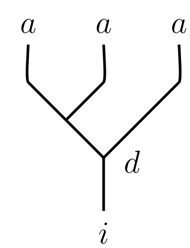
$$= \quad \textstyle\sum_d [F_i^{aaa}]_{d,c}$$



$$= \quad \textstyle\sum_d [F_i^{aaa}]_{d,c} R_d^{aa}$$



$$= \quad \textstyle\sum_{d,e} [F_i^{aaa}]_{d,c} R_d^{aa} [F_i^{aaa}]^{-1}_{e,d}$$



$$= \quad \textstyle\sum_{d,e} [F_i^{aaa}]_{d,c} R_d^{aa} [F_i^{aaa}]^{-1}_{e,d} \quad |a,a,a;i;e\rangle$$

Exercise 2 *Show that*

$$\rho(\sigma_2) = [F_i^{aaa}]^{-1} \rho(\sigma_1) [F_i^{aaa}]$$

and use this to show that the representation of $\mathcal{B}_3$ *on the morphism spaces* $\mathrm{Hom}(i, a \otimes a \otimes a)$ *of a UMTC is unitary.*

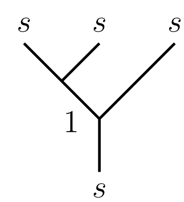
The image of the 3-strand braid group associated to n anyons of type a with total charge i in a UMTC $\mathcal{C} = \{N_c^{ab}, R_c^{ab}, [F_d^{abc}]_{ef}\}$ is generated by the two matrices $\rho(\sigma_1)$ and $\rho(\sigma_2)$.

Whether the subgroup of $d \times d$ unitary matrices realized by braiding anyons is finite, infinite, or dense depends on the UMTC at hand.

9 Examples

9.1 Semion UMTC

Since $s \otimes s \otimes s = s$, the total charge is fixed to be s and there is only one admissibly labeled fusion tree from s to $s \otimes s \otimes s$.



So $\dim(V_s^{s^{\otimes 3}}) = 1$ and we get a 1-dimensional representation of $\mathcal{B}_3$.

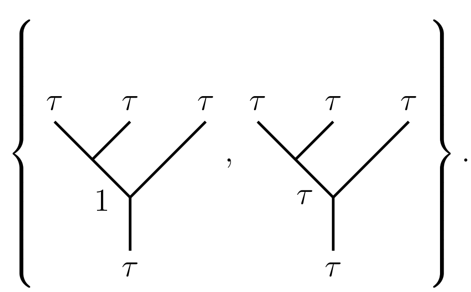
One can check that $\rho(\sigma_1) = R_1^{ss} = i$ and $\rho(\sigma_2)[F_s^{sss}]_{11} R_1^{ss}([F_s^{sss}]^{-1})_{11} = i$.

While as a linear representation the image is isomorphic to $\mathbb{Z}_4$, the projective image (up to a U(1) factor) is trivial.

9.2 Fibonacci UMTC

The fusion rule $\tau \otimes \tau = 1 \oplus \tau$ allows for two possibilities for the total charge of three Fibonacci anyons, either 1 or τ. One can check that the vector space $\mathrm{Hom}(1, \tau \otimes \tau \otimes \tau)$ is 1-dimensional, and $\mathrm{Hom}(\tau, \tau \otimes \tau \otimes \tau)$ is 2-dimensional.

A left-associated fusion basis is given by



With respect to this basis the matrix representation of $\mathcal{B}_3$ is determined by

$$\rho(\sigma_1) = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}$$

$$\rho(\sigma_2) = \begin{pmatrix} \phi^{-1}e^{4\pi i/5} & \phi^{-1/2}e^{-3\pi i/5} \\ \phi^{-1/2}e^{-3\pi i/5} & -\phi^{-1} \end{pmatrix}$$

Theorem 3 (Freedman, Larsen, Wang) *The Fibonacci representation on $V_\tau^{\tau^{\otimes 3}}$ is dense in U(2):*

$$\overline{\rho(\mathcal{B}_3)} \supset U(2).$$

References

[1] M. Barkeshli, P. Bonderson, M. Cheng, Z. Wang, *Symmetry, defects, and gauging of topological phases.* arXiv:1410.4540. (2014).

[2] Z. Wang. *Topological quantum computation.* CBMS Regional Conference Series in Mathematics. Vol. 112, No. 201. (2010). `http://web.math.ucsb.edu/~zhenghwa/data/course/cbms.pdf`

[3] P. Etinghof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories.* AMS. Mathematical surveys and monographs. Vol. 205. (2015). `http://www-math.mit.edu/~etingof/egnobookfinal.pdf`

10 Glossary

Anyon	Quasiparticle excitation in a gapped (2+1)D topological phase of matter, modeled mathematically by a simple object in a UMTC.
Anyon type	Also called the topological charge of an anyon, corresponds to the isomorphism class of a simple object in a UMTC.
Braid groups $\mathcal{B}_n$	The n-strand braid group $\mathcal{B}_n$ is the fundamental group of the n-punctured disk. Since the evolution of the quantum state of a collection of anyons under exchange depends only on the topology of the spacetime trajectories of the anyons, there is an action of the n-strand braid group on the Hilbert space of states associated to a collection of n anyons of type a.
Boson (physical)	A particle is a boson if its exchange statistics are trivial, i.e. performing a full exchange does not change the state. A bosonic (2+1)D TPM is a TPM whose underlying physical particles are bosons, as opposed to fermions.
Boson (emergent)	An anyon in a (2+1)D topological phase of matter anyon is a boson if its exchange statistics are trivial, i.e. performing a full exchange does not change the state. A simple object b in a UMTC is a boson if it has trivial quantum dimensions and twists, $d_b = 1$ and $\theta_b = 1$.
BFC	Braided fusion category
Central charge	An invariant of a RFC which is determined by the quantum dimensions and twists. For MTCs the central charge measures the framing anomaly of the corresponding (2+1)D TQFT.
CFT	Conformal field theory
Fermion (physical)	A particle is a fermion if performing a full exchange of two such particles changes the state by -1.
Fermion (emergent)	An abelian anyon in a (2+1)D TPM is a fermion its exchange statistics are -1. An anyon f in a UMTC is a fermion if $d_f = 1$ and $\theta_f = -1$.
Modular data	The modular data of an MTC is the set $\{S_{ab}, T_a\}$ consisting of the invariants associated to the Hopf link and the once-twisted unknot respectively.
MTC	Modular tensor category
SFC	Spherical fusion category
TO	Topological order
TPM	Topological phase of matter
TQC	Topological quantum computation
TQFT	Topological quantum field theory
UMTC	Unitary modular tensor category
χCFT	Chiral conformal field theory