Feynman path integrals on phase space and the metaplectic representation

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1 Introduction

According to the WKB method a solution of Schrödinger's equation has the form

$$f(t_1, x_1) = \int_{x_0 \in \mathbb{R}^n} e^{i\,S(t_1,x_1,t_0,x_0)/\hbar} A(t_1, x_1, t_0, x_0) f(t_0, x_0)\, dx_0 \qquad (1)$$

asymptotically as $\hbar \to 0$. The exponent S satisfies the Hamilton-Jacobi equation from classical mechanics. According to the principal of stationary phase the major conribution to such an integral occurs at the critical points of the exponent.

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Feynman in [9] (see also [10]) argued similarly. He argued that integrals

$$f(t_1, x_1) = \int\limits_{x(t_1)=x} e^{i\,I(x)/\hbar} f(x(t_0))\mathcal{D}\,x \tag{2}$$

over the space of all curves $x : [t_0, t_1] \to \mathbb{R}^n$ are fundamental. He was led to integrals of this type by physical considerations. He assigned a phase $e^{iI(x)/\hbar}$ to each classical path x and summed over all paths x. The exponent $I(x)$ is the action integral from classical mechanics:

$$I(x) = \int_{t_0}^{t_1} K(\dot{x}) - V(x)\,dt$$

where K is the kinetic energy and V is the potential energy. The Euler-Lagrange equations of I are Newton's equations of motion. Hence by the principal of stationary phase the major contribution to (2) should occur at the classical trajectories.

The generating function S of equation (1) is obtained from the action integral $I(x)$ of equation (2) by evaluating $I(x)$ at 'the' classical trajectory x such that $x(t_0) = x_0$ and $x(t_1) = x_1$. If Feynman's integral were finite dimensional one might integrate out all the variables except x_0 and arrive at (1). Something analogous occurs in Hörmander's theory [13] of Fourier integral operators where the phase function which appears in the expression for a Fourier integral operator can be replaced by another phase function which defines the same symplectic relation. In fact, this is almost exactly what Feynman did. He replaced the integral $I(x)$ by a finite dimensional approximation and evaluated the resulting finite dimensional integral to get something of form (1).

In this paper we do something similar to what Feynman did. Unlike Feynman, we use paths in phase space rather than configuration space and use the symplectic action integral rather than the (classical) Lagrangian integral. We eventually restrict to (inhomogeneous) quadratic Hamiltonians so that the finite dimensional approximation to the path integral is a Gaussian integral. In evaluating this Gaussian integral the signature of a quadratic form appears. This quadratic form is a discrete approximation to the second variation of the action integral. We obtain the formula of Leray [16] for the Metaplectic representation.

For Lagrangians of the form kinetic energy minus potential energy, evaluated on curves in configuaration space, the index of the second variation is well-defined and, via the Morse Index Theorem, related to the Maslov Index of the corresponding linear Hamiltonian system. The second variation of the symplectic action has both infinite index and infinite coindex. However, this second variation does have a well-defined signature via the aforementioned discrete approximation. This signature can be expressed in terms of the Maslov index of the corresponding linear Hamiltonian system. This is a symplectic analog of the Morse Index Theorem.

Our topic has a vast literature. Our formula for the metaplectic representation appears in [16] where it is obtained by other arguments. Souriau [27] found an explicit solution for the quantum harmonic oscillator involving the Maslov index (thus correcting Feynman's original formula which is valid only for short times). Keller [14] first noticed the phase shift due to the Maslov index in Theorem 8.5 below and for this reason the Maslov index is sometimes called the *Keller-Maslov index*. Duistermaat's article [8] explains how to interpret the Morse index in terms of the Maslov index but in the situation studied here the Morse index is undefined. The article [1] explains how Feynman and Dirac [5] were motivated by using the method of stationary phase to obtain classical mechanics as the limit (as $\hbar \to 0$) of quantum mechanics. Daubechies and Klauder [6] (see also [7]) have formulated a theory of path integrals on phase space where the Hamiltonian function can be any polynomial. They remark that the 'time slicing' construction used by Feynman does not generalize. However, our Hamiltonians are at worst quadratic and Feynman's original method is adequate.

2 Affine Hamiltonian Mechanics

An affine symplectomorphism has the form

$$\psi(z) = \Psi(z) + w$$

where $w \in \mathbb{R}^{2n}$ and $\Psi \in \mathrm{Sp}(2n)$. Here the linear part $\Psi \in \mathrm{Sp}(2n)$ has block matrix form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3}$$

Then the equation $z' = \psi(z)$ takes the form

$$\begin{array}{rcl} x' & = & Ax + By + u \\ y' & = & Cy + Dx + v \end{array} \tag{4}$$

where $z = (x, y)$, $z' = (x', y')$, and $w = (u, v)$. A **quadratic generating function** on $\mathbb{R}^{2n}$ has the form

$$S(x, x') = \tfrac{1}{2}\langle \alpha x, x\rangle + \langle \beta x, x'\rangle + \tfrac{1}{2}\langle \gamma x', x'\rangle + \langle a, x\rangle + \langle b, x'\rangle + c \tag{5}$$

where $\alpha, \beta, \gamma \in \mathbb{R}^{n\times n}$ are square matrices with $\alpha = \alpha^T$ and $\gamma = \gamma^T$ symmetric, $a, b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. The affine symplectic relation $\mathrm{Rel}(S)$ generated by S is

$$\mathrm{Rel}(S) = \left\{ (x, y, x', y') : -y = \frac{\partial S}{\partial x}, \quad y' = \frac{\partial S}{\partial x'} \right\}.$$

In matrix notation the equations defining $\mathrm{Rel}(S)$ are

$$\begin{array}{rcl} -y & = & \alpha x + \beta x' + a \\ y' & = & \beta^T x + \gamma x' + b. \end{array}$$

Remark 2.1 Let $R \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ be a linear symplectic relation. Then

(1) $R = \mathrm{Gr}(\Psi)$ for some Ψ iff R is transverse to $0_{2n} \times \mathbb{R}^{2n}$.

(2) $R = \mathrm{Rel}(S)$ for some S iff R is transverse to $0_n \times \mathbb{R}^n \times 0_n \times \mathbb{R}^n$.

An affine symplectic relation is a graph (resp. admits a generating function) iff its translate through the origin is (resp. does).

Remark 2.2 An affine symplectomorphism ψ as in equation (4) admits a generating function S iff $\det(B) \neq 0$. In this case $\mathrm{Gr}(\psi) = \mathrm{Rel}(S)$ where S is defined by equation (5) with

$$\alpha = B^{-1}A, \qquad \beta = -B^{-1}, \qquad \gamma = DB^{-1},$$

$$a = B^{-1}u, \qquad b = v - DB^{-1}u$$

and the additive constant $c \in \mathbb{R}$ is arbitrary.

Remark 2.3 A quadratic generating function S as in (5) defines an affine symplectomorphism ψ iff $\det(\beta) \neq 0$. In this case $\mathrm{Rel}(S) = \mathrm{Gr}(\psi)$ where ψ is defined by equation (4) with

$$\begin{array}{ll} A = -\beta^{-1}\alpha, & B = -\beta^{-1}, \\ C = \beta^T - \gamma\beta^{-1}\alpha, & D = -\gamma\beta^{-1}, \\ u = -\beta^{-1}a, & v = b - \gamma\beta^{-1}a. \end{array}$$

Example 2.4 A shear

$$\Psi(x, y) = (x + By, y)$$

is symplectic iff B is symmetric. The generating function is given by

$$S(x, x') = \tfrac{1}{2}\langle B^{-1}(x - x'), (x - x')\rangle.$$

The symplectic shears form a subgroup: to compose two shears, add the corresponding off-diagonal blocks.

Remark 2.5 Suppose that ψ_{10} and ψ_{21} are affine symplectomorphisms admitting generating functions S_{01} and S_{12} respectively. Then the composition $\psi_{20} = \psi_{21} \circ \psi_{10}$ admits a generating function S_{02} iff the inhomogeneous linear system

$$\frac{\partial S_{01}}{\partial x_1} + \frac{\partial S_{12}}{\partial x_1} = 0 \tag{6}$$

has a unique solution $x_1 = g(x_0, x_2)$ for each choice of (x_0, x_2). The equation

$$S_{02}(x_0, x_2) = S_{01}(x_0, g(x_0, x_2)) + S_{12}(g(x_0, x_2), x_2) \tag{7}$$

defines a generating function for ψ_{20} when one exists. This generating function satisfies

$$S_{01}(x_0, x_1) + S_{12}(x_1, x_2) = S_{02}(x_0, x_2) + \tfrac{1}{2}\langle Q\xi_1, \xi_1\rangle \tag{8}$$

where $\xi_1 = x_1 - g(x_0, x_2)$. The form Q is called the **composition form** of ψ_{10} and ψ_{21}.

Remark 2.6 In the notation of equation (5) ψ_{20} admits a generating function iff $\alpha_{12} + \gamma_{01}$ is invertible. This is because equation (6) has the form $(\alpha_{12} + \gamma_{01})x_1 = f(x_0, x_2)$. An explicit formula for Q is

$$Q = \gamma_{01} + \alpha_{12}.$$

This was found by equating the homogeneous quadratic terms in x_1 in (8). With B_{kj} as in equation (3) we have $\alpha_{12} = B_{21}^{-1}A_{21}$, $\gamma_{01} = D_{10}B_{10}^{-1}$, and hence

$$Q = B_{21}^{-1}B_{20}B_{10}^{-1}.$$

3 Symplectic action

A **quadratic Hamiltonian** is a function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of form

$$\begin{aligned} H(x,y) &= \tfrac{1}{2}\langle H_{xx}x, x\rangle + \langle H_{yx}x, y\rangle + \tfrac{1}{2}\langle H_{yy}y, y\rangle \\ &\quad + \langle H_x, x\rangle + \langle H_y, y\rangle + H_0. \end{aligned} \tag{9}$$

where H_{xx}, H_{yx}, H_{yy} are $n \times n$ matrices with H_{xx} and H_{yy} symmetric, $H_x, H_y \in \mathbb{R}^n$ and $H_0 \in \mathbb{R}$. We define $H_{xy} = H_{yx}^T$. Do not confuse H_x and $\partial_x H = H_x + H_{xx}x + H_{xy}y$.

Fix a smooth time dependent quadratic Hamiltonian

$$\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (t,x,y) \mapsto H(t,x,y).$$

Denote the solution of Hamilton's equations

$$\dot{x} = \partial_y H(t,x,y), \qquad \dot{y} = -\partial_x H(t,x,y) \tag{10}$$

satisfying the initial condition $x(t_0) = x_0$, $y(t_0) = y_0$ by

$$(x(t), y(t)) = \psi_{t_0}^t(x_0, y_0). \tag{11}$$

Thus each $\psi_{t_0}^t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is an affine symplectomorphism. These form an **evolution system** meaning that

$$\psi_{t_1}^{t_2} \circ \psi_{t_0}^{t_1} = \psi_{t_0}^{t_2}, \qquad \psi_{t_0}^{t_0} = \mathbb{1}.$$

We shall denote the linear part of $\psi_{t_0}^{t_1}$ by $\Psi_{t_0}^{t_1}$. These form the evolution system for the *homogeneous* quadratic Hamiltonian which results from H by discarding the lower order terms.

When a symplectomorphism ψ admits a generating function S, this function S is determined only up to an additive constant. However, for a curve of symplectomorphisms generated by a time dependent Hamiltonian we have a natural choice for the additive constant. We now explain this. The time dependent Hamiltonian determines a one-form σ_H on $\mathbb{R}^{2n+1}$ via

$$\sigma_H = \langle y, dx\rangle - H\,dt \tag{12}$$

The form σ_H is called the **action form**. For each smooth curve $c : [t_0, t_1] \to \mathbb{R}^n \times \mathbb{R}^n$ the integral

$$I(c) = \int_c \sigma_H$$

of the action form along c is called the **action integral**. A more explicit formula is

$$I(c) = \int_{t_0}^{t_1} \Big(\langle y(t), \dot{x}(t) \rangle - H(t, x(t), y(t)) \Big)\, dt.$$

where $c(t) = (x(t), y(t))$.

Proposition 3.1 *The Euler-Lagrange equations for the action integral are Hamilton's equations (10).*

Proof: Let

$$c_\lambda = (x_\lambda, y_\lambda) : [t_0, t_1] \to \mathbb{R}^m \times \mathbb{R}^n$$

be a curve of curves and denote the derivatives by

$$\xi(t) = \frac{\partial}{\partial \lambda} x_\lambda \bigg|_{\lambda=0}, \qquad \eta(t) = \frac{\partial}{\partial \lambda} y_\lambda \bigg|_{\lambda=0}, \qquad \hat{I} = \frac{\partial}{\partial \lambda} I(c_\lambda) \bigg|_{\lambda=0}.$$

By differentiation under the integral sign and integration by parts we obtain

$$\begin{aligned} \hat{I} &= \int_{t_0}^{t_1} \langle \eta, \dot{x} - \partial_y H \rangle\, dt - \int_{t_0}^{t_1} \langle \dot{y} + \partial_x H, \xi \rangle\, dt \\ &\quad + \langle y(t_1), \xi(t_1) \rangle - \langle y(t_0), \xi(t_0) \rangle. \end{aligned} \tag{13}$$

The Euler-Lagrange equations assert that the two integrals vanish for all (ξ, η). □

Assume $\psi_{t_0}^{t_1}$ admits a generating function. By Remark 2.1 this means that for each pair $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$ there are unique y_0 and y_1 such that $\psi_{t_0}^{t_1}(x_0, y_0) = (x_1, y_1)$. Then $c = (x, y)$ defined by

$$c(t) = \psi_{t_0}^{t}(x_0, y_0)$$

is the unique solution of the Euler-Lagrange-Hamilton equations such that $x(t_0) = x_0$ and $x(t_1) = x_1$. This determines an affine map

$$\mathbb{R}^n \times \mathbb{R}^n \to C^\infty([t_0, t_1], \mathbb{R}^n \times \mathbb{R}^n) : (x_0, x_1) \mapsto c$$

which might be called the *affine Green's function*. Define

$$S(x_0, x_1) = I(c).$$

Proposition 3.2 *The function S is a generating function for $\psi_{t_0}^{t_1}$.*

Proof: Choose (x_0, x_1) and (ξ_0, ξ_1). For $\lambda \in \mathbb{R}$ let $c_\lambda(t) = (x_\lambda(t), y_\lambda(t))$ denote the unique solution of Hamilton's equations (10) with boundary condition

$$x_\lambda(t_0) = x_0 + \lambda\xi_0, \qquad x_\lambda(t_1) = x_1 + \lambda\xi_1.$$

Differentiate the formula $S(x_0 + \lambda\xi_0, x_1 + \lambda\xi_1) = I(c_\lambda)$ with respect to λ. From equation (13) we get

$$\frac{\partial S}{\partial x_0}\xi_0 + \frac{\partial S}{\partial x_1}\xi_1 = \langle y_1, \xi_1\rangle - \langle y_0, \xi_0\rangle$$

where $y_j = y_0(t_j)$. Since $\psi(x_0, y_0) = (x_1, y_1)$ this says that S is a generating function for ψ. □

Remark 3.3 A generating function for $\psi_{t_0}^{t_1}$ is determined only up to an additive constant. To distinguish the generating function of Proposition 3.2 we may call it the **generating function determined by** H on the interval $[t_0, t_1]$. By the addition property for definite integrals this generating function satisfies (7) for $t_0 < t_1 < t_2$. Hence also the composition formula (8) of Remark 2.5 holds.

Remark 3.4 Fix $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ and let $S(t, x)$ denote the generating function determined by H on $[t_0, t]$ evaluated at (x_0, x). Then S satisfies the **Hamilton-Jacobi equation**

$$\partial_t S + H(t, x, \partial_x S) = 0.$$

To prove this let $(x(\cdot), y(\cdot))$ be the solution of the Hamiltonian differential equation (10) with $x(t_0) = x_0$ and $x(t) = x$. Differentiate the identity

$$S(t, x(t)) = \int_{t_0}^{t} \Big(\langle y(s), \dot{x}(s)\rangle - H(s, x(s), y(s))\Big)\, ds$$

with respect to t and use $y = \partial S/\partial x$.

Assume that $\psi_{t_0}^{t_1}$ admits a generating function. Fix x_0, x_1 and let $c = (x, y) : [t_0, t_1] \to \mathbb{R}^n \times \mathbb{R}^n$ be the unique critical point of the action integral I with boundary condition $x(t_0) = x_0$ and $x(t_1) = x_1$. Expand $I(c)$ by Taylor's formula about c. The result is

$$I(c + \gamma) = S(x_0, x_1) + \tfrac{1}{2}\langle W\gamma, \gamma\rangle \tag{14}$$

for $\gamma = (\xi, \eta) : [t_0, t_1] \to \mathbb{R}^n \times \mathbb{R}^n$ with $\xi(t_0) = \xi(t_1) = 0$. The inner product on the right is the L^2 inner product and $W = W_{t_0 t_1}$ is the self-adjoint operator

$$W(\xi, \eta) = (-\dot{\eta} - H_{xx}\xi - H_{xy}\eta, \dot{\xi} - H_{yx}\xi - H_{yy}\eta)$$

on $L^2([t_0, t_1], \mathbb{R}^n \times \mathbb{R}^n)$ with dense domain

$$\mathcal{W}(t_0, t_1) = H_0^1([t_0, t_1], \mathbb{R}^n) \times H^1([t_0, t_1], \mathbb{R}^n).$$

The coefficients H_{xx} etc. on the right are independent of c since H is quadratic in (x, y). The operator W is called the **second variation** of the action from t_0 to t_1. We denote by $\Psi_{t_0}^{t_1} \in \mathrm{Sp}(2n)$ the linear part of the affine symplectomorphism $\psi_{t_0}^{t_1}$. The matrices $\Psi_{t_0}^{t_1}$ form the evolution system corresponding to the linear Hamiltonian system

$$\dot{\xi} = H_{yx}\xi + H_{yy}\eta, \qquad -\dot{\eta} = H_{xx}\xi + H_{xy}\eta. \tag{15}$$

We shall consider the boundary conditions

$$\xi(t_0) = \xi_0, \qquad \xi(t_1) = \xi_1. \tag{16}$$

The operator $W_{t_0 t_1} : \mathcal{W}(t_0, t_1) \to L^2([t_0, t_1], \mathbb{R}^n \times \mathbb{R}^n)$ is invertible iff $\Psi_{t_0}^{t_1}$ admits a generating function iff the boundary value problem (15-16) has a unique solution for every choice of (ξ_0, ξ_1). The function

$$G : \mathbb{R}^n \times \mathbb{R}^n \to H^1([t_0, t_1], \mathbb{R}^n \times \mathbb{R}^n)$$

which assigns to each pair $(\xi_0, \xi_1) \in \mathbb{R}^n \times \mathbb{R}^n$ the unique solution of (15-16) is called the **Green's function** on the interval $[t_0, t_1]$.

Proposition 3.5 *Fix $t_0 < t_1 < t_2$ such that $\psi_{t_j}^{t_k}$ admits a generating function. There is an isomorphism*

$$\mathcal{W}(t_0, t_2) \to \mathcal{W}(t_0, t_1) \oplus \mathbb{R}^n \oplus \mathcal{W}(t_1, t_2) : \gamma_{02} \mapsto \gamma_{01} \oplus \xi_1 \oplus \gamma_{12}$$

such that

$$\langle W_{t_0t_2}\gamma_{02},\gamma_{02}\rangle = \langle W_{t_0t_1}\gamma_{01},\gamma_{01}\rangle + \langle W_{t_1t_2}\gamma_{12},\gamma_{12}\rangle + \langle Q\xi_1,\xi_1\rangle$$

Q is the composition form as in (8).

Proof: Let G_{jk} denote the Green's function on $[t_j, t_k]$. The isomorphism sends $\gamma_{02} = (\xi, \eta)$ to

$$\gamma_{01} = \gamma|[t_0,t_1] - G_{01}(0,\xi_1), \qquad \xi_1 = \xi(t_1), \qquad \gamma_{12} = \gamma|[t_1,t_2] - G_{12}(\xi_1,0).$$

□

4 Discrete Hamiltonian mechanics

By a **partition** of $\mathbb{R}$ we mean an infinite discrete subset $\mathcal{T} \subset \mathbb{R}$ extending to infinity in both directions. Every $t \in \mathcal{T}$ has a unique **successor** $t^+ \in \mathcal{T}$ and **predecessor** $t^- \in \mathcal{T}$ defined by

$$t^- = \sup \mathcal{T} \cap (-\infty, t), \qquad t^+ = \inf \mathcal{T} \cap (t, \infty).$$

A **discrete evolution system** is a two-parameter family ϕ_s^t of affine symplectomorphisms defined for $s, t \in \mathcal{T}$ such that

$$\phi_{t_1}^{t_2} \circ \phi_{t_0}^{t_1} = \phi_{t_0}^{t_2}, \qquad \phi_{t_0}^{t_0} = 1\!\!1.$$

Such a system is uniquely determined by its generators $\phi_t^{t^+}$ where $t \in \mathcal{T}$.

Fix a time dependent quadratic Hamiltonian $H(t,x,y)$ as in section 3 and a partition $\mathcal{T}$. We shall assume that the partition satisfies the condition

$$(t^+ - t)|H_{xy}(t)| < 1 \tag{17}$$

for $t \in \mathcal{T}$. The **discrete Hamiltonian equations** determined by H and $\mathcal{T}$ are

$$\begin{array}{rcl} x' - x &=& \partial_y H(t,x',y)(t^+ - t) \\ y' - y &=& -\partial_x H(t,x',y)(t^+ - t). \end{array} \tag{18}$$

These equations define (x', y') implicitly in terms of (x, y): condition (17) implies that $1\!\!1 - (t^+ - t)H_{xy}(t)$ is invertible. The next proposition shows why the right hand side is evaluated at (x', y) rather than (x, y).

Proposition 4.1 *The equations (18) define an affine symplectomorphism* $(x', y') = \phi_t^{t^+}(x, y)$.

Proof: The generating function is

$$S(x', x) = I(x', x, g(x', x))$$

where $y = g(x', x)$ is the unique solution of $\partial I/\partial y = 0$ and

$$I(x', x, y) = \langle y, x' - x\rangle - H(x', y)(t^+ - t).$$

This works when $\partial_y^2 H$ is nondegenerate. The general case holds by an approximation argument. □

Let $t_0, t_1 \in \mathcal{T}$ with $t_0 < t_1$. Define the space

$$\mathcal{P}^{\mathcal{T}}(t_0, t_1) = \{c = (x, y) \, : \, x : \mathcal{T} \cap [t_0, t_1] \to \mathbb{R}^n, \, y : \mathcal{T} \cap [t_0, t_1) \to \mathbb{R}^n\}$$

of discrete paths in $\mathbb{R}^{2n}$. These discrete paths are finite sequences of length N and $N - 1$ where N is the cardinality of the finite set $\mathcal{T} \cap [t_0, t_1]$. The **discrete action functional** $I^{\mathcal{T}} : \mathcal{P}^{\mathcal{T}}(t_0, t_1) \to \mathbb{R}$ is defined by

$$I^{\mathcal{T}}(c) = \sum_{\substack{t \in \mathcal{T} \\ t_0 \le t < t_1}} \Big(\langle y(t), x(t^+) - x(t)\rangle - H(t, x(t^+), y(t))(t^+ - t) \Big).$$

Proposition 4.2 *The Euler-Lagrange equations of the discrete action with fixed endpoints* $x(t_0) = x_0$, $x(t_1) = x_1$ *are the discrete Hamiltonian equations (18).*

Proof:

$$\frac{\partial I}{\partial y(t)} = x(t^+) - x(t) - \partial_y H(t, x(t^+), y(t))(t^+ - t)$$

for $t_0 \le t < t_1$ and

$$\frac{\partial I}{\partial x(t)} = y(t^-) - y(t) - \partial_x H(t^-, x(t), y(t^-))(t - t^-)$$

for $t_0 < t < t_1$. (In equation (18) $y(t_1)$ is determined by $x(t_1)$ and $y(t_1^-)$.) □

The definitions and propositions for the rest of this section parallel the continuous time case. We shall omit the proofs.

As in the continuous time case, the affine symplectomorphism $\phi_{t_0}^{t_1}$ admits a generating function iff for each pair $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$ the Euler-Lagrange-Hamiltonian equations of the discrete action have a unique solution $c = (x, y) \in \mathcal{P}^{\mathcal{T}}(t_0, t_1)$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$. This determines an affine map

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}^{\mathcal{T}}(t_0, t_1) : (x_0, x_1) \mapsto c$$

which might be called the *discrete affine Green's function*. Define

$$S^{\mathcal{T}}(x_0, x_1) = I^{\mathcal{T}}(c).$$

Proposition 4.3 *The function $S^{\mathcal{T}}$ is a generating function for $\phi_{t_0}^{t_1}$. It is called the* **discrete generating function** *determined by $\mathcal{T}$ and H on the interval $[t_0, t_1]$. (Compare Remark 3.3.)*

Remark 4.4 Suppose $t_0 < t_1 < t_2$, that $\phi_{t_j}^{t_k}$ admits a generating function, and that $S_{jk}^{\mathcal{T}}$ is the discrete generating function from t_j to t_k determined by $\mathcal{T}$ and H. Then

$$S_{01}^{\mathcal{T}}(x_0, x_1) + S_{12}^{\mathcal{T}}(x_1, x_2) = S_{02}^{\mathcal{T}}(x_0, x_2) + \tfrac{1}{2}\langle Q^{\mathcal{T}}\xi_1, \xi_1\rangle$$

where $\xi_1 = x_1 - g^{\mathcal{T}}(x_0, x_2)$ as in Remark 2.5. Here $Q^{\mathcal{T}}$ is the composition form of $\phi_{t_0}^{t_1}$ and $\phi_{t_1}^{t_2}$.

For $t_0, t_1 \in \mathcal{T}$ with $t_0 < t_1$ define

$$\mathcal{W}^{\mathcal{T}}(t_0, t_1) = \left\{\gamma = (\xi, \eta) \in \mathcal{P}^{\mathcal{T}}(t_0, t_1) : \xi(t_0) = \xi(t_1) = 0\right\}.$$

This is a Hilbert space with the approximate L^2-norm

$$\|\gamma\|_{\mathcal{T}}^2 = \sum_{t_0 \le t < t_1} (|\xi(t^+)|^2 + |\eta(t)|^2)(t^+ - t).$$

Expand $I^{\mathcal{T}}$ by Taylor's formula about $c = (x, y)$. The result is

$$I^{\mathcal{T}}(c + \gamma) = S^{\mathcal{T}}(x_0, x_1) + \tfrac{1}{2}\langle W^{\mathcal{T}}\gamma, \gamma\rangle \tag{19}$$

for $\gamma \in \mathcal{W}(t_0, t_1)$ where $W^{\mathcal{T}} = W^{\mathcal{T}}_{t_0t_1} : \mathcal{W}^{\mathcal{T}}(t_0, t_1) \to \mathcal{W}^{\mathcal{T}}(t_0, t_1)$ is a (finite dimensional) operator. It is given by $W^{\mathcal{T}}(\xi, \eta) = (u, v)$ where

$$u(t) = -\frac{\eta(t) - \eta(t^-)}{t - t^-} - H_{xx}(t^-)\xi(t) - H_{xy}(t^-)\eta(t^-),$$

$$v(t) = \frac{\xi(t^+) - \xi(t)}{t^+ - t} - H_{yx}(t)\xi(t^+) - H_{yy}(t)\eta(t).$$

The operator $W^{\mathcal{T}}$ is called the **second variation** of the discrete action from t_0 to t_1. It is symmetric with respect to the above inner product. As in the continuous time case, the operator $W^{\mathcal{T}}_{t_0t_1}$ is invertible iff the affine symplectomorphism $\phi^{t_1}_{t_0}$ admits a generating function iff for every choice of (ξ_0, ξ_1) the discrete boundary value problem

$$\frac{\xi(t^+) - \xi(t)}{t^+ - t} = H_{yx}(t)\xi(t^+) + H_{yy}(t)\eta(t),$$

$$\frac{\eta(t^+) - \eta(t)}{t^+ - t} = -H_{xx}(t)\xi(t^+) - H_{xy}(t)\eta(t),$$

$$\xi(t_0) = \xi_0, \qquad \xi(t_1) = \xi_1$$

has a unique solution $(\xi, \eta) = G(\xi_0, \xi_1)$. We call G the **discrete Green's function** on $[t_0, t_1]$.

Proposition 4.5 *Fix $t_0 < t_1 < t_2$ such that $\phi^{t_k}_{t_j}$ admits a generating function. Then there exists an isomorphism*

$$\mathcal{W}^{\mathcal{T}}(t_0, t_2) \to \mathcal{W}^{\mathcal{T}}(t_0, t_1) \oplus \mathbb{R}^n \oplus \mathcal{W}^{\mathcal{T}}(t_1, t_2) : \gamma_{02} \mapsto \gamma_{01} \oplus \xi_1 \oplus \gamma_{12}$$

such that

$$\langle W^{\mathcal{T}}_{t_0t_2}\gamma_{02}, \gamma_{02}\rangle = \langle W^{\mathcal{T}}_{t_0t_1}\gamma_{01}, \gamma_{01}\rangle + \langle W^{\mathcal{T}}_{t_1t_2}\gamma_{12}, \gamma_{12}\rangle + \langle Q^{\mathcal{T}}\xi_1, \xi_1\rangle$$

where $Q^{\mathcal{T}}$ is the composition form for $\phi^{t_1}_{t_0}$ and $\phi^{t_2}_{t_1}$.

Corollary 4.6 *In the notation of Proposition 4.5*

$$\operatorname{sign} W^{\mathcal{T}}_{t_0t_2} = \operatorname{sign} W^{\mathcal{T}}_{t_0t_1} + \operatorname{sign} W^{\mathcal{T}}_{t_1t_2} + \operatorname{sign} Q^{\mathcal{T}}.$$

5 The Maslov index

In [17] we develop a theory of the Maslov index for paths of symplectic matrices. Our theory differs slightly from other treatments in the literature. Our Maslov index assigns a half integer $\mu(\Psi)$ to every path $\Psi : [a,b] \to \mathrm{Sp}(2n)$ of symplectic matrices. Here we summarize the results which are needed in the sequel.

Denote by $\mathrm{Sp}_0(2n)$ the open and dense set of all symplectic matrices which admit a generating function and by

$$\Sigma = \mathrm{Sp}(2n) \setminus \mathrm{Sp}_0(2n)$$

its complement. The space $\mathrm{Sp}_0(2n)$ has two components distinguished by the sign of the determinant of B in the block decomposition (3). The set Σ is called the **Maslov cycle**. It admits a natural co-orientation. The Maslov index of a path with endpoints in $\mathrm{Sp}_0(2n)$ is the intersection number of that path with Σ. For a path which begins and/or ends on Σ we add half the intersection number at the endpoints. The definition of this intersection number is not completely obvious because we allow paths which begin and end in the strata of the Maslov cycle of codimension bigger than 1. For example we allow paths which start at the identity. As in [3] this is motivated by Hamiltonian evolution systems.

For a time dependent quadratic Hamiltonian H denote by $\Psi(H)_{t_0}^t \in \mathrm{Sp}(2n)$ the evolution system generated by the linearized Hamiltonian differential equation. Define

$$\mu(t_0, t_1, H) = \mu(\Psi_{H,[t_0,t_1]})$$

where $\Psi_{H,[t_0,t_1]}$ denotes the path $[t_0, t_1] \to \mathrm{Sp}(2n) : t \mapsto \Psi(H)_{t_0}^t$. We use the Maslov index $\mu(t_0, t_1, H)$ only when $\Psi(H)_{t_0}^{t_1} \in \mathrm{Sp}_0(2n)$. Any time this notation is used this is implicitly assumed. In particular the homotopy H_λ mentioned below satisfies $\Psi(H_\lambda)_{t_0}^{t_1} \in \mathrm{Sp}(2n)$ for every λ. The Maslov index satisfies the following conditions.

(Homotopy) Two time dependent quadratic Hamiltonians are homotopic as above if and only if they have the same Maslov index.

(Composition) If $t_0 < t_1 < t_2$ then

$$\mu(t_0, t_2, H) = \mu(t_0, t_1, H) + \mu(t_1, t_2, H) + \tfrac{1}{2}\mathrm{sign}\, Q$$

where Q is the composition form of $\Psi(H)_{t_0}^{t_1}$ and $\Psi(H)_{t_1}^{t_2}$ as in Remark 2.6.

(Product) If $H(t,x',x'',y',y'') = H'(t,x',y') + H''(t,x'',y'')$ then

$$\mu(t_0,t_1,H) = \mu(t_0,t_1,H') + \mu(t_0,t_1,H'').$$

(Normalization) If $H(t,x,y) = \frac{1}{2}\langle H_{yy}(t)y,y\rangle$ so that

$$\Psi(H)_{t_0}^{t} = \begin{pmatrix} \mathbb{1} & B(t) \\ 0 & \mathbb{1} \end{pmatrix}$$

is a symplectic shear then

$$\mu(t_0,t_1,H) = -\tfrac{1}{2}\mathrm{sign}\, B(t_1).$$

(Determinant) The number $\mu(t_0,t_1,H) + n/2$ is an integer and

$$\mathrm{sign}\det\ B = (-1)^{\mu+n/2}$$

where $\mu = \mu(t_0,t_1,H)$ and B is the right upper block in the decompsition (3) of $\Psi(H)_{t_0}^{t_1}$.

6 Approximation

Fix a time dependent quadratic Hamiltonian as in section 3 and a partition $\mathcal{T}$ as in section 4 and denote by $\{\phi_s^t : t,s \in \mathcal{T}\}$ the associated discrete evolution system. Let $\{\psi_s^t : t,s \in \mathbb{R}\}$ be the evolution system of Section 3 and Ψ_s^t be its linear part.

To make the statements of our convergence theorems less awkward we shall interpolate this discrete evolution system by a 2-parameter family of affine symplectomorphisms. For $t \in \mathcal{T}$ and $t \le s, s' \le t^+$ the symplectomorphism

$$(x',y') = \phi_s^{s'}(x,y)$$

is defined implicitly by

$$x' - x = \partial_y H(t,x',y)(s'-s), \qquad y' - y = -\partial_x H(t,x',y)(s'-s).$$

For $t_0 \le s \le t_0^+$ and $t_1 \le t \le t_1^+$ with $t_j \in \mathcal{T}$ define

$$\phi_s^t = \phi_{t_1}^t \circ \phi_{t_0}^{t_1} \circ \phi_s^{t_0}.$$

(Because of the interpolation, $\{\phi_s^t : t, s \in \mathbb{R}\}$ is *not* an evolution system.) Denote by $|\mathcal{T}| = \sup_t |t^+ - t|$ the mesh of the partition.

Theorem 6.1 *We have*

$$\psi_s^t = \lim_{|\mathcal{T}| \to 0} \phi_s^t.$$

The convergence is uniform in every compact domain $t_0 \le s, t \le t_1$.

Proof: By Proposition 4.1 we have

$$\begin{array}{rcl} x' & = & x + \partial_y H(t,x,y)(s'-s) + O((s'-s)^2), \\ y' & = & y - \partial_x H(t,x,y)(s'-s) + O((s'-s)^2). \end{array}$$

for $(x', y') = \phi_s^{s'}(x, y)$ with $t \le s, s' \le t^+$. □

Corollary 6.2 *Suppose $\psi_{t_0}^{t_1}$ admits a generating function. Then the symplectomorphism $\phi_{t_0}^{t_1}$ admits a generating function for $|\mathcal{T}|$ sufficiently small and the limit of the discrete generating function determined by $\mathcal{T}$ and H*

$$\lim_{|\mathcal{T}| \to 0} S^{\mathcal{T}} = S$$

is the continuous generating function determined by H.

Theorem 6.3 *Assume that $\psi_{t_0}^{t_1}$ admits a generating function. Then for sufficiently small mesh the discrete second variation $W_{t_0 t_1}^{\mathcal{T}}$ is nonsingular and its signature is independent of the choice of the partition. This allows the definition*

$$\operatorname{sign} W_{t_0 t_1} = \lim_{|\mathcal{T}| \to 0} \operatorname{sign} W_{t_0 t_1}^{\mathcal{T}}.$$

Theorem 6.3 is an immediate consequence of the following relation between the signature of the Hessian and the Maslov index.

Theorem 6.4 *If $\psi_{t_0}^{t_1}$ admits a generating function then for sufficiently small mesh the signature of the discrete second variation from t_0 to t_1 is related to the Maslov index by*

$$\operatorname{sign} W_{t_0 t_1}^{\mathcal{T}} = 2\mu(t_0, t_1, H).$$

Proof: We first prove Theorem 6.4 in the case of a symplectic shear

$$H(t,x,y) = \tfrac{1}{2}\langle H_{yy}(t)y, y\rangle.$$

Then the discrete time evolution system is linear and given by

$$\phi_{t_0}^t = \begin{pmatrix} 1\!\!1 & B(t) \\ 0 & 1\!\!1 \end{pmatrix}$$

where $B(t)$ for $t \in \mathcal{T} \cap [t_0, t_1]$ is defined inductively by

$$B(t^+) = B(t) + H_{yy}(t)(t^+ - t), \qquad B(t_0) = 0, \qquad B^{\mathcal{T}} = B(t_1).$$

Moreover

$$\langle W^{\mathcal{T}}\gamma, \gamma\rangle = 2\sum_{t_0<t<t_1}\langle \eta(t^-) - \eta(t), \xi(t)\rangle - \sum_{t_0\le t<t_1}\langle H_{yy}(t)\eta(t), \eta(t)\rangle(t^+ - t).$$

Introduce new variables $u, v : \mathcal{T} \cap (t_0, t_1) \to \mathbb{R}^n$ and $\eta_0 \in \mathbb{R}^n$ defined by

$$\begin{aligned} u(t) &= \xi(t) - \tfrac{1}{2}(B(t) - B^{\mathcal{T}})(\eta(t) + \eta(t^-)), \\ v(t) &= -\frac{\eta(t) - \eta(t^-)}{t - t^-}, \\ \eta_0 &= \eta(t_0). \end{aligned}$$

for $t_0 < t < t_1$. Then the map $\gamma \mapsto (u, v, \eta_0)$ is an isomorphism. The L^2-inner product of u and v is given by

$$\begin{aligned} \langle u, v\rangle &= \sum_{t_0<t<t_1}\langle u(t), v(t)\rangle(t - t^-) \\ &= \sum_{t_0<t<t_1}\langle \xi(t), \eta(t^-) - \eta(t)\rangle \\ &\quad +\tfrac{1}{2}\sum_{t_0<t<t_1}\langle (B(t) - B^{\mathcal{T}})(\eta(t) + \eta(t^-)), \eta(t) - \eta(t^-)\rangle \\ &= \sum_{t_0\le t<t_1}\langle \xi(t), \eta(t^-) - \eta(t)\rangle \\ &\quad +\tfrac{1}{2}\sum_{t_0<t<t_1}\langle (B(t) - B^{\mathcal{T}})\eta(t)), \eta(t)\rangle \\ &\quad -\tfrac{1}{2}\sum_{t_0<t\le t_1}\langle (B(t) - B^{\mathcal{T}})\eta(t^-)), \eta(t^-)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t_0 \le t < t_1} \langle \xi(t), \eta(t^-) - \eta(t) \rangle \\
&\quad + \tfrac{1}{2} \sum_{t_0 \le t < t_1} \langle (B(t) - B(t^+))\eta(t), \eta(t) \rangle + \tfrac{1}{2} \langle B^{\mathcal{T}} \eta_0, \eta_0 \rangle \\
&= \sum_{t_0 \le t < t_1} \langle \xi(t), \eta(t^-) - \eta(t) \rangle \\
&\quad - \tfrac{1}{2} \sum_{t_0 \le t < t_1} \langle H_{yy}(t)\eta(t), \eta(t) \rangle (t^+ - t) + \tfrac{1}{2} \langle B^{\mathcal{T}} \eta_0, \eta_0 \rangle \\
&= \tfrac{1}{2} \langle W^{\mathcal{T}} \gamma, \gamma \rangle + \tfrac{1}{2} \langle B^{\mathcal{T}} \eta_0, \eta_0 \rangle.
\end{aligned}
$$

Hence the second variation $W^{\mathcal{T}}$ can be represented by the matrix

$$
W^{\mathcal{T}} = \begin{pmatrix} -B^{\mathcal{T}} & 0 & 0 \\ 0 & 0 & \mathbb{1} \\ 0 & \mathbb{1} & 0 \end{pmatrix}.
$$

By the normalization property of the Maslov index this shows that

$$
\operatorname{sign} W^{\mathcal{T}} = -\operatorname{sign} B^{\mathcal{T}} = 2\mu(t_0, t_1, H).
$$

in the case of a symplectic shear.

Now let H be any quadratic Hamiltonian with $\Psi_{t_0}^{t_1} \in \mathrm{Sp}_0(2n)$. Throughout the remainder of the proof we shall fix the interval $[t_0, t_1]$ and denote the second variation $W_H^{\mathcal{T}}$. Consider a new Hamiltonian $H_0 = H \oplus H'$ where

$$
H'(x', y') = \tfrac{1}{2} \langle H'_{yy} y', y' \rangle
$$

on $\mathbb{R}^{n'} \times \mathbb{R}^{n'}$ and H'_{yy} is a constant symmetric $n' \times n'$ matrix of signature zero. (The number n' is necessarily even.) Then the induced symplectomorphism of H' is a symplectic shear of Maslov index zero. By the first part of the proof the corresponding Hessian $W_{H'}^{\mathcal{T}}$ has signature zero:

$$
\operatorname{sign} W_{H'}^{\mathcal{T}} = 0
$$

(for every mesh). By the product property it suffices to prove the theorem for H_0. By the determinant and normalization properties of the Maslov index there exists, for n' sufficiently large, a symplectic shear H_1 on $\mathbb{R}^{2n+2n'}$ whose Maslov index agrees with that of H_0. By the homotopy property there exists a homotopy $\{H_\lambda\}_\lambda$ from H_0 to H_1 such that the symplectomorphism generated by H_λ admits a generating function for every λ. Hence

$$\begin{aligned}
\operatorname{sign} W_H^{\mathcal{T}} &= \operatorname{sign} W_{H_0}^{\mathcal{T}} \\
&= \operatorname{sign} W_{H_1}^{\mathcal{T}} \\
&= 2\mu(t_0, t_1, H_1) \\
&= 2\mu(t_0, t_1, H_0) \\
&= 2\mu(t_0, t_1, H).
\end{aligned}$$

The last identity follows the product property, the last but one from the homotopy property, and the last but two is the theorem for symplectic shears. □

Remark 6.5 A direct proof of Theorem 6.3 is slightly more difficult. It would proceed by showing that for sufficiently small mesh the signature of $W^{\mathcal{T}}$ remains unchanged under the introduction of a new mesh point. This argument requires a uniform estimate on the inverse of $W^{\mathcal{T}}$.

Theorem 6.6 *If the mesh of $\mathcal{T}$ is sufficiently small then the determinant of the second variation $W^{\mathcal{T}}$ from t_0 to t_1 is given by*

$$\det(W^{\mathcal{T}}) = (-1)^{Nn+n}(t_0^+ - t_0)^n \det(B^{\mathcal{T}}) \prod_{\substack{t\in\mathcal{T} \\ t_0\le t<t_1}} \frac{\det(\mathbb{1} - (t^+ - t)H_{xy}(t))}{(t^+ - t)^{2n}}$$

where $B^{\mathcal{T}}$ is the right upper block in the decomposition (3) of the affine symplectomorphism $\psi(\mathcal{T}, H)_{t_0}^{t_1}$, and $N = \#\{t \in \mathcal{T} : t_0 \le t < t_1\}$.

Proof: The formula for the absolute value of the determinant follows from the proof of Theorem 8.5 below. By the determinant property from section 5

$$\operatorname{sign}\det(B^{\mathcal{T}}) = (-1)^{\mu+n/2}$$

Hence

$$\begin{aligned}
\text{sign }\det(W^{\mathcal{T}}) &= (-1)^{(\operatorname{rank} W^{\mathcal{T}} - \operatorname{sign} W^{\mathcal{T}})/2} \\
&= (-1)^{nN+n/2-\mu} \\
&= (-1)^{nN+n/2-\mu} \\
&= (-1)^{nN+n}\text{sign }\det(B^{\mathcal{T}}).
\end{aligned}$$

□

7 Two Lie Algebras

We summarize some material from [12]. Denote the space of all complex linear differential operators on $\mathbb{R}^n$ with polynomial coefficients by $\mathrm{Op}^{\mathbb{C}}(n)$. This space forms an associative algebra under composition of operators and hence a Lie algebra. It is convenient to define generators for this associative algebra by

$$(P_j f)(x) = -i\hbar\partial_j f(x), \qquad (Q_j f)(x) = x_j f(x), \tag{20}$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$. Here $\hbar$ is is a fixed positive number called **Planck's constant**. These operators satisfy the **Heisenberg relations**

$$[P_j, P_k] = [Q_j, Q_k] = 0, \qquad [Q_j, P_k] = i\hbar\delta_{jk}\mathbb{1} \tag{21}$$

In multi-index notation a typical element $A \in \mathrm{Op}^{\mathbb{C}}(n)$ has form

$$A = \sum a_{\alpha,\beta} Q^\alpha P^\beta$$

where the coefficients $a_{\alpha,\beta}$ are complex numbers. This operator has a formal adjoint defined by

$$A^* = \sum \bar{a}_{\alpha,\beta} P^\beta Q^\alpha.$$

Denote the (formally) skew-adjoint operators by

$$\mathrm{Op} = \mathrm{Op}(n) = \{A \in \mathrm{Op}^{\mathbb{C}}(n) : A + A^* = 0\}.$$

Then $\mathrm{Op}^{\mathbb{C}}(n)$ is the complexification of $\mathrm{Op}(n)$:

$$\mathrm{Op}^{\mathbb{C}}(n) = \mathrm{Op}(n) + i\mathrm{Op}(n).$$

Introduce a filtration of Op via

$$\mathrm{Op}_r = \bigoplus_{s\le r} \mathrm{Op}^{(s)}, \qquad \mathrm{Op}^{(s)} = \mathrm{span}\{Q^\alpha P^\beta : |\alpha| + |\beta| = s\}.$$

This gives

$$[\mathrm{Op}_r, \mathrm{Op}_s] \subset \mathrm{Op}_{r+s-2}.$$

In particular, Op_1 and Op_2 are Lie algebras and Op_1 acts on Op_2 by derivations.

The **polynomial Poisson algebra** $\mathcal{F} = \mathcal{F}(2n)$ in $2n$ dimensions is the Lie algebra of all real valued polynomials $f : \mathbb{C}^n = \mathbb{R}^{2n} \to \mathbb{R}$ with the bracket operation defined by

$$\{f_1, f_2\} = \sum_{j=1}^{n} \left(\frac{\partial f_1}{\partial x_j}\frac{\partial f_2}{\partial y_j} - \frac{\partial f_1}{\partial y_j}\frac{\partial f_2}{\partial x_j} \right)$$

where $z_j = x_j + iy_j$. This algebra is graded

$$\mathcal{F} = \bigoplus_{r=1}^{\infty} \mathcal{F}^{(r)}$$

and

$$\{\mathcal{F}^{(r)}, \mathcal{F}^{(s)}\} \subset \mathcal{F}^{(r+s-2)}$$

where $\mathcal{F}^{(r)}$ denotes the subspace of homogeneous polynomials of degree r. Denote by

$$\mathcal{F}_r = \bigoplus_{s \le r} \mathcal{F}^{(s)}$$

the corresponding filtration.The co-ordinate functions

$$q_j(x, y) = x_j, \qquad p_j(x, y) = y_j, \tag{22}$$

for $j = 1, 2, \ldots, n$ form a vector space basis for $\mathcal{F}^{(1)}$. They satisfy the relations

$$[p_j, p_k] = [q_j, q_k] = 0, \qquad [q_j, p_k] = \delta_{jk} \tag{23}$$

The monomials $q^\alpha p^\beta$ with $|\alpha| + |\beta| = r$ form a basis for $\mathcal{F}^{(r)}$.

Theorem 7.1 *There is an isomorphism* $\mathcal{F}_2 \to \mathrm{Op}_2$.

Proof: Equations (21) and (23) show that the linear map

$$1 \mapsto \frac{1}{i\hbar}\mathbb{1}, \qquad p_j \mapsto \frac{1}{i\hbar}P_j, \qquad q_k \mapsto \frac{1}{i\hbar}Q_k,$$

is an isomorphism $\mathcal{F}_1 \to \mathrm{Op}_1$. It extends to to an isomorphism $\mathcal{F}_2 \to \mathrm{Op}_2$ via

$$q_j q_k \mapsto \frac{1}{i\hbar}Q_j Q_k, \qquad p_j p_k \mapsto \frac{1}{i\hbar}P_j P_k, \qquad q_k p_j \mapsto \frac{1}{i\hbar}\left(Q_k P_j - \frac{i\hbar}{2}\delta_{jk}\mathbb{1} \right).$$

The right hand side is achieved by replacing q_j and p_j with Q_j and P_j, taking the self-adjoint part, and dividing by $i\hbar$. The following multiplication table proves this for $n = 1$:

$$
\begin{array}{rcrrcrl}
\{p^2, q\} & = & -2p & [P^2, Q] & = & -2i\hbar P & \\
\{q^2, p\} & = & 2q & [Q^2, P] & = & 2i\hbar Q & \\
\{qp, p\} & = & p & [QP, P] & = & i\hbar P & \\
\{qp, q\} & = & -q & [QP, Q] & = & -i\hbar Q & \\
\\
\{p^2, q^2\} & = & -4qp & [P^2, Q^2] & = & -4i\hbar QP & -2\hbar^2 \mathbb{1} \\
\{p^2, qp\} & = & -2p^2 & [P^2, QP] & = & -2i\hbar P^2 & \\
\{q^2, qp\} & = & 2q^2 & [Q^2, QP] & = & 2i\hbar Q^2 &
\end{array}
$$

□

Remark 7.2 This isomorphism does not extend to a Lie algebra homomorphism $\mathcal{F} \to \mathrm{Op}$. See [12].

8 Feynman path integrals

In this section the triple

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$$

made from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the Hilbert space $L^2(\mathbb{R}^n)$, and the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions plays the role of a kind of Gelfand triple. By a **unitary Schwartz automorphism** we mean a unitary automorphism of $L^2(\mathbb{R}^n)$ which restricts to a toplinear automorphism of $\mathcal{S}(\mathbb{R}^n)$ and hence extends uniquely to a toplinear automorphism of $\mathcal{S}'(\mathbb{R}^n)$. Some unitary Schwartz automorphisms are the Fourier transform

$$\mathcal{F}(f)(y) = (2\pi\hbar)^{-n/2} \int e^{-i\langle y,x\rangle/\hbar} f(x)\, dx,$$

the inverse Fourier transform

$$\bar{\mathcal{F}}(g)(x') = (2\pi\hbar)^{-n/2} \int e^{i\langle y,x'\rangle/\hbar} g(y)\, dy,$$

the multiplication operator

$$\mathcal{M}(N)g(y) = e^{-iN(y)/\hbar}g(y)$$

where $N : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a polynomial, the translation operator

$$\mathcal{R}(y_0)g(y) = g(y + y_0)$$

where $y_0 \in \mathbb{R}^n$, and the composition operator

$$\mathcal{K}(B)g(x') = |\det B|^{1/2}g(Bx')$$

where $B \in \mathrm{GL}(n, \mathbb{R})$.

A time independent quadratic Hamiltonian of the form (9) determines a linear map $\mathcal{T}(H) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ via

$$\mathcal{T}(H)f(x') = (2\pi\hbar)^{-n}\det(\mathbb{1} - H_{yx})^{1/2}\int\int e^{i(\langle y,x'-x\rangle - H(x',y))/\hbar}f(x)\,dx\,dy$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. (It is required that H_{yx} be sufficiently small so that the determinant is non-zero.)

Proposition 8.1 *The operator $\mathcal{T}(H)$ is a unitary Schwartz automorphism.*

Proof: It factors as

$$\mathcal{T}(H) = \mathcal{M}(L) \circ \mathcal{K}(B) \circ \bar{\mathcal{F}} \circ \mathcal{M}(N) \circ \mathcal{F}.$$

where $L(x) = H_0 + \langle H_x, x\rangle + \frac{1}{2}\langle H_{xx}x, x\rangle$, $B = \mathbb{1} - H_{yx}$, and $N(y) = \langle H_y, y\rangle + \frac{1}{2}\langle H_{yy}y, y\rangle$. Each factor is a unitary Schwartz automorphism. $\square$

Let $H(t, x, y) = H_t(x, y)$ be a smooth time dependent quadratic Hamiltonian, $\mathcal{T}$ be a partition of the interval $[t_0, t_1]$, and $\mathcal{P}(\mathcal{T})$ be the finite dimensional space of paths as in section 4. For $t, t_0 \in \mathcal{T}$ define the **Feynman product** $\mathcal{U}^{\mathcal{T}}(t, t_0, H)$ inductively by

$$\mathcal{U}^{\mathcal{T}}(t^+, t_0, H) = \mathcal{T}((t^+ - t)H_t) \circ \mathcal{U}^{\mathcal{T}}(t, t_0, H),$$

$$\mathcal{U}^{\mathcal{T}}(t_0, t_0, H) = \mathbb{1}.$$

This has the appearance of an integral over paths:

$$\mathcal{U}^{\mathcal{T}}(t_1, t_0, H)f(x) = \int\limits_{\substack{c\in\mathcal{P}^{\mathcal{T}} \\ x(t_1)=x}} e^{i\, I^{\mathcal{T}}(c)/\hbar} f(x(t_0))\mathcal{D}\, c \tag{24}$$

where

$$\mathcal{D}\, c = \prod_{t_0\le t<t_1} (2\pi\hbar)^{-n} \det(\mathbb{1} - (t^+ - t)H_{yx}(t))^{1/2}\, dx(t)dy(t).$$

The order of integration is the time order, i.e. first $dx(t_0)$, then $dy(t_0)$, then $dx(t_0^+)$ etc. The notation $\mathcal{D}\, c$ hides the normalization which makes the Feynman product a unitary operator. The integral does not converge absolutely as an integral in all its variables. Interchanging the order of integration requires justification.

Theorem 8.2 *The limit*

$$\mathcal{U}(t_1, t_0, H) = \lim_{|\mathcal{T}|\to 0} \mathcal{U}^{\mathcal{T}}(t_1, t_0, H)$$

exists in the strong operator topology. It is a unitary Schwartz automorphism. Here $|\mathcal{T}| = \max_j(s_j - s_{j-1})$ *is the mesh of the partition and the partitions partition the interval* $[t_0, t_1]$*.*

Corollary 8.3 $\mathcal{U}(t_2, t_1, H) \circ \mathcal{U}(t_1, t_0, H) = \mathcal{U}(t_2, t_0, H)$.

Theorem 8.4 *The operators* $\mathcal{U}(t, t_0, H)$ *are the evolution operators of the time dependent partial differential equation*

$$i\hbar\frac{\partial\phi}{\partial t} = H_t(Q, P)\phi$$

where Q_j *and* P_j *denote the self-adjoint operators of equation (20) and* $(i\hbar)^{-1}H_t(Q, P) \in \mathrm{Op}_2$ *is the image of* $H_t \in \mathcal{F}_2$ *under the Lie algebra homomrphism of Theorem 7.1.*

We now give an explicit formula for the operator $\mathcal{U}(t, t_0, H)$. Let ψ denote the evolution system determined by H as in equation (11).

Theorem 8.5 *If $\psi_{t_0}^{t_1}$ admits a generating function then $\mathcal{U}(t_1, t_0, H)$ is given by*

$$\mathcal{U}(t_1, t_0, H)f(x) = \frac{e^{i\pi\mu(t_0,t_1,H)/2}}{(2\pi\hbar)^{n/2}|\det B|^{1/2}} \int_{\mathbb{R}^n} e^{i\,S(x,x_0)/\hbar} f(x_0)\,dx_0.$$

where $S(x, x_0)$ is the generating function from t_0 to t_1 as in Remark 3.3, $\mu = \mu(t_0, t_1, H)$ is the Maslov index as in section 5, and $B = B(t_1, t_0)$ is the right upper block in the block decomposition (3) of $\Psi_{t_0}^{t_1}$.

The previous theorem shows that the unitary operator $\mathcal{U}(t_1, t_0, H)$ depends, up to multiplication by a complex number of modulus 1, only on the symplectomorphism $\psi_{t_0}^{t_1}$ generated by the quadratic Hamiltonian H but not on H itself. If H is chosen with constant term $c = 0$ then $\mathcal{U}(t_1, t_0, H)$ is determined by $\psi_{t_0}^{t_1}$ up to a sign. The sign is determined by the Maslov index.

Proof of Theorem 8.2 and Theorem 8.5: By equations (24) and (19)

$$\mathcal{U}^{\mathcal{T}}(t_1, t_0, H)f(x) = \int_{\mathcal{W}^{\mathcal{T}}(t_0,t_1)} \int_{\mathbb{R}^n} e^{i\,S^{\mathcal{T}}(x,x_0)/\hbar} f(x_0)\,dx_0\, e^{i\langle W^{\mathcal{T}}\gamma,\gamma\rangle/2\hbar}\,\mathcal{D}\gamma$$

where

$$\mathcal{D}\gamma = (t_0^+ - t_0)^{n/2} \prod_{t_0 \le t < t_1} (2\pi\hbar(t^+ - t))^{-n} \det(\mathbb{1} - (t^+ - t)H_{yx}(t))^{1/2}\,d\gamma,$$

and

$$d\gamma = (t_0^+ - t_0)^{n/2} d\eta(t_0) \prod_{t_0 < t < t_1} (t^+ - t)^n d\xi(t) d\eta(t).$$

Recall that $\mathcal{W}^{\mathcal{T}}(t_0, t_1)$ is a finite dimensional Hilbert space and $d\gamma$ is the Euclidean volume form. The Gaussian integral is

$$\int_{\mathcal{W}^{\mathcal{T}}(t_0,t_1)} e^{i\langle W^{\mathcal{T}}\gamma,\gamma\rangle/2\hbar}\,d\gamma = \frac{(2\pi\hbar)^{nN-n/2}}{|\det W^{\mathcal{T}}|^{1/2}} e^{i\pi\,\mathrm{sign}\,W^{\mathcal{T}}/4}$$

where

$$N = N(t_0, t_1) = \#\{t \in \mathcal{T} : t_0 \le t < t_1\}.$$

By Theorem 6.4

$$\int_{\mathcal{W}^{\mathcal{T}}(t_0,t_1)} e^{i\langle W^{\mathcal{T}}\gamma,\gamma\rangle/2\hbar}\,\mathcal{D}\gamma = \lambda \frac{(2\pi\hbar)^{-n/2}}{|\det B^{\mathcal{T}}|^{1/2}} e^{i\pi\mu(t_1,t_0,H)/2}.$$

where

$$\lambda = \frac{|\det B^{\mathcal{T}}|^{1/2}}{|\det W^{\mathcal{T}}|^{1/2}} (t_0^+ - t_0)^{n/2} \prod_{t_0 \le t < t_1} (t^+ - t)^{-n} \det(\mathbb{1} - (t^+ - t) H_{xy}(t))^{1/2}$$

Hence

$$\mathcal{U}^{\mathcal{T}}(t_1, t_0, H) f(x) = \lambda \frac{(2\pi\hbar)^{-n/2}}{|\det B^{\mathcal{T}}|^{1/2}} e^{i\pi\mu(t,t_0,H)/2} \int_{\mathbb{R}^n} e^{i\, S^{\mathcal{T}}(x,x_0)/\hbar} f(x_0)\, dx_0.$$

Since $\mathcal{U}^{\mathcal{T}}(t_1, t_0, H)$ is a unitary operator it follows that $\lambda = 1$. Now let the mesh go to zero and use Theorem 6.1 and Proposition 4.3. □

Proof of 8.3: This is immediate from Theorem 8.2. We give another proof using Theorem 8.5. Fix $t_0 < t_1 < t_2$. For $j, k = 0, 1, 2$ abbreviate

$$\psi_{kj} = \psi_{t_j}^{t_k}, \qquad \Psi_{kj} = \Psi_{t_j}^{t_k}, \qquad \mu_{kj} = \mu(t_k, t_j, H)$$

and write Ψ_{kj} in block matrix notation

$$\Psi_{kj} = \begin{pmatrix} A_{kj} & B_{kj} \\ C_{kj} & D_{kj} \end{pmatrix}.$$

Assume $\Psi_{kj} \in \mathrm{Sp}_0(2n)$ and denote by S_{kj} the generating function of ψ_{kj} given by the action functional. Define

$$K(x_2, x_0) = \int_{\mathbb{R}^n} e^{iS_{21}(x_2,x_1)/\hbar + iS_{10}(x_1,x_0)/\hbar} dx_1.$$

By equation (8) and Remark 3.3

$$S_{21}(x_2, x_1) + S_{10}(x_1, x_0) = S_{20}(x_2, x_0) + \tfrac{1}{2}\langle Qv, v\rangle$$

where $v = x_1 + \ell(x_0, x_2)$. The Gaussian integral is given by

$$\int_{\mathbb{R}^n} e^{i\langle v, Qv\rangle/2\hbar}\, dv = (2\pi\hbar)^{n/2} |\det(Q)|^{-1/2} e^{i\pi\,\mathrm{sign}\,Q/4}.$$

Combining these gives another formula for $K(x_2, x_0)$:

$$K(x_2, x_0) = (2\pi\hbar)^{n/2} |\det Q|^{-1/2} e^{i\pi \mathrm{sign}\,Q/4 + iS_{20}(x_2,x_0)/\hbar}.$$

By the composition property of the Maslov index

$$\mu_{21} + \mu_{10} = \mu_{20} - \tfrac{1}{2}\mathrm{sign}\, Q$$

Moreover, $Q = B_{21}^{-1} B_{20} B_{10}^{-1}$ and hence

$$|\det\, Q| \cdot |\det\, B_{21} B_{10}| = |\det\, B_{20}|.$$

Using these last three identities and the formula of Theorem 8.5 gives

$$\begin{aligned}
&\mathcal{U}(t_2, t_1) \circ \mathcal{U}(t_1, t_0) f(x_2) \\
&\quad = e^{i\pi(\mu_{21}+\mu_{10})/2} (2\pi\hbar)^{-n} |\det B_{21} B_{10}|^{-1/2} \int_{\mathbb{R}^n} K(x_2, x_0) f(x_0)\, dx_0 \\
&\quad = e^{i\pi(\mu_{20})/2} (2\pi\hbar)^{-n/2} |\det B_{20}|^{-1/2} \int_{\mathbb{R}^n} e^{iS_{20}(x_2,x_0)/\hbar} f(x_0)\, dx_0 \\
&\quad = \mathcal{U}(t_2, t_0) f(x_2)
\end{aligned}$$

as required. □

Proof of Theorem 8.4: Assume that $\psi_{t_0}^t$ admits a generating function and let $S(t, x, x_0)$ be given by the action. Let $B(t)$ denote the right upper block in the block decomposition of $\Psi_{t_0}^t = d\psi_{t_0}^t$ and abbreviate $\lambda = e^{i\pi\mu(t,t_0,H)/2} (2\pi\hbar)^{-n/2}$. Then

$$u(t, x) = \mathcal{U}(t, t_0, H) f(x) = \lambda |\det B(t)|^{-1/2} \int_{\mathbb{R}^n} e^{iS(t,x,x_0)/\hbar} f(x_0)\, dx_0.$$

Differentiating with respect to x gives

$$P_j u = \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} \frac{\partial S}{\partial x_j} e^{iS/\hbar} f$$

and

$$P_j P_k u = -i\hbar \frac{\partial^2 S}{\partial x_j \partial x_k} u + \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} \frac{\partial S}{\partial x_j} \frac{\partial S}{\partial x_k} e^{iS/\hbar} f$$

Hence the right hand side of the equation is

$$\begin{aligned}
H(t, Q, P)u \quad &= \quad -i\hbar\tfrac{1}{2}\mathrm{tr}\,(H_{yx} + H_{yy} D B^{-1})\, u \\
&\qquad + \lambda |\det B|^{-1/2} \int_{\mathbb{R}^n} H(t, x, \partial_x S) e^{iS/\hbar} f.
\end{aligned}$$

Here we have used the identity $\partial^2 S/\partial x^2 = DB^{-1}$ where $D = D(t)$ is the lower right block in the block decomposition (3) of $\Psi_{t_0}^t$. Now

$$\begin{aligned}\frac{d}{dt}|\det B|^{-1/2} &= -\tfrac{1}{2}\mathrm{tr}\,(\dot{B}B^{-1})|\det B|^{-1/2} \\ &= -\tfrac{1}{2}\mathrm{tr}\,(H_{yx} + H_{yy}DB^{-1})|\det B|^{-1/2}\end{aligned}$$

and hence

$$i\hbar\frac{\partial u}{\partial t} = -i\hbar\tfrac{1}{2}\mathrm{tr}\,(H_{yx} + H_{yy}DB^{-1})\,u - \lambda|\det B|^{-1/2}\int\limits_{\mathbb{R}^n}\frac{\partial S}{\partial t}e^{iS/\hbar}f.$$

Since S satisfies the Hamilton-Jacobi equation $\partial_t S + H(t, x, \partial_x S) = 0$ this proves the statement whenever $\psi_{t_0}^t$ admits a generating function. The general case follows since both sides of the equation depend continuously on H.

Here is an alternative proof. Since $\mathcal{U}(t, t_0)$ is a strongly continuous evolution operator it suffices to prove

$$\left.\frac{d}{dt}\right|_{t=t_0}\mathcal{U}(t, t_0)f = \frac{1}{i\hbar}H_{t_0}(Q, P)f$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\mathcal{U}(t_0 + \tau, t_0)f - \mathcal{T}(H_{t_0}\tau)f = O(\tau^2)$$

it suffices to prove

$$\lim_{\tau\to 0}\frac{\mathcal{T}(H\tau)f - f}{\tau} = \frac{1}{i\hbar}H(Q, P)f \tag{25}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and any quadratic Hamiltonian H. The limit is in the L^2-norm.

First assume that H either depends only on x or only on y or consists only of a mixed term. If $H = H(x)$ depends only on x then $\mathcal{T}(H\tau) = \mathcal{M}(H\tau)$ and hence

$$\begin{aligned}\lim_{\tau\to 0}\frac{\mathcal{T}(H\tau)f(x) - f(x)}{\tau} &= \lim_{\tau\to 0}\frac{e^{-i\tau H(x)/\hbar} - 1}{\tau}f(x) \\ &= \frac{1}{i\hbar}H(x)f(x).\end{aligned}$$

If $H = H(y)$ depends only on y then $\mathcal{T}(H\tau) = \bar{\mathcal{F}} \circ \mathcal{M}(H\tau) \circ \mathcal{F}$. Hence

$$\begin{aligned}
\lim_{\tau\to 0} \mathcal{F}\,\frac{\mathcal{T}(H\tau)f(y) - f(y)}{\tau} &= \lim_{\tau\to 0} \frac{e^{-i\tau H(y)/\hbar} - 1}{\tau}\mathcal{F}f(y) \\
&= \frac{1}{i\hbar} H(y)\mathcal{F}f(y).
\end{aligned}$$

If $H(x, y) = \langle H_{yx}x, y\rangle$ then $\mathcal{T}(H\tau) = \mathcal{K}(\mathbb{1} - \tau H_{yx})$ and hence

$$\begin{aligned}
\lim_{\tau\to 0} \frac{\mathcal{T}(H\tau)f(x) - f(x)}{\tau} &= \lim_{\tau\to 0} \frac{\det(\mathbb{1} - \tau H_{yx})^{1/2} f(x - \tau H_{yx}x) - f(x)}{\tau} \\
&= -\langle H_{yx}x, \nabla f(x)\rangle + \tfrac{1}{2}\operatorname{tr} H_{yx} \\
&= \frac{1}{i\hbar} H(Q, P)f(x).
\end{aligned}$$

In either case equation (25) is satisfied.

The general case follows by decomposing the operator $\mathcal{T}(H\tau)$ as in the proof of Proposition 8.1. More precisely, define for each $k = 0, 1, \ldots$ the Hilbert space $\mathcal{S}^k(\mathbb{R}^n)$ to be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_k^2 = \sum_{\mu,\nu} \|Q^\nu P^\mu f\|_{L^2}^2$$

where the sum is over all pairs of multi-indices μ and ν with $|\mu|+|\nu| \le k$. The $\mathcal{S}^0$-norm is the L^2-norm. Fourier transform is a Hilbert space isomorphism of each $\mathcal{S}^k$. The operator $\mathcal{T}(H)$ is also an isomorphism of $\mathcal{S}^k$ and satisfies estimates

$$\|\mathcal{T}(H\tau)f\|_k \le c\,\|f\|_k$$

and

$$\|\mathcal{T}(H\tau)f - f\|_k \le \tau c\,\|f\|_{k-2}$$

for $0 < \tau \le 1$ where the constant $c = c(k, H) > 0$ is independent of f and τ and depends continuously on the coefficients of H. This proves strong convergence in (25) for $f \in \mathcal{S}^2(\mathbb{R}^n)$. □

9 Geometric Quantization

A time dependent Hamiltonian H on $\mathbb{R}^{2n}$ determines an evolution system

$$(z_1, u_1) = g_{t_0}^{t_1}(z_0, u_0)$$

on $W = \mathbb{R}^{2n} \times \mathrm{U}(1)$ via the formula

$$z_1 = \psi_{t_0}^{t_1}(z_0), \qquad u_1 = u_0 e^{iI(c)/\hbar},$$

for $(z_0, u_0) \in W = \mathbb{R}^{2n} \times \mathrm{U}(1)$ where

$$I(c) = \int_{t_0}^{t_1} (\langle y, \dot{x} \rangle - H(t, x, y))\, dt,$$

$$c(t) = (x, y) = \psi_{t_0}^{t}(x_0, y_0), \qquad c(t_0) = z_0, \quad c(t_1) = z_1,$$

and $\psi_{t_0}^{t_1}$ is the evolution system generated by H. If the generating function S of Proposition 3.2 is defined then

$$g_{t_0}^{t_1}(z_0, u_0) = \left(\psi_{t_0}^{t_1}(z_0), u_0 e^{iS(x_0, x_1)/\hbar}\right) \tag{26}$$

where $z_j = (x_j, y_j)$. The group $\mathrm{ESp}(W, \hbar)$ of all diffeomorphisms of W of form $g_{t_0}^{t_1}$ where H runs over the time dependent (inhomogeneous) quadratic Hamiltonians $\mathbb{R} \to \mathcal{F}_2$ is called the **extended symplectic group**.[1] The various groups $\mathrm{ESp}(W, \hbar)$ depend set-theoretically on $\hbar$ but are isomorphic as abstract groups. There is a central extension

$$1 \to \mathrm{U}(1) \to \mathrm{ESp}(W, \hbar) \to \mathrm{ASp}(\mathbb{R}^{2n}) \to 1$$

where $\mathrm{ASp}(\mathbb{R}^{2n})$ denotes the **affine symplectic group**; the projection is given by $g_{t_0}^{t_1} \mapsto \psi_{t_0}^{t_1}$ and the $\mathrm{U}(1)$ subgroup consists of those $g_{t_0}^{t_1}$ where H is constant.

If the Hamiltonian H is time independent then the corresponding evolution systems $\psi_{t_0}^{t_1}$ and $g_{t_0}^{t_1}$ are flows: denote by X_H and Y_H the vector fields generating these flows. Then X_H is the Hamiltonian vector field[2] of H, and Y_H is a lift of X_H to W. The Lie algebra to $\mathrm{ASp}(\mathbb{R}^{2n})$ is the image of $\mathcal{F}_2$ under the representation $H \mapsto X_H$ but this representation is not faithful as the constant Hamiltonians $\mathcal{F}_0$ map to zero. However the representation $H \mapsto Y_H$ *is* faithful. Differentiating gives the following

Proposition 9.1 *The vector field Y_H on W is given by*

$$Y_H(z, u) = \left(X_H(z), uis_H/\hbar\right), \qquad s_H = \langle y, \partial_y H \rangle - H.$$

[1] Souriau [27] would probably call it the *affine quantomorphism group*.

[2] That is, the vector field whose integral curves are the solutions of Hamilton's equations (10).

Following Souriau [26] and Kostant [15] we describe the extended symplectic group as a group of bundle automorphisms. View W as (the total space of) a principal U(1) bundle over $\mathbb{R}^{2n}$ and define a connection form on W by

$$\alpha = -\frac{i}{\hbar}\langle y, dx\rangle + u^{-1}du.$$

The curvature form is $F = d\alpha = i\omega/\hbar$ a multiple of the standard symplectic form. Denote by $\nu(z,u) = (0, ui)$ the generator of the U(1) action and by $\widetilde{X}_H$ the horizontal lift of X_H defined by

$$\widetilde{X}_H = (X_H, \dot{u}), \qquad \alpha(\widetilde{X}_H) = 0.$$

In other words

$$\dot{u} = u\frac{i}{\hbar}\langle y, \partial_y H\rangle.$$

The following is routine.

Lemma 9.2 *$Y_H = \widetilde{X}_H - (H/\hbar)\nu$.*

Corollary 9.3 *The extended symplectic group* $\mathrm{ESp}(W,\hbar)$ *is the group of all automorphisms of the* U(1) *bundle W which preserve the connection α and cover an affine symplectomorphism of* $\mathbb{R}^{2n}$.

Proof: Let $\iota(X)$ denote interior multiplication by a vectorfield X and $\ell(X)$ denote Lie differentiation. Then $\iota(\widetilde{X}_H)\alpha = 0$ and $\iota(\widetilde{X}_H)d\alpha = idH/\hbar$ so $\ell(\widetilde{X}_H)\alpha = idH/\hbar$ by Cartan's formula. Since $\iota(\nu)\alpha = i$ and $\iota(\nu)d\alpha = 0$ Cartan's formula gives $\ell((H/\hbar)\nu)\alpha = (dH/\hbar)i$. Hence $\ell(Y_H)\alpha = 0$. This shows that the elements of the group $\mathrm{ESp}(W,\hbar)$ preserve α. The definition shows that the elements $g_{t_0}^{t_1}$ of $\mathrm{ESp}(W,\hbar)$ commute with the $U(1)$ action and cover elements $\psi_{t_0}^{t_1}$ of $\mathrm{ASp}(\mathbb{R}^{2n})$. The bundle automorphisms which cover the identity of $\mathbb{R}^{2n}$ are given by the $U(1)$ action and are the elements of $\mathrm{ESp}(W,\hbar)$ corresponding to constant Hamiltonians. □

10 Representations

The Extended Metaplectic representation

The set of all unitary Schwartz automorphisms

$$U = \mathcal{U}(t_1, t_0, H) \tag{27}$$

where H runs over the time dependent quadratic Hamiltonians and t_1, t_0 range over the real numbers forms a group $\mathrm{EMp}(2n)$ called the **extended metaplectic group**. The **metaplectic group** $\mathrm{Mp}(2n)$ is the subgroup of all elements of form (27) where H is a time dependent homogeneous quadratic Hamiltonian. Let $g(H)_{t_0}^{t_1}$ denote the evolution system on W defined in section 9.

Proposition 10.1 *The formula*

$$\mathrm{EMp}(2n) \to \mathrm{ESp}(W, \hbar) : \mathcal{U}(t_1, t_0, H) \mapsto g_{t_0}^{t_1}(H)$$

gives a well-defined group homomorphism. It is a double cover.

Proof: By Theorem 8.5 and equation (26) the map is well-defined and two-to-one. It is a group homomorphism by Corollary 8.3. $\square$

Remark 10.2 It follows that $\mathrm{Mp}(2n)$ is a nontrivial double cover of $\mathrm{Sp}(2n)$. Since $\pi_1(\mathrm{Sp}(2n)) = \mathbb{Z}$ the double cover is unique up to isomorphism.

Decompose the Lie algebra Op_1 as

$$\mathrm{Op}_1 = \mathcal{M} \oplus \mathrm{Op}_0$$

where $\mathcal{M}$ is spanned by $P_1, \ldots, P_n, Q_1, \ldots, Q_n$. The linear version of the following theorem appears in [16].

Theorem 10.3 *The extended metaplectic group is precisely the set of all unitary Schwartz automorphisms U such that*

$$U \mathrm{Op}_1 U^{-1} = \mathrm{Op}_1.$$

The subgroup of all U such that

$$U \mathcal{M} U^{-1} = \mathcal{M}$$

is isomorphic to $U(1) \times \mathrm{Mp}(2n)$.

The Heisenberg representation

The subgroup $\mathrm{HG}(W, \hbar)$ of the extended symplectic group $\mathrm{ESp}(W, \hbar)$ consisting of those elements which cover translations is called the **Heisenberg group**. Its Lie algebra is the image of $\mathcal{F}_1$ in the representation by vector fields of Proposition 9.1. The restriction of the isomorphism $\mathcal{F}_2 \to \mathrm{Op}_2$ of Theorem 7.1 gives an isomorphism $\mathcal{F}_1 \to \mathrm{Op}_1$ called the **Heisenberg representation**. A typical element $H \in \mathcal{F}_1$ is a Hamiltonian of degree one

$$H(x, y) = \langle H_x, x\rangle + \langle H_y, y\rangle + H_0.$$

Such a Hamiltonian never admits a generating function. Hence Theorem 8.5 cannot be used to give a formula for the limit $\mathcal{U}(t, t_0, H)$.

Theorem 10.4 *For a time independent affine Hamiltonian H we have*

$$\mathcal{U}(t, t_0, H) = \mathcal{U}((t - t_0)H)$$

where

$$\begin{aligned} \mathcal{U}(H)f(x) &= e^{-iw(H,x)/\hbar} f(x - H_y), \\ w(H, x) &= H_0 + \langle H_x, x\rangle - \langle H_x, H_y\rangle/2. \end{aligned}$$

Proof: By direct computation

$$\mathcal{T}(H)f(x) = e^{-iu(H,x)/\hbar} f(x - H_y), \qquad u(H, x) = H_0 + \langle H_x, x\rangle.$$

Choose a partition $t_0 < t_1 < \cdots < t_N = t$ and define $\tau_j = t_j - t_{j-1}$. By induction on N

$$\begin{aligned} \mathcal{U}^{\mathcal{T}}(t, t_0, H) &= \mathcal{T}(\tau_N H) \circ \cdots \circ \mathcal{T}(\tau_1 H) f(x) \\ &= e^{-iw^{\mathcal{T}}(t_0,t,H,x)/\hbar} f(x - (t - t_0)H_y) \end{aligned}$$

where

$$w^{\mathcal{T}}(t_0, t, H, x) = w((t - t_0)H, x) + \tfrac{1}{2}\langle H_x, H_y\rangle \sum_{j=0}^{N} \tau_j^2.$$

Now let the mesh $|\mathcal{T}| = \max|\tau_j|$ go to zero. Alternatively the statement follows from Theorem 8.4 $\square$

The Lie algebra of the Heisenberg group is the space $\mathcal{F}_1$. It is isomorphic to the vector space

$$\mathfrak{hg}(2n) = \mathbb{R}^{2n} \times \mathbb{R}$$

with bracket operation

$$[(v_1, c_1), (v_2, c_2)] = (0, \omega(v_1, v_2)).$$

for $(v_j, c_j) \in \mathfrak{hg}(2n)$. This space is called the **Heisenberg algebra**. An explicit isomorphism $\mathfrak{hg}(2n) \to \mathcal{F}_1 : (v, c) \mapsto H$ is given by

$$H_0 = c, \qquad (H_x, H_y) = v \tag{28}$$

with H defined as above. The universal cover of the Heisenberg group $\mathrm{HG}(2n)$ is the central extension

$$\widetilde{\mathrm{HG}}(2n) = \mathbb{R}^{2n} \times \mathbb{R}$$

of $\mathbb{R}^{2n}$ with group operation

$$(v_1, c_1) \cdot (v_2, c_2) = (v_1 + v_2, c_1 + c_2 + \tfrac{1}{2}\omega(v_1, v_2)).$$

An explicit covering map $\widetilde{\mathrm{HG}}(2n) \to \mathrm{HG}(2n)$ is given by

$$\widetilde{\mathrm{HG}}(2n) \to \mathrm{HG}(2n) : (v, c) \mapsto \mathcal{U}(H)$$

where H is given by (28). We may think of this covering map as an irreducible unitary representation of $\widetilde{HG}(2n)$. If Ψ is a symplectic matrix then the map

$$\widetilde{\mathrm{HG}}(2n) \to \mathrm{HG}(2n) : (v, c) \mapsto \mathcal{U}(H \circ \Psi)$$

is another such representation corresponding to the same value of Planck's constant $\hbar$. By the Stone-von Neumann theorem both representations are unitarily isomorphic. In other words there exists a unitary operator $U : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, unique up to multiplication by a complex number of modulus 1, such that

$$\mathcal{U}(H \circ \Psi) = U^{-1} \circ \mathcal{U}(H) \circ U.$$

These operators U are the elements of the metaplectic group. This is apparently how the metaplectic representation was discovered (see [24]). The elements of the metaplectic representation are thus viewed as intertwining operators of various incarnations of the Heisenberg representation. See [20] for an exposition in terms of co-adjoint orbits and polarizations.

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