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Aspects of D-branes in supergravity

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Dissertation presented in partial fulfillment of the requirements for the degree of Doctor in Science

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Abstract

Branes play an important role in String Theory as dynamical, extended objects where open strings can attach their endpoints. An effective description of string theory at low energies is provided by supergravity. By looking at the effective description, the branes become very massive and they appear as non-perturbative objects, no longer partaking in any of the dynamics and only playing a role in shaping the background. In this thesis we look at two different aspects of branes in supergravity.

First, we will consider a D3-brane world-volume set in various supergravity backgrounds. Because the brane world-volume inherits part of the symmetries of the background as global symmetries, different backgrounds lead to different world-volume theories. Our motivation for studying these set-ups stems from the study of higher derivative terms in supergravity and their construction. Recently, the D3-brane world-volume theory embedded in a ten-dimensional Minkowski background was studied and used to construct supersymmetric higher invariants. This was done by deforming the action and supersymmetry transformation rules of the $D = 4$, $\mathcal{N} = 4$ Maxwell multiplet. The resulting theory has 16 deformed Maxwell multiplet supersymmetries and 16 Volkov-Akulov type non-linear supersymmetries. To extend this rigid supersymmetric result to supergravity one would like to use superconformal methods, however, in order to use these methods, we need to determine the superconformal transformation rules of the deformed Maxwell multiplet. An interesting question that arises immediately is how the Volkov-Akulov supersymmetry of the deformed $16 + 16$ Maxwell multiplet are related to the S-supersymmetry of the conformal Maxwell multiplet. The superconformal transformation rules can be obtained by embedding the world-volume in an $AdS_5 \times S^5$ background. We investigate and establish this relation, and, in addition, propose a method for constructing higher-derivative invariants by using this relation between S-supersymmetry and Volkov-Akulov symmetry.

Secondly, we consider the effect of branes on their background. Our motivation originates in applying the AdS/CFT correspondence and other gauge/gravity dualities. They provide an excellent framework to study strongly coupled quantum field theories in terms of their (weakly coupled) gravitational duals. Recent work, in the context of 5-dimensional gauge theories and their gravity duals, has compared the partition function of the gauge theory with the result from the gravity dual, as well as the vacuum expectation value of the half-BPS Wilson line for totally symmetric and anti-symmetric representations. In either case the vacuum expectation value on the gravity side can be well approximated by probe branes, branes that do not affect their background. To go further than the probe approximation, one must include the effect of the branes on the background, their backreaction. In terms of branes, the background 5-dimensional gauge theory arises as the low energy limit of a configuration of branes consisting of D4-branes and D8-branes along with an O8 orientifold projection. Introducing a Wilson line in the fundamental representation corresponds to introducing a fundamental string perpendicular to the D4/D8-brane system. Rank M symmetric representations arise from introducing an additional D4-brane and stretching M fundamental strings between the D4-brane and the D4/D8-brane stack. Rank M anti-symmetric representations arise by introducing a perpendicular D4-brane and M fundamental strings. In general, the BPS-Wilson line reduces the superconformal symmetry of the 5-dimensional gauge theory from $F(4;2) \times SU(2)$ to $D(2,1;2;1) \times SO(4)$. With the task of finding backreacted geometries describing these Wilson lines in mind, we study general solutions of massive IIA supergravity with $D(2,1;\gamma;1) \times SO(4)$ symmetry. We give a partial reduction and integration of the BPS equations, including obtaining algebraic expressions for the metric factors in terms of spinor bilinears as well as solutions in special cases of symmetry enhancement.

Beknopte samenvatting

Branen spelen een belangrijke rol in snaartheorie als dynamische objecten waar open snaren op kunnen eindigen. Een effectieve beschrijving van snaartheorie wordt gegeven door supergravitatie. Door een effectieve beschrijving te beschouwen worden de branen zeer massief en komen ze voor in supergravitatie als niet-perturbatieve objecten. Branen nemen niet langer deel aan de dynamica van de theorie maar geven wel nog vorm aan de achtergrond. In deze thesis kijken we naar twee verschillende aspecten van branen in supergravitatie. We beginnen echter met de introductie van wat achtergrond materiaal.

We beschrijven p-branen als supergravitatie-oplossingen, de acties van hun wereldvolume en de relatie tussen hun inbedding in de achtergrond en het behoud van een deel van de symmetrieën van de achtergrond op het wereldvolume. We beschouwen de constructie van doorkruisende braan configuraties, in het bijzonder het D4/D8-braan systeem dat we verderop in deze thesis bestuderen. Deze supergravitatie theorieën bestaan in tien of elf ruimte-tijd dimensies. We zullen kort in gaan op de relatie tussen deze theorieën en snaartheorie, zodat we in staat zijn om de AdS/CFT-correspondentie te formuleren in zijn originele vorm: een dualiteit tussen type IIB snaartheorie op $AdS_5 \times S^5$ en $D = 4$, $\mathcal{N} = 4$ super Yang Mills theorie met ijkgroep $SU(N)$. Het doel hier is om een aantal concepten van de AdS/CFT-correspondentie te introduceren door een paar voorbeelden te bekijken. We formuleren de algemenere ijk/gravitatie correspondentie, inclusief de identificatie van velden en operatoren, en bij wijze van voorbeeld berekenen we de 2-punts functie van een veldentheorie gebruikmakende van de duale gravitatie theorie. Tot slot, bespreken we Wilson-lijnen omdat deze een cruciaal deel uit maken van de motivatie voor een deel van het werk dat verderop gepresenteerd wordt.

Zoals hierboven vermeld zijn we geïnteresserd in twee verschillende aspecten van branen in supergravitatie. Eerst zullen we D3-branen in verschillende supergravitatie achtergronden beschouwen. Omdat het wereldvolume van het braan een deel van de symmetrieën van het braan overneemt als

globale symmetrieën, leiden de verschillende achtergronden tot verschillende wereldvolume theorieën. Onze motivatie voor het bestuderen van deze set-up vind zijn oorsprong in het bestuderen van hogere orde afgeleide termen in supergravitatie en hun constructie. Recent werd het D3-braan wereldvolume bestudeerd in een tien-dimensionale Minkowski achtergrond en gebruikt om supersymmetrische hogere orde invarianten te construeren door de actie en de supersymmetrietransformaties van het $D = 4$, $\mathcal{N} = 4$ Maxwell multiplet te vervormen. De resulterende theorie heeft 16 vervormde Maxwell multiplet supersymmetrieën en 16 niet-lineaire supersymmetrieën van het Volkov-Akulov type. Om deze rigide resultaten uit te breiden naar supergravitatie zou men superconforme methoden willen gebruiken. Maar om deze methoden te gebruiken hebben we de superconforme transformatie-regels nodig van het vervormde Maxwell multiplet. Een interessante vraag die meteen opduikt is hoe de Volkov-Akulov supersymmetrie van het vervormde 16+16 Maxwell multiplet gerelateerd is met de S-supersymmetrie van het conforme Maxwell multiplet. De superconforme transformatieregels kunnen we vinden door het wereldvolume van het braan in een $AdS_5 \times S^5$ achtergrond te plaatsen. We bestuderen en vinden dit verband, en gebaseerd op dit verband tussen S-supersymmetrie en Volkov-Akulov symmetrie, stellen ook een methode voor om hogere-orde invarianten te construeren.

Hiernaast beschouwen we ook het effect van branen op hun achtergrond. Onze motivatie kadert in de toepassing van de AdS/CFT-correspondentie en andere ijk/gravitatie theorieën. Deze geven een ideale toolkit om sterk gekoppelde kwantumveldentheorieën te bestuderen in termen van hun (zwak gekoppelde) duale gravitatie-theorieën. Recent werk in de context van 5-dimensionale ijktheorieën en hun duale gravitatie-theorieën vergelijkt de partitie functie van de ijktheorie met het resultaat van de gravitatie-theorie. Ook de vacuüm-verwachtingswaarde van de half-BPS Wilson lijn voor totaal symmetrische en totaal antisymmetrische representaties zijn met succes vergeleken. In beide gevallen kan de vacuum verwachtingswaarde aan de gravitationele kant goed benaderd worden door het gebruik van probe-branen, branen die geen invloed hebben op hun achtergrond. Om een exacter resultaat te verkrijgen moet men rekening houden met de invloed van de branen op de achtergrond, de backreaction. In termen van branen verkrijgen we de 5-dimensionale ijktheorie als de lage energie limiet van een configuratie van branen die bestaat uit D4-branen en D8-branen samen met een O8-oriëntifold projectie. De introductie van een Wilson lijn in de fundamentele representatie correspondeert met de introductie van een fundamentele snaar loodrecht op het D4/D8-braan systeem. Rang M symmetrische representaties ontstaan door het introduceren van een extra D4-braan en dan M fundamentele snaren op te spannen tussen dit D4-braan en het D4/D8-braan systeem. De rang M antisymmetrische representaties verkrijgen we door de introductie van D4-braan dat loodrecht staat op de D4/D8 configuratie en dan hiertussen M fundamentele snaren op te spannen. In het algemeen reduceert de introductie van de Wilson lijn de superconforme symmetrie van de 5-dimensionale ijktheorie van $F(4;2) \times SU(2)$ naar $D(2,1;2;1) \times SO(4)$. Met de taak van het zoeken naar achtergronden die de backreaction van deze configuraties bevatten in het achterhoofd, bestuderen we algemene oplossingen van massieve IIA supergravitatie met $D(2,1;\gamma;1) \times SO(4)$ symmetrie. We geven een partiële reductie en integratie van de BPS vergelijkingen, inclusief algebraïsche uitdrukkingen voor de factoren in de metriek in termen van spinor bilineairen en oplossingen in speciale gevallen van versterkte symmetrie.

Contents

Chapter 1

Introduction

1.1 Twentieth Century Physics

In the twentieth century, physicists have been extremely successful in formulating physical theories that are capable of describing, explaining and predicting physical phenomena, ranging from the subatomic scale to the scale of galaxies and our universe. The formulation of General Relativity by Einstein, as a more complete picture of Newtonian gravity in the first half of the century, changed the way we view time and space by combining them in a geometric picture. The discovery and development of Quantum Mechanics, inspired by the need for quantised energy in studies of black body radiation and the photo-electric effect, followed not long after. Motivated by the unification of the electric and magnetic forces by Maxwell, physicists set out to unify the four fundamental forces of nature. Of these four fundamental forces, gravity and electromagnetism are the most familiar to everyday life. Gravity is well known to all of us, after all it keeps us sticking to this lump of matter we call earth, and electromagnetism is crucial in all sorts of everyday appliances that make life easier. The remaining two fundamental forces are lesser known. The weak interaction governs the radioactive decay of particles, while the strong interaction regulates interactions at the level of the nucleus and ensures the stability of ordinary matter. Quantum Mechanics was combined with notions of special relativity to form relativistic quantum field theories. Electromagnetism was the first fundamental force to be realised as a quantum field theory, Quantum Electro Dynamics (QED), with great success. Calculations in QED made extremely accurate predictions of quantities like the anomalous magnetic moment of the electron and the Lamb shift of the energy levels of hydrogen, the former being confirmed to within ten parts in a billion, making this one of the most precise physical theories ever. It was found that QED could be unified with the weak interaction within the framework of quantum field theories, to form an electro-weak theory. Subsequently the strong interaction was incorporated with the electroweak theory, culminating in the Standard Model of physics. Together the Standard Model and General Relativity represent the pinnacle of twentieth century physics. These theories are able to describe the universe in terms of a remarkable small number of elementary particles and the four fundamental interactions between them. Both theories have been verified to extreme high accuracies, they have explained observed phenomena and in turn predicted new phenomena that were later observed. Our thirst for unification has not been sated yet though. Both theories and in particular attempts at combining both viewpoints run into questions and problems that beg for an overarching theory, unifying gravity and the three other fundamental forces. To understand why, we first discuss both theories in some more detail.

The Standard Model of Physics

The *Standard Model* (SM) describes the interaction of particles under the three fundamental forces: electromagnetism, the weak interaction and the strong interaction. The fundamental objects are quantum fields, and hence the SM is called a *Quantum Field Theory* (QFT). Particles are represented by fluctuations of these fields. Every type of fundamental particle corresponds to a quantum field and the propagation of these fields through spacetime is described by the QFT Lagrangian. Interactions between these particles are represented by interaction terms for the fields in the QFT Lagrangian.

Combining the principles of special relativity and quantum mechanics leads to the idea that particles can only interact with each other by exchanging other particles. The elementary particles can be classified into matter particles and force carriers. The force carriers are exchanged by interacting matter particles. The matter particles are fermions, carrying fractional spin 1/2, while the force carriers have spin 1 and are bosons.

A major realisation in the formulation of the SM was that the fundamental interactions correspond to symmetries of the QFT Lagrangian, called *gauge symmetries*. These gauge symmetries are symmetries that are realised locally, meaning that they can act independently at every point in spacetime. Every gauge symmetry has a corresponding set of gauge fields, or in the terminology above, force carriers. Each fundamental interaction is determined by a specific symmetry and so mediated by their own force carriers. Electromagnetism is represented by the emission and absorption of a photon, the weak interaction is governed by the massive $W^{\pm}$ and Z bosons, while the strong interaction is mediated by a set of 8 gauge fields, the gluons.

The SM has a gauge symmetry group $SU(3) \times SU(2) \times U(1)$, where the $SU(3)$-factor represents the symmetry group of the strong interaction, while $SU(2) \times U(1)$ encompasses the unified electro-weak interactions. This last group is broken to a $U(1)$ by the mechanism of spontaneous symmetry breaking, giving rise to QED. Matter particles are incorporated in the SM as a set of quantum fields that sit in a specific representation of the gauge group. Each representation transforms in a particular way under the action of the gauge group and this transformation determines how the particles corresponding to the fields interact by exchanging gauge particles. The matter particles are split into two groups. The particles that couple to gluons are referred to as the quarks, and particles that do not interact with gluons are called the leptons. Each of these two branches then consists of three generations of two particles. The quarks are the matter particles that constitute for instance the proton and neutron, whereas the leptons contain the electrons. With the discovery of the Brout-Englert-Higgs particle in 2012 at the LHC[1] in CERN [1], all of the particles in the SM have been experimentally observed.

Gravity

While the SM describes how particles interact (quantum mechanically) at small scales, General Relativity describes effects at large scales. Gravity is quite different from the other fundamental forces. For one, as counter-intuitive as it seems at first glance, it is far weaker than the other forces. Imagine picking up a small metal ball using a magnet. The magnetic force of the magnet exerted on the ball overcomes the gravitational force exerted on the ball by the entire earth! To make this more precise we can consider the ratio of the gravitational attraction to the electric force between two electrons. This ratio is an astonishingly small number of the order 10^{-43}.

Even though it is so weak, gravity still plays a major role in our lives due to its shear abundance. Gravity is universal, and by that we mean that all forms of matter, the other gauge fields and even the gravitational field itself, interact gravitationally. This happens because the charge to which gravity couples is energy, and everything of physical relevance carries energy. This is in sharp contrast with the other forces, where we know that they couple to a certain number of fields while they don't interact with others. For example, quarks are charged under the strong interaction and so they interact with each other by

[1] The world's largest and most powerful particle accelerator.

exchanging gluons, while an electron is not charged under the strong interaction and consequently does not interact with other particles by exchanging gluons.

A second feature that enhances the influence of gravity is that it is always attractive. Contrary to gravity, electromagnetic (and even more so for the strong force) sources align themselves in neutral combinations. For electromagnetism this is the attraction between opposite charges, for the strong interaction this is known as confinement. Furthermore, the fields that mediate the weak force are massive, making the weak force exponentially smaller with distance. So in spite of its weakness with respect to the other fundamental forces, at large enough scales, gravity is the only relevant force, as on this scale all objects are neutral to the other forces. Combine this with the universality and it is clear that gravity dominates the large scale interactions.

In the early twentieth century, Einstein altered the way we consider space and time in his theory of special relativity, they are no longer to be considered as separate concepts but are combined in a unified geometrical picture, spacetime. A few years after his formulation of special relativity, he made a connection between gravity and the structure of spacetime in his theory of *General Relativity* (GR), hereby generalising Newton's law of gravitation. In connecting space and time in a unified picture also energy and mass became related, leading to the famous equation $E = mc^2$. Einstein's equations form a set of non-linear differential equations that determine the relation between the dynamical field of general relativity (the metric) and the content of spacetime (the matter). The metric is a classical field and determines the way we measure distances in space and time, as well as the concept of a straight line (called geodesics) in spacetime. The matter enters these field equations through an object called the stress-energy tensor and is seen as arising from a different theory. In textbooks many examples for the stress-energy tensor are given such as perfect fluids, non-interacting matter distributions, etc. The field equations show how matter curves spacetime (the stress-energy tensor determines the metric) and, in turn spacetime tells matter how to move (the metric determines the geodesics).

Trouble in paradise

At first glance we seem to have two highly successful theories that describe the world as we know it to a satisfactory level, however, we are left with some questions and problems. From the SM perspective some of the questions that we would like to see explained are why there are three generations and why the gauge group is $SU(3) \times SU(2) \times U(1)$. The SM has 19 free parameters that have to be fixed experimentally, such as the relative masses of leptons and quarks, and the relative interaction strengths of the fundamental forces.

This is still quite a large number for a 'fundamental' theory.[2] Furthermore it does not provide an explanation for matter/antimatter asymmetry or the dark matter/dark energy content observations of the universe, and we seem to have a fine-tuning problem related to the lightness of the Higgs particle.[3] Finally, the SM does not incorporate gravity, which is reinforced by the separation of the internal symmetries from the spacetime symmetries. This separation of internal and spacetime symmetries is enforced by the Coleman-Mandula theorem[2] which can be formulated under some standard physical assumptions (locality, causality,...).

A more fundamental problem has to do with the quantisation of gravity. At first glance it seems unclear why we would need a quantum theory of gravity since it is negligible at microscopic scales. Indeed, at energy scales that are available to present day accelerators gravity is so weak compared to the other forces that its effects can be ignored. However, at higher and higher energies, gravity is expected to become more important. This is related to the running of the coupling of the gauge theories in the SM. The coupling constants of the fundamental forces in the SM are not really constants, they depend on the energy scale used to probe the theory, and it is found that they roughly converge at energy scales of 10^{16} GeV. A similar analysis can be done for gravity, by studying perturbations on the metric called gravitons. It is found that the strength of the gravitational interaction becomes comparable to the strengths of the SM interactions at the Planck scale corresponding to energies of 10^{19} GeV. This means that at this scale we can no longer neglect gravitational effects, and we would need a quantum theory of gravity. Furthermore, one could think of other situations where quantum gravity effects would become important such as the very early stages of the universe or in black holes, where strong gravitational effects take place on very small scales.

Developing a theory of quantum gravity runs into some problems. It seems that gravity unlike the SM is non-renormalisable. In both quantum theories there are divergences that arise in computations of loop integrals. In the SM one can reabsorb these infinities by a renormalisation procedure, where one redefines a finite number of parameters. To do the same for gravity one would

[2] Although one could argue we are doing quite well compared to the parameters one needs as input for Chemistry. Each atom requires several parameters describing for example the mass, ionisation energy, electron affinity, and electro-negativity. Just counting the number of atoms, this already leads to a much larger set of parameters than we need for the SM.

[3] It seems that quantum corrections to the Higgs mass pile up to be very large. Since the observed Higgs-mass is relatively small, we would need a very large bare mass for the Higgs particle such that the difference between the bare mass and the quantum corrections is a very small (observable) number. Since we are subtracting two enormous numbers to get a very precise small result, it looks there is an awful (and suspicious) lot of fine-tuning needed to make this happen exactly the way we observe it.

need to introduce an infinite number of parameters, and in the process lose all predictability.

It is clear that the SM and the theory of general relativity are incomplete, and cannot be fundamental theories, instead we consider them to be *effective theories*. An effective theory is a theory that is valid up to a certain energy scale. It masks the microscopic details by considering coarse grained variables in their regime of validity. As an example consider Newtonian mechanics. This is a valid theory for velocities that are small compared to the speed of light. When the velocity approaches the speed of light, we need to add corrections, and a better description is found in special relativity. In this picture, Newtonian mechanics form an effective theory and special relativity provides the overarching framework. One can go one step further and considers what happens when we allow for large masses. Special relativity is now to be considered as an effective theory, valid for small masses or energies, whereas the corrections one needs for large energies are described by GR. Considering a theory of quantum gravity as the overarching theory means that it should provide us with the quantum corrections to GR, as well as the gravitational corrections to the SM. Since we seem unable to construct a quantum theory of gravity by directly quantizing gravity like we did for the other forces, the question remains what this overarching theory could be.

1.2 Enter string theory

The prime candidate for a theory of quantum gravity is *String Theory*. An early version was originally developed in the early 1970s as a quantum field theory to describe the strong interaction, but it was cast aside with the advent of Quantum Chromo Dynamics (QCD). It regained interest not long after as a possible theory of quantum gravity. String theory replaces the notion of point particles with those of tiny one-dimensional objects, strings. The observed particles in nature are then just different vibrational patterns of the fundamental string. The obvious advantage is that there may be many elementary particles, but there is only one fundamental string, whose various excitations could reproduce the entire particle zoo. In fact, studying the spectrum of closed strings one finds a massless spin 2 particle, the graviton. Aside from the graviton, the spectrum of open strings also contains massless spin-1 particles, suggesting that the theory might be able to unify gravity and the other SM forces. Indeed, the endpoints of an open string can join to form a closed string, and so open string theories cannot exist without also including closed strings. Which in turn implies that string theories automatically include the graviton, and by extension gravity.

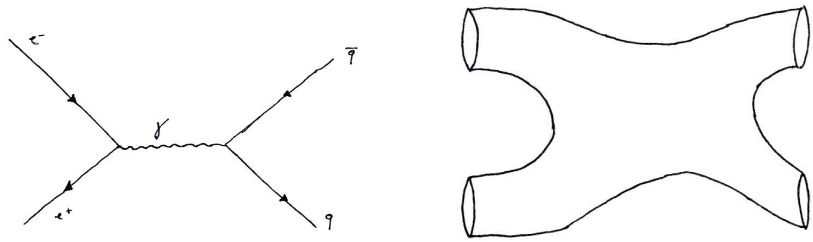
Similar to point particles living on a world-line in spacetime, the time evolution of a string sweeps out a 2-dimensional surface called a world-sheet in a higher dimensional spacetime. As the strings are extended objects they have a characteristic length-scale associated to them, called the string length l_s. The string length sets the tension of the strings and this is the only parameter of the theory. As we expect a quantum theory of gravity to live at the Planck scale, we can estimate the stringy effects and strings to appear at lengths comparable to the Planck length. This sets the string length at about 10^{-35}m, let us give up the compact, scientific notation for a moment, write this number out in full

$$l_S \sim 0.000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,1\,\mathrm{m},$$

and see that it is an astonishingly small number. The energy scale associated to the string length is $M_p \sim 10^{19}$GeV, much higher than the energies probed at current accelerators (10^4GeV). Hence, the accelerators only see point particles, explaining the amazing success of QFTs. Only at higher energies the particle description breaks down by the appearance of stringy effects, and we need a description in terms of a theory of quantum gravity.

The interaction between various strings can be envisioned by their world-sheets joining and then splitting (figure 1.1(b)). The strength of the interactions is set by the string coupling g_s. The string coupling is related to a massless scalar string excitation called the dilaton $g_s = e^{\phi}$. This is an important point because it means that the string coupling is not an adjustable (dimensionless) parameter since the dilaton is a dynamical field of the theory. An interacting string reduces to the theory of two-dimensional Riemann surfaces and the interactions are determined by different topologies of free strings. The smoothness of this interaction means that strings tend to smear out the short-distance behaviour of Feynman diagrams in QFT (figure 1.1), giving the hope of a theory free of UV divergencies.

These advantages come at a price, however, and it is not possible to formulate a consistent theory of strings in four spacetime dimensions (three space dimensions plus one time dimension), one needs to include another six dimensions, bringing the total to ten dimensions. This means that we have to find a way to deal with these six extra dimensions. A scheme for this exists in terms of Kaluza-Klein compactifications. We consider the six extra dimensions to be very small and compact, and the remaining four should correspond to a four-dimensional Minkowski spacetime. In this way, the extra six dimensions are not directly visible to an observer in the four-dimensional theory. The effect of these compact dimensions is visible in the presence of the Fourier modes of the ten-dimensional fields along the compactified dimensions. They are observed as a tower of massive fields in the four-dimensional theory, their masses inversely related to the size of the compact directions. The size of these extra dimensions then



(a) (b)

Figure 1.1: Depicted in (a) is a Feynman diagram of interacting point-particles. In (b) we have two strings joining and splitting, effectively smearing out the interaction.

has to be sufficiently small to explain why these massive modes have not been observed in experiment.

Supersymmetry and supergravity

A crucial ingredient for the consistency of string theory is the introduction of supersymmetry. Supersymmetry is a spacetime symmetry that connects bosonic and fermionic states. The anti-commuting generators of supersymmetry are necessary to cure the theory of unphysical tachyonic modes in the spectrum of the bosonic string.

Supersymmetry had been considered independent of the string theory context in the early 1970s. A number of people considered the possibility of a supersymmetry as global spacetime symmetry in four-dimensional quantum field theories. Incorporating supersymmetry in the SM introduces superpartners for each particle (named sleptons, squarks and gauginos), which has the benefit of removing the hierarchy problem because contributions of particles and their superpartners to the quantum corrections of the Higgs mass cancel, eliminating the need for a large bare mass and the corresponding fine-tuning. Additionally, it provides natural candidates for dark matter particles[4] and has a favourable influence on the unification of the SM coupling constants.

However, supersymmetry cannot be realised as an unbroken symmetry. Unbroken supersymmetry would imply that for every bosonic particle there is a corresponding fermionic particle of the same mass and vice versa. Since

[4]The lightest superpartner would be a stable particle, only interacting gravitationally with the particle content of the SM.

these superpartners are not observed, supersymmetry has to be broken at the scale of current detectors. The mechanism for symmetry breaking associates a particle to a broken global symmetry. A bosonic global symmetry gives rise to a massless Goldstone particle (a boson), while broken global supersymmetry gives rise to a massless fermion, the Goldstino. Since no such particles haven observed either, supersymmetry cannot be realised as a broken global symmetry. Instead, we should consider a local realisation of supersymmetry. A broken gauge symmetry does not lead to a massless Goldstone particle, instead, the corresponding gauge field becomes massive. In the case of supersymmetry, this leads to a massive vector-spinor field (spin 3/2). If the supersymmetry breaking scale is well above the SM scale this could explain why no such particle has been observed yet.

Field theories in which supersymmetry is realised as a gauge theory were first constructed in [3]. A remarkable effect of requiring supersymmetry to be local is the automatic inclusion of gravity. This is due to the appearance of translations in the supersymmetry algebra, which means that a theory that is invariant under local supersymmetry has to be invariant under local translations. Local translations correspond to the principle of general covariance of GR, and so a theory with local supersymmetry automatically includes gravity. For this reason the field theories in which supersymmetry is realised locally are called *supergravity* theories. Furthermore, with the help of the fermionic generators of supersymmetry, supergravity theories manage to circumvent the Coleman-Mandula theorem and allow for the unification of Poincaré spacetime group and the internal symmetry group. One of the main motivations to study supergravity theories was the hope that these theories would be renormalisable and could provide a theory of quantum gravity. However, soon enough it was realised that the problems remained. The divergences in the loop integrals were softened but were still present and so supergravity could not be the final answer.

Our understanding of string theory is currently incomplete, in part due to the complications of the massive modes[5] that appear at the Planck scale. However, there are interesting features that appear when considering low energies. The spectrum of quantised strings contains a number of massless particles. For the closed string one finds the metric $g_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and the dilaton ϕ, while for the open string one encounters gauge vectors A_μ. The massive modes have masses proportional to $1/l_S$. Since l_S is of the order of the Planck length, the lowest energy modes are already extremely massive and thus can be ignored in an effective description well below the Planck scale. The effective field theory describing the massless modes of superstring theory is a 10-dimensional supergravity theory, and in fact, string compactifications

[5]These modes appear due to the quantisation of the string action, and are not to be confused with the tower of massive states obtained by compactification.

can effectively be described by supergravity theories in dimensions lower than 10. This provides a strong connection between string theory and supergravity theories, and motivates the study of supergravity theories.

Nevertheless, there is great merit to studying supergravity theories *an sich* and not just consider them as low energy, effective actions for string theory. It turns out that the inclusion of supersymmetry is quite restrictive, limiting the spacetime dimensions in which one can construct gravity theories to a maximum of $D = 11$. Supergravity theories can be classified according to the number of spacetime dimensions D and the amount of supersymmetry $\mathcal{N}$ realised in the theory. Basic supergravity theories are theories for which the action only consists of kinetic terms plus interaction terms that are completely determined by supersymmetry. These undeformed supergravity theories are determined completely by supersymmetry if they have more than 16 supersymmetries. Undeformed theories with less than 16 supersymmetries leave some wriggle room and depend on the number of matter multiplets that are coupled to the theory. A complete classification of these undeformed theories has been obtained, and we refer to [4] for a review.

It is more interesting to look at the deformed theories. A possible deformation which is of interest to this thesis is the addition of higher derivative terms. In standard supergravity theories, the maximum number of spacetime derivatives in a term appearing in the action is two. Allowing terms with more than two derivatives allows for the inclusion of next to leading order terms in the low energy effective action of string theory. They are also important as counterterms for UV divergences that appear when studying the quantum behaviour of supergravity theories. The incorporation of higher derivative terms in a supersymmetric framework is highly non-trivial, and a complete classification of such higher derivative supergravities is still a long way off. Part of the research presented in this thesis investigates the construction of such higher derivative terms.

Branes and dualities

Before the mid 1990s most of the research in string theory involved perturbative calculations, using the string coupling g_S as a small parameter. However, since g_S is dynamical and corresponds to the expectation value of the dilaton, there is no reason to assume that it would be small, and so we cannot neglect non-perturbative effects in the full string theory. Furthermore, there was not just one string theory, but five different versions, a rather uncomfortable situation for a 'fundamental theory'. There are type IIA and type IIB string theory containing 32 supercharges, whereas type I string theory, the heterotic $SO(32)$ and heterotic $E_8 \times E_8$ string theory only contain 16 supercharges.

A resolution of these issues came with the realisation that string theory is not just a theory of strings. A more careful description of open strings shows the existence of higher dimensional generalisations of strings. Like the string generalises the point-particle by adding an extra dimension, we can add extra dimensions to a string and consider the resulting branes. Collectively these objects are referred to as p-branes, and a specific type of p-branes are the D-branes[6]. At zero string coupling, $g_s = 0$, these D-branes are rigid hypersurfaces in spacetime on which the endpoints of open strings are restricted to move on. When g_s becomes non-zero these objects become dynamical through the strings that end on them. They behave as very heavy objects with a mass of order $1/g_s$. At large g_s where the perturbative treatment of strings breaks down, these objects become light and should provide the right perturbative degrees of freedom. The full string theory should then incorporate both strings and branes, and the perturbative use of strings to describe the full theory is only valid in a specific regime of the theory.

The above example represents a *duality* in the theory. When two theories that seemingly describe different physics turn out to describe the same physics but in different regimes we call them dual. The concept of duality is not specific to string theory and indeed, there exists the more familiar electromagnetic duality in Maxwell theory. The electromagnetic duality leaves the free Maxwell equations invariant under the interchange of electric and magnetic fields

$$\vec{E} \to \vec{B}, \quad \vec{B} \to -\vec{E}. \tag{1.1}$$

In the presence of electric sources q_e this invariance is lost, unless one accepts the existence of magnetic monopoles q_m. In a consistent quantum mechanical picture they have to satisfy the Dirac quantisation condition

$$q_e q_m = 2\pi\hbar n, \qquad n \in \mathbb{N}, \tag{1.2}$$

and one needs to include dionic solutions, carrying both electric and magnetic charge. Magnetic monopoles appear naturally in supersymmetric Yang-Mills theories, they are solitonic solutions of the classical field equations with localised energy. Their mass goes like the inverse of the coupling constant (electric charge in this case). In other words, they are very heavy in the perturbative regime but become relevant when one deals with non-perturbative effects.

As the non-perturbative effects and dualities in string theory were better understood it was realised that there was an intricate web of dualities relating

[6]The D stands for Dirichlet and stems from the fact that strings ending on these branes have Dirichlet boundary conditions in directions transverse to the brane, effectively restricting the movement of the endpoints to the world-volume of the brane.

the five string theories. The simplest example of these dualities is T-duality. T-duality relates strings on a background of flat space where one direction is compactified with radius R with strings on a background that is now compactified on a circle with radius l_S^2/R. T-duality relates type IIA and type IIB as well as heterotic $SO(32)$ and heterotic $E_8 \times E_8$. The existence of the various dualities has led to the idea that the five different string theories are just different descriptions of one fundamental theory, called M-theory. What M-theory precisely is remains an open question.

1.3 Overview of this thesis

Personal work

In this thesis we will consider two slightly different topics. Both of them are related to D-branes and both are treated in a supergravity context.

Higher derivative terms in supergravity and their construction are an interesting topic. Recently in [5], the D3-brane world-volume theory embedded in a ten-dimensional Minkowski background was studied, and it was used to construct supersymmetric higher derivative invariants by deforming the action and supersymmetry transformation rules of the $D = 4$, $\mathcal{N} = 4$ Maxwell multiplet. The resulting theory has 16 deformed Maxwell multiplet supersymmetries and 16 Volkov-Akulov type non-linear supersymmetries. To extend this rigid supersymmetric result to supergravity one would like to use superconformal methods [6, 7, 8, 9, 10, 4]. In order to use these methods, we need to determine the superconformal transformation rules of the deformed Maxwell multiplet. An interesting question that arises immediately is how the Volkov-Akulov supersymmetry of the deformed 16+16 Maxwell multiplet are related to the S-supersymmetry of the conformal Maxwell multiplet. The superconformal transformation rules can be obtained by embedding the world-volume in an $AdS_5 \times S^5$ background. We investigate and establish this relation, and in addition propose a method for constructing higher-derivative invariants by using this relation between S-supersymmetry and VA-symmetry. This work was presented in [11] and was performed in collaboration with Frederik Coomans. It forms the topic of chapter 4.

The AdS/CFT correspondence and other gauge/gravity dualities provide an excellent framework to study strongly coupled quantum field theories in terms of their (weakly coupled) gravitational duals. Of particular interest is the study of defects, interfaces or boundaries in field theories that can lead to many interesting phenomena. For example, by adding a boundary in a system describing a superconductor one can study boundary superconductivity through the addition of a defect in the dual supergravity theory. Similarly, a Wilson line operator, which measures the interaction energy or self-energy of charged particles, can also be considered as a defect. Recent work, in the context of 5-dimensional gauge theories and their gravity duals, has compared the partition function of the gauge theory with the result from the gravity dual [12], as well as the vacuum expectation value of the half-BPS Wilson line for totally symmetric and anti-symmetric representations [13]. In either case the vacuum expectation value on the gravity side can be well approximated by probe branes. To go further, one must include the backreaction of the probe branes. In terms of branes, the background 5-dimensional gauge theory arises as the low energy limit of a configuration of branes consisting of D4-branes and D8-branes along with an O8 orientifold projection. Introducing a Wilson line in the fundamental representation corresponds to introducing a fundamental string perpendicular to the D4/D8-brane system. Rank M symmetric representations arise from introducing an additional D4-brane and stretching M fundamental strings between the D4-brane and the D4/D8-brane stack. Rank M anti-symmetric representations arise by introducing a perpendicular D4-brane and M fundamental strings. In general, the BPS-Wilson line reduces the superconformal symmetry of the 5-dimensional gauge theory from $F(4;2) \times SU(2)$ to $D(2,1;2;1) \times SO(4)$. With the task of finding backreacted geometries describing these Wilson lines in mind, we study general solutions of massive IIA supergravity with $D(2,1;\gamma;1) \times SO(4)$ symmetry. We give a partial reduction and integration of the BPS equations, including obtaining algebraic expressions for the metric factors in terms of spinor bilinears as well as solutions in special cases of symmetry enhancement. This work forms the topic of chapter 5 and was performed in [14] in collaboration with John Estes and Darya Krym.

Note: Aside from the work presented in this thesis, the author has also contributed to research on an effective description of charged black branes. In collaboration with Marco M. Caldarelli and Roberto Emparan, the blackfold approach [15, 16, 17] was extended to study new classes of higher-dimensional rotating black holes with electric and magnetic charges in theories of gravity coupled to a 2-form or 3-form field strength and to a dilaton with arbitrary coupling. This work was performed in [18].

Overview of the following chapters

In chapter 2, we will describe p-brane solutions in the context of supergravity. We take a brief look at supersymmetry and supergravity before introducing the theories in which we will consider these p-branes. These theories are supergravity theories in ten or eleven spacetime dimensions, and they are (related to) low energy effective theories of string theory. We will briefly touch upon their relation with string theory, in a way that will allow us to formulate the AdS/CFT correspondence in its original form in the following chapter. We discuss the p-brane solutions and their world-volume actions as well as the relation between their embedding and the conservation of some of the background symmetry. We consider the construction of intersecting branes and discuss the construction of the $D4/D8$-brane system with fundamental strings that is of interest to this thesis.

The AdS/CFT correspondence is introduced in chapter 3. We use the discussion of the previous chapter to introduce the form it was originally formulated in, a duality between type IIB string theory on $AdS_5 \times S^5$ and $D = 4$, $\mathcal{N} = 4$ super Yang Mills theory with gauge group $SU(N)$. The goal of this chapter is to introduce some of the concepts associated with the AdS/CFT correspondence by considering a few small examples. We formulate the statement of the more general gauge/gravity correspondence, including the identification of fields and operators, and as an example we use this to calculate the two-point function of the field theory using the gravity dual. We discuss Wilson loops as they form an intricate part of the motivation for some of the work presented in this thesis. We end the chapter with a discussion of five-dimensional gauge theories and their gravity duals, the checks that have been performed, and recent work that further motivates our work in chapter 5.

In chapter 4, we consider the embedding of a D3-brane in two different backgrounds, ten-dimensional Minkowski spacetime and its own near-horizon geometry $AdS_5 \times S^5$ spacetime. Embedded in the Minkowski background the world-volume theory inherits a different symmetry group than the embedding in $AdS_5 \times S^5$, this is made explicit by a coset construction of the transformation rules. Furthermore, a relation between these different sets of transformation rules is established by considering a large R-limit of the $AdS_5 \times S^5$ background. A connection is established with the VA symmetry of [5] and finally a proposal for the construction of higher derivative invariants using this connection is formulated. To make the steps in this chapter more clear, we have moved some of the constructions and calculations for this chapter to the appendices.

Chapter 5 discusses the construction of solutions to massive type IIA supergravity with symmetry group $D(2, 1; \gamma; 1) \times SO(4)$. An invariant metric ansatz is formulated and with this ansatz the BPS-equations are reduced to a two-dimensional system, including algebraic expressions for the warpfactors in terms of the spinor bilinears. In order to better understand the structure of the BPS system, we study the special cases of enhanced supersymmetry, which corresponds to setting certain fluxes to zero and setting γ to specific values. In general there are two distinct cases of enhanced supersymmetry, one given by setting $\gamma = -1/2, -2$ and the second given by setting $\gamma = 1$. In the first case, we show the most general solution is given by the AdS_6 geometries of [19], which is simply the dual of the 5-dimensional gauge theory without the half-BPS Wilson line. The second case corresponds to fundamental strings ending on D8-branes.[7] We identify three types of solutions. The first, given in section 5.3.1, we interpret as a stack of fundamental strings in the presence of D8-branes, i.e. in a background with $F_{(0)} \neq 0$. The other two solutions, given in section 5.3.2, we interpret as fundamental strings ending on a stack of D8-branes or an O8-plane. In all three cases the geometry contains an asymptotically flat region. Naively, the geometry does not admit a decoupling limit. However, we note that the string coupling goes to zero in the asymptotically flat regions, which may be sufficient for a valid decoupling limit. We also consider solutions where $F_{(0)}$ is allowed to jump across an interface, corresponding to the presence of a stack of D8-branes. This allows for a large family of solutions, parametrised by the number of such jumps. However, we find that there is no way to glue D8-brane caps or O8-plane caps together. Consequently, we argue that there are no solutions dual to $1+0$-dimensional CFTs.

The thesis is also supplemented by several appendices. Appendix A contains a small summary of Clifford algebras and presents the various algebras used throughout the thesis. Appendix B presents the $SU(2,2|4)$ superalgebra and its decomposition in terms of AdS-variables on the one hand and conformal variables on the other hand. It also presents the relation between these two decompositions. In appendix C we consider $AdS_5 \times S^5$ as a coset space and derive the isometries using a coset construction. Appendix D provides a sample calculation of the coset construction for transformation rules in chapter 4. Appendix E collects elements of the reduction of BPS-equations omitted in chapter 5. We also partially solve these reduced equations and show that this solution is indeed a solution to the equations of motion.

[7] We note this system was studied in [20] without the assumption of conformal symmetry.

Chapter 2

Supergravity and p-branes

First we give a brief introduction to what supersymmetry and supergravity are, and we outline a general strategy to find supersymmetric solutions in section 2.1. In section 2.2, we introduce eleven-dimensional supergravity and some of its ten-dimensional incarnations, type IIA, massive type IIA and type IIB. We discuss p-branes as solutions to supergravity as well as their symmetries in section 2.3. We also take a brief look at the near-horizon geometries of D3-, M2- and M5-branes. Next, in section 2.4, we define the D-brane world-volume actions that couple to a supergravity background and use a probe brane to revisit the symmetries of D-branes. We make contact with a stringy interpretation of the branes by discussing the world-volume field theory. Finally, section 2.5 concludes this chapter with a discussion of intersecting branes. We investigate the preserved supersymmetry of intersecting brane configurations as well as the construction of a class of intersecting brane solutions using the harmonic function rule. We finish by discussing a particular configuration of intersecting D4/D8-branes, the configuration of interest in chapter 5.

2.1 What is supergravity?

To answer the question what is supergravity, we have to start by introducing supersymmetry. *Supersymmetry* is defined as a Fermion-Boson symmetry, i.e. transformations that mix bosonic and fermionic degrees of freedom and leave the physics invariant. An excellent introduction to supersymmetry and supergravity can be found in the book [4], as well as the review [21].

A theorem by Coleman and Mandula states [2, 22] that if both Poincaré (Lorentz symmetry plus translations) and internal symmetry are present, they must have a trivial mixing, i.e. the full symmetry group should be a direct product of both (and the algebra would be their direct sum). However, superalgebras circumvent some of the hypothesis in the Coleman-Mandula theorem, instead being governed by the Haag-Łopuszańsky-Sohnius theorem [23, 22]. The algebra of symmetries now admits spinor charges[1] Q^I_α, and the number ($\mathcal{N}$) of spinor charges determines which kind of (extended) supersymmetry one deals with. Supersymmetric theories realise the most symmetry possible within the framework of these theorems, as well as uniting bosons and fermions, the two classes of particles found in nature.

The supersymmetry algebra can schematically be written as

$$[P,P]=0, \quad [P,M]=P, \quad [M,M]=M,$$

$$[P,Q^I]=0, \quad [M,Q^I]=Q^I,$$

$$\{Q^I,\bar{Q}^J\}=P\delta^{IJ}, \quad \{Q^I,Q^J\}=Z^{IJ}, \quad \{\bar{Q}^I,\bar{Q}^J\}=Z^{IJ}, \tag{2.1}$$

where the P stand for translations, M for Lorentz generators, Q^I and $\bar{Q}^I$ for the supersymmetry generators, and Z^{IJ} are the central charges.[2] The indices $I,J=1,\ldots,\mathcal{N}$ label different sets of supersymmetry generators.

The supersymmetry generators transform as (spin 1/2) spinors under Lorentz transformations and, as a consequence, we have that under a supersymmetry transformation bosons transform into fermions and vice versa. Additionally, an irreducible representation of a supersymmetry algebra will correspond to several particles (including both bosons and fermions), forming what is called a supermultiplet. It can be shown that a supermultiplet always contains the same number of bosonic and fermionic degrees of freedom. Furthermore, as supersymmetry transformations commute with momentum generators, we have in particular that $[P^2,Q]=0$, and consequently, that all particles in the same supermultiplet must have the same mass. At this point one might think that supersymmetry is far from reality because if there were supersymmetric particles with the same mass as the usual ones, we would have certainly observed them by now. The only way out is to say that if supersymmetry exists, it has to be broken at a scale of energy at least as high as the energies probed in current accelerators.

[1] We will often neglect writing the spinor index α throughout this thesis.

[2] Central charges are operators which commute with all other operators and are therefore simply numbers.

We see from the superalgebra (2.1) that local supersymmetry transformations lead to local translations and Lorentz transformations. Hence a gauge theory of supersymmetry will include gravity (the gauge theory for translations). For this reason a gauge theory of supersymmetry is called *supergravity*, and it will include the metric g_{MN} and a set of $\mathcal{N}$ Rarita-Schwinger fields, the gravitini ψ_M, that act as the gauge fields for supersymmetry.

The precise field content of supergravity theories depends strongly on the number of spacetime dimensions. Conventional supergravities cannot exist in more than 11 dimensions. The reason for this is that the smallest Lorentz spinor in 11 dimensions has 32 components, while in higher dimensions it will necessarily have more components. Through dimensional reduction arguments this minimal spinor in eleven dimensions corresponds to eight 4-component spinors in four dimensions. A supermultiplet in four dimensions with this amount of supersymmetry ($\mathcal{N} = 8$) contains the metric, gravitini, gauge fields, spinors and scalars. We can have a maximum of 8 gravitini if we want a finite number of interacting fields, requiring more supersymmetry will result in a supermultiplet containing higher spin fields (spin > 2). These fields cannot consistently couple to themselves. If we consider spinors with more than 32 components in higher dimensions, we obtain a contradiction if we view things from a four-dimensional point of view (for instance by considering only a dependency on four coordinates). The 32-component spinor in higher dimensions would imply particles with spin > 2 in four dimensions. The bosonic fields of supergravity comprise the graviton field through the metric tensor, a number of antisymmetric gauge fields ($(p + 1)$-forms), and a set of scalar fields. The fermionic content of the supergravity theory consist of one or more ($\mathcal{N}$) gravitino fields and a number of ordinary Lorentz spinors.

One of the advantages of including supersymmetry in gravity theories is that they simplify the process of finding (supersymmetric) solutions. Finding solutions in general relativity requires solving a set of non-linear, second order differential equations, which in general is a very complicated system of equations. Supersymmetry simplifies this greatly. Instead of having to solve this set of second order equations, they present us with a system of first order equations which automatically solve the second order problem. Supersymmetric solutions will always satisfy such a first order system (of course there exist also non-supersymmetric solutions that cannot be found following this strategy). We will illustrate how this process works.

We will look for classical configurations. This means that the expectation value of the fermionic fields should be zero (otherwise, a Lorentz symmetry would not conserve the vacuum). As explained earlier, supercharges are spin 1/2 fields and they turn bosons into fermions and vice versa. Schematically we have

$$\delta_{\text{SUSY}} F = f(B), \qquad \delta_{\text{SUSY}} B = g(F), \tag{2.2}$$

where $f(B)$ and $g(F)$ are some functions of the bosonic and fermionic fields respectively, and δ_{SUSY} represents our supersymmetry transformation. As the fermions are zero (which implies $g(F) = 0$), the invariance of the bosonic fields describing the solutions is guaranteed ($\delta_{\text{SUSY}} B = 0$). In order to preserve supersymmetry, the fermionic fields should also not vary ($\delta_{\text{SUSY}} F = 0$), hence

$$f(B) = 0, \tag{2.3}$$

which gives a system of equations, first order in derivatives.

The strategy to solve this set of first order equations is to start with an ansatz for the bosonic fields. Then, (2.3) leads to a set of equations from which the functions in the ansatz can be computed. Usually for these equations to be solvable, one must impose some projections on the spinor that parametrises the transformation. When this happens, not all the supercharges present in the supergravity theory are preserved by the solution. These projections are of the type $\epsilon = \mathcal{P}\epsilon$ ($\mathcal{P}$ being some function of the gamma matrices depending on the solution). Generally, each independent projection reduces the number of preserved supercharges by half.

In the following sections we will describe several supergravity theories and a particular solution to these theories, brane solutions.

2.2 Supergravity actions and supersymmetry transformations

In this section we gather expressions for several supergravity theories that will be used throughout the thesis. We will mostly be interested in bosonic supersymmetric configurations, so the only sector of the action we will need is the bosonic sector. We also give the supersymmetry transformations of the fermionic fields, which, as mentioned in the previous section, can be used to find supersymmetric solutions. A review on eleven- and ten-dimensional supergravities, the relation among them, solutions from branes and many other topics on gravity and its relation with strings can be found in [24]. We refer to chapter 12 of [4] for a review of the basic supergravity theories and their deformations.

$D = 11$, $\mathcal{N} = 1$ supergravity

The unique eleven-dimensional supergravity theory was first constructed in [25]. It is the low-energy effective description of strongly coupled string theory in 11 dimensions. The number of supercharges is 32, corresponding to one Majorana spinor.

The bosonic content of the theory includes the metric g_{MN}, and a 3-form[3] potential $C_{(3)}$, with a 4-form field strength $F_{(4)} = dC_{(3)}$. The bosonic part of the action of these fields is

$$\begin{aligned} S_{11\mathrm{D}} &= \frac{1}{2\kappa_{(11)}^2}\int d^{11}x\sqrt{-g}\left[R - \frac{1}{2\cdot 4!}F_{MNPQ}F^{MNPQ}\right] \\ &\quad + \frac{1}{2\kappa_{(11)}^2}\int \frac{1}{6}F_{(4)}\wedge F_{(4)}\wedge C_{(3)}, \end{aligned} \tag{2.5}$$

where $\kappa_{(11)}$ is related to the 11-dimensional Newton's constant by $2\kappa_{(11)}^2 = 16\pi G_{11}$.

The only fermionic degrees of freedom are those corresponding to a Rarita-Schwinger field, the gravitino ψ_M. Its supersymmetry variation is given by

$$\delta\psi_M = D_M\epsilon + \frac{1}{288}F^{(4)}_{M_1\ldots M_4}\left(\Gamma_M^{\ M_1\ldots M_4} - 8\delta_M^{M_1}\Gamma^{M_2\ldots M_4}\right)\epsilon. \tag{2.6}$$

$D = 10$, type IIA supergravity

Eleven-dimensional supergravity can be dimensionally reduced yielding a maximal non-chiral supergravity in ten dimensions. The resulting theory is called type IIA supergravity [26, 27, 28] and it is the low energy limit of type IIA string theory.

The Kaluza-Klein reduction ansatz for the metric is

$$ds_{11}^2 = e^{-\frac{2}{3}\phi}ds_{10}^2 + e^{\frac{4}{3}\phi}\left(R_{11}d\psi + C_{(1)}\right)^2, \tag{2.7}$$

[3]Througout the text we will denote forms by both their components as well as their full form. These are related by

$$\omega_{(p)} = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}. \tag{2.4}$$

where ψ has period 2π. One also has to reduce the eleven-dimensional three-form (or equivalently the four form field strength),

$$F^{(11\mathrm{D})}_{(4)} = F_{(4)} + R_{11} H_{(3)} \wedge d\psi \qquad (2.8)$$

which generates a three-form and a two-form in ten dimensions, depending on whether or not the reduction direction is part of the original form. Therefore the bosonic content of this theory consists of a metric g_{MN}, a dilaton ϕ and a Ramond-Ramond (RR) one-form $C_{(1)}$ coming from the reduction of the metric, and a Neveu-Schwarz (NS) two-form $B_{(2)}$ and an RR three-form $C_{(3)}$ coming from the reduction of the three-form. The supergravity action in string-frame takes the from[4]

$$\begin{aligned} S_{\mathrm{IIA}} &= \frac{1}{2\kappa^2_{(10)}} \int d^{10}x \sqrt{-g} \left[e^{-2\phi} \left(R + 4|\nabla\phi|^2 - \frac{1}{2\cdot 3!} H^2_{(3)} \right) \right. \\ & \left. -\frac{1}{2} \sum_{n=2,4} \frac{1}{n!} F^2_{(n)} \right] + \frac{1}{2\kappa^2_{(10)}} \int \frac{1}{2} dC_{(3)} \wedge dC_{(3)} \wedge B_{(2)}, \qquad (2.9) \end{aligned}$$

where

$$H_{(3)} = dB_{(2)}, \qquad F_{(2)} = dC_{(1)}, \qquad F_{(4)} = dC_{(3)} + C_{(1)} \wedge H_{(3)}. \qquad (2.10)$$

We can write this action in the more familiar Einstein-frame[5] where the gravitational part takes the standard Einstein-Hilbert form by rescaling the string-frame metric g_{MN}. In terms of the Einstein-frame metric

$$g^{(E)}_{MN} \equiv e^{-\frac{1}{2}\phi} g_{MN}, \qquad (2.11)$$

the string action takes the form

$$\begin{aligned} S_{\mathrm{IIA}} &= \frac{1}{2\kappa^2_{(10)}} \int d^{10}x \sqrt{-g^{(E)}} \left[R - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2\cdot 3!} e^{-\phi} H^2_{(3)} \right. \qquad (2.12) \\ & \left. -\frac{1}{2} \sum_{n=2,4} \frac{1}{n!} e^{\frac{5-n}{2}\phi} F^2_{(n)} \right] + \frac{1}{2\kappa^2_{(10)}} \int \frac{1}{2} dC_{(3)} \wedge dC_{(3)} \wedge B_{(2)}, \end{aligned}$$

where all quantities (e.g. R and $F^2_{(n)}$) are now calculated using the Einstein-frame metric.

[4] $\kappa^2_{(11)} = \kappa^2_{(10)} R_{11}$

[5] In the Einstein-frame the Lagrangian contains the gravitational term $\sqrt{-g}R$, whereas in the string-frame the gravitational term in the Lagrangian is $\sqrt{-g}e^{-2\phi}R$.

The fermionic content of the theory comprises two Majorana-spinors: a gravitino ψ_μ and a dilatino λ, each decomposable into two Majorana-Weyl components. Their supersymmetry variations are (Einstein-frame)

$$\begin{aligned}
\delta\lambda &= \frac{1}{2\sqrt{2}} D_M\phi\Gamma^M\Gamma^{11}\epsilon + \frac{3}{16\sqrt{2}} e^{\frac{3\phi}{4}} F^{(2)}_{M_1M_2}\Gamma^{M_1M_2}\epsilon \\
&\quad + \frac{i}{24\sqrt{2}} e^{-\frac{\phi}{2}} H^{(3)}_{M_1\dots M_3}\Gamma^{M_1\dots M_3}\epsilon - \frac{i}{192\sqrt{2}} e^{\frac{\phi}{4}} F^{(4)} M_1 \dots M_4 \Gamma^{M_1\dots M_4}\epsilon, \\
\delta\psi_M &= D_M\epsilon + \frac{1}{64} e^{\frac{3\phi}{4}} F^{(2)}_{M_1M_2}\left(\Gamma_M{}^{M_1M_2} - 14\delta_M^{M_1}\Gamma^{M_2}\right)\Gamma^{11}\epsilon \\
&\quad + \frac{1}{96} e^{-\frac{\phi}{2}} H^{(3)}_{M_1\dots M_3}\left(\Gamma_M{}^{M_1\dots M_3} - 9\delta_M^{M_1}\Gamma^{M_2M_3}\right)\Gamma^{11}\epsilon \\
&\quad + \frac{i}{256} e^{\frac{\phi}{4}} F^{(4)}_{M_1\dots M_4}\left(\Gamma_M{}^{M_1\dots M_4} - \frac{20}{3}\delta_M^{M_1}\Gamma^{M_2\dots M_4}\right)\Gamma^{11}\epsilon, \qquad (2.13)
\end{aligned}$$

where the chirality operator Γ^{11} is defined as $\Gamma^{11} = i\Gamma^0\Gamma^1\dots\Gamma^9$.

$D = 10$, massive type IIA supergravity

Massive type IIA supergravity was first discovered by Romans in [29]. The bosonic field content of massive type IIA supergravity consists of the metric g_{MN}, a dilaton ϕ, a NSNS two-form $B_{(2)}$ and an RR three-form $C_{(3)}$.

The action of massive type IIA supergravity is[6]

$$\begin{aligned}
S_{\mathrm{mIIA}} \quad = \quad & \frac{1}{2\kappa^2_{(10)}} \int d^{10}x\sqrt{-g}\left[R - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2\cdot 3!}e^{-\phi}H^2_{(3)}\right. \\
& \left. - \frac{1}{2\cdot 4!}e^{\phi/2}F^2_{(4)} - \frac{F^2_{(0)}}{4}e^{\frac{3\phi}{2}}B^2_{(2)} - \frac{F^2_{(0)}}{2}e^{\frac{5\phi}{2}}\right]
\end{aligned} \qquad (2.14)$$

[6]We follow the conventions of [29], with the following replacements $\phi \to -\phi/2$, $\xi \to e^{-\phi/4}$, $G_{MNP} \to H_{MNP}/2$, $F_{MNPQ} \to F_{MNPQ}/2$ and $m \to F_{(0)}$.

$$+\frac{1}{4\kappa_{(10)}^2}\int\Big(dC_{(3)}\wedge dC_{(3)}\wedge B_{(2)}$$

$$+\frac{1}{3}F_{(0)}dC_{(3)}\wedge B_{(2)}\wedge B_{(2)}\wedge B_{(2)}$$

$$+\frac{1}{20}F_{(0)}^2 B_{(2)}\wedge B_{(2)}\wedge B_{(2)}\wedge B_{(2)}\wedge B_{(2)}\Big)\,. \tag{2.15}$$

$F_{(0)}$ is a constant called the Romans mass and the field strengths are related to the gauge potentials by

$$H_{(3)}=dB_{(2)},\qquad F_{(4)}=dC_{(3)}+\frac{F_{(0)}}{2}B_{(2)}\wedge B_{(2)}, \tag{2.16}$$

with Bianchi-identities

$$dH_{(3)}=0,\qquad dF_{(4)}=F_{(0)}B_{(2)}\wedge H_{(3)}. \tag{2.17}$$

The fermionic field content of the theory is the same as in the previous section, two Majorana-spinors: a gravitino ψ_μ and a dilatino λ, each decomposable into two Majorana-Weyl components. Their supersymmetry variations in Einstein-frame are given by

$$\begin{aligned}\delta\lambda =&\Big[(D_M\phi)\Gamma^M+\frac{5}{4}F_{(0)}e^{\frac{5}{4}\phi}+\frac{1}{96}e^{\frac{\phi}{4}}(F^{(4)}_{MNPQ}\Gamma^{MNPQ})\\ &-\frac{3}{8}F_{(0)}e^{\frac{3\phi}{4}}B^{(2)}_{MN}\Gamma^{MN}\Gamma_{11}-\frac{1}{12}e^{-\frac{\phi}{2}}H^{(3)}_{MNP}\Gamma^{MNP}\Gamma_{11}\Big]\epsilon\ ,\\ \delta\psi_M =&\Big[D_M-\frac{1}{32}F_{(0)}e^{\frac{5}{4}\phi}\Gamma_M+\frac{1}{128}\frac{e^{\frac{\phi}{4}}}{2}F^{(4)}_{NPQR}(\Gamma_M{}^{NPQR}-\frac{20}{3}\delta_M{}^N\Gamma^{PQR})\\ &-\frac{1}{32}F_{(0)}\frac{e^{\frac{3\phi}{4}}}{2}B^{(2)}_{NP}(\Gamma_M{}^{NP}-14\delta_M{}^N\Gamma^P)\Gamma_{11}\\ &+\frac{1}{48}\frac{e^{-\frac{\phi}{2}}}{2}H^{(3)}_{NPQ}(\Gamma_M{}^{NPQ}-9\delta_M{}^N\Gamma^{PQ})\Gamma_{11}\Big]\epsilon\ .\end{aligned} \tag{2.18}$$

Since the $B_{(2)}$ field has a mass in massive IIA supergravity, the theory is no longer invariant under gauge transformations of $B_{(2)}$. In [29], this fact has been used to absorb the Ramond-Ramond 2-form, $F_{(2)}$ into the definition of $B_{(2)}$. In order to connect back to the massless IIA supergravity, we make the field redefinitions

$$B_{(2)}\to B_{(2)}-F_{(0)}^{-1}F_{(2)},\qquad C_{(3)}\to C_{(3)}-(2F_{(0)})^{-1}A_{(1)}\wedge F_{(2)}, \tag{2.19}$$

with $dF_{(2)} = 0$ and $F_{(2)} = dA_{(1)}$. Note that $F_{(4)}$ is invariant under these combined transformations. The theory then enjoys the symmetry

$$B_{(2)} \to B_{(2)} + d\Lambda \,, \qquad F_{(2)} \to F_{(2)} + F_{(0)}\, d\Lambda \,,$$

$$C_{(3)} \to 2\Lambda \wedge F_{(2)} + 2C_{(1)} \wedge d\Lambda + 2F_{(0)}\, \Lambda \wedge d\Lambda \,, \tag{2.20}$$

where Λ is a 1-form. To obtain massless IIA supergravity, we simply take $F_{(0)} = 0$.

$D = 10$, type IIB supergravity

There is another maximal supergravity that can be constructed in ten dimensions. This type IIB supergravity theory [30, 31, 32] is chiral and cannot be obtained by dimensional reduction from eleven dimensions. However, it is related to type IIA supergravity by T-duality, specifically, type IIA theory compactified on a circle of radius R is T-dual to the IIB theory compactified on a circle of radius $1/R$. This duality is rooted in the uniqueness of the $D = 9$, $\mathcal{N} = 2$ theory, both the IIA and IIB theory are mapped to the same theory in $D = 9$. Duality between the IIA and IIB theories can be treated in a 'democratic' formulation, where all the fields are introduced together with their magnetic duals [33, 34], but we will not consider this here.

The bosonic degrees of freedom are the metric g_{MN}, the dilaton ϕ, a NSNS two-form $B_{(2)}$, an RR scalar $C_{(0)}$, two-form $C_{(2))}$ and four-form $C_{(4)}$. The action for these fields reads[7]

$$\begin{aligned} S_{\text{IIB}} \;=\; & \frac{1}{2\kappa^2_{(10)}} \int d^{10}x\sqrt{-g}\,\Big[R - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2\cdot 3!}e^{-\phi}H^2_{(3)} \\ & -\frac{1}{2}e^{2\phi}|\nabla C_{(0)}|^2 - \frac{1}{2\cdot 3!}e^{\phi}F^2_{(3)} - \frac{1}{2\cdot 5!}e^{\phi}F^2_{(5)}\Big] \\ & + \frac{1}{2\kappa^2_{(10)}} \int \frac{1}{2} dC_{(4)} \wedge F_{(3)} \wedge H_{(3)}, \end{aligned} \tag{2.22}$$

[7] The term action is a misnomer for type IIB supergravity. In fact the self-duality of the five-form causes a bit of a problem, the term

$$F^2_{(5)} \sim F_{(5)} \wedge \star F_{(5)} \tag{2.21}$$

vanishes identically since $F_{(5)} = \star F_{(5)}$, and $F_{(5)} \wedge F_{(5)} = 0$. One should really consider the action (2.22) as a tool for deriving the equations of motion as follows. Use (2.22) to vary with respect to the fields and obtain the field equations, after which we impose the self-duality constraint.

where

$$H_{(3)} = dB_{(2)}, \qquad F_{(3)} = dC_{(2)} - C_{(0)}H_{(3)}, \tag{2.23}$$

and,

$$F_{(5)} = dC_{(4)} + C_{(2)} \wedge H_{(3)}. \tag{2.24}$$

One also has to impose the self-duality condition[8] $F_{(5)} = \star F_{(5)}$.

The fermionic content again comprises two spinors, a dilatino λ and a gravitino ψ_M. As mentioned before, type IIB supergravity is a chiral theory. The spinor ϵ is composed by two Majorana-Weyl spinors ϵ_{L1} and ϵ_{L2} of well-defined ten-dimensional chirality. We can arrange these two chiralities as a two-component vector in the form

$$\epsilon = \begin{pmatrix} \epsilon_{L1} \\ \epsilon_{L2} \end{pmatrix}, \tag{2.26}$$

but we can also use complex spinors instead of the real spinor of (2.26). The complex spinor is simply

$$\epsilon = \epsilon_{L1} + i\epsilon_{L2}. \tag{2.27}$$

To pass from one notation to the other we use

$$\epsilon^* \leftrightarrow \sigma^3\epsilon, \qquad i\epsilon^* \leftrightarrow \sigma^1\epsilon, \qquad i\epsilon \leftrightarrow -i\sigma^2\epsilon, \tag{2.28}$$

where σ^i are the Pauli matrices. In the complex notation, the transformation rules for the fermions are

$$\begin{aligned} \delta\lambda &= iP_M\Gamma^M\epsilon^* - \frac{i}{24}F_{M_1\ldots M_3}\Gamma^{M_1\ldots M_3}\epsilon \\ \delta\psi_M &= D_M\epsilon - \frac{i}{1920}F^{(5)}_{M_1\ldots M_5}\Gamma^{M_1\ldots M_5}\Gamma_M\epsilon \\ &\quad + \frac{1}{96}F_{M_1\ldots M_3}\left(\Gamma_M^{\ M_1\ldots M_3} - 9\delta_M^{M_1}\Gamma^{M_2M_3}\right)\epsilon^*, \end{aligned} \tag{2.29}$$

where P_M and $F_{M_1\ldots M_3}$ are given by

$$\begin{aligned} P_M &= \frac{1}{2}\partial_M\phi + \frac{i}{2}e^{\phi}\partial_M C_{(0)}, \\ F_{M_1\ldots M_3} &= e^{-\frac{\phi}{2}}H^{(3)}_{M_1\ldots M_3} + ie^{\frac{\phi}{2}}F^{(3)}_{M_1\ldots M_3}. \end{aligned} \tag{2.30}$$

[8] The Hodge-dual of a p-form is a $(D-p)$-form, defined as

$$\star\,\omega_{(p)} = \frac{\sqrt{-g}}{p!(D-p)!}\epsilon_{\mu_{p+1}\ldots\mu_D\mu_1\ldots\mu_p}g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\omega_{\nu_1\ldots\nu_p}dx^{\mu_{p+1}}\wedge\ldots dx^{\mu_D}, \tag{2.25}$$

where D is the dimension of spacetime and $\epsilon_{\mu_{p+1}\ldots\mu_D\mu_1\ldots\mu_p}$ is the totally anti-symmetric Levi-Civita tensor in D-dimensions.

2.3 Supersymmetric solutions: p-branes

We start out by considering the description of parallel branes as supersymmetric solutions of supergravity. We describe a few of the simplest cases here in detail. Next, we consider the conserved supersymmetry by the brane solutions and we end the section by looking at the near-horizon geometry of D3-branes, as well as M2 and M5-branes. For a detailed derivation of the p-brane solutions and other details from the supergravity perspective we refer to the review [35]. For more details about D-branes in string theory we recommend consulting the lecture notes [36, 37, 38].

2.3.1 Brane solutions in supergravity

We have seen that the various supergravity theories include $(p+1)$-form gauge potentials. These gauge potentials couple naturally to $(p+1)$-dimensional objects and thus produce electric branes that are charged with respect to the $(p+2)$-form field strength $F_{(p+2)}$. As time will be a direction tangent to this $(p+1)$-dimensional object, we can consider this object to be the world-volume of a p-dimensional brane, and hence we will call them *p-branes*.

If the brane carries no other charges then it is a solution of an Einstein-Maxwell-type gravity theory with action

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-g} \left(R - \frac{1}{2(p+2)!} F^2_{(p+2)} \right) \tag{2.31}$$

where κ_D is related to the D-dimensional Newton's constant G_D and Planck length l_P by

$$2\kappa_D^2 = 16\pi G_D = (2\pi)^{D-3} l_P^{D-2}. \tag{2.32}$$

For $D = 11$ and $p = 2$ this is the bosonic part of the eleven-dimensional supergravity action (2.5) (apart from the Chern-Simons term in the last line).

In order to obtain a (supersymmetric) solution, one usually makes some simplifying assumptions about it. Since we would like to retain some supersymmetries we will also need (some) translational symmetry (as imposed by the algebra) along the brane. Finally we will require isotropy in the directions transverse to the brane. Together this leads to a bosonic symmetry group of the form Poincaré$_{p+1} \times SO(D-p-1)$. We will not derive the solution here[9] but rather just present the solutions obtained using this ansatz.

[9] For a detailed derivation we refer to [35].

A solution to the equations of motion of the action (2.31) is

$$
\begin{aligned}
ds^2 &= H^{-\frac{D-p-3}{D-2}} dx^2_{(1,p)} + H^{\frac{p+1}{D-2}} dx^2_{(D-p-1)}, \\
F_{(p+2)} &= c\,(dH) \wedge \omega_{(1,p)}, \\
H &= 1 + \frac{c_p N}{r^{D-p-3}},
\end{aligned}
\tag{2.33}
$$

where $dx^2_{(1,p)}$ is the $(p+1)$-dimensional Minkowski metric with volume form $\omega_{(1,p)}$, and

$$
dx^2_{(D-p-1)} = dr^2 + r^2 d\Omega^2_{(D-p-2)} \tag{2.34}
$$

is the $(D-p-1)$-dimensional Euclidian metric with radial coordinate r. This solution is interpreted as N coincident branes with a $(p+1)$-dimensional Minkowski world-volume located at $r=0$. Branes and anti-branes differ in the sign $(c = \pm 1)$ of $F_{(p+2)}$.

The equations of motion reduce to the condition that H satisfies the Laplace equation in the transverse space with a source term at $r=0$. These source terms can be understood as coming from the coupling of a p-brane action (see section 2.4, later in this chapter) to the supergravity action

$$
S = S_{\text{supergravity}} + S_{p\text{-brane}}. \tag{2.35}
$$

This leads to a relation between the constant c_p appearing in the harmonic function H and the p-brane tension T_p (for the case of M2-branes see [39])

$$
c_p = \frac{2\kappa_D^2 T_p}{(D-p-3)V(S^{D-p-2})}, \tag{2.36}
$$

where $V(S^{D-p-2})$ is the volume of a $(D-p-2)$ unit-sphere.

We can also consider magnetic $(D-p-4)$-brane solutions which are magnetically charged under $F_{(p+2)}$. At the classical level we can consider either $F_{(p+2)}$ or its Hodge dual $F_{(D-p-2)} \equiv \star F_{(p+2)}$ to be the fundamental field strength. Conventionally, $F_{(p+2)}$ with $(p+2) \leq D/2$ is considered to be the fundamental field strength. Other than this, there is no distinction between electric and magnetic solutions at the level of classical solutions of supergravity. As with electric particles and magnetic monopoles in four dimensions, there is a Dirac quantisation condition relating the charges (and tensions) of electric and magnetic branes charged under the same field strength. In terms of the brane tensions this takes the form [40]

$$
2\kappa_D^2 T_p T_{D-p-4} = 2\pi n, \quad n \in \mathbb{Z}. \tag{2.37}
$$

This is satisfied for the branes we consider with $n = 1$.

Brane solutions in eleven-dimensional supergravity

For eleven-dimensional supergravity there is only one field strength, $F_{(4)}$. This means that the only brane solutions are electric 2-branes ($p = 2$, called M2-branes) or magnetic 5-branes ($p = 5$, called M5-branes). The tensions of these branes are related by the Dirac quantisation condition (2.37). Due to the absense of a dilaton, the M2-brane tension can be fixed in terms of the eleven-dimensional Planck length [41]

$$T_{\mathrm{M2}} = \frac{1}{4\pi^2 l_P^3}. \tag{2.38}$$

Having reviewed the possible p-branes in eleven-dimensional supergravity, we turn to ten-dimensional type IIA and type IIB supergravity theories.

Brane solutions in ten-dimensional supergravity

In type IIA supergravity we have the RR field strengths $F_{(2)}$ and $F_{(4)}$. The field strength $F_{(2)}$ allows for electric 0-branes (called D0-branes) and magnetic 6-branes (D6-branes), while $F_{(4)}$ allows for magnetic 2-branes (D2-branes) and electric 4-branes (D4-branes).

Type IIB supergravity contains the RR field strengths $F_{(1)} = dC_{(0)}$, $F_{(3)}$ and the self-dual five-form $F_{(5)}$. They lead to electric -1-branes (D(-1)-branes and magnetic 7-branes (D7-branes), electric 1-branes (D1-branes) and magnetic 5-branes (D5-branes), and, 3-branes (D3-branes) which are both electric and magnetic due to the self-duality of the five-form. The D(-1)-brane is a solution localised at a point in spacetime.

From a string theory point of view Dp-branes in type IIA and type IIB are $(p+1)$-dimensional submanifolds on which open strings can end.[10] As a consequence, a $(p+1)$-dimensional gauge theory describes the low energy dynamics of Dp-branes. The tension of a Dp-brane can be calculated from a 1-loop open string amplitude [42]

$$T_{\mathrm{D}p} = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{2.39}$$

There are also non-dynamical RR-charged objects known as orientifold p-planes (as opposed to D-branes, which are dynamical objects). These are fixed planes of a $\mathbb{Z}_2$ action which consists of a reflection of the $9-p$ transverse directions

[10] In fact, Dirichlet boundary conditions for the strings in the directions transverse to the brane fix the endpoints to stick to the world-volume of the brane. These Dirichlet boundary conditions lend their name to the branes, D-branes.

together with a reversal of the orientation of the string worldsheet. The charge and tension of these orientifold p-planes is given by

$$T_{\mathrm{O}p} = \pm 2^{p-5} T_{\mathrm{D}p}. \tag{2.40}$$

In both type IIA and type IIB there is also a NS-NS three-form field strength $H_{(3)}$, and so there are also electric 1-brane and magnetic 5-brane solutions associated to this. The 1-branes correspond to fundamental strings and are referred to as F1-branes (or NS1-branes). The 5-branes are called NS5-branes. These objects are not D-branes (they are not endpoints of open strings). The tension of the fundamental string

$$T_{\mathrm{F1}} = \frac{1}{2\pi l_s^2}, \tag{2.41}$$

defines the string length $l_s = \sqrt{\alpha'}$. This string length is related to the ten-dimensional Newton's constant by

$$2\kappa_{10}^2 = (2\pi)^7 g_s^2 l_s^8, \tag{2.42}$$

where also the string coupling constant g_s appears. The string coupling constant $g_s \equiv e^{\phi_\infty}$ is related to the asymptotic value of the dilaton. The dilaton can be shifted to vanish at infinity, bringing out explicit factors of g_s.

Finally, there also exist D8-branes which are domain walls in ten dimensions. These are solutions of massive IIA supergravity and they are predicted to exist in string theory by T-duality from other D-branes, however it is unclear how they are related to an eleven-dimensional theory. T-duality changes Dirichlet with Neumann boundary conditions and vice versa, this means that if we perform T-duality in a direction transverse or tangent to a Dp-brane we obtain a D$(p+1)$- or D$(p-1)$ brane respectively. This then also implies the existence of space-filling D9-branes in the type IIB theory.

Since type IIA supergravity is related to eleven-dimensional supergravity by a Kaluza-Klein reduction on a circle of radius R_{11}, there are relations between the types of branes in both theories. Fundamental strings and D2-branes are simply M2-branes wrapped or not wrapped on the eleventh dimension, while D4-branes and NS5-branes both correspond to M5-branes in eleven dimensions. The field strength $F_{(2)}$ is the Kaluza-Klein gauge field strength and so D0 branes are Kaluza-Klein particles while D6-branes are Kaluza-Klein monopoles.

We now present the supergravity solutions to equations of motion derived from the actions (2.9) or the string-frame version of (2.22). The solution for N

coincident Dp-branes is (string-frame)

$$\begin{aligned} ds^2 &= H^{-1/2} dx^2_{(1,p)} + H^{1/2} dx^2_{(9-p)}, \\ F_{(p+2)} &= -d(H^{-1}) \wedge \omega_{(1,p)}, \\ e^{\phi} &= H^{\frac{3-p}{4}}, \\ H &= 1 + \frac{c_p N}{r^{7-p}}, \end{aligned} \tag{2.43}$$

for fundamental strings

$$\begin{aligned} ds^2 &= H^{-1} dx^2_{(1,1)} + dx^2_{(8)}, \\ H_{(3)} &= -d(H^{-1}) \wedge \omega_{(1,1)}, \\ e^{\phi} &= H^{-\frac{1}{2}}, \\ H &= 1 + \frac{2^5 \pi^2 g_s^2 l_s^6 N}{r^6}, \end{aligned} \tag{2.44}$$

and for NS5-branes

$$\begin{aligned} ds^2 &= dx^2_{(1,5)} + H dx^2_{(4)}, \\ H_{(3)} &= \star \left(d(\ln H) \wedge \omega_{(1,5)} \right), \\ e^{\phi} &= H^{\frac{1}{2}}, \\ H &= 1 + \frac{l_s^2 N}{r^2}. \end{aligned} \tag{2.45}$$

In each case r is the radial coordinate in the directions transverse to the branes. In all cases the brane solution is determined by a harmonic function H.

As a final remark, we can also describe *multi-centred brane solutions*, solutions consisting of separated parallel branes. They are the same as in equations (2.33), (2.43), (2.44) or (2.45), but with H replaced by a multi-centred harmonic function

$$H_{\text{m.c.}} = 1 + \sum_{i=1}^{N} \frac{K(p)}{|\vec{r} - \vec{r}_i|^{\Delta(p)}}, \tag{2.46}$$

where $\vec{r}$ and $\vec{r}_i$ are vectors in the transverse space, and the constants $K(p)$ and $\Delta(p)$ depend on the type of brane we consider. For Dp-branes $K(p) = c_p$ and $\Delta(p) = 7 - p$. The solutions correspond to N parallel branes, where the i-th brane is centered at $\vec{r}_i$ in the transverse space.

2.3.2 Supersymmetry of brane solutions

An important property of the brane solutions is that they preserve half the supersymmetry of the supergravity theory. To see this explicitely we need to consider the form of the supersymmetry transformations in the appropriate supergravity theory. In all cases, since we are considering bosonic solutions, the supersymmetry transformations of all bosonic fields vanish. We are left with the supersymmetry transformations of the fermionic fields. The symmetries preserved by the solution are given by the subset of all allowed transformations which vanish for this particular solution. We will use eleven-dimensional supergravity from section 2.2 as an example. The supersymmetry transformation of the gravitino was given in equation (2.6). It can be easily checked that for the M2- and M5-brane solutions of the previous section, these supersymmetry variations vanish with an arbitrary choice of half of the components of the spinor ϵ. More precisely, in each case these supersymmetry variations vanish when ϵ is some specific function multiplying a constant spinor ϵ_0

$$\epsilon = f(r)\epsilon_0, \tag{2.47}$$

which satisfies a projection condition that depends on the brane under consideration.

For an M2-brane (equation (2.33) with $p = 2$) the solution admits Killing spinors of the form (2.47) and

$$\hat{\Gamma}_{012}\epsilon_0 = c\epsilon_0 \qquad c = \pm 1, \tag{2.48}$$

where $\hat{\Gamma}_{0\ldots p} = \hat{\Gamma}_0 \ldots \hat{\Gamma}_p$ is the product of $p + 1$ distinct Gamma matrices in an orthonormal frame. Using that $\left(\hat{\Gamma}_{012}\right)^2 = 1$ and $\mathrm{Tr}\left(\hat{\Gamma}_{012}\right) = 0$, we see that the projection eliminates half of the spinor-components. We conclude that the M2-brane solution has 16 Killing spinors and preserves half of the supersymmetry.

Similarly, the M5-brane solution allows for Killing spinors satisfying

$$\hat{\Gamma}_{0\ldots 5}\epsilon_0 = c\epsilon_0, \tag{2.49}$$

for M5-branes with world-volume directions 012345. Because of the tracelessness and the unipotency of $\hat{\Gamma}_{0\ldots 5}$, there are again 16 Killing spinors and thus the solution preserves half of the supersymmetry.

Similar results hold for the brane solutions of type IIA and type IIB supergravity. We will make this a bit more formal in section 2.4, where we discuss the use of a probe brane to identify the remaining supersymmetry.

2.3.3 Asymptotic and near-horizon geometries of M2, M5 and D3-branes

In this section we take a closer look at the geometries of D3-branes as well as M2- and M5-branes. We will particularly be interested in the symmetry group associated with the near-horizon regions of their geometries. All of these have a near-horizon geometry of the form AdS×Sphere.

All of the solutions presented in section 2.3.1 are supergravity solutions, which means that they are valid as long as their typical length scale is much larger than the string scale l_s. For Dp-branes this length scale is defined by $R_p^{7-p} = c_p N$ and is found in the harmonic function of the solution. Requiring supergravity to be a valid limit of string theory then imposes $R_p/l_s \gg 1$ (or equivalently $g_s N \gg 1$).

Now consider the regime where $r \gg R_p$. In this regime the harmonic functions approaches unity $H \simeq 1$ and the metric reduces to that of a flat space plus small corrections of the order $(R_p/r)^{7-p}$. We can write

$$\left(\frac{R_p}{r}\right)^{7-p} \sim \frac{g_s N \alpha'^{7-p}}{r^{7-p}} \sim \frac{G_N M}{r^{7-p}}, \tag{2.50}$$

where we defined $M = N\tau_p$ and τ_p is the tension of a single Dp-brane. The gravitational effect of N Dp-branes is similar to that of a point particle of mass M in $9-p$ dimensions. We see that the radius R_p sets the range of the gravitational influence of N Dp branes. This regime is called the *asymptotic limit* of the geometry.

The metric of the D3-brane was given in (2.43), which we repeat here, for $p = 3$

$$\begin{aligned} ds^2 &= H^{-1/2} dx_{(1,3)}^2 + H^{1/2} dx_{(6)}^2, \\ F &= -\frac{1}{H}\partial_r(H) dr \wedge \omega_{(1,3)}, \\ e^{\phi} &= 1, \\ H &= 1 + \frac{c_3 N}{r^4}. \end{aligned} \tag{2.51}$$

Note that the dilaton vanishes. The typical length scale for the case $p = 3$ is defined by $R_3^4 = N c_3 \sim g_s N \alpha'^2$.

Gravitational effects are strong in the region where $r/R_3 \ll 1$. In this limit, the metric in (2.51) reduces to

$$ds^2 = \left(\frac{r}{R_3}\right)^2 dx^2_{(1,3)} + \left(\frac{R_3}{r}\right)^2 \left(dr^2 + r^2 d\Omega^2_{(5)}\right). \tag{2.52}$$

This geometry describes $\mathrm{AdS}_5 \times \mathrm{S}^5$, where the AdS-factor is expressed in coordinates (3.26) (explained in section 3.2 later on in this thesis), and it has the same radius for the AdS and sphere factors. Since r is small compared to the characteristic length scale R_3 this limit is called the *near-horizon limit* of the geometry (2.51).

We have found two asymptotic regions that are connected by an infinite throat and the solution interpolates between flat space at $r \to \infty$ and a geometry that asymptotically tends to $AdS_5 \times S^5$ near the horizon, $r \to 0$. Furthermore, it turns out that there is an enhancement of supersymmetry in the asymptotic regions and both regions are maximally supersymmetric solutions to type IIB supergravity having 32 real supercharges. Hence, the D3-brane solution is a half BPS solution that interpolates between two maximally supersymmetric solutions, ten-dimensional Minkowski-space and $AdS_5 \times S^5$. The bosonic symmetry of the near-horizon region is given by the product of the symmetry groups of the constituent factors of the metric, an $SO(2,4)$-factor for AdS_5 and a factor $SO(6)$ for the S^5.

The energy E_r as measured by an observer at fixed radial position r is not the same as the energy E_∞ measured at infinity, in the asymptotically flat region. The two are related by a red-shift factor $E_\infty = E_r/H^{1/4}$ due to the curvature of the geometry. As seen by an observer in the bulk, there are two kinds of closed string excitations that remain in the low energy limit. The first group of excitations is given by massless excitations that propagate in the asymptotic flat region. The second group consists of the full spectrum of closed type IIB string excitations in the near-horizon part of the geometry $r/R_3 \to 0$. In fact, the closer the excitations are to the horizon, the higher their energy E_r can be while keeping $E_\infty = E_r r/R_3$ fixed.

Let us express all energy and lengths in string units by introducing factors of α' and then take the low-energy limit $\alpha' \to 0$. We find that $E_r\sqrt{\alpha'} = E_\infty\sqrt{\alpha'}R_3/r$. Taking the low-energy limit $\alpha' \to 0$, while keeping E_∞, R_3 and $r/\sqrt{\alpha'}$ fixed shows that the excitations of arbitrary finite energy $E_r\sqrt{\alpha'}$ are allowed in the near-horizon region. Furthermore, these excitations with finite energy cannot escape to the asymptotic region due to the shape of the gravitational potential. The massless excitations that propagate in the near-horizon region and the asymptotically flat region cannot interact and decouple in the limit $\alpha' \to 0$.

A similar story holds for the M2-and M5-branes, the near-horizon geometry is given by

$$ds^2 = \left(\frac{r}{R}\right)^{2\omega} dx^2_{(1,p)} + \left(\frac{R}{r}\right)^2 dr^2 + R^2 d\Omega^2_{(D-p-2)}, \tag{2.53}$$

where for the M2-branes $R = (c_2 N)^{1/6}$ and $\omega = 2$, and for the M5-branes $R = (c_2 N)^{1/3}$ and $\omega = 1/2$. With a coordinate change $\rho = \left(\frac{r}{R}\right)^\omega$ one obtains the horospherical form

$$ds^2 = \rho^2 dx^2_{(1,p)} + \left(\frac{R}{\omega}\right)^2 \frac{d\rho^2}{\rho^2} + R^2 d\Omega^2_{(D-p-2)}, \tag{2.54}$$

such that in the case of M2-branes we have an $\mathrm{AdS}_4 \times \mathrm{S}^7$-geometry with radii of curvature $2R_{\mathrm{AdS}} = R = R_S$, and for the case of M5-branes we have an $\mathrm{AdS}_7 \times \mathrm{S}^4$-geometry with radii of curvature $\frac{1}{2}R_{\mathrm{AdS}} = R = R_S$. Both these near-horizon geometries have enhanced supersymmetry and allow for 32 real supercharges. The M2-brane has a bosonic symmetry group $SO(2,3) \times SO(8)$ while the M5-branes near-horizon geometry has a symmetry $SO(2,6) \times SO(5)$.

2.4 D-brane actions

The (bosonic) world-volume action of a generic probe Dp-brane consists of two parts, the $(p+1)$-dimensional Dirac-Born-Infeld (DBI) action

$$S_{\mathrm{DBI}} = -T_{\mathrm{D}p} \int d^{p+1}\sigma e^{-\phi} \sqrt{-\det(G_{\mu\nu} + \mathcal{F}_{\mu\nu})}\,, \tag{2.55}$$

together with Wess-Zumino couplings

$$S_{\mathrm{WZ}} = T_{\mathrm{D}p} \int \sum_n \hat{C}_{(n)} \wedge e^{\mathcal{F}}. \tag{2.56}$$

The world-volume $\mathcal{M}_{(p+1)}$ is parametrised by $p+1$ coordinates σ^μ. The functions $X^M(\sigma)$ describe the embedding and shape of the Dp-brane world-volume in spacetime. A frequent choice of coordinates σ is the static gauge in which $\sigma^\mu = X^\mu$. S_{DBI} contains the induced metric

$$G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN}, \tag{2.57}$$

and $\mathcal{F} = 2\pi l_s^2 F - \hat{B}_{(2)}$ is a linear combination of the pullback of the background NS-NS 2-form potential $B_{(2)}$ and a world-volume 2-form field strength F. In the WZ terms, $\hat{C}_{(n)}$ is the pullback of $C_{(n)}$ onto the brane world-volume, n is to be understood to run over all RR-potentials present in the given supergravity theory and the integral includes only wedge products that have support on the wordvolume of the brane ($(p+1)$-forms). In particular note that S_{WZ} always contains a term of the form $\int C_{(p+1)}$, thus the Dp-brane is a source for $F_{(p+2)}$ as we claimed before. The full Dp-brane action is

$$S_{\mathrm{D}p} = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}. \tag{2.58}$$

The two terms in (2.58) describe the coupling of the probe brane to the two parts of the background geometry. S_{DBI} describes the 'gravitational' coupling to the background metric and S_{WZ} describes the 'electromagnetic' coupling to the background forms. We first continue the discussion on the supersymmetry preserved by the branes and then comment on a few simple limits of the brane action.

2.4.1 World-volume (super)symmetry

Let us consider a D-brane as a probe brane, a brane placed in a fixed background as a test brane. This means that the backreaction of the brane on the background is ignored. If the background is generated by the same type and orientation of branes, then we expect that the probe brane does not break any of the (super)symmetries that are preserved by the background. We look at the case of $N+1$ parallel branes where we consider one to be a probe brane. The backreaction is an $\mathcal{O}\left(\frac{1}{N}\right)$-effect but the supersymmetries preserved are independent of N and so should not be affected by neglecting the backreaction of this one brane. A convenient way to work with supersymmetry is given by *superspace*. In a superspace, the usual spacetime-geometry is extended by supplementing the bosonic coordinates with a set of fermionic coordinates. Coordinate transformations can now mix bosonic and fermionic coordinates in this superspace, hence the name 'super'. The supersymmetric extension of (2.58) can be obtained by embedding the brane world-volume in superspace. This superspace has 10 or 11 bosonic spacetime coordinates X^M and 32 fermionic coordinates Θ. The background coordinates X^M become fields $X^M(\sigma)$ on the world-volume and describe the bosonic embedding of the brane in spacetime, similarly the fermionic supercoordinates Θ on the world-volume are now also fields $\Theta(\sigma)$. By construction the probe brane actions are invariant under the background superisometries. These symmetries are now symmetries acting on fields, i.e. they depend on the world-volume coordinates σ^μ through $\{X^M(\sigma), \Theta(\sigma)\}$, and act as global symmetries.

In order to have world-volume supersymmetry the number of on-shell degrees of freedom in the bosonic and fermionic sector must match. This requires a (local) fermionic symmetry of the world-volume action linking the two, called

κ-symmetry [43, 44]. This κ-symmetry will project out half of the components of Θ on the brane world-volume. κ-symmetry is very similar to supersymmetry and we just need to find a way to link this κ-symmetry on the world-volume to the supersymmetry of the background. The κ-symmetry transformations take the form

$$\delta_\kappa \Theta = \frac{1}{2}\left(1+\Gamma\right)\kappa, \tag{2.59}$$

while the supersymmetry transformation on the background is

$$\delta_\epsilon \Theta = \epsilon. \tag{2.60}$$

The form of Γ depends on the type of brane, but it always satisfies $\Gamma^2 = 1$ such that we can construct projection operators $\mathcal{P}_\pm = \frac{1}{2}\left(1 \pm \Gamma\right)$. Also Γ is traceless such that each projection operator projects out exactly half the components of an arbitrary spinor. Decomposing Θ under these projections we find

$$\delta_\kappa\left(\mathcal{P}_-\Theta\right) = 0, \qquad \delta_\kappa\left(\mathcal{P}_+\Theta\right) = \mathcal{P}_+\kappa, \tag{2.61}$$

and

$$\delta_\epsilon\left(\mathcal{P}_-\Theta\right) = \mathcal{P}_-\epsilon, \qquad \delta_\epsilon\left(\mathcal{P}_+\Theta\right) = \mathcal{P}_+\epsilon. \tag{2.62}$$

We see that we can consistently set $\left(1+\Gamma\right)\Theta = 0$ to fix κ-symmetry, leaving $\mathcal{P}_-\Theta$ as the world-volume fermionic degrees of freedom. The gauge fixing procedure boils down to setting $\left(1+\Gamma\right)\Theta = 0$ and then preserving this gauge choice under symmetry transformations by compensating a supersymmetry transformation with a κ-symmetry transformation with parameter $\kappa(\epsilon) = -\mathcal{P}_+\epsilon$. This last equality is called a *decomposition law* and tells us that the κ-symmetry is now a composite symmetry. Having removed $\mathcal{P}_+\Theta$, leaves us just $\mathcal{P}_-\Theta$ with supersymmetry transformation $\delta_\epsilon\left(\mathcal{P}_-\Theta\right) = \mathcal{P}_-\epsilon$. Requiring preservation of the world-volume supersymmetry is then $\delta_\epsilon\left(\mathcal{P}_-\Theta\right) = 0$, or

$$\Gamma\epsilon = \epsilon. \tag{2.63}$$

Since Γ depends on the world-volume fields we see that the brane locally preserves half of the background global supersymmetries. The brane preserves at most 16 supersymmetries.

2.4.2 (Non-Abelian) Gauge theories on D-branes

We choose a Minkowski background spacetime ($G = \eta$, $B = 0$, $\phi = 0$) and we consider the brane to be an infinite extended brane by taking the *static gauge*

$$\sigma^\mu = \delta^\mu_M X^M. \tag{2.64}$$

With these choices the DBI action reduces to the Born-Infeld action

$$S_{\rm BI} = -T_{{\rm D}p} \int d^{p+1}x \sqrt{-\det\left(\eta_{\mu\nu} + \partial_\mu X^i \partial_\nu X^i + 2\pi\alpha' F_{\mu\nu}\right)}, \qquad (2.65)$$

where $i = 1, \ldots 9-p$ runs over the transverse directions. Expanding it at second order in derivatives[11]

$$S_{\rm BI} \simeq -T_{{\rm D}p} \int d^{p+1}x \left[1 + \frac{1}{2}(2\pi\alpha')^2 \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{4}(2\pi\alpha')^2 F_{\mu\nu}F^{\mu\nu}\right], \qquad (2.66)$$

we recover the standard kinetic terms for $9-p$ scalars living on the Dp-brane world-volume (we defined $X^i = 2\pi\alpha'\phi^i$). Similarly, the quadratic term in $F_{\mu\nu}$ is the kinetic term of the gauge vector A_μ.

When we consider the presence of more than one Dp-brane, there emerges a fundamental aspect from the string theory perspective related to the presence of open strings. In presence of multiple D-branes, the endpoints of a string can still lie on a single brane, however, the string can also stretch between two different branes. When the string stretches between two non-coincident branes it has a minimum finite length and, because of its tension, gives rise to modes with finite mass.

Let us consider the case of two-branes. Strings with endpoints on the same brane give rise to two U(1)-gauge fields $(A_\mu)_1^1$ and $(A_\mu)_2^2$, where the indices indicate on which brane the string starts and ends. Open strings connecting different branes give rise to another two additional vector fields $(A_\mu)_2^1$ and $(A_\mu)_1^2$, whose mass is proportional to the distance between the branes. If the Dp-branes are brought on top of each other the modes related to strings stretching between the branes become massless. Altogether, they constitute the gauge field of a non-abelian $U(2)$ gauge group. Similarly, the scalars can be grouped in a matrix $(\vec{\phi})_n^m$, which transforms in the adjoint representation of the $U(2)$ gauge group. The case of two branes is easily generalised to a stack of N coincident Dp-branes. In this case the symmetry is enhanced to $U(N)$. Moving branes apart corresponds to a symmetry breaking from $U(N)$ to $U(1)^N$, with the off-diagonal degrees of freedom in A_μ becoming massive.

It turns out that it is a highly non-trivial problem to generalise the Dp-brane action (2.58) to multiple Dp-branes and only the first few orders in α' of the non-Abelian extension of the full action have been obtained. In Minkowski spacetime with no background fields, the leading order of the effective action is the dimensional reduction to $(p+1)$ dimensions of the ten-dimensional super

[11] Note that this is an expansion in α'.

Yang-Mills action. At the two derivative level, the bosonic sector reads

$$S = -\frac{1}{g_{YM}^2}\int d^{p+1}x \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi^i D^\mu\phi^i - \frac{1}{2}[\phi^i,\phi^j][\phi_i,\phi_j]\right), \tag{2.67}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \qquad vD_\mu\phi^i = \partial_\mu\phi^i + i[A_\mu,\phi^i]. \tag{2.68}$$

Comparing with (2.66) shows that the Yang-Mills coupling constant is

$$g_{\mathrm{YM}}^2 = \frac{1}{2\pi(\alpha'^2)T_{\mathrm{D}p}}. \tag{2.69}$$

The vacuum structure of the theory matches the Dp-brane picture above. The scalar field potential in (2.67) is a sum of squares and its vacua are given by configurations that satisfy $[\phi_i,\phi_j]=0$. Both A_μ and ϕ^i are $N\times N$ matrices, and, if commuting they can be simultaneously diagonalised by a gauge transformation. In the vacuum, the commuting scalar fields can take on an expectation value. The diagonal entries of the scalar fields $\phi^i = \mathrm{diag}(\phi_1^i, \ldots \phi_N^i)$ are interpreted as the position of the N Dp-branes in spacetime. When the eigenvalues differ, the gauge group $U(N)$ is broken by the Brout-Englert-Higgs mechanism to $U(1)^N$ since the diagonal components of the A_μ remain massless, while the off-diagonal ones pick up a mass through the terms $\mathrm{Tr}[A_\mu,\phi^i]^2$. This matches the D-brane picture above where the diagonal gauge fields corresponded to strings with both endpoints on the same brane and off-diagonal ones to open strings stretching between different branes. The diagonal $U(1)$ subgroup of $U(N)$ describes the motion of the centre of mass of the whole rigid system of N branes. Because of translational invariance, its dynamics decouples from the internal dynamics of the N D-branes, and the gauge group reduces to $SU(N)$.

The low energy effective action (2.67) receives higher derivative corrections suppressed by $\alpha' E^2$, at energy scale E.[12] The full type IIB String Theory also contains closed strings which propagate and interact between themselves as well as with the open strings in the ten-dimensional space. The interaction strength of closed strings is set by Newton's constant (which has dimension of length to the power 8 in ten dimensions), or equivalently, by a dimensionless coupling $G_N E^8 \propto g_s^2(\alpha' E^2)^4$. This dimensionless coupling vanishes in the IR, and closed strings become non-interacting as expected from a theory of gravity. Since gravity couples universally to all forms of matter, the same parameter should control the interactions with open strings, and this in turn means that the open and closed string sectors decouple at low energies. The higher derivative

[12] An easy way to see this, is by noting that it is the generalisation of (2.66), which is an expansion in α'

corrections vanish as well in this limit and we are left with $\mathcal{N} = 4$ $SU(N)$ SYM in $3+1$ dimensions on the world-volume of the brane.

As an example we present the non-abelian gauge theory obtained from a stack of N D3-branes. The action is given by (2.67) with $p = 3$. The field content of the theory consists of a vectorfield A_μ, six scalars ϕ^i and four Weyl fermions, all of which are in the adjoint representation of $U(N)$. At the two derivative level, the world-volume theory is $\mathcal{N} = 4$ super Yang-Mills in $3+1$ dimensions with $U(N)$ gauge group. The $\mathcal{N} = 4$ signals that it is a supersymmetric theory with 16 real supercharges in 4 dimensions. The branes preserve a global $SO(1,3) \times SO(6)$ symmetry, matching the Lorentz group on the brane world-volume and the R-symmetry group of the theory $SO(6) \sim SU(4)$. Four-dimensional $\mathcal{N} = 4$ SYM is also conformal. The full symmetry of the theory is the superconformal group $SU(2,2|4)$, which has bosonic subgroup $SU(2,2) \times SU(4)$. The 16 real supercharges combine with another 16 conformal supercharges, which are needed to close the algebra, to obtain 32 real supersymmetry generators.

2.5 Intersecting branes

In this section we discuss properties of intersecting branes and branes ending on branes. In particular, we want to touch upon the supersymmetry preserved by intersecting branes. We briefly discuss more general configurations of intersecting branes. Finally, we conclude with a discussion of an M2/M5 brane-configuration in M-theory, as well as a related D4/D8-brane configuration that will motivate the brane configuration considered in chapter 5. For a more detailed discussion of intersecting branes we refer to the reviews [45, 46].

2.5.1 Supersymmetry of orthogonally intersecting branes

All essential features of orthogonally intersecting branes can be understood by looking at two intersecting branes (or two intersecting stacks of branes). Let us consider a $(p+q_1)$-brane and a $(p+q_2)$-brane embedded in a D-dimensional space as displayed in table 2.1. Here $d = D - p - q_1 - q_2 - 1$ is the dimension

	$p+1$	q_1	q_2	d
$(p+q_1)$-brane	X	X		
$(p+q_2)$-brane	X		X	

Table 2.1: Intersecting $(p+q_1)$ and $(p+q_2)$-branes.

of the space transverse to both branes. The two branes have a common $(p+1)$-dimensional world-volume when they intersect, i.e. when they are at the same position in the transverse d-dimensional space. The branes will not actually intersect unless they are at the same location in the overall transverse space, although we will refer to all configurations containing non-parallel branes as intersecting brane configurations. The important features are related to the structure of the relative transverse space, comprised of the directions labelled by q_1 and q_2. We will now consider the amount of supersymmetry preserved by such a configuration and discover it is the orientation and not the position of the branes that plays a role here.

We know which supersymmetries are preserved by a single brane (or a single stack of branes). For two branes we simply have to see which supersymmetries survive both projection conditions. For the first brane we have the projection condition $\Gamma^{(1)}\epsilon = \epsilon$ and the second brane has condition $\Gamma^{(2)}\epsilon = \epsilon$. The matrices $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are a product of Γ-matrices, with specific expressions determined by the supergravity theory and the brane under consideration. There are two options, either $\Gamma^{(1)}\Gamma^{(2)} = \Gamma^{(2)}\Gamma^{(1)}$ or $\Gamma^{(1)}\Gamma^{(2)} = -\Gamma^{(2)}\Gamma^{(1)}$ (there is a third option where $\Gamma^{(1)} = \Gamma^{(2)}$ which preserves half the supersymmetry). In both cases we have $\mathrm{Tr}\left(\Gamma^{(1)}\Gamma^{(2)}\right) = 0$. In the case where $\Gamma^{(1)}$ and $\Gamma^{(2)}$ anti-commute, we have

$$\epsilon = \Gamma^{(1)}\epsilon = \Gamma^{(1)}\Gamma^{(2)}\epsilon = -\Gamma^{(2)}\Gamma^{(1)}\epsilon = -\Gamma^{(1)}\epsilon = -\epsilon, \tag{2.70}$$

so no supersymmetry is preserved. In the case where the projections commute, we have that one quarter supersymmetry is preserved. This can be seen as follows. Because $\Gamma^{(1)}$ and $\Gamma^{(2)}$ commute they can be simultaneously diagonalised, and because both are traceless and square to one they have equal numbers of $+1$ and -1 eigenvalues (16 each). If we denote by n_{+-} the amount of simultaneous eigenstates of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with eigenvalues $+1$ and -1 respectively, and similarly n_{++}, n_{--} and n_{-+} we have

$$n_{++} + n_{+-} = n_{-+} + n_{--} = n_{+-} + n_{--} = n_{-+} + n_{++} = 16. \tag{2.71}$$

Since $\Gamma^{(1)}\Gamma^{(2)}$ is traceless it also has equal numbers of eigenvalues $+1$ and -1

$$n_{++} + n_{--} = n_{+-} + n_{-+} = 16, \tag{2.72}$$

such that

$$n_{--} = n_{++} = n_{+-} = n_{-+} = 8, \tag{2.73}$$

and the number of preserved supersymmetries is 8, one quarter of the original 32 supersymmetries.

If we have more than two types of orthogonally intersecting branes then we can analyse the amount of supersymmetry preserved in a similar manner. The result will depend on the type and orientation of the branes. If the whole configuration of m orthogonally intersecting branes is to preserve any supersymmetry, then a necessary condition is that each pair of branes must preserve one quarter supersymmetry. By doing a similar analysis of the simultaneous eigenstates of the m operators $\Gamma^{(i)}$, provided the product of any number of these distinct operators is traceless, then precisely $1/2^m$ supersymmetry is preserved. It is possible that some product of these operators is plus or minus the identity instead of a traceless product of Γ-matrices. In the case of a plus sign, one of the projection conditions is already imposed by the others and does not further break supersymmetry. The case with a minus sign breaks all supersymmetry, but the sign can be changed by reversing the orientation of one of the branes.

Examples of orthogonally intersecting branes

We start by considering the D-branes in type IIA or type IIB theories. For any two branes we have that the relative transverse space has a dimension $q_1 + q_2$ which is a multiple of two. There are two distinct cases for intersecting branes and they have

$$\begin{aligned} [\Gamma^{(1)}, \Gamma^{(2)}] = 0 \quad &\text{if} \quad q_1 + q_2 = 0 \pmod 4, \\ \{\Gamma^{(1)}, \Gamma^{(2)}\} = 0 \quad &\text{if} \quad q_1 + q_2 = 2 \pmod 4. \end{aligned} \tag{2.74}$$

So we see that the condition for preserving one quarter supersymmetry is that the branes have 0, 4 or 8 transverse directions.

As a second example we consider three sets of orthogonally intersecting branes, two M2-branes and one M5-brane. We take the orientations to be in the 012345, 016 and 027 hyperplanes. This solution is a special triple intersection. The product of the three Gamma-matrix projections gives another projection

$$\hat{\Gamma}_{012345}\hat{\Gamma}_{016}\hat{\Gamma}_{027} = -\hat{\Gamma}_{034567}, \tag{2.75}$$

corresponding to an M5-brane in the directions 034567. This means we can obtain an intersection of an M2-brane with another M2-brane and two (non-parallel) M5-branes that preserves 1/8 supersymmetry (and not 1/16 as one might expect for 4 intersecting branes) as long as we choose the polarisation of the fourth brane (M5-brane number two) to be determined by the polarisations of the others.

2.5.2 Smeared intersections

In section 2.3.1 we saw that solutions for parallel branes are described by harmonic functions with singularties at the location of the branes. It turns out that a large class of intersecting brane solutions can be described in a similar fashion by following a set of rules for combining the harmonic functions associated to each type of brane, called the *harmonic function rule.* As a guiding example we will consider the case of intersecting M2-branes. We will consider a set of parallel M2-branes with world-volume directions 012 intersecting with another set of parallel M2-branes with world-volume directions 034. The constituent parallel branes (with world-volume directions either 012 or 034) would be described in terms of harmonic functions $H_{(1)}$ and $H_{(2)}$, respectively, as

$$\begin{aligned} ds^2 &= -H_{(1)}^{-\frac{2}{3}} dt^2 + H_{(1)}^{-\frac{2}{3}} \left(dx_1^2 + dx_2^2\right) + H_{(1)}^{\frac{1}{3}} \left(dx_3^2 + dx_4^2\right) + H_{(1)}^{\frac{1}{3}} dx_\perp^2, \\ F &= -d(H_{(1)}^{-1}) \wedge dt \wedge dx_1 \wedge dx_2 \end{aligned} \tag{2.76}$$

and

$$\begin{aligned} ds^2 &= -H_{(2)}^{-\frac{2}{3}} dt^2 + H_{(2)}^{\frac{1}{3}} \left(dx_1^2 + dx_2^2\right) + H_{(2)}^{-\frac{2}{3}} \left(dx_3^2 + dx_4^2\right) + H_{(2)}^{\frac{1}{3}} dx_\perp^2, \\ F &= -d(H_{(2)}^{-1}) \wedge dt \wedge dx_3 \wedge dx_4. \end{aligned} \tag{2.77}$$

The solution for the intersecting M2-branes is simply given by multiplying the metric factors and adding the field strengths

$$\begin{aligned} ds^2 &= -H_{(1)}^{-\frac{2}{3}} H_{(2)}^{-\frac{2}{3}} dt^2 + H_{(1)}^{-\frac{2}{3}} H_{(2)}^{\frac{1}{3}} \left(dx_1^2 + dx_2^2\right) \\ &\quad + H_{(1)}^{\frac{1}{3}} H_{(2)}^{-\frac{2}{3}} \left(dx_3^2 + dx_4^2\right) + H_{(1)}^{\frac{1}{3}} H_{(2)}^{\frac{1}{3}} dx_\perp^2, \\ F &= -d(H_{(1)}^{-1}) \wedge dt \wedge dx_1 \wedge dx_2 - d(H_{(2)}^{-1}) \wedge dt \wedge dx_3 \wedge dx_4. \end{aligned} \tag{2.78}$$

The equations of motion for F are satisfied provided that $H_{(1)}$ and $H_{(2)}$ are harmonic funtions of the coordinates transverse to both types of M2-branes. From the perspective of the 012 M2-branes we can interpret the form of $H_{(1)}$ as describing a continuous distribution of M2-branes in the 34 directions. We say that these branes are smeared in the 34 directions. So the solution corresponds to the intersection of M2-branes oriented in the 012 and 034 directions, smeared over their relative transverse directions. In general, a brane solution whose harmonic function is independent of a number of transverse coordinates is said to be *delocalised, averaged or smeared* over those directions.

The generalisation of the above example is very obvious. The constituent brane solutions each have a harmonic function associated to them. To construct the intersecting solution we simply combine the solutions like we did in the M2-brane example. For every constituent brane we add together the field strengths and multiply the components of the diagonal metric. When we are dealing with branes in type IIA or IIB supergravity there is also a dilaton, which for the intersecting solution is given as the sum of the constituent solutions for the dilaton. This procedure provides a solution describing intersecting branes on the condition that the configuration of the intersecting branes is supersymmetric.

In general, when one checks the equations of motion for such an ansatz, one finds that for each pair of (sets of parallel) branes, at least one of them needs to be smeared over the world-volume directions of the other. This gives several choices for the smearing and the choice of this smearing determines on which coordinates each harmonic function can depend. These harmonic functions then have to satisfy the curved-space Laplace equation following from $d \star F = 0$, with appropriate source terms. There is an exception to this rule, when two branes intersect with eight relative transverse dimensions. In this case one can allow the harmonic function to depend on the relative transverse coordinates provided they are independent of the overall transverse direction.

The curved-space harmonic functions improve the localisation of the branes, even allowing full localisation in some cases, but usually it is not possible to find explicit solutions. However, taking a near-horizon limit makes the problem more tractable and explicit solutions can often be found, examples include [47, 48, 49], and the geometry of the near-horizon limits of several semi-localised solutions was studied in [50]. The smeared solutions constructed with the harmonic function rule are all of the same general form, but it is not clear whether the different fully localised solutions will have such a similar description. It is still unclear what the properties of such localised solutions are for general (supersymmetric) configurations of branes, even in the near-horizon limit.

General intersections

So far we have only considered orthogonally intersecting branes, but one could ask what the most general configuration of intersecting branes is that preserves some of the supersymmetry. The problem essentially is a more complicated version of what we discussed here. A large class of intersecting brane solutions can be described in a similar way by combining harmonic functions following the simple rules of the previous section and then relate them to non-orthogonal intersections by performing boosts and dualities [51]. For example, consider the case of two planar M5-branes, with one of them having directions 012345. The embedding of the second M5-brane is related to the first by a rotation of the spatial 5-plane in the ten-dimensional space. This can be parametrised in terms of angles describing the rotations in each of the 2-planes spanning e.g. directions 16, 27, ... The supersymmetry projection conditions can be analysed with the result that for various constraints on the angles the possible fractions of supersymmetry which can be preserved are [52]

$$\frac{1}{32}, \frac{1}{16}, \frac{3}{32}, \frac{1}{8}, \frac{5}{32}, \frac{3}{16}, \frac{1}{4}, \frac{1}{2}. \tag{2.79}$$

Application to Black Holes

The harmonic function rules above provide a method of constructing large classes of solutions, which are related to intersecting branes. However, the smearing of the branes over the relative transverse directions means that it is not obvious what happens at the intersection and there are important features that cannot be described by these solutions, such as the relative separations of the branes in directions over which they are smeared. These parameters are important for intersecting brane configurations that describe gauge theories.

The smearing of the branes is not important when we want to compactify the directions along which the branes are smeared. When we perform such a reduction we end up with a p-brane solution of a lower dimensional supergravity, where $p+1$ is the number of common world-volume dimensions of the intersecting branes. We can further compactify some or all of these p directions, and when we compactify all p directions we end up with a particle. These solutions describe black holes with various charges specified by the constituent intersecting branes.

The advantage of constructing black hole solutions from intersecting branes is that we automatically have a string theory interpretation. In particular, the interpretation of a black hole as a particular configuration of branes allows us to calculate the entropy of the black hole by considering the number of massless degrees of freedom in string theory. Comparing this to the area of the lower dimensional black hole horizon provides a microscopic derivation [53] of the Bekenstein-Hawking black hole entropy. This is quite a large subject and we will not consider it any further. For the interested reader, we refer to [54, 55, 56, 57] for a review.

Gauge theories and intersecting branes

Intersecting branes that preserve supersymmetry can be used to study the infrared dynamics of supersymmetric gauge theories [58, 59]. One considers different types of branes intersecting in some chosen arrangement. The low-energy dynamics on the world-volume of one type of brane is then associated with a supersymmetric quantum field theory that one wishes to study. By considering the low-energy dynamics from the point of view of different branes and allowing the branes to move around, enables one to determine the low-energy effective dynamics of the field theory in some cases. This brings us in the domain of the AdS/CFT correspondence and other gauge-gravity dualities. This is the subject of the next chapter and we will come back to this subject there. We will mention here that often the gauge theory is used to determine the symmetries of the intersecting brane solution. One starts with the most general form of the metric and fields that adhere to these symmetries, and then use the conditions for preservation of supersymmetry to constrain the metric and fields further and obtain a supergravity solution describing an intersecting brane configuration. Once a solution has been obtained, one can then try to describe the gravity dual of the field theory.

2.5.3 Intersecting M2/M5 branes and intersecting D4/D8-branes

We have seen that we can construct configurations of branes that intersect and still preserve some supersymmetry. In their near-horizon geometry these configurations often have a bosonic symmetry enhancement wherein a conformal subgroup appears, allowing for an AdS-factor in the geometry, as well as supersymmetry enhancement, restoring some of the supersymmetry lost due to the addition of branes.[13]

Intersecting M2/M5-branes

We consider intersecting M2-branes and M5-branes, with the configuration given in table 2.2. From the discussion above we know that we are more likely to solve the system of equations and find a localised solution if we look at the near-horizon geometry of this configuration. We will assume there is a conformal symmetry enhancement in this limit, i.e. there is a conformal factor that appears as a subgroup of the bosonic symmetry. The intersection

[13] i.e. the D3-brane configuration in their near-horizon limit of section 2.3.3.

preserves 16 Poincaré supersymmetries.[14] The counting goes as follows. The original eleven-dimensional supergravity has 32 supersymmetries. The addition of the stacks of M2- and M5-branes halves the amount of supersymmetry each. However, the projector of the M5' stack is not independent of the others, and does not reduce the amount of supersymmetry further, leaving us with 8 supersymmetries.[15] These 8 supersymmetries are then enhanced to 16 in the near-horizon limit. The bosonic symmetry preserved by the brane configuration is $ISO(1,1) \times SO(4) \times SO(4)$ and is enhanced to $SO(2,2) \times SO(4) \times SO(4)$ in the conformal limit.[16] The corresponding superconformal group with $SO(2,2) \times SO(4) \times SO(4)$ bosonic symmetry and 16 conformal supersymmetries is not unique but comes in a one-parameter family, $D(2,1;\gamma;1) \times D(2,1;\gamma;1)$.

	0	1	2	3	4	5	6	7	8	9	10
M2	X	X									X
M5	X		X	X	X	X					X
M5'	X						X	X	X	X	X

Table 2.2: Intersecting M2/M5-branes, preserving half of the maximal number of supersymmetries.

In [60], the problem of finding solutions of M-theory with $D(2,1;\gamma;1) \times D(2,1;\gamma;1)$ symmetry was reduced to a single linear partial differential equation. At three special points, the $D(2,1;\gamma;1)$ supergroup reduces to a classical

[14]The origin of the name Poincaré supersymmetry is found in the fact that we only have a Poincaré supergroup in the original eleven-dimensional theory, the only supersymmetry generators are the generators Q, which are sometimes referred to as Poincaré supersymmetry generators. If we had a superconformal supergroup, we would have both Q- and S-supersymmetry, counting both is referred to as *conformal supersymmetry*, counting only the Q-supersymmetry is referred to as *Poincaré supersymmetry*.

[15]The symmetry projections of the branes in Tabel 2.2 are given by

$$\text{M2-brane} \quad \epsilon = \Gamma_{23456789}\epsilon, \quad \text{M5-brane} \quad \epsilon = \Gamma_{16789}\epsilon, \quad \text{M5'-brane} \quad \epsilon = \Gamma_{12345}\epsilon. \qquad (2.80)$$

[16]We refer to the conformal limit as a near-horizon limit where we assume the appearance of a conformal factor in the symmetry group. In terms of the AdS/CFT correspondence of chapter 3 this corresponds to having an IR fixed point in the field theory dual, which necessitates conformal symmetry, or in terms of the gravity side, an AdS-factor. Some brane configurations will admit such an enhancement and others will not, whether we are dealing with the former or the latter will be a result of solving the BPS equations and obtaining a solution that realises this conformal symmetry. It turns out that in this case such an enhancement is possible.

	0	1	2	3	4	5	6	7	8	9
F1	X	X								
D4	X		X	X	X	X				
D4'	X						X	X	X	X
O8/D8	X		X	X	X	X	X	X	X	X
D0	X									

Table 2.3: Intersecting D4-branes and F1-strings, preserving half of the maximal number of supersymmetries.

supergroup

$$D(2,1;\gamma;1) = OSp(4^*|2) \qquad \gamma = -1/2, -2\,,$$
$$D(2,1;\gamma;1) = OSp(4|2,\mathbf{R}) \qquad \gamma = 1\,. \tag{2.81}$$

Furthermore, at each of these three special points, the superconformal group becomes a subgroup of a larger superconformal group. This group structure leads to a large and interesting family of solutions.

We consider first the special values $\gamma = -1/2, -2$. In this case we have $D(2,1;-1/2,0) \times D(2,1;-1/2;0) \subset OSp(8^*|4)$ and $D(2,1;-2;0) \times D(2,1;-2;0) \subset OSp(8^*|4)$. The extended symmetry $OSp(8^*|4)$ corresponds to having only a single stack of M5-branes and is realised as the superisometry group of the near-horizon geometry $AdS_7 \times S^4$. For the special value $\gamma = 1$, the extended symmetry is $OSp(8|4,\mathbf{R})$ and corresponds to having a stack of M2-branes, whose near-horizon geometry is $AdS_4 \times S^7$.

The general M-theory solutions with $D(2,1;\gamma;1) \times D(2,1;\gamma;1)$ symmetry were constructed in [60, 61].

Intersecting D4/D8-branes

The above story can be reduced to type IIA string theory after compactifying and dimensionally reducing along the 10-direction. The resulting type IIA brane configuration is given in the top half of table 2.3. The M2-branes become a fundamental string, while the M5-branes become D4-branes.

2.5.4 Intersecting F1/D4/D8-branes in massive type IIA supergravity

We now introduce the system of interest to this thesis (chapter 5). Starting with the above IIA configuration, we introduce D8-branes[17], along with an orientifold projection, as in the lower half of table 2.3.[18] The D8-branes/O8-plane reduce the supersymmetry further and only 8 supersymmetries remain. The D8-branes are magnetically charged under the Romans mass and signal that we now work in the massive version of type IIA supergravity. Note that we may also introduce D0-branes without further reducing the symmetry. The bosonic symmetry in the conformal limit is given by $SO(1,2) \times SO(4) \times SO(4)$. The corresponding superconformal group again comes in a one parameter family, $D(2,1;\gamma;1) \times SO(4)$, where we have an extra bosonic symmetry which is not part of the supergroup. As in M-theory, for three special values of γ, we find that $D(2,1;\gamma;1) \times SO(4)$ is a subgroup of an extended supergroup:

$$\begin{aligned} &D(2,1;\gamma;1) \times SO(4) \subset F(4;2) \times SO(3) \qquad && \gamma = -1/2, -2\,, \\ &D(2,1;\gamma;1) \times SO(4) \subset OSp(8|2,\mathbf{R}) \qquad && \gamma = 1\,. \end{aligned} \tag{2.82}$$

We see a group structure analogous to the one encountered for M-theory and it is a natural question to ask what the enhanced symmetry points correspond to this time.

For the special values $\gamma = -1/2, -2$, the extended symmetry $F(4;2) \times SO(3)$ corresponds to a single stack of D4-branes in addition to the D8-branes/O8-plane. These solutions of massive IIA supergravity have been constructed in [63, 19].

We now consider what happens when we introduce the fundamental strings. Introducing the fundamental string reduces the supersymmetry by half. The

[17] As we shall see later, the D8-branes and O8-plane are important in the AdS/CFT correspondence where it was argued in [62] that their presence is necessary to have a five-dimensional fixed point.

[18] The symmetry projections of the branes given in table 2.3 are given by

$$\begin{aligned} &\text{F1}-\text{string} \qquad && \epsilon = \Gamma_{23456789}\epsilon\,, \qquad && \text{D4}-\text{brane} \qquad && \epsilon = \Gamma_{16789}\epsilon\,, \\ &\text{D4}'-\text{brane} && \epsilon = \Gamma_{12345}\epsilon\,, && \text{D8}-\text{brane} && \epsilon = \Gamma_{1}\epsilon\,, \\ &\text{D0}-\text{brane} && \epsilon = \Gamma_{123456789}\epsilon\,. \end{aligned}$$

The fundamental string projection breaks the number of supersymmetries from 32 to 16, while the D4-brane projection further breaks the number of supersymmetries to 8. Introducing the D8-branes again reduces the supersymmetries by half, so that the resulting theory preserves 4 Poincaré supersymmetries. In the conformal limit, the total number of supersymmetries doubles to 8.

superconformal symmetry is broken to $OSp(4^*|2) \subset F(4;2)$, while the $SO(3)$ bosonic symmetry remains unbroken. These supergravity solutions have yet to be constructed but fall within the ansatz studied in chapter 5 of this thesis.

The special value of $\gamma = 1$ corresponds to the extended symmetry $OSp(8|2, \mathbf{R})$. This symmetry arises as the superconformal symmetry associated with fundamental strings ending on or intersecting the D8-branes/O8-plane. This can be seen in table 2.3, where the $SO(8)$ symmetry acts on the directions 2 through 8, while $Sp(2, \mathbf{R}) \sim SO(1,2)$ arises as the superconformal group of time translations. Additionally, we may include the D0-branes without loss of symmetry. In this case, the fundamental strings can be stretched between the D8-branes and the D0-branes.

The configuration and discussion presented in this section will form the basis of chapter 5. We will use the symmetry considerations discussed here to formulate an ansatz for the solutions. Using this ansatz we will then solve the BPS-equations for the cases of enhanced symmetry.

Chapter 3

AdS/CFT correspondence

In this chapter we will give a concise introduction to the AdS/CFT correspondence focusing on basic examples to explain some of the key ideas. These ideas hold in a more general context but the details are more involved. We will try to avoid these unnecessary details, instead trying to focus on the notions that will help elucidate the results that motivate the research presented in chapter 5.

Let's start with some terminology. What is the AdS/CFT correspondence and why is it interesting? The *AdS/CFT correspondence* is a conjectured duality between two theories, a string theory and a gauge theory. The prototype example of the correspondence was first conjectured by Maldacena in a historical paper [64]. It postulates the exact equivalence of type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four-dimensional Minkowski spacetime. The name of the correspondence, AdS/CFT, originates from the simplest examples which involve a duality between AdS-spaces and Conformal Field Theories (CFT). Since then, many other examples of the correspondence have been found, including gravity theories without AdS-spaces and gauge theories that are not conformal, leading to the general name *gauge/gravity duality*. On the one side of the duality, the gauge theory is a quantum field theory in d dimensions, while on the other side we have a gravity theory in $d+1$ dimensions that has an asymptotic boundary that is d-dimensional. Describing a $(d+1)$-dimensional gravity theory in terms of a lower dimensional system reminds us of an optical hologram that stores a three-dimensional image on a two-dimensional photographic plate. For this reason, the correspondence is also referred to as a *holographic correspondence*.

The idea of holography [65, 66] has its origin in the study of the thermodynamics of black holes. It was shown by Bekenstein and Hawking that black holes can be viewed as thermodynamical systems with a temperature and an entropy [67, 68, 69]. The temperature is related to the black body radiation emitted by the black hole while the entropy is given by the Bekenstein-Hawking entropy formula

$$S_{\mathrm{BH}} = \frac{A_h}{4G_N}, \tag{3.1}$$

where A_h denotes the surface area of the event horizon of a black hole. Since in statistical physics entropy is a measure for the number of degrees of freedom in a theory, it is quite surprising to see that the entropy of a black hole is proportional to the area of the horizon. One would have expected a proportionality to the volume. A consistent picture is reached if gravity in d dimensions is somehow equivalent to a local field theory in $d-1$ dimensions. Both would have an entropy proportional to the area in d dimensions, which corresponds to a volume in $d-1$ dimensions. The AdS/CFT correspondence is a concrete realisation of this.

The correspondence in its full generality has not been proven, one of the reasons being that it would require a complete understanding of string theory. A weaker form of the AdS/CFT correspondence is obtained by restricting to low energies at the string theory side. At low-energies type IIB string theory on $AdS_5 \times S^5$ reduces to type IIB supergravity on $AdS_5 \times S^5$. On the gauge theory side one has to take the corresponding limit and the gauge theory side has a large effective coupling λ and N limit, where N is the rank of the $U(N)$ gauge group. It is important to note that the gravity theory cannot cover the entire gauge theory but only a certain regime. This weaker form of the equivalence has been well tested by now, for instance, by matching correlation functions on both sides of the correspondence (see [70] for a discussion). The general attitude, is to assume that the correspondence holds, as long as it does not lead to contradictions.

An interesting property of the AdS/CFT correspondence is that the duality is a strong/weak duality. Both theories describe the same physics through a dictionary that relates quantities in one theory to the quantities in the other, so both can be used to calculate the same physical quantities. But, since one of the theories is strongly coupled and the other theory is weakly coupled, this allows for calculating quantities in a strongly coupled regime of one theory by doing a calculation of the desired dual quantity in the weakly coupled regime of the other theory.

A famous example is the calculation of the viscosity to entropy density ratio for $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ in the limit of strong coupling [71], the result of the gravity calculation was found to be $\frac{\eta}{s} = \frac{1}{4\pi}$. This limit is a lower bound for the ratio in strongly coupled field theories [72], and actually it is subject to a few corrections due to the inclusion of higher derivative terms, modifying the value slightly [73, 74, 75, 76]. Measurements at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory, where two heavy atomic nuclei are collided in order to produce a quark-gluon plasma just above the confinement scale, place the viscosity to entropy density ratio of the QCD plasma (a strongly coupled system) within the range $1/4\pi \simeq 0.08 < \frac{\eta}{s} < 0.3$ [77, 78].

There are two viewpoints towards the AdS/CFT correspondence. The first involves string theory constructions. In this case, we have alternative descriptions of the same object (such as D3-branes) and the AdS/CFT correspondence follows from string theory. Here, we typically have a precise duality between two theories. In the second case, we think of the AdS/CFT correspondence as giving some effective description of the system (similar to effective fields theories like Landau-Ginzburg theory describing superconductors). In this case, either the effective description is good or not. More precisely, we could define a QFT by its gravitational dual description and then use the AdS/CFT correspondence to calculate all that we wish to know about the QFT without ever needing an explicit expression from the QFT.

It is clear that the AdS/CFT correspondence and its generalisations provide a perfect tool to study strongly coupled field theories through their gravity duals, in fact, it has been used to describe field theories which show behaviours similar to superconductors [79, 80], including a critical temperature and evidence for pair formation.

We will start this chapter by stating some facts about the basic building blocks of the AdS/CFT correspondence, conformal field theories and AdS-spaces. We consider the arguments that led Maldacena to formulate the correspondence between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills, and show that by comparing the parameters of the dual theories we indeed obtain a strong/weak duality. This will involve a comparison of the two pictures of D3-branes discussed in chapter 2, the closed string picture provided by the near-horizon geometry of D3-branes and the open string picture provided by the world-volume theory of D3-branes. Having introduced some relevant terminology and facts, we give a general formulation of the correspondence. We will consider a few of the possible checks of the correspondence one could perform, and as an example we will calculate the 2-point function of an operator in the field theory side, using a computation on the gravity side. We introduce Wilson loops, and discuss their relation with branes. Finally, we turn to a discussion of the papers that motivate the work in chapter 5.

For more detailed discussions on the AdS/CFT correspondence we refer to the original papers [64, 81, 82] and the reviews [83, 84, 70].

3.1 Conformal Field theories

In this section we will introduce some concepts related to conformal field theories. A review of conformal field theories can be found in the review [83], book [85], and lecture notes [86].

A conformal transformation from the point of a view of a D-dimensional spacetime is a change of coordinates that rescales the metric as

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}. \tag{3.2}$$

In particular this rescaling of the metric rescales length in spacetime but preserves angles and thus the conformal structure of spacetime. As a consequence lengths have no meaning and we have a scale invariant spacetime.

A simple example of scale invariance occurs in fractals. Zooming in or out on a fractal configuration will give exactly the same configuration, the length-scale at which we study the fractal system is irrelevant. Another (field theory) example of a scale invariant theory is the massless scalar field with only quartic interaction

$$S = -\int dx^4 \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{g}{4!}\phi^4. \tag{3.3}$$

Due to the absence of scales (dimensionful parameters) the action is invariant under a simultaneous rescaling of the spacetime coordinates ($x^\mu \to \lambda x^\mu$) and the field with a specific weight

$$\phi(x) \to \lambda^{-\Delta}\phi(\lambda x), \tag{3.4}$$

Δ is called the scaling dimension of the field, and here we have that $\Delta = 1$. The same theory would not be invariant if we added a mass term. Also, classical gauge theories in 4 dimensions are conformally invariant theories[1].

Conformal transformations

In $d > 2$ dimensions the conformal transformations consist of translations

$$\delta x^\mu = a^\mu, \tag{3.5}$$

[1] In particular, in the example of the correspondence that we will consider, the conformal field theory is a supersymmetric gauge theory in 4 dimensions.

with generators P_μ, and Lorentz transformations

$$\delta x^\mu = \lambda^{[\mu\nu]} x_\nu, \tag{3.6}$$

with generators $M_{\mu\nu} = -M_{\nu\mu}$. We note here the infinitesimal forms of the transformations. Both translations and Lorentz transformations are conformal transformations with $\Omega^2 = 1$. In addition, there are dilations

$$\delta x^\mu = \alpha x^\mu, \tag{3.7}$$

with generator D. They correspond to the transformation in example (3.3) with $\Omega^2 = e^{2\alpha}$. Finally, there are special conformal transformations

$$\delta x^\mu = x^2 c^\mu - 2x^\mu (c \cdot x) \tag{3.8}$$

with generators by K_μ.

The generators satisfy the following algebra (only the non-zero commutators are shown)

$$[M_{\mu\nu}, M_{\rho\sigma}] = 4\eta_{[\mu[\rho} M_{\sigma]\nu]},$$

$$[P_\mu, M_{\nu\rho}] = 2\eta_{\mu[\nu} P_{\rho]}, \qquad [K_\mu, M_{\nu\rho}] = 2\eta_{\mu[\nu} K_{\rho]},$$

$$[D, P_\mu] = P_\mu, \qquad [D, K_\mu] = -K_\mu,$$

$$[P_\mu, K_\nu] = 2\eta_{\mu\nu} D + M_{\mu\nu}.$$

The first line corresponds to the algebra of the Lorentz group $SO(1, D-1)$, the second line states that P_μ and K_μ are vectors and D is a scalar (since it commutes with the Lorentz generator), the third line establishes P_μ and K_μ as ladder operators for D, increasing and decreasing its eigenvalue respectively, and the last line states closure of the algebra. The number of generators (or equivalently parameters) is $(D+1)(D+2)/2$. This coincides with the number of generators of the group $SO(D, 2)$, and indeed we can make the isomorphism between the groups explicit by defining

$$M_{MN} = \begin{pmatrix} M_{\mu\nu} & \frac{1}{2}(P_\mu - K_\nu) & \frac{1}{2}(P_\mu + K_\nu) \\ -\frac{1}{2}(P_\mu - K_\nu) & 0 & -D \\ -\frac{1}{2}(P_\mu + K_\nu) & D & 0 \end{pmatrix}, \tag{3.9}$$

The generators M_{MN} satisfy the $SO(D, 2)$-algebra

$$[M_{MN}, M_{PQ}] = 4\eta_{[M[P} M_{Q]N]}, \tag{3.10}$$

with diagonal metric $\eta = \mathrm{diag}(-1, 1, \dots, 1, -1)$.

The conformal group can be enhanced to a supergroup by adding supercharges Q_I^α and the R-symmetry that rotates these supercharges. For consistency (closure of the algebra) one then also needs to add conformal supercharges S_I^α.

Representations

Since the Casimir operator P^2 no longer commutes with all the operators in the algebra (in particular with D), it makes little sense to use energy as an identifying label for particles. Mass and energy can be rescaled by a conformal transformation and so if a representation contains a state with a given energy, it will contain states with arbitrary energy obtained by applying a succession of dilation transformations. Instead of labelling our representations by their energy, we will consider operators and fields that are eigenfunctions of the dilation operator D with scaling (conformal) dimension Δ

$$\mathcal{O} \to \lambda^{-\Delta}\mathcal{O}. \tag{3.11}$$

For gauge theories, the physical objects are gauge invariant operators with a given conformal dimension. Since the generators P_μ and K_μ act as a raising and a lowering operator for the conformal dimension, we can use them to construct a tower of operators. We define a primary operator as an operator that is annihilated by K_μ. We can then construct a representation by acting with P_μ on a primary operator.

Constraints from conformal symmetry

Conformally invariant theories are constrained by the conformal symmetry. In particular the stress-energy tensor is required to be conserved (due to translations), symmetric (Lorentz transformations) and traceless (scale invariance).

In addition, the high degree of symmetry completely determines some physical quantities. Vacuum one-point functions vanish since a non-zero expectation value would break dilation invariance. Vacuum two- and three-point functions for operators $\mathcal{O}_i$ of scaling dimension Δ_i are fixed as well. Two-point functions have the form

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \begin{cases} \frac{C_i}{|x-y|^{2\Delta_i}} & \Delta_i = \Delta_j, \\ 0 & \Delta_i \neq \Delta_j, \end{cases} \tag{3.12}$$

while three-point functions are restricted to[2]

$$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|y-z|^{\Delta_j+\Delta_k-\Delta_i}|z-x|^{\Delta_k+\Delta_i-\Delta_j}}. \tag{3.13}$$

Conformal symmetry also strongly constrains the form of Green's functions involving the products of more operators, but it does not completely fix them.

In a quantum theory, conformal invariance is typically broken by the introduction of a renormalisation scale. However, conformally invariant quantum theories can arise in the infrared (IR) as fixed points in the renormalisation group flow, or, as finite theories when the beta function vanishes for all values of the coupling. A standard example of the latter is four-dimensional $\mathcal{N}=4$ SYM with gauge group $SU(N)$.

3.2 Anti-de Sitter spacetime

Anti-de Sitter spacetime is a maximally symmetric solution of the Einstein equations with a cosmological constant. The Einstein-Hilbert action in D dimensions with a cosmological constant is

$$S = \frac{1}{2\kappa_D^2}\int dx^D \sqrt{-g}\,(\mathcal{R}-\Lambda)\,, \tag{3.14}$$

and its equations of motion are

$$\mathcal{R}_{\mu\nu} - \frac{\mathcal{R}-\Lambda}{2} g_{\mu\nu} = 0. \tag{3.15}$$

We write the cosmological constant as

$$\Lambda = -\frac{(D-1)(D-2)}{R^2}, \tag{3.16}$$

where R has dimensions of length. The Ricci tensor is proportional to the metric

$$\mathcal{R}_{\mu\nu} = -\frac{D-1}{R^2} g_{\mu\nu}, \tag{3.17}$$

[2]Actually, in principle, we can use conformal symmetry to relate higher point functions to lower point functions. Thus if we were to give all the weights Δ_i and coefficients C_{ijk}, the theory is uniquely determined. This works well if the system is finite, so there are a finite number of operators and a finite number of C_{ijk}. In practice everything is usually infinite, so we cannot really explicitly determine everything.

identifying the space as an Einstein space. Furthermore, AdS is an example of a maximally symmetric spacetime. This means that the Riemann curvature tensor can be written in terms of the metric $g_{\mu\nu}$ as

$$\mathcal{R}_{\mu\nu\rho\sigma} = k\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right), \tag{3.18}$$

where k is a constant of dimension $1/(\text{length})^2$. For AdS this constant is negative, $k = -1/R^2$, and the overall scale of the spacetime is set by R. Note that equation (3.17) automatically follows from equation (3.18).

In Euclidean signature, the maximally symmetric solution with positive cosmological constant is the sphere S^D with isometry $SO(D+1)$ and the one with negative curvature is the hyperboloid H^D with isometry $SO(1,D)$. In Minkowskian signature, the maximally symmetric solution with $\Lambda > 0$ is called de-Sitter space (dS_D) and the one with $\Lambda < 0$ is Anti-de Sitter space (AdS_D).

A maximally symmetric spacetime of dimension D can be embedded as a hyperboloid in flat spacetime of dimension $D+1$. The metric $g_{\mu\nu}$ is then the induced metric. We will discuss this in some more detail next but first we provide some references. For standard expositions of AdS-spacetimes we refer to the books [4, 87] and reviews [70, 83, 84].

AdS_D from embedding in $\mathbb{R}^{D-1,2}$

Let us look at the embedding space with metric $\eta_{AB} = \text{diag}(-++\ldots+-)$ and Cartesian coordinates Y^A, $A = 0,\ldots,D$. The hyperboloid that defines AdS_D is the surface

$$Y^A\eta_{AB}Y^B = -(Y^0)^2 + \sum_{i=1}^{D-1}(Y^i)^2 - (Y^D)^2 = -R^2. \tag{3.19}$$

From this defining equation it is obvious that the isometry group of AdS_D is $SO(2,D-1)$, identical to the conformal group in $D-1$ dimensions. Given a parametrisation $Y^A(x^\mu)$ one finds the induced metric

$$g_{\mu\nu} = \frac{\partial Y^A}{\partial x^\mu}\frac{\partial Y^B}{\partial x^\nu}\eta_{AB}. \tag{3.20}$$

We will introduce several sets of coordinates that will be useful throughout the following sections and the rest of the thesis.

We start with a set of coordinates defined by

$$Y^0 = R\cosh\lambda\sin\tau, \quad Y^D = R\cosh\lambda\cos\tau, \quad Y^i = R\sinh\lambda\,\hat{x}_i, \tag{3.21}$$

with $i = 1, \dots, D-1$ and

$$\sum_{i=1}^{D-1} \hat{x}_i^2 = 1, \tag{3.22}$$

such that the $\hat{x}_i$ will describe the embedding of a $(D-2)$-sphere with unit radius. The induced metric reads

$$ds^2 = R^2 \left(-\cosh^2 \lambda d\tau^2 + d\lambda^2 + \sinh^2 \lambda \, d\Omega^2_{(D-2)} \right), \tag{3.23}$$

where $d\Omega^2_{(D-2)}$ is the line element of a $(D-2)$-sphere. This coordinate system is *global*, the entire hyperboloid is covered once as the radial variable varies in the range $0 \leq \lambda < \infty$, the angular variables cover S^{D-2} and the time-coordinate τ ranges between $0 \leq \tau \leq 2\pi R$. Note that time is periodic and therefore we have closed time-like curves. To avoid this we take the universal cover, where we allow $-\infty < \tau < +\infty$, and we also consider τ to be no longer related to the Y^A from before.[3] We shall always refer to AdS_D as the universal cover.

A second set of coordinates given by a $(D-1)$-dimensional Lorentz vector x^μ and a final coordinate u, defined in terms of the embedding coordinates as

$$\begin{aligned}
Y^0 &= Rux^0, \\
Y^i &= Rux^i, \qquad i = 1, \dots, D-2, \\
Y^{D-1} &= \frac{1}{2u} \left(-1 + u^2 (R^2 - x^2) \right), \\
Y^D &= \frac{1}{2u} \left(1 + u^2 (R^2 + x^2) \right), \\
x^2 &= -(x^0)^2 + \sum_i (x^i)^2.
\end{aligned} \tag{3.24}$$

The time coordinate x^0 and the spatial coordinates x^i range from $-\infty$ to $+\infty$ and $0 < u < \infty$. The induced metric is

$$ds^2 = R^2 \left[\frac{du^2}{u^2} + u^2 \left(-(dx^0)^2 + \sum_i (dx^i)^2 \right) \right]. \tag{3.25}$$

This metric has slices isomorphic to $(D-1)$-dimensional Minkowski spacetime, and for this reason these coordinates are called *Poincaré coordinates*. The $(D-1)$-dimensional space is foliated over u, and the Minkowski metric is multiplied by a warpfactor u^2 which means that an observer living on a Minkowski slice sees

[3] We consider the space to be described by τ and σ, and no longer by Y^A.

all lengths rescaled by a factor u according to its position in the u-direction. The plane at $u \to \infty$ is referred to as the boundary of AdS_D. Note that for $u \to \infty$ the metric blows up and mathematically speaking $u \to \infty$ is a conformal boundary, i.e. it is the conformally equivalent metric $d\tilde{s}^2 = ds^2/u^2$ that has a boundary $\mathbb{R}^{1,D-1}$ at $u \to \infty$. The plane $u = 0$ is instead a horizon, the timelike Killing vector ∂_{x^0} has zero norm at $u = 0$. The Poincaré coordinates only cover half of the hyperboloid and $u = 0$ does not correspond to a singularity as the metric can be extended through the horizon (using for example global coordinates).

There are other forms of the metric in Poincaré coordinates that are commonly used. They all differ by a redefinition of the last coordinate u. We mention a few examples. For the first example we define $u = \frac{\rho}{R}$ to obtain

$$ds^2 = R^2 \frac{d\rho^2}{\rho^2} + \frac{\rho^2}{R^2}\left(-(dx^0)^2 + \sum_i (dx^i)^2\right). \tag{3.26}$$

A second and third example are related by $u = 1/z = e^r$

$$ds^2 = R^2 \left(\frac{dz^2 + dx_\mu dx^\mu}{z^2}\right) = R^2 \left(dr^2 + e^{2r} dx_\mu dx^\mu\right). \tag{3.27}$$

The boundary is now at $z = 0$ and $r = \infty$, and the horizon at $z = \infty$ and $r = -\infty$, respectively.

To conclude this section, we mention a particular coordinate choice that we will use for an AdS_6-space. We solve the constraint (3.19) by introducing the coordinates

$$\begin{aligned}
Y^0 &= R\frac{\sinh(\psi)}{\sin(\rho)}\cosh(x), \\
Y^1 &= R\frac{\cosh(\psi)}{\sinh(\rho)}\cosh(x), \\
Y^i &= R\sinh(x)\hat{x}^i, \qquad i = 2, \dots 5, \qquad \sum_i \hat{x}^i = 1, \\
Y^6 &= R\coth(\rho)\cosh(x),
\end{aligned} \tag{3.28}$$

where the $\hat{x}^i$ describe a unit S^3. The induced metric is

$$\begin{aligned}
ds^2 &= R^2\left(\frac{\cosh^2(x)}{\sinh^2(\rho)}(d\rho^2 - d\psi^2) + \sinh^2(x) ds^2_{S^3} + dx^2\right) \\
&= R^2\left(\cosh^2(x) ds^2_{AdS_2} + \sinh^2(x) ds^2_{S^3} + dx^2\right),
\end{aligned} \tag{3.29}$$

which represents an $AdS_2 \times S^3$ slicing of the AdS_6-space, where the AdS_2- and S^3-parts in the metric are preceded by an x-dependent warpfactor. The AdS_2-factor is represented by the choice of embedding coordinates

$$Y^0_{\mathrm{AdS}_2} = R\tfrac{\sinh(\psi)}{\sin(\rho)}, \quad Y^1_{\mathrm{AdS}_2} = R\tfrac{\cosh(\psi)}{\sinh(\rho)}, \quad Y^2_{\mathrm{AdS}_2} = R\coth(\rho). \tag{3.30}$$

It is worth noting that at the boundary at $x \to \infty$ the metric approaches

$$ds^2_{\mathrm{bdry}} = \frac{R^2}{4}e^{2x}\left(ds^2_{AdS_2} + ds^2_{S^3}\right) + R^2 dx^2. \tag{3.31}$$

3.3 General aspects of the correspondence

In chapter 2, we saw how the low-energy limit $\alpha' \to 0$ of the open and closed string description of a stack of D3-branes leads two different systems. In section 2.4.2, we looked at D3-branes as hyperplanes in a flat background spacetime and discussed the gauge theories arising on the world-volume of the brane due to the presence of open strings in string theory. The low energy limit decouples this open string picture from the closed string background and it reduces to a gauge theory, in this case $\mathcal{N} = 4$ $SU(N)$ SYM. While in section 2.3.3, we considered the closed string picture as we discussed the backreaction of the branes on spacetime and the resulting geometry. The low energy limit consists of the type IIB supergravity description on $AdS_5 \times S^5$, the near-horizon geometry of the branes.

The AdS/CFT correspondence formulated in its original incarnation by Maldacena [64] conjectures the equivalence between the type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in four-dimensional Minkowski spacetime. Let us recall here some basics of the dictionary of this correspondence. The SYM coupling and the string coupling are related by

$$g^2_{\mathrm{YM}} = 2\pi g_s, \tag{3.32}$$

the radii of the $AdS_5 \times S^5$ factors in the metric are given by

$$R^4_{\mathrm{AdS}_5} \sim g_s N \alpha'^2, \tag{3.33}$$

and the rank of the gauge group is set by the five-form flux through the sphere

$$\int_{S^5} \star F_5 \sim N. \tag{3.34}$$

In terms of the effective gauge theory coupling (the 't Hooft coupling $\lambda \equiv g^2_{\mathrm{YM}} N$) we have

$$2\pi g_s = \frac{\lambda}{N}, \qquad \frac{R_{\mathrm{AdS}_5}}{l_s} \sim \lambda^{1/4}. \tag{3.35}$$

The two descriptions are useful to characterise different regimes. When $\lambda \ll 1$, or equivalently $R_{\mathrm{AdS}_5}/l_s \ll 1$, the field theory in the picture of D3-branes in a flat background is weakly coupled and a perturbative expansion of a non-abelian gauge theory in terms of the 't Hooft coupling λ is available. While in the gravity picture, where the string scale is large, the theory becomes strongly curved and one needs a string theory prescription. On the other hand, when $\lambda \gg 1$, or equivalently $R_{\mathrm{AdS}_5}/l_s \gg 1$, the gravity picture suppresses higher derivative terms and massive string modes. If we also suppress string loops by $g_s \to 0$, type IIB string theory reduces to a far more tractable theory, type IIB supergravity. On the gauge theory side this corresponds to a strong coupling ($\lambda \gg 1$) regime and the large N limit, where a perturbative approach is not available.

From the discussion above, it is clear that the AdS/CFT correspondence is a weak/strong coupling duality. When the gauge theory is weakly coupled the other side of the correspondence is strongly coupled (high curvature) and vice versa, making the correspondence very useful to study strongly coupled regimes in one theory using the weakly coupled regime of the other. The correspondence is commonly used to study complicated interacting systems at strong coupling by their dual treatment in terms of a theory of gravity in the weak curvature regime[4]. However, it can also be used in the opposite sense to try and gain some insight into the physics of strongly curved or even singular gravity through perturbative field theory methods.

A basic consistency check for the correspondence is to match the symmetries of the dual theories. For the example above, the matching of symmetries is as follows. $\mathcal{N}=4$ SYM is a conformal theory in $3+1$ dimensions. It has isometry group $SO(2,4)$ which matches with the isometries of AdS_5. The gauge theory is maximally supersymmetric, meaning it has 16 Poincaré supercharges. These are supplemented by 16 superconformal charges which make the theory invariant under the superconformal group $SU(2,2|4)$, with 32 conformal supercharges. The $AdS_5 \times S^5$ background has the same amount of supersymmetry, it is a maximally supersymmetric solution to type IIB supergravity, containing 32 supersymmetries. The supercharges of the gauge theory are rotated into each other by an $SU(4)$ R-symmetry. This symmetry is present in the string theory side in the form of the isometry group of S^5.

[4]I once attended a PhD school where a small course on the AdS/CFT correspondence was given. The course concluded with a rough estimate of the amount of papers related to proving the correspondence (10%), extending holography to different spacetimes and field theories (10%), implications of holography for gravity (5-10%), and applying the AdS/CFT correspondence to model strongly interacting QFTs (most of the papers).

UV/IR correspondence

An important aspect of the AdS/CFT dictionary is the identification of the radial position in the bulk with the energy scale of the boundary theory. As we saw in section 3.2, AdS_5 in Poincaré coordinates (3.27)

$$ds^2 = \frac{R^2_{\mathrm{AdS}_5}}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right), \tag{3.36}$$

has conformal boundary at $z = 0$ (obtained with the conformal rescaling $ds^2 \to z^2/R^2_{\mathrm{AdS}_5} ds^2$ in the limit $z \to 0$). The conformal boundary geometry corresponds to the background geometry where the dual field theory lives, which for this reason is referred to as the boundary theory. Similarly for any fixed z slice in (3.36), the geometry is the same as the four-dimensional boundary geometry, the only difference being the presence of the warpfactor $R^2_{\mathrm{AdS}_5}/z^2$. One can loosely think there is a copy of the boundary theory at any point along the radial direction. The theory at z is then related to the boundary theory by conformal rescaling

$$x^\mu \to \frac{z}{R_{\mathrm{AdS}_5}} x^\mu, \tag{3.37}$$

which means that a process with energy E_z taking place at z corresponds to a process with energy

$$E_{\mathrm{boundary}} = \frac{R_{\mathrm{AdS}_5}}{z} E_z \tag{3.38}$$

on the boundary theory. Seen the other way around, the scaling $x^\mu \to u x^\mu$ in the field theory, which connects a state of energy E to a state with energy E/u corresponds to the $SO(2,4)$ transformation $(z, x^\mu) \to (uz, ux^\mu)$ in the bulk which sends a point close to the boundary $z = 0$, towards the interior.

In summary, we can identify a process involving energies up to a scale μ in the boundary theory with a process that takes place at $z \sim \frac{1}{\mu}$ in the bulk. The boundary region of AdS corresponds to the UV regime of the field theory. Conversely, the physics far in the bulk interior is associated to the IR regime of the gauge theory. Notice that in a conformal field theory there is no IR cut-off, and excitations of arbitrarily low energy are present. This matches with the fact that the bulk geometry extends all the way to $z \to \infty$. The correspondence is holographic in the sense that the four x^μ coordinates are identified with the field theory ones, while the fifth coordinate z corresponds to the energy scale in the CFT, parametrizing the renormalisation group flow of the boundary theory.

This relation is referred to as the *UV/IR correspondence*, since the high (low) energy field theory regime is determined by the behavior of the bulk solution near the boundary (interior).

In this section we have described the realisation of the AdS/CFT correspondence in its best known form, the duality between type IIB String Theory on $AdS_5 \times S^5$ and the superconformal field theory on the world-volume of the branes, $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. Similar dualities hold between other theories. We have seen in section 2.3.3 that the near-horizon geometries of a stack of N M2-branes (M5-branes) correspond to a geometry $AdS_4 \times S^7$ ($AdS_7 \times S^4$). These geometries would be the gravity duals of a superconformal field theory on the world-volume of N M2-branes (M5-branes). For M2-branes the dual gauge theory is a three-dimensional $\mathcal{N} = 8$ superconformal field theory, while for M5-branes the dual gauge theory is a six-dimensional $\mathcal{N} = (2, 0)$ theory. Each time the symmetry group of the AdS factor matches the conformal symmetry group of the gauge theory, the R-symmetry group that rotates the charges in the dual gauge theory is the same group as the symmetry group of the sphere, and there is a match of the supersymmetries. There is a plethora of correspondences, involving all sorts of generalisations that not always deal with AdS-spaces or CFTs. We refer to the review [83] for an overview.

3.4 General formulation of the correspondence

So far we have discussed the statement of the conjecture, the dictionary between the parameters of the two theories, and the field theory interpretation of the bulk dimension. However, since the two theories are equivalent, we have to be more precise and describe the mapping between quantities in the bulk and in the boundary theory. This will be the topic of this section. The AdS/CFT correspondence in the form in which it was originally proposed [64] did not provide a detailed map between the quantities in both theories. Such a map was given in [81, 82], and relates fields in the gravity side to operators in the gauge theory side.

3.4.1 Field/operator correspondence

We will refer to the fields in the gravity theory as bulk fields, and we assume that their interaction is described by the String Theory action

$$S_{\mathrm{ST}}(g_{\mu\nu}, A_\mu, \phi, \ldots) \tag{3.39}$$

with an AdS_d vacuum. The fields in the CFT are referred to as boundary fields and we will call L_{CFT} the Lagrangian of this theory. The spectrum of the CFT is specified by a complete set of primary operators in the CFT.

We consider a field $\phi(x,z)$ in the bulk with the boundary condition

$$\phi(x,z)|_{z=0} \simeq \phi_0(x) z^{\Delta-d}, \tag{3.40}$$

where $\phi_0(x)$ denotes the boundary value.

The field $\phi(x,z)$ is associated to an operator $\mathcal{O}$ in the CFT with the same quantum numbers and they know about each other via boundary conditions. The relation between the field in the bulk and the operator on the boundary is made clear in the CFT point of view. The value of ϕ at the boundary (ϕ_0) acts as a source for the operator $\mathcal{O}$

$$L_{\mathrm{CFT}} + \int d^d x \phi_0(x)\mathcal{O}(x). \tag{3.41}$$

The partition function of the conformal field theory computed in presence of a set of classical sources $\phi_0(x)$, that are in one to one correspondence with the gauge invariant conformal operators $\mathcal{O}(x)$ is

$$Z_{\mathrm{CFT}}[\phi_0(x)] = \langle e^{\int \phi_0(x)\mathcal{O}(x)} \rangle_{\mathrm{CFT}} \tag{3.42}$$

A massive scalar field in AdS

We consider the Euclidean action for a massive scalar field

$$S^E_\phi = -\frac{1}{2}\int d^4x dz \sqrt{g}\left(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right), \tag{3.43}$$

in the background metric (3.27). The equation of motion reads

$$\partial_z\left(\frac{1}{z^3}\partial_z\phi\right) + \partial_\mu\left(\frac{1}{z^3}\partial^\mu\phi\right) = \frac{1}{z^5}m^2\phi. \tag{3.44}$$

Let us first consider the z behaviour and look at a mode independent of x^μ. The equation reduces to

$$\partial_z\left(\frac{1}{z^3}\partial_z\phi\right) = \frac{1}{z^5}m^2\phi. \tag{3.45}$$

which has two independent power-like solutions[5]

$$z^{4-\Delta}, \quad \text{and} \quad z^{\Delta}, \tag{3.46}$$

[5] For $\Delta = 2$, the independent power-like solutions presented here coincide. Instead we have the following independent solutions: z^2 (as expected) and the logarithmic solution $z^2 \log z$.

with

$$m^2 = \Delta(\Delta - 4). \tag{3.47}$$

Under a dilation $z \to \lambda z$ and $x^\mu \to \lambda x^\mu$, Δ corresponds to a scaling dimension for the field and will be identified with the conformal dimension of the dual operator $\mathcal{O}$. This means that the conformal dimension of the operator $\mathcal{O}$ is set by the mass of the dual scalar field by equation (3.47). Assuming that Δ is the larger of the two values Δ and $4 - \Delta$, we can write the solution as

$$\phi = \phi_1 z^{4-\Delta} + \phi_2 z^\Delta. \tag{3.48}$$

The coefficients ϕ_1 and ϕ_2 correspond to two linearly independent solutions of the second order equation of motion, and can be distinguished by the fact that the solution corresponding to ϕ_1 is not normalisable at the boundary ($z = 0$)

$$\int_0^{z_0} d^4x dz \sqrt{g} |\phi|^2 = \int_0^{z_0} z^{-5} |\phi_1|^2 z^{8-2\Delta} d^4x dz = \int_0^{z_0} z^{3-2\Delta} |\phi_1|^2 d^4x dz = \infty, \tag{3.49}$$

while the one corresponding to ϕ_2 is[6].

Reinstating the x^μ dependence modifies the previous behaviour to

$$\phi(z, x) \sim (\phi_1(x) + \mathcal{O}(z))\, z^{4-\Delta} + (\phi_2(x) + \mathcal{O}(z))\, z^\Delta, \tag{3.50}$$

where we can still identify the coefficients $\phi_1(x)$ and $\phi_2(x)$ of the two linearly independent solutions, which still grow as $z^{4-\Delta}$ and z^Δ with corrections depending on both z and x^μ.

At the boundary the leading term of a solution of the equation of motion can be singular if $\Delta > 4$ or vanishes if $\Delta < 4$. It approaches a constant only for the case $\Delta = 4$. In order to have a consistent prescription we need to impose that at the boundary ($z = 0$)

$$\phi(z, x) \to z^{4-\Delta} \phi_1(x). \tag{3.51}$$

This identifies $\phi_1(x)$ as the boundary value of our field, and also as the source of the dual operator $\mathcal{O}$. $\phi_2(x)$ will be determined by the regularity conditions at the center of AdS_5 and by imposing the equations of motion as a functional of $\phi_1(x)$. Once the value of $\phi_1(x)$ is specified, we have a unique regular solution that extends to all of AdS_5. Furthermore, the mode $\phi_2(x)$ determines the vacuum expectation value (vev) of the dual operator. Normally, the vev is not something we get to specify and indeed this is true here. The relation between the modes allows us to specify the source for the operator and then compute the corresponding vev.

[6] The cases $1 \leq \Delta \leq 3$ require special care. In these cases, one has to choose which mode is to be identified with the source. The different choices correspond to different field theory duals.

3.4.2 Statement of the correspondence

Having established a relation between fields in the bulk and operators in the boundary, we are now ready to make the fundamental identification between both theories that allows for the comparison of physical quantities. The statement of the correspondence is summarised in the formula

$$Z_{\mathrm{ST}}[\phi(x,z)|_{z=0}] = Z_{\mathrm{CFT}}[\phi_0(x)], \tag{3.52}$$

where Z_{ST} is the Euclidean string theory partition function of the bulk string theory. In the $\lambda \gg 1$, large N limit, the left-hand side of this equation can be evaluated in the saddle point approximation to be

$$Z_{\mathrm{ST}}[\phi(x,z)|_{z=0}] \quad \rightarrow \quad Z_{\mathrm{bulk}}[\phi(x,z)|_{z=0}] \simeq e^{-S^E_{\mathrm{sugra}}[\phi^E_c]}, \tag{3.53}$$

in terms of the on-shell Euclidean supergravity action S^E_{sugra}, i.e. the classical Euclidean action evaluated on the solution to its equations of motion ϕ^E_c subject to the boundary condition (3.40) and to appropriate regularity conditions in the interior of spacetime. Since the knowledge of $Z_{\mathrm{CFT}}[\phi_0(x)]$ for all possible sources of composite operators determines the CFT completely, equation (3.52) states the required equivalence between the CFT and the $(d+1)$-dimensional theory. Note that the equivalence is between an off-shell d-dimensional theory and an on-shell $(d+1)$-dimensional theory of gravity.

There is an important point that we have not mentioned so far. The partition function, as well as the generating functional of the CFT are divergent quantities. This is clear from the quantum field theory point of view, where it is known that the action suffers from UV divergences that need to be regularised and renormalised in order to give finite physical quantities. These UV divergences on the boundary correspond to IR divergencies in the bulk on-shell gravitational action, consistent with the UV/IR relation between the theories. The bulk suffers from infinite volume effects due to integration close to the boundary of AdS. In fact, in the (asymptotically) AdS geometry each point in the bulk is infinitely distant from the boundary $z = 0$, as reflected in the warp factor $1/z^2$ of the (asymptotic) metric. In order to have finite quantities on the two sides of (3.52) (and (3.54) further down), one needs to work with renormalised quantities. The interested reader can find more information on these renormalisation procedures in [88].

Connected n-point functions of the gauge theory are obtained through

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle = \left. \frac{\delta^n S^E_{\mathrm{sugra}}[\phi^E_c]}{\delta\phi_0(x_1)\dots\delta\phi_0(x_n)} \right|_{\phi_0=0} . \tag{3.54}$$

Each variation, which brings down an insertion $\mathcal{O}$, sends a particle in the bulk. Correlation functions have been considered in great detail in the context of the AdS/CFT correspondence [70]. They provide a way of checking the correspondence by comparing results from both theories. There is a problem in the sense that one is required to do a strong coupling computation on the CFT side, and so the cases where one can actually make a comparison of results are limited. The set of correlation functions that can be compared are the ones that satisfy non-renormalisation theorems, so that the strong coupling answer is a straightforward extrapolation of the weak coupling answer. Such correlation functions have been compared to the string theory results (so far) with great success. We will illustrate this by calculating a 2-point function using the prescription defined by equations (3.52) and (3.54). We will perform the calculation on the gravity side, expecting to reproduce the result (3.12), which we know must hold for conformal invariant theories.

3.4.3 Example calculation of a 2-point function

Our aim is to calculate the two-point function of an operator $\mathcal{O}$, dual to a bulk scalar field. We want to show that we indeed reproduce the result (3.12). Since we only need to take two functional derivatives of the Euclidean action to work out the two-point function, we can neglect interactions and work with the scalar field action (3.43). It is convenient to perform a Fourier decomposition of modes on $\mathbb{R}^4$

$$\phi(z,x) = \int \frac{d^4p}{(2\pi)^4}\phi_p(z)e^{ipx}. \tag{3.55}$$

The Fourier mode $\phi_p(z)$ has to satisfy the equation of motion

$$z^5\partial_z\left[z^{-3}\partial_z\phi_p(z)\right] - p^2z^2\phi_p(z) = m^2\phi_p(z) = \Delta(\Delta-4)\phi_p(z) \tag{3.56}$$

which, defining $\phi_p(z) = (pz)^2y(pz)$, reduces to a Bessel equation

$$(pz)^2\frac{d^2y}{d(pz)^2} + (pz)\frac{dy}{d(pz)} - ((pz)^2 + (\Delta-2)^2)y = 0, \tag{3.57}$$

whose general solution is $y(pz) = A_pI_{\Delta-2}(pz) + B_pK_{\Delta-2}(pz)$. The asymptotic behaviour of the Bessel functions is given by

$$I_a(x) \sim x^a + \ldots, \quad K_a(x) \sim \tfrac{1}{x^a}\left(1 + \ldots + a_ax^{2a} + c_ax^{2a}\log x + \ldots\right) \tag{3.58}$$

for $x \to 0$, and

$$I_a(x) \sim \frac{e^x}{\sqrt{x}}, \quad K_a(x) \sim \frac{e^{-x}}{\sqrt{x}} \tag{3.59}$$

for $x \to \infty$. Near the horizon we then have

$$\phi_p \sim B_p(1+\ldots) + A_p(z^{\Delta-2}+\ldots) \tag{3.60}$$

as expected from (3.50). $K_{\Delta-2}$ is the non-normalisable solution and $I_{\Delta-2}$ is the normalisable one. Since $I_{\Delta-2}(pz)$ is exponentially growing for large z, regularity of the solution in AdS_5 requires $A_p = 0$, and we are left with $K_{\Delta-2}(pz)$ which is exponentially small for large z.

In general computations in AdS_5, various quantities diverge for $z \to 0$ and it is convenient to introduce a cut off and impose boundary conditions at $z = \epsilon$ instead. At the end of the computation one sends ϵ to zero. This allows to keep track of divergent pieces of the effective action and it is a general prescription for computing correlation functions in AdS_5. Equation (3.51) sets the asymptotic value of the solution equal to $\phi_1 \epsilon^{\Delta-4}$. We impose

$$\phi_p(z=\epsilon) \equiv \phi^1_p \epsilon^{4-\Delta}, \tag{3.61}$$

such that the solution is

$$\phi_p(z) = \frac{(pz)^2 K_{\Delta-2}(pz)}{(p\epsilon)^2 K_{\Delta-2}(p\epsilon)} \phi^1_p \epsilon^{4-\Delta}. \tag{3.62}$$

We are now ready to compute the on-shell Euclidean action of the bulk. The computation of the on-shell Euclidean action can be simplified using a standard trick

$$S^E_\phi = -\frac{1}{2}\int_{\text{boundary}} d^4x\sqrt{g}\phi\partial^n\phi - \frac{1}{2}\int d^4x dz\sqrt{g}\phi(-\Box + m^2)\phi \tag{3.63}$$

the second term is zero on-shell, and the action reduces to a boundary contribution. In our case,

$$\begin{aligned}
S^E_\phi\big|_{\text{on-shell}} &= \frac{1}{2}\int_{z=\epsilon} d^4x\sqrt{g}g^{zz}\phi(x,z)\partial_z\phi(x,z) \\
&= \frac{R^3}{2}\int d^4x \left.\frac{\phi(x,z)\partial_z\phi(x,z)}{z^3}\right|_{z=\epsilon} \\
&= \frac{R^3}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4} e^{ipx} \int \frac{d^4p'}{(2\pi)^4} e^{ip'x} \left.\frac{\phi_p(z)\partial_z\phi_p(z)}{z^3}\right|_{z=\epsilon} \\
&\sim \int d^4x d^4p d^4p' \delta^{(4)}(p+p')\epsilon^{5-2\Delta}\phi^1_p\phi^1_{p'} \left.\frac{\partial_z\left[(pz)^2K_{\Delta-2}(pz)\right]}{(p\epsilon)^2K_{\Delta-2}(p\epsilon)}\right|_{z=\epsilon} \\
&\sim \int d^4x d^4p d^4p' \delta^{(4)}(p+p')\epsilon^{1-\Delta}p^{\Delta-4}\phi^1_p\phi^1_{p'}\, \partial_z\left[(pz)^2K_{\Delta-2}(pz)\right]\big|_{z=\epsilon},
\end{aligned} \tag{3.64}$$

where we have inserted the solution of the equations of motion, and in the last line we used that for small ϵ we have $(p\epsilon)^2 K_{\Delta-2}(p\epsilon) \sim (p\epsilon)^{4-\Delta}$. We now focus on the derivative term, it becomes

$$\begin{aligned}
&\partial_z \left[(pz)^2 K_{\Delta-2}(pz)\right]\Big|_{z=\epsilon} \\
&\quad = \partial_z \left[(pz)^{-(\Delta-4)} \ldots + a_{\Delta-2}(pz)^{\Delta} + c_{\Delta-2}(pz)^{\Delta}\log(pz) + \ldots\right]\Big|_{z=\epsilon} \\
&\quad \sim p(p\epsilon)^{3-\Delta} + \ldots + p(p\epsilon)^{\Delta-1} + p(p\epsilon)^{\Delta-1}\log(p\epsilon) + \ldots
\end{aligned} \tag{3.65}$$

where the final line is a schematic presentation of the terms. In the integral we then get terms

$$\sum_k \frac{1}{\epsilon^k}(\text{polynomial in p}) + p^{2\Delta-4}\log(p\epsilon) + p^{2\Delta-4} + \mathcal{O}(\epsilon) \tag{3.66}$$

We see that there are divergent terms in ϵ, terms going as $1/\epsilon^k$ and a logarithmic divergene $\log(p\epsilon)$. These divergent terms can be removed in a quantum field theory by adding suitable counterterms [88]. In the $\epsilon \to 0$ limit the relevant contribution is

$$\langle \mathcal{O}(p)\mathcal{O}(p')\rangle \sim \delta^{(4)}(p+p')p^{2\Delta-4}, \tag{3.67}$$

which after a Fourier transformation[7] back to coordinate space becomes

$$\langle \mathcal{O}(x)\mathcal{O}(x')\rangle \sim \frac{1}{|x-x'|^{2\Delta}}, \tag{3.69}$$

in agreement with CFT expectations for an operator of conformal dimension Δ, equation (3.12).

[7] The inverse Fourier transformation goes as

$$\begin{aligned}
\int d^4p d^4p' \delta^{(4)}(p+p') e^{-ipx-ip'y} p^{2\Delta-4} &\sim \int dp \int_{-1}^{1} d\xi e^{-ip|x-y|\xi} p^{2\Delta-1} \\
&\sim \int dp \int_{-1}^{1} d\xi e^{-ip|x-y|\xi} p^{2\Delta-1} \\
&\sim \int_{-1}^{1} \frac{1}{|x-y|^{2\Delta}\xi^{2\Delta}} d\xi \sim \frac{1}{|x-y|^{2\Delta}},
\end{aligned} \tag{3.68}$$

giving the desired result.

3.5 Wilson loops

Another interesting quantity to compare is the vacuum expectation value of a Wilson loop. This was first discussed in [89]. In the gauge theory, the Wilson loop is defined for a closed contour $\mathcal{C}$ and a representation R of the gauge group by the path ordered integral of the holonomy of the gauge field along $\mathcal{C}$

$$W_R(\mathcal{C}) = \mathrm{Tr}_R \mathcal{P} \exp \left[i \int_{\mathcal{C}} A^a_\mu \mathbf{T}^a dx^\mu \right], \tag{3.70}$$

where $\mathbf{T}^a$ are the generators in the representation R and $\mathcal{P}$ indicates the path ordering.

A Wilson loop can be interpreted as follows. Given a pure gauge theory, we introduce external massive sources (quarks) transforming in a representation R of the gauge group. The loop $\mathcal{C}$ corresponds to the path of a quark and antiquark from their creation to their annihilation and measures the free energy of this configuration. For a rectangular Wilson loop in Euclidean space with length L in space and height $T \to \infty$ in time,

$$W_R(\mathcal{C}) = e^{-TE_I(L)} \tag{3.71}$$

where $E_I(L)$ is the energy of a pair of quarks at distance L. The Wilson loop provides a natural tool for defining the gauge theory potential energy between a pair of test charges in a gauge theory.

We can define an analogous quantity in AdS. An external source is inserted at the boundary and we may attach it to a string. This is a very natural thing to do in the explicit realisations of the AdS/CFT correspondence where the gravitational background is embedded in a string vacuum. We are lead to consider a string whose endpoints lies on a contour on the boundary. The fundamental string is described by the Nambu-Goto action at the classical level with fermions put to zero. The Nambu-Goto action is proportional to the area of the world-sheet of the string

$$S \sim \int d\sigma d\tau \sqrt{\det \left(g_{MN} \partial_a X^M \partial_b X^N \right)}, \tag{3.72}$$

where g_{MN} is the background metric in string-frame, $X^M(\tau, \sigma)$ are the embedding coordinates, and, τ and σ are the coordinates on the string world-sheet. We can now define a very natural observable in AdS

$$-\log\langle W(\mathcal{C})\rangle = (\text{minimal surface area with boundary } \mathcal{C}), \tag{3.73}$$

giving a geometrical interpretation to the Wilson loop. This is then identified with the expectation value of some Wilson loop in the dual CFT.

The Wilson loop is a signal for confinement if it grows as the area of the loop $\mathcal{C}$. In a confining theory, external quarks have an energy which grows linearly with distance $E = m_q + m_{\bar{q}} + E_I$ with $E_I = \tau L$ since they are connected by a colour flux tube, or QCD string, with tension τ. It then follows that for a rectangular loop $W = e^{-TE_I} = e^{-\tau TL}$, and more generally

$$W(\mathcal{C}) \sim e^{-\tau A(\mathcal{C})}, \tag{3.74}$$

where $A(\mathcal{C})$ is the area of the loop, or equivalently the area of the world-sheet for a propagating string. In this picture, the quarks are considered as external non-dynamical sources (for example quarks with a very large mass) and $W(\mathcal{C})$ just captures the dynamics of the gauge fields in the theory.

In a flat spacetime, the surface of minimal area with rectangular $\mathcal{C}$ would lie entirely on the boundary, giving an obvious confining behaviour $S \sim LT$. Things are different in AdS, however. Let us take the coordinates (3.25), where the boundary is located at $u = \infty$. We see that the metric diverges on the boundary and so it is energetically favourable for the string to enter inside AdS, where the gravitational interaction is weaker.

Following [89], let's perform a small calculation for a time invariant configuration of two external sources separated by a distance L. We can choose coordinates $\tau = t$, $\sigma = x$, and we have that $u(\sigma) = u(x)$. Let us also, for simplicity, set $R = 1$. The Nambu-Goto action (3.72) becomes

$$S \sim \int_{-L/2}^{L/2} \int_0^T dtdx \sqrt{(\partial_x u)^2 + u^4} \sim T \int_{-L/2}^{L/2} dx \sqrt{(\partial_x u)^2 + u^4}. \tag{3.75}$$

Finding the minimal area is simply a classical exercise in Euler-Lagrange equations. Since the action does not depend on x explicitely the solution satisfies

$$\frac{u^4}{\sqrt{(\partial_x u)^2 + u^4}} = \text{constant}. \tag{3.76}$$

Calling u_0 the minimum value of u, which by symmetry is at $x = 0$, we have

$$\frac{u^4}{\sqrt{(\partial_x u)^2 + u^4}} = u_0^2. \tag{3.77}$$

Rewriting this equation then leads to a differential equation for u

$$u' = u^2 \sqrt{\frac{u^4}{u_0^4} - 1}, \tag{3.78}$$

which can be easily solved by

$$x = \int_0^x dx = \frac{1}{u_0} \int_1^{u/u_0} \frac{dy}{y^2 \sqrt{y^4 - 1}}, \tag{3.79}$$

where we defined $y = u/u_0$. Note that at this point we can obtain a relation between L and u_0. At the boundary $u = \infty$ and $x = L/2$ such that

$$\frac{L}{2} = \frac{1}{u_0}\int_1^\infty \frac{dy}{y^2\sqrt{y^4-1}} \sim \frac{1}{u_0}. \tag{3.80}$$

We can see from equation (3.80) that for large separation between the sources the turning point u_0 goes to the center of AdS. This has a very natural holographic interpretation: probing large distances in quantum field theory means probing the horizon. More generally, from $L \sim 1/u_0$, we see that the field theory UV computations $L \ll 1$ take contributions from the region with large u, whereas IR computations $L \gg 1$ get contributions from region with small u, in accordance with the interpretation of u as an energy scale and the UV/IR duality.

We can evaluate the action on the solution by plugging equation (3.78) into the action

$$S = 2u_0 T\int_1^\infty \frac{y^2 dy}{\sqrt{y^4-1}}. \tag{3.81}$$

This integral is linearly divergent. This is to be expected as we are really computing the energy of a pair of quarks, including their large renormalised self-energy $m_q + m_{\bar{q}} + E_I$. The energy of a single quark can be estimated by a long linear string from $u = \infty$ to $u = 0$. We are only interested in the potential energy of the quarks, and so we can subtract two linearly divergent contributions and obtain a finite result

$$S = 2Tu_0\int_1^\infty \left(\frac{y^2}{\sqrt{y^4-1}} - 1\right) dy \sim Tu_0 \sim \frac{T}{L} \tag{3.82}$$

Note that the area of the string does not grow linearly as the distance between the endpoints. This is consistent with the fact that the theory is conformal and not confining. By dimensional reasons in absence of dimensionful parameters, the potential energy should go like $1/L$. If we restore the factors of the AdS radius R and the tension τ of the string, the final result is $E_I \sim \tau R^2/L \sim (\lambda)^{1/2}/L$. Notice that the energy goes as $\lambda^{1/2}$ indicating that it is a strong coupling result. At weak coupling we would find $E \sim \frac{\lambda}{L}$.

We have described the bulk description of a Wilson loop in the fundamental representation of the gauge group. This was done in terms of a fundamental string propagating in the bulk and ending at the boundary of AdS along the curve $\mathcal{C}$.

It has been shown that all half-BPS[8] Wilson loop operators in $\mathcal{N} = 4$ SYM have a gravitational description in terms of D3-branes (or D5-branes) in $AdS_5 \times S^5$

[8]The terminology half-BPS indicates that the object under consideration conserves half of the supercharges of the background theory it is embedded in. The BPS-terminology

[90]. It turns out that supersymmetry restricts the half-BPS Wilson line to be a straight line, such that the Wilson line is solely determined by the choice of irreducible representation of $U(N)$. The choice of representation $U(N)$ can be summarised in a Young tableau. The data in this Young tableau can be precisely encoded in the AdS bulk description by a certain configuration of D5- or D3-branes. It can be shown that a configuration of D5-branes corresponds to a half-BPS Wilson line in an antisymmetric product representation, while a configuration of D3-branes corresponds to one in a symmetric representation. We will sketch this relation between brane configurations and Young tableaux.

Brane configurations and Young tableaux

For higher rank representations we have to consider multiple fundamental strings. These strings are identical particles and we can represent a configuration of identical particles by a Young tableau. In general, irreducible representations of groups are also represented by Young tableaux. Thus a Young tableaux provides a map from irreducible representations of groups to configurations of strings. In a Young tableau one can represent the symmetry or anti-symmetry under exchange of the particles as follows, boxes in a same line indicate that the representation is symmetric under the interchange of the corresponding particles, while boxes in the same column indicate anti-symmetry under the interchange of the corresponding particles. For example, the following configuration

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array} \quad , \tag{3.83}$$

is symmetric under the interchange of 1,2 and 3, as well as 4 and 5, but antisymmetric when interchanging 1, 4 and 6, or 2 and 5.

Let us consider the configuration in table 3.1 where we have a stack of N D3-branes as well as a single D3-brane parallel to this stack (D3') and a single fundamental string stretching between the stack and the single brane. This string introduces a $0+1$ dimensional defect on the stack of N D3-branes. This is a co-dimension 3 defect which corresponds to the Wilson line in the CFT. In [90, 91] it was argued that when we integrate out the degrees of freedom associated with the bulk D3-brane we can make the correspondence between bulk branes and the Wilson loop-operator explicit. Integrating out the degrees

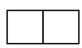
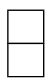
finds its origin in that supersymmetric objects, such as extremal black holes, satisfy some kind of supersymmetric bound, in the case of black holes this extremality constraint is the Bogomol'nyi–Prasad–Sommerfield (BPS) bound that relates the mass and charge of the black hole.

	0	1	2	3	4	5	6	7	8	9
D3	X	X	X	X						
D3’	X	X	X	X						
D5	X				X	X	X	X	X	X

Table 3.1: Configuration of a stack of N D3-branes, a parallel $D3$-brane (D3’) or an orthogonal $D5$-brane.

of freedom of this extra brane leaves us with some degrees of freedom localised on the $0+1$ dimensional defect that are in in the fundamental representation of $SU(N)$. This means that the representation for this string in terms of a Young tableau would be a single box.

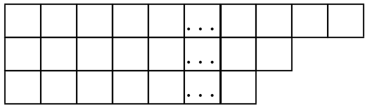
Now consider stretching not one but two fundamental strings between the stack and single brane (figure 3.1(a)). We could wonder how to represent the strings as a Young tableau. We would need two boxes, but would it be symmetric or anti-symmetric arrangement of the boxes,

 or  ?

Things get slightly more complicated here as there is a unique option in $0 + 1$ dimensions to choose how to quantise the local degrees of freedom. Choosing to quantise the degrees of freedom as bosons, leading to the symmetric representation turns out to correspond to a bulk D3-brane [90, 91]. Choosing to quantise the degrees of freedom as fermions and obtaining the anti-symmetric representations corresponds to stretching strings between a stack of N D3-branes and a single D5-brane with a configuration as in table 3.1 (figure 3.1(b)).

The construction above gives a D-brane interpretation for Young tableaux. The boxes are represented by strings stretched between a stack of N D3-branes and parallel D3-branes or orthogonal D5-branes to form symmetric or anti-symmetric representations respectively. We can also consider multiple extra branes. For example, consider a stack of D3-branes with 3 extra, non-coinciding D3-branes, and N_1 strings stretching between the stack and the first of these extra branes, N_2 strings between the stack and the second brane, and N_3 strings between the stack and the third brane. This configuration is depicted in figure 3.1(c). Considering these extra branes to be probe branes and letting $N_i \gg 3$

$(i = 1, 2, 3)$, this configuration can be interpreted as a Young tableau



where the length N_i $(i = 1, 2, 3)$ of the rows is much larger than the number of rows. Similarly one could replace the extra D3-branes by extra D5-branes to obtain Young tableaux for which the number of columns is much smaller than their length. Wilson lines of these configurations (totally symmetric or anti-symmetric representations) have been considered in [90]. To study Young tableaux of arbitrary size, we need backreacted solutions where we no longer consider the extra branes to be probe-branes.

We have established that we can obtain higher representations by considering multiple strings. Using a single string stretched between a stack of branes and an extra probe-brane will not be able to tell us if this probe-brane is a D3-brane or D5-brane. However, stretching multiple strings between the stack and the extra brane gives different representations depending on the extra brane being either a D3-brane or D5-brane, and will allow us to see a difference in the value for the Wilson-line. In the case of a single string we used the Nambu-Goto action to do a gravity calculation. However, the Nambu-Goto action only describes a single string. For multiple strings we use the DBI-action. This action encodes the number of strings M through the flux on the brane, and whether we use a symmetric or anti-symmetric representation is encoded in the choice of brane, D3 versus D5. Of course, the brane has to intersect the boundary on a time-like curve, defining the Wilson loop in the boundary field theory. The problem of calculating higher representation Wilson loops through a gravity dual reduces to finding an appropriate brane configuration that describes these Wilson loops. The brane configuration needs to intersect the boundary in the time-like curve related to the Wilson loop and needs to preserve the same symmetries in the gravity theory as the Wilson loop does in the gauge theory. Using the brane action instead of the Nambu-Goto action, we can then calculate the Wilson loop in the desired representation.

3.6 Five-dimensional gauge theories and the AdS/CFT correspondence

The version of the AdS/CFT correspondence that interests us in this thesis, arises in the context of 5-dimensional gauge theories [19]. The superconformal algebra in five dimensions is unique, and is given by $F(4;2) \times SU(2)$ with bosonic

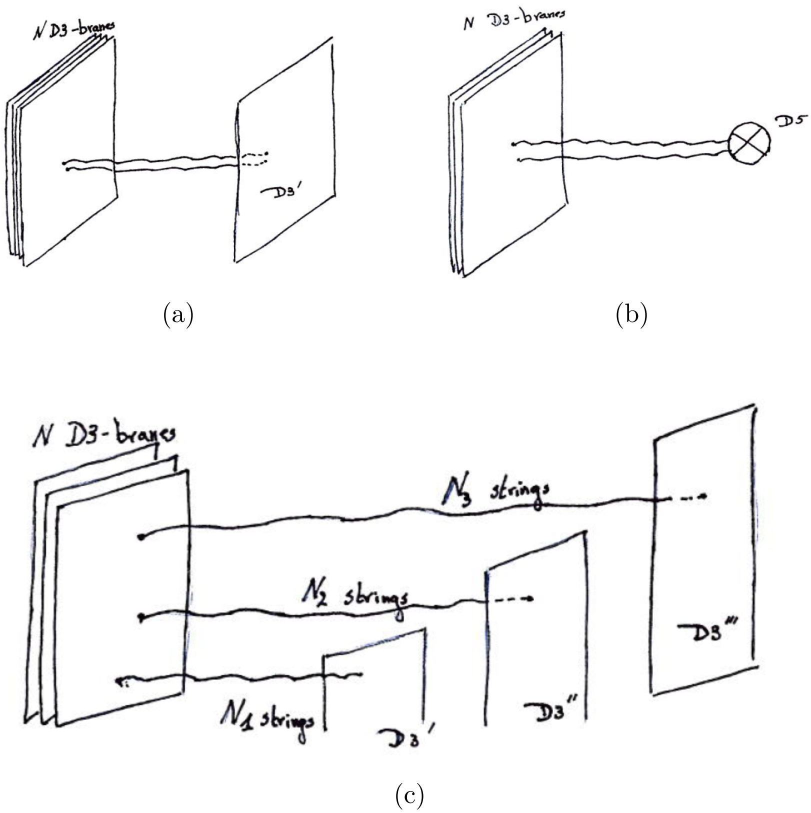


Figure 3.1: Sketch of strings stretching between a stack of D3 branes and (a) a parallel D3-brane, (b) an orthogonal D5-brane, (c) multiple parallel D3 branes.

subgroup $SO(2,5) \times SO(4)$. The dual gravity system is obtained as the near-horizon geometry of a D4-D8 brane system in massive type IIA supergravity, with a configuration of branes as in table 3.2. The near-horizon geometry is a fibration of AdS_6 over S^4 and has the isometry group $SO(2,5) \times SO(4)$. Normally, a four-sphere has isometry group $SO(5)$ but due to the warped product structure of the geometry this is reduced to $SO(4)$. The background 5-dimensional gauge theory arises as the low energy limit of the parallel D4-branes and D8-branes along with an O8 orientifold projection. The D8-branes and the O8-plane are a necessary ingredient to have a five-dimensional fixed point [62]. The resulting $AdS_6 \times S^4$ geometry is singular at the position of the O8-plane.

	0	1	2	3	4	5	6	7	8	9
D4	X		X	X	X	X				
D8/O8	X		X	X	X	X	X	X	X	X
F1	X	X								
D4’	X						X	X	X	X

Table 3.2: Intersecting D4/D8-brane configuration giving rise to a fibration of AdS_6 over S^4 in the near-horizon geometry. The second part shows the configuration of the perpendicular D4-brane (D4’) as well as the fundamental strings stretching between the branes.

Dimensionally reducing the massive type IIA supergravity on the warped S^4 gives $F(4)$ gauged supergravity in 6 dimensions. Introducing a Wilson line in the fundamental representation corresponds to introducing a fundamental string perpendicular to the D4/D8-brane system. Rank M symmetric representations arise from introducing an additional parallel D4-brane and stretching M fundamental strings between the D4-brane and the D4/D8-brane stack. Rank M anti-symmetric representations arise by introducing a perpendicular D4-brane and M fundamental strings. The brane configurations are summarised in table 3.2.

In general, the BPS-Wilson line reduces the superconformal symmetry[9] of the 5-dimensional gauge theory from $F(4;2) \times SU(2)$ to $D(2,1;2;1) \times SO(4)$, with bosonic subgroup $SO(1,2) \times SO(4) \times SO(4)$.

Intersecting M2/M5-branes

As discussed in section 2.5.3, an analogous system exists in M-theory which consists of intersecting M2/M5-branes with supergroup $D(2,1;\gamma;1) \times D(2,1;\gamma;1)$. In fact, the M-theory system is the M-theory uplift of the above configuration after removing the D8-branes and O8-plane. In this case the 5-dimensional gauge theory of the D4-branes becomes a 6-dimensional CFT at the UV fixed point and the Wilson line operators in the gauge theory become surface operators in the 6-dimensional CFT. As discussed in section 2.5.3, special values of $\gamma = 1, -1/2, -2$ correspond to cases of enhanced symmetry. These special values have a nice interpretation in terms of brane configurations, however, in general the precise relation between γ and the brane configuration is unclear. We consider first the special values $\gamma = -1/2, -2$. In this case we have $D(2,1;-1/2,0) \times D(2,1;-1/2;0) \subset OSp(8^*|4)$ and

[9] For supergroups, we follow the conventions of [92].

$D(2,1;-2;0) \times D(2,1;-2;0) \subset OSp(8^*|4)$. The extended symmetry $OSp(8^*|4)$ corresponds to having only a single stack of M5-branes and is realised as the superisometry group of the near-horizon geometry $AdS_7 \times S^4$. It is conjectured that the dual field theory is given by the 6d $(2,0)$ SCFT with $OSp(8^*|4)$ symmetry, although no completely satisfactory construction of this theory exists. It is believed that this theory admits supersymmetric surface operators (also known as the self-dual string), which preserve half of the supersymmetries and break the symmetry to $OSp(4^*|2) \oplus OSp(4^*|2)$. These operators arise from introducing the M2-branes of table 2.2. A single M2-brane produces a surface operator in the fundamental representation of the gauge group, while multiple M2-branes produce a surface operator in a higher rank representation. Dual supergravity solutions describing the surface operators have been constructed in [93].

For the special value $\gamma = 1$, the extended symmetry is $OSp(8|4, \mathbf{R})$ and corresponds to having a stack of M2-branes, whose near-horizon geometry is $AdS_4 \times S^7$. The dual field theory is given by a 3d $\mathcal{N} = 8$ SCFT. A Lagrangian description of this theory is given by ABJM theory, although only $\mathcal{N} = 6$ supersymmetry is manifest in the Lagrangian. Consider now the brane configuration of table 2.2. Naively, the intersecting M5-branes give rise to a 2d defect preserving half of the supersymmetries, although such a superconformal defect has yet to be constructed either directly in the field theory or in the dual supergravity solutions. Solutions corresponding to the effective theory on the M2-branes were found, these correspond to the Janus solutions of [94, 95]

Instead of inserting the M5-branes as a defect, we can also consider the case where the M2-branes end on the M5-branes. In the dual field theory, this corresponds to putting ABJM theory on a half-space with supersymmetric boundary conditions, possibly coupled to 2d degrees of freedom localised on the boundary. Such supersymmetric boundary conditions were studied in [96, 97, 98] and dual supergravity solutions were constructed in [99], although a precise holographic dictionary has yet to be constructed. Since the M2-branes end on the M5-branes, the supersymmetry does not necessarily need to be a subgroup of $OSp(8|4, \mathbf{R})$, as there is no way to remove the M5-branes and recover the full M2-brane theory. Indeed the solutions of [99] have $D(2,1;\gamma,0) \times D(2,1;\gamma,0)$ symmetry with $\gamma < 0$.

Intersecting F1/D4/D8-branes

We have discussed the relation of these configurations of M2/M5-branes to configuration of F1/D4-branes in section 2.5.3. A natural question is whether there are analogous solutions after introducing the D8-branes and possibly the O8-plane as in section 2.5.4.

For the special values $\gamma = -1/2, -2$, the extended symmetry $F(4;2) \times SO(3)$ corresponds to a single stack of D4-branes in addition to the D8-branes/O8-plane. The low energy field theory consists of a 5-dimensional $\mathcal{N} = 2$ gauge theory with gauge group $USp(2N)$, a single antisymmetric hypermultiplet and N_f fundamental hypermultiplets, where there are $2N$ D4-branes and N_f D8-branes.[10] It is believed that this theory is ultraviolet complete with a 5d $\mathcal{N} = 2$ ultraviolet superconformal fixed point. Field theory arguments for this conjecture are given in [62, 100, 101]. Dual solutions of massive IIA supergravity have been constructed in [63, 19], which provide further evidence for the existence of the ultraviolet fixed point.

We now consider what happens when we introduce the fundamental strings. Introducing the fundamental string reduces the supersymmetry by half and in the field theory corresponds to introducing a half-BPS Wilson line. The superconformal symmetry is broken to $OSp(4^*|2) \subset F(4;2)$, while the $SO(3)$ bosonic symmetry remains unbroken, since the Wilson line is neutral. To see this, note that a time-like Wilson line preserves an $SO(4) = SO(3) \times SO(3)$ rotational symmetry and translations in time, which in the conformal limit are enhanced to $SO(1,2)$. This gives a full bosonic symmetry $SO(1,2) \times SO(4) \times SO(4)$, which can be seen in table 3.2. The dual supergravity solutions have yet to be constructed but fall within the anstaz studied in chapter 5.

The special value of $\gamma = 1$ corresponds to the extended symmetry $OSp(8|2, \mathbf{R})$. This symmetry arises as the superconformal symmetry associated with fundamental strings ending on or intersecting the D8-branes/O8-plane. This can be seen in table 3.2, where the $SO(8)$ symmetry acts on the directions 2 through 9, while $Sp(2, \mathbf{R}) \sim SO(1,2)$ arises as the superconformal group of time translations. The putative dual field theory would correspond to a 2-dimensional boundary or defect CFT and would be the massive IIA analogue of the M2-brane near-horizon geometry. Additionally, we may include the D0-branes without loss of symmetry. In this case, the fundamental strings can be stretched between the D8-branes and the D0-branes.

With both the task of finding backreacted geometries describing these Wilson lines, as well as determining the existence of a decoupling limit for $\gamma = 1$ in mind, we study general solutions of massive IIA supergravity with $D(2,1;\gamma;1) \times SO(4)$ symmetry in chapter 5.

[10] The antisymmetric hypermultiplets arise from D4-D4 strings stretched across the O8-plane, while the N_f fundamental hypermultiplets come from D4-D8 strings.

Chapter 4

D3-brane world-volume theories

Supergravities can be considered as gauge theories of the super-Poincaré group, which for some combinations of dimension D and supersymmetry $\mathcal{N}$ is a subgroup of a bigger supergroup, the superconformal group. The superconformal group is much more restrictive than the super-Poincaré group and so it might be useful to construct theories that are invariant under local superconformal symmetry and then extract from these theories the super-Poincaré invariant ones. This methodology was dubbed *superconformal tensor calculus*, and it was initially developed for minimal $D = 4$ supergravity in [7, 8, 10, 9]. Extensions to other theories are available for several theories with no more than 16 supersymmetries. The interesting advantage of this method is that the super-Poincaré theories obtained through the superconformal method are off-shell theories. This means that their supersymmetry algebra closes on the fields without using the equations of motion, as opposed to on-shell theories where this is a necessary evil. Some of the fields that are needed for the superconformal formulation remain present in the super-Poincaré theory as auxiliary fields, whose field equations are algebraic. Without superconformal methods it is often very difficult to find the auxiliary fields needed to make the algebra close off-shell.

The superconformal method is based on the gauge equivalence program [6]. This program requires the use of more symmetry for the construction of a theory than one wants in the final theory. The extra symmetry is used as a tool and is no longer present in the final theory. To start we define the symmetry group G_f we want in our final theory. Together with the extra symmetries we want to use throughout the procedure, these symmetries form a (super)group

G_i. We introduce a gauge field for every generator in G_i. These fields form a representation of G_i, which we will call the gauge multiplet.[1] In general this representation is reducible and one can impose curvature constraints to obtain irreducibility of the *gauge multiplet*. One then introduces compensator multiplets. These are matter multiplets defined in the background of the gauge multiplet fields. These compensators are then used to construct actions that are invariant under G_i using superconformal tensor calculus in the background of the gauge multiplet. This construction boils down to a construction of (a combination) of terms that transform into a total derivative under G_i, and thus they can be used as a Lagrangian in the action of a theory with symmetry G_i (provided the term has the right dimensions). In the final step G_i is broken to G_f by fixing the values of (some of) the fields. In general these gauge fixings introduce dependencies among the symmetry parameters (called *decomposition laws*) and deform the transformation laws. One ends up with a theory that is only invariant under G_f. A fairly simple example is the construction of the Einstein-Hilbert action from a locally conformal action of a scalar field [10]. The initial symmetry group is the conformal group, whereas the final one only has Poincaré symmetry. The scalar field represents the compensator, and gauge fixing it will break dilation symmetry. Special conformal transformations are broken by a gauge fixing within the Weyl multiplet and one is left with the Einstein-Hilbert action. For an excellent description of the workings of this process we refer to [4], where the construction of four-dimensional $\mathcal{N} = 1, 2$ (gauged) supergravity theories using these superconformal methods is worked out in great detail.

One can deform a basic supergravity theory, determining only kinetic terms of the specified matter fields, in several ways. One can consider gauging (part of) the internal symmetries which often leads to a potential for the scalar fields. This process is determined by the embedding tensor formalism and unlike for the basic theories, there is not a full classification of possible gauged theories yet [102, 103, 104]. We will not be interested in deformations by gauging internal symmetries here, but we will consider a different kind of deformation, namely, the addition of higher derivative terms to the Lagrangian. Higher derivative terms are terms that contain more than two derivatives if the term is purely bosonic or more than one derivative if the term also contains fermions. Higher derivative terms generically give rise to nonlinear terms in the field equations, and incorporating them in a theory requires the introduction of a dimensionful parameter such that the higher derivative term can have the right mass dimension.

In the early days of supergravity it was hoped that the supergravity theories were

[1] In cases where G_i is the superconformal group this gauge multiplet is known as the Weyl multiplet.

renormalisable. However, rather quickly it was realised that even though the UV divergencies were softened by the additional constraints provided by local supersymmetry, supergravity theories were still non-renormalisable. Nowadays, supergravity theories are considered as effective field theories, describing the low energy physics of a full string theory. The divergent loop integrals no longer pose any problems, since they can be cut off at an energy scale where supergravity loses its validitiy and at which stringy effects become important. Higher derivative terms appear as counterterms in the quantum effective action of supergravities, and the corresponding dimensionful parameter is the inverse of the cutoff scale in momentum space. The expansion of a (compactified) string theory Lagrangian in powers of small α' (low energies) amounts to a supergravity Lagrangian at lowest order with extra higher derivative terms at higher orders of α'. These extra terms are important to make the connection between the microscopic description of string theory and the macroscopic supergravity description. In light of the AdS/CFT correspondence these higher derivative corrections parametrised by α', introduce order $\mathcal{O}\left(\frac{1}{N}\right)$ effects in the dual gauge theory. They can have significant consequences and are responsible for the adjustment of the famous shear viscosity over entropy bound mentioned in chapter 3 [73, 74, 75]. There is no complete catalogue of allowed higher derivative deformations for a particular $\mathcal{N}$ and D, showing the importance of studying the possibilities of higher derivative deformations of supergravity theories.

In [5] new ways for constructing supersymmetric higher derivative invariants were investigated in supergravity settings where there are no known off-shell formulations. In particular, the action and supersymmetry transformation rules of the $D = 4$, $\mathcal{N} = 4$ Maxwell multiplet were deformed with higher derivative terms. This was done in such a way that at each order of the deformation the theory has 16 deformed Maxwell multiplet supersymmetries and 16 Volkov-Akulov (VA) type non-linear supersymmetries. The results were obtained by studying the world-volume theory of the gauge-fixed D3-superbrane in a ten-dimensional flat background.

The world-volume theory of a brane embedded in a certain background inherits some of the symmetry of the background (depending on the embedding). We already saw how branes can preserve some of the supersymmetry of the background in section 2.4.1. Depending on the symmetry of the background and the embedding we can obtain different world-volume theories. It is known that the world-volume theory of a brane in its own near-horizon background is a superconformal theory [105, 106, 107]. Consider as an example a D3-brane in its own near-horizon background. Studying the world-volume theory, we saw in section 2.4.2 that it is indeed a superconformal theory.

If we want to extend the rigid supersymmetric results of [5] to supergravity by using superconformal methods [6, 7, 8, 9, 10, 4], we need to determine the superconformal transformation rules of the deformed Maxwell multiplet. These can be obtained by studying the D3 brane in an $AdS_5 \times S^5$ background. As a first step, it is an interesting question to find out how the S-supersymmetry of the conformal Maxwell multiplet relates with the VA supersymmetry of the $16 + 16$ deformed Maxwell multiplet. Once the deformed superconformal Maxwell multiplet is constructed, it could then be used as a compensator in the superconformal construction of $D = 4$, $\mathcal{N} = 4$ supergravity.

The topic of this chapter will be the relation between the S-supersymmetry of the conformal Maxwell multiplet and the VA supersymmetry of the $16 + 16$ deformed Maxwell multiplet, and presents the work of [11]. Since both cases arise from the embedding of a D3-brane in a background, it is natural to look at the relation between the backgrounds. One can obtain the Minkowski background as a large R limit of the $AdS_5 \times S^5$ background, where R is the scale factor of the latter. This already suggests a possible relation between the world-volume theories by carrying over this limiting procedure.

This chapter is organised as follows. At the end of this introduction we present the conventions we used for the exposition of this chapter. There are a lot of indices going around here and it is convenient to have a set of conventions handy. In section 4.1 we introduce the tools of the trade for this chapter. We aim to derive the world-volume transformation rules in a notation that allows for an easy comparison between the different backgrounds such that a relation might be deduced. The framework wherein we will derive these relations is provided by Coset superspaces, and form the topic of this section. We will then apply this machinery and derive the background vielbein and isometries for a flat 10-dimensional superspace as well as an $AdS_5 \times S^5$ superspace. We want to embed a D3-brane in these backgrounds and then look at the symmetries that are induced on the world-volume by these background geometries. A discussion on the symmetries and embedding, as well as the presentation of the resulting world-volume symmetries in both backgrounds is provided in section 4.2. Section 4.3 provides the comparison of both sets of transformations. We specify a limiting procedure to compare both backgrounds and establish a relation between the transformation rules. Finally, section 4.4 presents some conclusions and we formulate a proposal for the construction of higher derivative invariants for the D3-brane in the Minkowski background.

In the interest of clarity, some of the finer details and calculations performed to obtain the results in this chapter have been relegated to some of the Appendices. They will be referred to where needed in the text.

Notational conventions

We use the following conventions for indices

$\bar{M}$	label for the coset generators in $\mathbf{K}$
$\bar{I}$	label for the stability group generators in $\mathbf{H}$
$\Lambda = \{\bar{M}, \bar{I}\}$	label for generators of the superalgebra $\mathbf{G} = \mathbf{K} \oplus \mathbf{H}$
$\mathcal{A}$	label for the collection of bosonic generators in $\mathbf{G}$
$\tilde{a}, \tilde{b}, \tilde{c} = 0, \ldots, 4$	AdS_5 tangent space index
$a, b, c = 0, \ldots, 3$	part of the AdS_5 tangent space index such that $\tilde{a} = \{a, 4\}$
$a', b', c' = 5, \ldots, 9$	S^5 tangent space index
$A, B, C = 0, \ldots, 9$	10D tangent space index such that $A = \{\tilde{a}, a'\}$
$\tilde{m}, \tilde{n}, \tilde{p} = 0, \ldots, 4$	5D spacetime index, associated with the AdS_5 space
$m, n, p = 0, \ldots, 3$	part of the 5D spacetime index such that $\tilde{m} = \{m, 4\}$
$m', n', p' = 5, \ldots, 9$	5D spacetime index, associated with the S^5 space
$M, N, P = 0, \ldots, 9$	10D spacetime index such that $\{M = \tilde{m}, m'\}$
$\alpha, \beta, \gamma = 1, \ldots, 4$	$so(2,4)$ spinor index projected on RH chiral subspace (AdS_5)
$i, j, k = 1, \ldots, 4$	$so(6)$ spinor index projected on RH chiral subspace (S^5)
$\hat{\alpha}, \hat{\beta}, \hat{\gamma} = 1, \ldots, 32$	$D = 10$ Majorana-Weyl spinor index
$I, J, K = 1, \ldots 4$	$so(6)$ spinor index
$\mu = 0, \ldots 3$	coordinate index of the world-volume of the D3-brane

4.1 Coset superspaces

In this section we briefly recap the formalism of Cartan forms on coset superspaces [108, 109, 110, 111]. We then use this formalism to write down the superisometries of Minkowski superspace and $AdS_5 \times S^5$ superspace.

We consider the coset manifold G/H, where G is a supergroup and $H \subset G$ is a subgroup. Each coset is represented by a coset representative $\mathcal{G}(Z)$, labelled by super-coordinates $Z^M = \{X^M, \theta^\alpha\}$. Left-invariant Cartan 1-forms are defined as

$$L(Z) \equiv \mathcal{G}(Z)^{-1} d\mathcal{G}(Z). \tag{4.1}$$

Since $L(Z)$ is a group element close to the identity it is a $\mathbf{G}$ valued super 1-form

$$L(Z) = L^\Lambda \mathbf{T}_\Lambda = dZ^M L_M^\Lambda \mathbf{T}_\Lambda, \tag{4.2}$$

where $\mathbf{T}_\Lambda$ are the generators of the superalgebra $\mathbf{G}$ associated to G.
We consider two decompositions which will be useful. First there is the coset decomposition of the algebra, defined by $\mathbf{G} = \mathbf{K} \oplus \mathbf{H}$ where $\mathbf{H}$ is the Lie-algebra associated with the stability group H of G, $\mathbf{G}$ is the Lie-algebra of G, and $\mathbf{K}$ collects the coset generators. We introduce the split of labels $\Lambda = (\bar{M}, \bar{I})$, where $\bar{M}$ are the directions in $\mathbf{K}$ and $\bar{I}$ are the directions in $\mathbf{H}$. The second decomposition that we consider is a boson-fermion split of the algebra $\mathbf{G} = \mathbf{B} \oplus \mathbf{F}$, where $\mathbf{B}$ contains the bosonic generators B_A and $\mathbf{F}$ the fermionic generators F_α, and define the split of a $\mathbf{G}$-valued object A as

$$A = A^\Lambda \mathbf{T}_\Lambda = A^{\mathbf{B}} + A^{\mathbf{F}} = A^{\mathcal{A}} \mathbf{T}_{\mathcal{A}} + A^\alpha \mathbf{F}_\alpha. \tag{4.3}$$

For the coset representative we choose the parametrisation $\mathcal{G}(Z) = g(X) e^\Theta$, where $g(X)$ represents the bosonic coset representative of the coset space and

$$\Theta = \Theta^\alpha \mathbf{F}_\alpha = \theta^{\dot{\alpha}} e_{\dot{\alpha}}{}^\alpha(X) \mathbf{F}_\alpha, \tag{4.4}$$

where $e_{\dot{\alpha}}{}^\alpha(X)$ determines the choice of fermionic coordinates.
In [108, 109] the complete geometric superfields $L(Z)$ and Killing superfields $\Sigma(Z)$ for a generic maximally supersymmetric superspace were constructed independent of the choice of coordinates (to all orders in θ), we repeat their results here. The Cartan 1-forms and the parameters Σ (defining the superisometries) are split as follows

$$\begin{aligned} L =& E + \Omega = E^{\bar{M}} \mathbf{K}_{\bar{M}} + \Omega^{\bar{I}} \mathbf{H}_{\bar{I}}, \\ \Sigma =& \hat{\Xi} + \hat{\Lambda} = \hat{\Xi}^{\bar{M}} \mathbf{K}_{\bar{M}} + \hat{\Lambda}^{\bar{I}} \mathbf{H}_{\bar{I}}, \end{aligned} \tag{4.5}$$

where the parameters $\{\hat{\Xi}^{\bar{M}}, \hat{\Lambda}^{\bar{I}}\}$ are defined in terms of the superisometries $\{\Xi^M, \Lambda^{\bar{I}}\}$ as

$$\hat{\Xi}^{\bar{M}} = \Xi^M E_M^{\bar{M}}, \qquad \hat{\Lambda}^{\bar{I}} = \Lambda^{\bar{I}} + \Xi^M \Omega_M^{\bar{I}}. \tag{4.6}$$

We will be interested in maximally supersymmetric superspaces where $\mathbf{F} \subset \mathbf{K}$ or $\mathbf{F} \cap \mathbf{H} = 0$. Both the Minkowski and $AdS_5 \times S^5$ backgrounds fall in this category. The bosonic generators are split into $\mathbf{B} = \{\mathbf{P}_a, \mathbf{M}_i\}$, with $\mathbf{P}_a \in \mathbf{K}$ and $\mathbf{M}_i \in \mathbf{H}$. We also consider the gravitino $L_0^{\mathbf{F}} = L^{\mathbf{F}}(\theta = 0)$ to be vanishing. Splitting $\bar{M}$ into bosonic a and fermionic α, the supervielbein is given by [108, 109]

$$E_M{}^{\bar{M}} = \begin{pmatrix} e_\mu{}^b(X) & 0 \\ 0 & e_{\dot{\alpha}}{}^\beta(X) \end{pmatrix} \begin{pmatrix} \delta_b{}^a + (UA\mathcal{Y})_b{}^a & (UAB)_b{}^\alpha \\ (A\mathcal{Y})_\beta{}^a & (AB)_\beta{}^\alpha \end{pmatrix}, \tag{4.7}$$

where

$$\begin{aligned} A_\alpha{}^\beta =& 2\left(\frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2}\right)_\alpha{}^\beta, \qquad B_\alpha{}^\beta = (\mathcal{M} \coth \mathcal{M}/2)_\alpha{}^\beta, \\ \mathcal{Y}_\alpha{}^a =& -\Theta^\delta f_{\delta\alpha}{}^a, \qquad \mathcal{M}_\alpha{}^\beta = f_{\alpha\gamma}{}^A \Theta^\gamma \Theta^\delta f_{\delta A}{}^\beta, \end{aligned} \tag{4.8}$$

and $f_{\Lambda\Sigma}^{\Gamma}$ are the structure constants of the algebra $\mathbf{G}$. The $e_\mu{}^a$ form the vielbein of the bosonic space and $e_{\dot{\alpha}}{}^\alpha$ is the matrix introduced in the boson-fermion parametrisation of the coset representative. The matrix $U_a{}^\alpha$ and Θ^α depend on the spinorial gauge choice $e_{\dot{\alpha}}{}^\beta$

$$U_a{}^\alpha = e_a{}^\mu \left[\theta^{\dot{\alpha}} \partial_\mu e_{\dot{\alpha}}{}^\alpha + (L_0^A)_\mu \theta^{\dot{\beta}} e_{\dot{\beta}}{}^\beta f_{A\beta}{}^\alpha\right]. \tag{4.9}$$

The superisometries, $\Sigma(Z) = \mathcal{G}^{-1}(Z)\Upsilon_0\mathcal{G}(Z)$, in general are determined completely in terms of the $\theta = 0$ Killing superfields Σ_0^Λ, which we denote here by

$$\Sigma_0^\Lambda T_\Lambda = \tilde{\xi}^a \mathbf{P}_a + \tilde{\epsilon}^\alpha \mathbf{F}_\alpha + \tilde{l}^i \mathbf{M}_i, \tag{4.10}$$

where

$$\tilde{\xi}^a = \xi^\mu e_\mu{}^a, \qquad \tilde{\epsilon}^\alpha = \epsilon^{\dot{\alpha}} e_{\dot{\alpha}}{}^\alpha, \qquad \tilde{l}^i = l^i + \xi^\mu \omega_\mu{}^i. \tag{4.11}$$

In terms of the structure constants of $\mathbf{G}$, one can show [108, 109] that the superisometries are

$$\begin{aligned} \Xi^\mu =& \xi^\mu + \tilde{\epsilon}^\beta (\mathcal{M}^{-1} \tanh \mathcal{M}/2)_\beta{}^\alpha \mathcal{Y}_\alpha{}^a e_a^\mu, \\ \Xi^{\dot{\alpha}} =& \left(\Theta^\beta \tilde{\xi}^a f_{a\beta}{}^\alpha + \Theta^\beta \tilde{l}^i f_{i\beta}{}^\alpha - \xi^a U_a{}^\alpha\right) e_\alpha{}^{\dot{\alpha}} \\ &+ \tilde{\epsilon}^\beta (\mathcal{M} \coth \mathcal{M})_\beta{}^\alpha e_\alpha{}^{\dot{\alpha}} - \tilde{\epsilon}^\gamma \left(\mathcal{M}^{-1} \tanh \mathcal{M}/2\right)_\gamma{}^\beta (\mathcal{Y}U)_\beta{}^\alpha e_\alpha{}^{\dot{\alpha}}. \end{aligned} \tag{4.12}$$

The variations of the superspace coordinates are given by

$$\delta X^\mu = -\Xi^\mu, \qquad \delta\theta^{\dot{\alpha}} = -\Xi^{\dot{\alpha}}. \tag{4.13}$$

In the next subsections we will use equations (4.7) and (4.13) to write down the supervielbein and superisometries of the Minkowski and $AdS_5 \times S^5$ background superspaces.

4.1.1 Flat 10-dimensional superspace

As a warm-up we derive the isometries and vielbein of the Minkowski background. We start from the super Poincaré group G. The algebra is given by

$$\begin{aligned}
[M_{AB}, M_{CD}] &= \eta_{A[C} M_{D]B} - \eta_{B[C} M_{D]A}, \\
[P_A, M_{BC}] &= \eta_{A[B} P_{C]}, \\
[M_{AB}, Q_{\hat{\alpha}}] &= -\frac{1}{4} (\Gamma_{AB} Q)_{\hat{\alpha}}, \\
\{Q_{\hat{\alpha}}, Q_{\hat{\beta}}\} &= (\Gamma^A)_{\hat{\alpha}\hat{\beta}} P_A.
\end{aligned} \tag{4.14}$$

We make the split

$$\mathbf{H} = \{M_{AB}\}, \qquad \text{and} \qquad \mathbf{K} = \{P_A, Q_{\hat{\alpha}}\}. \tag{4.15}$$

This means that the indices of the previous section are chosen to be $\Lambda = \{A, [AB], \hat{\alpha}\}$, $\mathcal{A} = \{A, [AB]\}$, $\bar{I} = \{[AB]\}$, and $\bar{M} = \{a, \hat{\alpha}\}$. The spacetime fields are given by

$$e_M^A = \delta_M^A, \qquad \psi_M = 0, \qquad \omega_M^{AB} = 0, \tag{4.16}$$

and the solutions to the spacetime Killing equations ($\theta = 0$) are

$$\xi^M = a^M + \lambda_{(M)}^{MN} x_N, \qquad \epsilon^{\hat{\alpha}}(x) = \varepsilon_0^{\hat{\alpha}}, \qquad l^{AB} = \lambda_{(M)}^{MN} \delta_M^A \delta_N^B, \tag{4.17}$$

where a^M, $\lambda_{(M)}^{MN}$ and $\varepsilon_0^{\hat{\alpha}}$ are constant parameters. The matrix $\mathcal{M}$ vanishes, and the matrix $e_{\dot{\hat{\alpha}}}{}^{\hat{\beta}} = \delta_{\dot{\hat{\alpha}}}{}^{\hat{\beta}}$. The supervielbein (4.7) is then given by

$$E^{\hat{\alpha}} = d\theta^{\hat{\alpha}}, \qquad E^A = dx^A + \bar{\theta} \hat{\Gamma}^A d\theta, \tag{4.18}$$

where we suppressed the spinor indices in $\bar{\theta} \hat{\Gamma}^A d\theta = \theta^\alpha (\hat{\Gamma}^A)_\alpha{}^\beta d\theta_\beta$.
Plugging everything in (4.12), we obtain the well-known superisometries

$$\begin{aligned}
\delta x^M &= -\Xi^M = -a^M - \lambda_{(M)}^{MN} x_N - \frac{1}{2} (\bar{\varepsilon}_0 \hat{\Gamma}^M \theta + \text{h.c.}), \\
\delta \theta^{\hat{\alpha}} &= -\Xi^{\hat{\alpha}} = -\varepsilon_0^{\hat{\alpha}} - \frac{1}{4} \lambda_{(M)}^{MN} (\hat{\Gamma}_{MN} \theta)^{\hat{\alpha}}.
\end{aligned} \tag{4.19}$$

To facilitate things later, we introduce projectors $\mathcal{P}_{Q,S} = \frac{1}{2}(1 \mp \gamma_5) \otimes I_8$ (this is similar to what we will do for the $AdS_5 \times S^5$ case (see also appendix B)) such that

$$\mathcal{P}_Q \theta = \theta^i_\alpha, \qquad \mathcal{P}_S \theta = \vartheta^i_\alpha, \tag{4.20}$$

and we make a similar split for ε_0 into ϵ^i_α and η^i_α respectively. In terms of these refined variables, we have for the transformations

$$\begin{aligned}
\delta x^M &= -a^M - \lambda^{MN}_{(M)} x_N - \frac{1}{2}\Big[(\bar{\epsilon}_i \gamma^m \theta^i + \bar{\eta}_i \gamma^m \vartheta^i)\delta^M_m + (\bar{\epsilon}_i \vartheta^i - \bar{\eta}_i \theta^i)\delta^M_4 \\
&\qquad + (\bar{\epsilon}_i \vartheta^j + \bar{\eta}_i \theta^j)\left(\gamma'^{m'}\right)_j{}^i \delta^M_{m'} + \text{h.c.}\Big], \\
\delta\theta^i &= -\epsilon^i - \frac{1}{4}\lambda^{mn}_{(M)}\gamma_{mn}\theta^i - \frac{1}{4}\lambda^{m'n'}_{(M)}\left(\gamma'_{m'n'}\right)_j{}^i \theta^j + \frac{1}{2}\lambda^{m4}_{(M)}\gamma_m \vartheta^i \\
&\qquad - \frac{1}{2}\lambda^{m'4}_{(M)}\left(\gamma'_{m'}\right)_j{}^i \theta^j - \frac{1}{4}\lambda^{mn'}_{(M)}\gamma_m\left(\gamma'_{n'}\right)_j{}^i \vartheta^j, \\
\delta\vartheta^i &= -\eta^i - \frac{1}{4}\lambda^{mn}_{(M)}\gamma_{mn}\vartheta^i - \frac{1}{4}\lambda^{m'n'}_{(M)}\left(\gamma'_{m'n'}\right)_j{}^i \vartheta^j - \frac{1}{2}\lambda^{m4}_{(M)}\gamma_m \theta^i \\
&\qquad + \frac{1}{2}\lambda^{m'4}_{(M)}\left(\gamma'_{m'}\right)_j{}^i \vartheta^j - \frac{1}{4}\lambda^{mn'}_{(M)}\gamma_m\left(\gamma'_{n'}\right)_j{}^i \theta^j.
\end{aligned} \tag{4.21}$$

This form of the isometries will be used to compare with the large R limit of the $AdS_5 \times S^5$ isometries. For the $AdS_5 \times S^5$ background, however, there is no mixing between the first five and last five directions, this means that $\lambda^{mn'}_{(M)} = \lambda^{4n'}_{(M)} = 0$. For this reason we will set these equal to zero from here on out.

4.1.2 $AdS_5 \times S^5$ superspace

To construct this superspace we start from the superconformal group $G = SU(2,2|4)$, which has $SO(4,2) \times SO(6)$ as its bosonic subgroup. The superalgebra is presented in more detail in appendix B. For this supercoset the stability group H is the product group $SO(4,1) \times SO(5)$, which is purely bosonic. The $30+32$ generators of $SU(2,2|4) \supset SO(4,2) \times SO(6)$ are decomposed into $5+5$ translations $\tilde{P}_{\tilde{m}}$ and $P'_{m'}$, $10+10$ Lorentz generators $\tilde{M}_{\tilde{m}\tilde{n}}$ and $M'_{m'n'}$, and $16+16$ supersymmetries $Q_\alpha{}^i$ and $S_\alpha{}^i$. This superspace has $(10|32)$ coordinates (5 coordinates $x^{\tilde{m}} = \{x^m, \rho\}$ of AdS, 5 coordinates $z^{m'}$ of the sphere and 32 fermionic coordinates θ_α and $\vartheta_\alpha{}^i$). We have made the split

$$\mathbf{H} = \{\tilde{M}_{\tilde{m}\tilde{n}}, M'_{m'n'}\}, \qquad \mathbf{K} = \{\tilde{P}_{\tilde{m}}, P'_{m'}, Q_\alpha{}^i, S_\alpha{}^i\}. \tag{4.22}$$

This supercoset is an example of a maximally supersymmetric coset, i.e. all fermionic generators are in **K**.

Some of the details and calculations (though important) for deriving the supervielbein and superspace isometries would provide too much clutter and we have moved some of these technical details to the appendices. In particular, we refer to appendix C for a detailed discussion of the construction of the bosonic part of the $AdS_5 \times S^5$ coset space. Appendix C also contains a discussion on the choice of fermionic coordinates $e_{\dot{\hat{\alpha}}}{}^{\hat{\beta}}$ and the choice we make is given in equation (C.23). The supervielbein and superspace isometries are found by using the formulas (4.7) and (4.13) with the coset constructed in appendix C) and the conformal decomposition of the $SU(2,2|4)$-algebra presented in (B.17) as input. We will give the resulting supervielbein and superspace isometries here. For an example of the type of calculation involved in deriving these results, we refer to appendix D where we calculate the supersymmetry transformation of x^m.

The metric of $AdS_5 \times S^5$ is given by the sum of (C.2) and (C.15)

$$ds^2 = \rho^2 dx^2 + \left(\frac{R}{\rho}\right)^2 d\rho^2 + \frac{4R^2}{(1+z^2)^2} dz^2. \tag{4.23}$$

The supervielbein (4.5) of the geometry

$$E = E^{\bar{M}} K_{\bar{M}} = E^m \tilde{P}_m + E^\rho \tilde{P}_\rho + E^{m'} P'_{m'} + (\bar{Q}_i E^i_Q + \text{h.c.}) + (\bar{S}_i E^i_S + \text{h.c.}), \tag{4.24}$$

has components

$$\begin{aligned}
E^m = \rho & \left[dx^n \left(\delta_n{}^m - \frac{1}{2}\left(\frac{R}{\rho}\right)^2 \bar{\vartheta}_i \gamma_n \vartheta^j \bar{\vartheta}_j \gamma^m \vartheta^i \right) + \left(\frac{1}{2} d\bar{\theta}_i \gamma^m \theta^i \right.\right. \\
& \left.\left. + \frac{1}{4}\bar{\theta}_i d\vartheta^j \bar{\theta}_j \gamma^m \theta^i + \text{h.c.} \right) + \left(\frac{R}{\rho}\right)^2 \left(\frac{1}{2} d\bar{\vartheta}_i \gamma^m \vartheta^i + \frac{1}{4}\bar{\vartheta}_i d\theta^j \bar{\vartheta}_j \gamma^m \vartheta^i + \text{h.c.} \right) \right] \\
& + \mathcal{O}(\theta \wedge \vartheta), \\
E^\rho = \frac{R}{\rho} & \left[d\rho - \frac{1}{2}\left(d\bar{\theta}_i \vartheta^i - d\bar{\vartheta}_i \theta^i + \text{h.c.} \right) \rho \right] + \mathcal{O}(\theta \wedge \vartheta),
\end{aligned}$$

$$
\begin{aligned}
E^{m'} &= e^{m'} - \frac{R}{2}\left(d\bar{\theta}_i\vartheta^j + d\bar{\vartheta}_i\theta^j + dx^m\bar{\vartheta}_i\gamma_m\vartheta^j + \text{h.c.}\right)\left(u\gamma'^{m'}u^{-1}\right)_j{}^i + \mathcal{O}(\theta\wedge\vartheta),\\
E_Q^i &= \rho^{1/2}\left[d\theta^j - dx^m\gamma_m\vartheta^j + \frac{1}{3}\theta^k\left(d\bar{\vartheta}_k\theta^j - \bar{\theta}_k d\vartheta^j\right)\right]u_j{}^i + \mathcal{O}(\theta\wedge\vartheta),\\
E_S^i &= \rho^{-1/2}\left[d\vartheta^j + \frac{1}{3}\left(2d\bar{\theta}_k\vartheta^j - \bar{\vartheta}_k d\theta^j\right) + dx^m\vartheta^k\bar{\vartheta}_k\gamma_m\vartheta^j\right]u_j{}^i + \mathcal{O}(\theta\wedge\vartheta).
\end{aligned}
\tag{4.25}
$$

Here $\mathcal{O}(\theta\wedge\vartheta)$ stands for terms containing both θ^i and ϑ^i. We do not include these terms because they will drop out when we discuss the D3-brane embedding and gauge-fixing in section 4.2, where our gauge choice will set $\theta^i = 0$. We have left the coordinates of the sphere unspecified here. They are coded in the coset representative u and given in appendix C.2.

The superisometries for the various coordinates are

$$
\begin{aligned}
\delta x^m &= -\xi_C^m(x) - \frac{1}{2}\left(\bar{\epsilon}_i(x)\gamma^m\theta^i + \text{h.c.}\right) - \frac{1}{4}\left(\bar{\eta}_i\theta^j\bar{\theta}_j\gamma^m\theta^i + \text{h.c.}\right)\\
&\quad - \left(\frac{R}{\rho}\right)^2\left[\lambda_{(K)}^m + \frac{1}{2}\left(\bar{\eta}_i\gamma^m\vartheta^i + \text{h.c.}\right) + \frac{1}{4}\left(\bar{\epsilon}_i(x)\vartheta^j\bar{\vartheta}_j\gamma^m\vartheta^i + \text{h.c.}\right)\right]\\
&\quad + \mathcal{O}(\theta\wedge\vartheta),\\
\delta\rho &= \Lambda_D(x)\rho - \frac{1}{2}\left(\bar{\epsilon}_i(x)\vartheta^i - \bar{\eta}_i\theta^i + \text{h.c.}\right)\rho + \mathcal{O}(\theta\wedge\vartheta)\\
\delta z^{m'} &= -\xi^{m'}(z) + \frac{(1-z^2)}{4}\left(\bar{\epsilon}_i(x)\vartheta^j + \bar{\eta}_i\theta^j + \text{h.c.}\right)\left(u\gamma'^{m'}u^{-1}\right)_j{}^i + \mathcal{O}(\theta\wedge\vartheta),\\
\delta\theta^i &= -\epsilon^i(x) - \frac{1}{2}\Lambda_D(x)\theta^i - \frac{1}{4}\Lambda_M(x)\cdot\gamma\theta^i - \frac{1}{4}\theta^j\Lambda_{SO(6)}^{IJ}\left(\hat{\gamma}'_{IJ}\right)_j{}^i\\
&\quad - \left(\frac{R}{\rho}\right)^2\left[\lambda_{(K)}^m + \frac{1}{2}\left(\bar{\eta}_j\gamma^m\vartheta^j + \text{h.c.}\right) + \frac{1}{4}\left(\bar{\epsilon}_j(x)\vartheta^k\bar{\vartheta}_k\gamma^m\vartheta^j + \text{h.c.}\right)\right]\gamma_m\vartheta^i\\
&\quad - \frac{2}{3}\theta^j\left(2\bar{\eta}_j\theta^i - \bar{\theta}_j\eta^i\right) + \mathcal{O}(\theta\wedge\vartheta),\\
\delta\vartheta^i &= -\eta^i + \lambda_{(K)}^m\gamma_m\theta^i + \frac{1}{2}\Lambda_D(x)\vartheta^i - \frac{1}{4}\Lambda_M(x)\cdot\gamma\vartheta^i - \frac{1}{4}\vartheta^j\Lambda_{SO(6)}^{IJ}\left(\hat{\gamma}'_{IJ}\right)_j{}^i\\
&\quad - \frac{2}{3}\vartheta^j\left(2\bar{\epsilon}_j(x)\vartheta^i - \bar{\vartheta}_j\epsilon^i(x)\right) + \mathcal{O}(\theta\wedge\vartheta)
\end{aligned}
\tag{4.26}
$$

These $AdS_5 \times S^5$ isometries have been written in terms of x-dependent combinations of the superconformal parameters a^m, $\lambda^{mn}_{(M)}$, $\lambda^m_{(K)}$ and λ_D as defined in (C.10). We have defined

$$\epsilon^i(x) = \epsilon^i + x^m \gamma_m \eta^i, \tag{4.27}$$

and the supersymmetries and special supersymmetries are parametrised by ϵ and η. $\Lambda^{IJ}_{SO(6)}$ are the parameters of the $SO(6)$ R-symmetry, $\xi^{m'}(z)$ is given in (C.21), and $\hat{\gamma}'_{IJ}$ are elements of the 6-dimensional Clifford algebra, realizing the translation between $SO(6)$ and $SU(4)$,

$$\Lambda^{IJ}_{SO(6)} = \frac{1}{2} \Lambda_{SU(4)i}{}^j (\hat{\gamma}'^{IJ})_j{}^i. \tag{4.28}$$

4.2 D3-brane World-volume Theory

We already discussed the world-volume actions to some extent in section 2.4. Here we will focus a little more on the symmetries the world-volume theory inherits from its embedding in the background. Let us briefly recap some facts about the world-volume theory. Recall that the world-volume action of a generic super D3-brane probe consists of two parts [112, 43, 44]

$$S = S_{\mathrm{DBI}} + S_{\mathrm{WZ}}. \tag{4.29}$$

The world-volume $\mathcal{M}_4$ is parametrised by 4 coordinates σ^μ. The background superspace coordinates are now fields on the world-volume $Z^M = Z^M(\sigma)$.
Both terms of the brane action are by construction (separately) invariant under the background superisometries. The background isometries are now symmetries acting on the fields, i.e. they depend on the world-volume coordinates σ through $Z^M(\sigma)$. Upon fixing the embedding of the brane in the background, the rigid background isometries will be realised on the remaining world-volume fields.

Local symmetries of the world-volume actions

The D3-brane actions not only have global symmetries due to the background isometries, they also come with local symmetries. The first set of local symmetries of this action are the world-volume diffeomorphisms. They act as Lie-derivatives on the fields

$$\delta_{\text{loc.diff.}} Z^M = \zeta^\mu(\sigma)\partial_\mu Z^M, \qquad \delta_{\text{loc.diff.}} \mathcal{F}_{\mu\nu} = \zeta^\rho(\sigma)\partial_\rho \mathcal{F}_{\mu\nu} - 2\partial_{[\mu}\zeta^\rho(\sigma)\mathcal{F}_{\nu]\rho}. \tag{4.30}$$

The second local symmetry is called κ-symmetry [43, 44], which is a local fermionic symmetry. Its parameter is a 10-dimensional spinor κ, depending on the world-volume coordinates. The variations δZ^M of the world-volume fields are commonly defined in terms of the supervielbein, but by using the inverse vielbein they can be inverted to obtain $\delta_\kappa X^\mu$ and $\delta_\kappa \theta^\alpha$ as [108, 109]

$$\begin{aligned} \delta_\kappa X^\mu &= -\kappa^\beta \left(1+\Gamma^{\mathcal{C}}\right)_\beta{}^\alpha \left(\mathcal{M}^{-1}\tanh\frac{\mathcal{M}}{2}\Upsilon\right)_\alpha{}^a e_a^\mu, \\ \delta_\kappa \theta^{\dot\alpha} &= \kappa^\gamma \left(1+\Gamma^{\mathcal{C}}\right)_\gamma{}^\beta \left(\mathcal{M}\sinh^{-1}\mathcal{M} + \mathcal{M}^{-1}\tanh\frac{\mathcal{M}}{2}\Upsilon U\right)_\beta{}^\alpha e_\alpha{}^{\dot\alpha}. \end{aligned} \quad (4.31)$$

The matrix Γ appears here as its charge conjugate $\Gamma^{\mathcal{C}}$ and it is an element of the 10-dimensional Clifford algebra, satisfying $\Gamma^2 = 1$, $\mathrm{Tr}\Gamma = 0$. It is a combination of gamma matrices and depends on the world-volume fields. For the probe $D3$-brane, Γ is given by

$$\Gamma = \begin{pmatrix} 0 & \beta_- \\ -\beta_+ & 0 \end{pmatrix}, \quad (4.32)$$

with

$$\begin{aligned} \beta_- &= \frac{1}{\sqrt{-\det(G_{\mu\nu}+\alpha\mathcal{F}_{\mu\nu})}} \left(\sum_{k=0}^{2} \frac{(-\alpha)^k}{2^k k!}\gamma^{\mu_1\nu_1\ldots\mu_k\nu_k}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_k\nu_k}\right)\Gamma^{D_3}, \\ \beta_+ &= \frac{1}{\sqrt{-\det(G_{\mu\nu}+\alpha\mathcal{F}_{\mu\nu})}} \left(\sum_{k=0}^{2} \frac{\alpha^k}{2^k k!}\gamma^{\mu_1\nu_1\ldots\mu_k\nu_k}\mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_k\nu_k}\right)\Gamma^{D_3}, \end{aligned} \quad (4.33)$$

where $\Gamma^{D_3} = \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_{\mu\nu\rho\sigma}$ and γ^μ are the pullback of the 10-dimensional gamma matrices.

Comparing (4.31) with (4.12) we see that they almost act as supersymmetries, the difference being in the higher order fermion terms in $\delta_\kappa\theta^{\dot\alpha}$.

The irreducible κ symmetries are defined by the algebraic constraint

$$(1-\Gamma_{\mathrm{cl}})\kappa = 0, \quad (4.34)$$

where Γ_{cl} is the value of Γ at the classical value of the fields, compatible with the gauge fixing and brane wave equations. We can write the irreducible κ symmetries as

$$\kappa_+ \equiv (1+\Gamma)\kappa_*, \quad (4.35)$$

where κ_* is a solution to equation (4.34), $(1-\Gamma)\kappa_* = 0$.

The static gauge and the Q-gauge

The embedding of the brane in the background can be described by identifying some of the world-volume coordinates with the spacetime coordinates of the background. This 'gauge fixing' has to be admissible, which means that it has to be compatible with the equations of motion derived from the probe-brane action, the branewave equations. We will consider an infinite extended brane and will therefore take the *static gauge*

$$\sigma^\mu = \delta^\mu_m x^m, \tag{4.36}$$

where x^m are 4 coordinates of the background geometry. This gauge will only yield a stable configuration in specific backgrounds [106]. Two examples are the flat background and the $AdS \times S$ background where the x^m have to be the directions parallel to the boundary of AdS. The full transformation of the fields $Z^M(\sigma)$ is

$$\delta Z^M = \zeta^\mu \partial_\mu Z^M + \delta_{\rm global} Z^M + \delta_\kappa Z^M, \tag{4.37}$$

where $\delta_{\rm global} Z^M$ are the transformations in (4.19) or (4.26). In order to preserve the gauge choice (4.36) we need to impose the condition $\delta x^m = 0$, leading to a decomposition law for ζ^μ.
In fixing the κ-symmetry we will be guided by the effects of the $AdS_5 \times S^5$ background. There are two natural ways to gauge-fix the κ-symmetry and get rid of half of the fermionic gauge-degrees of freedom on the world-volume. We can either set $\vartheta^i = 0$ (S-gauge) or we can set $\theta^i = 0$ (Q-gauge). However, the S-gauge is not admissible for the infinite static branes in their own near-horizon geometry. The classical values of the fields in the static gauge are $x^m = \delta^m_\mu \sigma^\mu$, $\rho =$ constant, $z^{m'} =$ constant, $\theta^i = \vartheta^i = 0$, $\mathcal{F}_{\mu\nu} = 0$ leading to $\Gamma_{cl} = \hat{\gamma}_{ST}$, where this matrix $\hat{\gamma}_{ST}$ is precisely the one used in the projector to define Q and S supersymmetry (appendix B.2). This means that a gauge-fixing

$$0 = \vartheta^i = \frac{1}{2}\left(1 - \hat{\gamma}_{ST}\right) \Theta^i, \tag{4.38}$$

will not affect the irreducible κ symmetry and is not admissible. Since we are interested in the $AdS_5 \times S^5$ background, this leaves us with the 'natural' choice of the Q-gauge, $\theta^i = 0$. Imposing this condition will leave us with a decomposition law for κ_+.

4.2.1 D3-Brane world-volume in Minkowski Background

We consider the embedding of a D3-brane in a Minkowski background. The κ-symmetry transformation rules (4.31) become

$$\delta_\kappa x^M = -\frac{1}{2}\left[\left(\bar\kappa_{+Qi}\gamma^m\theta^i + \bar\kappa_{+Si}\gamma^m\vartheta^i\right)\delta^M_m + (\bar\kappa_{+Qi}\vartheta^i - \bar\kappa_{+Si}\theta^i)\delta^M_4 \right.$$
$$\left. +(\bar\kappa_{+Qi}\vartheta^j + \bar\kappa_{+Si}\theta^j)\left(\gamma'^{m'}\right)_j{}^i\delta^M_{m'} + \text{h.c.}\right],$$
$$\delta_\kappa\theta^i = \kappa^i_{+Q}, \qquad \delta_\kappa\vartheta = \kappa^i_{+S}, \tag{4.39}$$

where we have introduced the projections $\mathcal{P}_{Q,S}\kappa_+ = \kappa_{+Q,S}$.
As discussed in the previous section, the condition $\delta x^m = 0$, needed to preserve the static gauge, and the Q-gauge condition $\theta^i = 0$ (fixing the kappa gauge) give us two decomposition laws (up to cubic fermion terms)

$$\kappa^i_{+Q} = \epsilon^i - \frac{1}{2}\lambda^{m4}\gamma_m\vartheta^i, \tag{4.40}$$

and

$$\zeta^\mu(\sigma) = a^\mu + \lambda^{\mu N}_{(M)}x_N - \frac{1}{2}\left[\bar\vartheta_i\gamma^\mu\left(\eta^i - \beta_+\left(\epsilon^i - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^i\right)\right) + \text{h.c.}\right]. \tag{4.41}$$

The remaining fields then have as transformation laws

$$\delta x^4 = \xi^\mu\partial_\mu x^4 - \frac{1}{2}\left[\bar\vartheta_i\gamma^\mu\left(\eta^i - \beta_+\left(\epsilon^i - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^i\right)\right) + \text{h.c.}\right]\partial_\mu x^4$$
$$-\xi^4 - \left[\bar\vartheta_i\left(\epsilon^i - \frac{1}{4}\lambda^{m4}\gamma_m\vartheta^i\right) + \text{h.c.}\right],$$
$$\delta x^{m'} = \xi^\mu\partial_\mu x^{m'} - \frac{1}{2}\left[\bar\vartheta_i\gamma^\mu\left(\eta^i - \beta_+\left(\epsilon^i - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^i\right)\right) + \text{h.c.}\right]\partial_\mu x^{m'}$$
$$-\xi^{m'} - \left[(\bar\vartheta_i\epsilon^j - \frac{1}{4}\lambda^{m4}\vartheta_i\gamma_m\vartheta^j)\left(\gamma^{m'}\right)_j{}^i + \text{h.c.}\right], \tag{4.42}$$
$$\delta\vartheta^i = \xi^\mu\partial_\mu\vartheta^i - \frac{1}{2}\left[\bar\vartheta_j\gamma^\mu\left(\eta^j - \beta_+\left(\epsilon^j - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^j\right)\right) + \text{h.c.}\right]\partial_\mu\vartheta^i$$
$$-\frac{1}{4}\lambda^{mn}_{(M)}\gamma_{mn}\vartheta^i - \frac{1}{4}\lambda^{m'n'}_{(M)}\left(\gamma'_{m'n'}\right)_j{}^i\vartheta^j - \left[\eta^i + \beta_+\left(\epsilon^i - \frac{1}{2}\lambda^{m4}\gamma_m\vartheta^i\right)\right],$$

where we used that $\kappa_{+S} = -\beta_+\kappa_{+Q}$ and defined $\xi^M = a^M + \lambda^{MN}_{(M)}x_N$.

4.2.2 D3-Brane world-volume in $AdS_5 \times S^5$ Background

In this section we consider the D3-brane embedded in its own near-horizon background, $AdS_5 \times S^5$. Embedding a $D3$-brane in this background, the background coordinates are promoted to world-volume fields and their transformations under κ symmetry (4.31) are given by

$$\begin{aligned}
\delta_\kappa x^m &= -\frac{1}{2}\left(\bar{\kappa}_{+Qi}\gamma^m\theta^i + \text{h.c.}\right) - \frac{1}{4}\left(\bar{\kappa}_{+Si}\theta^j\bar{\theta}_j\gamma^m\theta^i + \text{h.c.}\right) \\
&\quad - \left(\frac{R}{\rho}\right)^2\left[\frac{1}{2}\left(\bar{\kappa}_{+Si}\gamma^m\vartheta^i + \text{h.c.}\right) + \frac{1}{4}\left(\bar{\kappa}_{+Qi}\vartheta^j\bar{\vartheta}_j\gamma^m\vartheta^i + \text{h.c.}\right)\right] \\
&\quad + \mathcal{O}\left(\theta\wedge\vartheta\right), \\
\delta_\kappa\theta^i &= \kappa^i_{+Q} - \left(\frac{R}{\rho}\right)^2\left[\frac{1}{2}\bar{\kappa}_{+Sj}\gamma^m\vartheta^j + \frac{1}{4}\bar{\kappa}_{+Qj}\vartheta^k\bar{\vartheta}_k\gamma^m\vartheta^j + \text{h.c.}\right]\gamma_m\vartheta^i \\
&\quad - \frac{1}{3}\theta^j\left(2\bar{\kappa}_{+Sj}\theta^i - \bar{\theta}_j\kappa^i_{+S}\right) + \mathcal{O}\left(\theta\wedge\vartheta\right), \\
\delta_\kappa\vartheta^i &= \kappa^i_{+S} - \frac{1}{3}\vartheta^j\left(2\bar{\kappa}_{+Qj}\vartheta^i - \bar{\vartheta}_j\kappa^i_{+Q}\right) + \mathcal{O}\left(\theta\wedge\vartheta\right), \\
\delta_\kappa\rho &= -\frac{1}{2}\left(\bar{\kappa}_{+Qi}\vartheta^i - \bar{\kappa}_{+Si}\theta^i\right)\rho + \text{ h.c} + \mathcal{O}\left(\theta\wedge\vartheta\right), \\
\delta_\kappa z^{m'} &= \frac{(1-z^2)}{4}\left(\bar{\kappa}_{+Qi}\vartheta^j + \bar{\kappa}_{+Si}\theta^j - \text{ h.c}\right)\left(u\gamma'^{m'}u^{-1}\right)_j^{\ i} + \mathcal{O}\left(\theta\wedge\vartheta\right).
\end{aligned} \tag{4.43}$$

Again, the conditions $\delta x^m = 0$ and $\theta^i = 0$ imply two decomposition laws (up to cubic fermion terms)

$$\begin{aligned}
\zeta^\mu(\sigma) &= \xi^\mu_C(\sigma) + \frac{1}{2}\left(\bar{\epsilon}_i(\sigma)\gamma^\mu\theta^i + \text{h.c.}\right) + \left(\frac{R}{\rho}\right)^2\left[\lambda^\mu_{(K)} + \frac{1}{2}\left(\bar{\eta}_i\gamma^\mu\vartheta^i + \text{h.c.}\right)\right] \\
&\quad + \frac{1}{2}\left(\bar{\kappa}_{+Qi}\gamma^\mu\theta^i + \text{h.c.}\right) + \left(\frac{R}{\rho}\right)^2\left[\frac{1}{2}\left(\bar{\kappa}_{+Si}\gamma^\mu\vartheta^i + \text{h.c.}\right)\right] + \mathcal{O}\left(\theta\wedge\vartheta\right),
\end{aligned} \tag{4.44}$$

and,

$$\kappa^i_{+Q} = \epsilon^i(x) + \left(\frac{R}{\rho}\right)^2\lambda^m_{(K)}\gamma_m\vartheta^i\,. \tag{4.45}$$

The remaining world-volume fields are then $\rho(\sigma), z^{m'}(\sigma)$ and $\vartheta(\sigma)$ and their transformation rules are the following (up to cubic fermion terms)

$$\delta\vartheta^i = \hat{\xi}^\mu_C(\sigma)\partial_\mu\vartheta^i - \frac{1}{4}\Lambda_M(\sigma)\cdot\gamma\vartheta^i + \frac{1}{2}\Lambda_D(\sigma)\vartheta^i - \frac{1}{4}\vartheta^j\Lambda^{IJ}_{SO(6)}\left(\hat{\gamma}'_{IJ}\right)_j{}^i$$
$$- \frac{\beta_+}{R}\left(\frac{R}{\rho}\right)^2\lambda^m_{(K)}\gamma_m\vartheta^i - \frac{\beta_+}{R}\epsilon^i(\sigma) - \eta^i\,,$$

$$\delta\rho = \hat{\xi}^\mu_C(\sigma)\partial_\mu\rho + \Lambda_D(\sigma)\rho - \left[\bar{\vartheta}_i\left(\epsilon^i + \frac{1}{2}\left(\frac{R}{\rho}\right)^2\lambda^m_{(K)}\gamma_m\vartheta^i\right) + \text{h.c}\right]\rho$$
$$+ \frac{1}{2}\left(\frac{R}{\rho}\right)^2\left[\bar{\eta}_i\gamma^\mu\vartheta^i + \bar{\vartheta}_i\gamma^\mu\frac{\beta_+}{R}\left(\epsilon^i(\sigma) + \left(\frac{R}{\rho}\right)^2\lambda^m_{(K)}\gamma_m\vartheta^i\right) + \text{h.c.}\right]\partial_\mu\rho,$$

$$\delta z^{m'} = \hat{\xi}^\mu_C(\sigma)\partial_\mu z^{m'} - \xi^{m'}(z)$$
$$+ \frac{(1-z^2)}{2}\left[\bar{\epsilon}_i(\sigma)\vartheta^j + \frac{1}{2}\left(\frac{R}{\rho}\right)^2\Lambda^m_K\bar{\vartheta}_i\gamma_m\vartheta^j + \text{h.c.}\right]\left(u\gamma'^{m'}u^{-1}\right)_j{}^i$$
$$+ \frac{1}{2}\left(\frac{R}{\rho}\right)^2\left[\bar{\eta}_i\gamma^\mu\vartheta^i + \bar{\vartheta}_i\gamma^\mu\frac{\beta_+}{R}\left(\epsilon^i(\sigma) + \left(\frac{R}{\rho}\right)^2\Lambda^m_K\gamma_m\vartheta^i\right) + \text{h.c.}\right]\partial_\mu z^{m'} \tag{4.46}$$

where

$$\hat{\xi}^\mu_C(\sigma) \equiv \xi^\mu_C(\sigma) + \left(\frac{R}{\rho}\right)^2\lambda^\mu_{(K)}\,. \tag{4.47}$$

4.3 From $AdS_5 \times S^5$ to Minkowski: The Large R limit

We want to compare the resulting world-volume transformations of the two backgrounds discussed in the previous section. Our aim is to establish a relation between the symmetries in $AdS_5 \times S^5$ background and the Volkov-Akulov supersymmetries in the Minkowski background of [5]. In order to make an identification, we need to take a suitable large R limit of the $AdS_5 \times S^5$ background. We start out with a discussion of the proper limit.
To take this limit, it is convenient to change (background) spacetime coordinates.

We define

$$\rho = e^{r/R}, \qquad z^{m'} = \frac{\tilde{z}^{m'}}{2R}. \tag{4.48}$$

The metric (4.23) then becomes

$$ds^2 = e^{2r/R} dx^\mu \eta_{\mu\nu} dx^\nu + e^{-2r/R} dr^2 + \frac{1}{(1+\tilde{z}^2/(4R^2))^2} d\tilde{z}^2 \,, \tag{4.49}$$

which becomes Minkowski space in the limit $R \to \infty$. We also need to make some redefinitions in the algebra. The algebra we have used to derive the transformation rules in the previous sections relied on the conformal decomposition (appendix B.2), but the AdS-decomposition (appendix B.1) is the one that we need for a reduction to a Poincaré sub-algebra. Equation (B.19) gives the relation between the various decompositions and is to be used to obtain the correct variables. In particular this means that we redefine

$$\tilde{\vartheta}^i = R\vartheta^i, \qquad \tilde{\eta}^i = R\eta^i \qquad \text{and} \qquad \tilde{\kappa}^i_{+S} = R\kappa^i_{+S}. \tag{4.50}$$

Applying these redefinitions and taking the large R limit nicely reduces the $AdS_5 \times S^5$ supervielbein and isometries, (4.25) and (4.26), to their Minkowski space equivalents, (4.18) and (4.19) (modulo the spacetime mixing requirement $\lambda^{mn'}_{(M)} = \lambda^{4n'}_{(M)} = 0$). We now apply this to the transformations of the world-volume fields in the $AdS_5 \times S^5$ background, (4.46), and take the limit $R \to \infty$ to obtain

$$\begin{aligned}
\delta r &= \xi^\mu \partial_\mu r - \frac{1}{2}\left[\bar{\tilde{\vartheta}}_i \gamma^\mu \left(\tilde{\eta}^i - \beta_+\left(\epsilon^i - \frac{1}{2}\tilde{A}^{mS}\gamma_m\tilde{\vartheta}^i\right)\right) + \text{h.c.}\right]\partial_\mu r \\
&\quad - \tilde{A}^S - \tilde{A}^{Sn}x_n - \left[\bar{\tilde{\vartheta}}^i \left(\epsilon^i - \frac{1}{4}\tilde{A}^{mS}\gamma_m\tilde{\vartheta}^i\right) + \text{h.c.}\right], \\
\delta\tilde{z}^{m'} &= \xi^\mu \partial_\mu \tilde{z}^{m'} - \frac{1}{2}\left[\bar{\tilde{\vartheta}}_i \gamma^\mu \left(\tilde{\eta}^i - \beta_+\left(\epsilon^i - \frac{1}{2}\tilde{A}^{mS}\gamma_m\tilde{\vartheta}^i\right)\right) + \text{h.c.}\right]\partial_\mu \tilde{z}^{m'} \\
&\quad - \tilde{\xi}^{m'} - \left[\left(\bar{\epsilon}_i - \frac{1}{4}\tilde{A}^{mS}\bar{\tilde{\vartheta}}_i\gamma_m\right)\tilde{\vartheta}^j (\gamma'^{m'})_j{}^i + \text{h.c.}\right], \\
\delta\tilde{\vartheta}^i &= \xi^\mu \partial_\mu \tilde{\vartheta}^i - \frac{1}{4}A^{mn}_{(M)}\gamma_{mn}\tilde{\vartheta}^i \\
&\quad - \frac{1}{4}\Lambda^{IJ}_{SO(6)}(\hat{\gamma}'_{IJ})_j{}^i\tilde{\vartheta}^j - \left[\tilde{\eta}^i + \beta_+\left(\epsilon^i - \frac{1}{2}\tilde{A}^{mS}\gamma_m\tilde{\vartheta}^i\right)\right]
\end{aligned} \tag{4.51}$$

where $\xi^\mu = \tilde{A}^\mu + r\tilde{A}^{\mu S} + \tilde{A}^{\mu n} x_n$.
Making the identifications

$$x^5_{\text{Mink}} = r, \quad x^{m'}_{\text{Mink}} = \tilde{z}^{m'}, \quad \vartheta^i_{\text{Mink}} = \tilde{\vartheta}^i,$$
$$\lambda^{m4}_{\text{Mink}} = \tilde{A}^{mS}, \quad \eta^i_{\text{Mink}} = \tilde{\eta}^i, \quad \lambda^{mn}_{(M),\text{Mink}} = \tilde{A}^{mn}_{(M)}, \tag{4.52}$$

and

$$\xi^4_{\text{Mink}} = \tilde{A}^S + \tilde{A}^{Sm} x_m, \tag{4.53}$$

where the subscript Mink refers to the quantities in (4.42), we can compare (4.51) with (4.42) and we find an exact match between the world-volume transformation rules.
However, there seems to be no way to link Minkowski background symmetries to the $AdS_5 \times S^5$ symmetries without introducing a length scale, not at all a surprising result. The reason for this is quite simple and can be found by looking at the conformal algebra. The algebra corresponding to our $AdS_5 \times S^5$ space was given in (B.17). We are interested in the anti-commutators of the fermionic generators which we repeat here for convenience

$$\{Q_\alpha{}^i, \bar{Q}_j{}^\beta\} = \delta_j{}^i (\gamma^a)_\alpha{}^\beta P_a, \qquad \{S_\alpha{}^i, \bar{S}_j{}^\beta\} = \delta_j{}^i (\gamma^a)_\alpha{}^\beta K_a,$$
$$\{Q_\alpha{}^i, \bar{S}_j{}^\beta\} = \delta_j{}^i \delta_\alpha{}^\beta D + \delta_j{}^i (\gamma^{ab})_\alpha{}^\beta M_{ab} - 2\delta_\alpha{}^\beta U_j{}^i. \tag{4.54}$$

Before we take the limit $R \to \infty$, we need to write these anticommutators in the notation of the AdS decomposition. Using the relations in section B.2 we find

$$\left\{ (\mathcal{P}_Q \tilde{\mathcal{Q}})_{\hat{\alpha}}{}^i, \left(\overline{\mathcal{P}_Q \tilde{\mathcal{Q}}}\right)_j{}^{\hat{\beta}} \right\} = -\frac{1}{2} \delta_j{}^i (\mathcal{P}_Q \hat{\gamma}^{aT})_{\hat{\alpha}}{}^{\hat{\beta}} \left(\tilde{P}_a + \frac{2}{R} \tilde{M}_{aS} \right),$$
$$\left\{ (\mathcal{P}_S \tilde{\mathcal{Q}})_{\hat{\alpha}}{}^i, \left(\overline{\mathcal{P}_S \tilde{\mathcal{Q}}}\right)_j{}^{\hat{\beta}} \right\} = -\frac{1}{2} \delta_j{}^i (\mathcal{P}_S \hat{\gamma}^{aT})_{\hat{\alpha}}{}^{\hat{\beta}} \left(\tilde{P}_a - \frac{2}{R} \tilde{M}_{aS} \right),$$
$$\left\{ (\mathcal{P}_Q \tilde{\mathcal{Q}})_{\hat{\alpha}}{}^i, \left(\overline{\mathcal{P}_S \tilde{\mathcal{Q}}}\right)_j{}^{\hat{\beta}} \right\} = -\frac{1}{2} \delta_j{}^i \, (\mathcal{P}_Q)_{\hat{\alpha}}{}^{\hat{\beta}} \, \tilde{P}_S + \frac{1}{2R} \delta_j{}^i (\mathcal{P}_Q \hat{\gamma}^{ab})_{\hat{\alpha}}{}^{\hat{\beta}} M_{ab}$$
$$- \frac{1}{R} \, (\mathcal{P}_Q)_{\hat{\alpha}}{}^{\hat{\beta}} \, U_j{}^i. \tag{4.55}$$

From these commutation relations it is clear that in the limit $R \to \infty$ the right hand side of the first two commutators reduces to a translation. In other words, the distinction between the operator P_a and the operator K_a disappears. The conformal structure is an $\mathcal{O}\left(\frac{1}{R}\right)$-effect, and requires a length scale used for

separation to work.
In light of this it is also clear why the major difference of the Volkov-Akulov supersymmetries of [5] and conformal supersymmetry rests in the Volkov-Akulov supersymmetries anti-commuting into translations. There simply is no length scale from the background available to make the distinction between translations and special conformal transformations. Let us look at the relation with the results from [5] a bit closer. In order to really compare with [5], we should write our transformations in a form that looks like (only considering the fermionic symmetries now)

$$\begin{aligned} \delta\phi^I &\sim \left(\bar{\lambda}_i \Gamma^\mu \epsilon^{2i} + \bar{\lambda}_i \Gamma^\mu \beta_+ \epsilon^{1i}\right) \partial_\mu \phi^I, \\ \delta\lambda^i &\sim \epsilon^{2i} + \beta_+ \epsilon^{1i}. \end{aligned} \tag{4.56}$$

Looking at the transformations (4.51), we find

$$\epsilon^{1i} = \epsilon^i - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^i, \quad \epsilon^{2i} = \eta^i + 2\epsilon^i - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^i. \tag{4.57}$$

In order to make the appearance of the Volkov-Akulov symmetry apparent, we define the parameters

$$\begin{aligned} \varepsilon^i &= \epsilon^{1i} = \epsilon^i - \frac{1}{2}\lambda^{n4}\gamma_n\vartheta^i, \\ \zeta^i &= \epsilon^{2i} - \epsilon^{1i} = \eta^i + \epsilon^i, \end{aligned} \tag{4.58}$$

suggesting that the generators for supersymmetry and Volkov-Akulov symmetry will be

$$(Q_{\mathrm{SUSY}})_{\hat{\alpha}}{}^i = \left(\mathcal{P}_Q \tilde{\mathcal{Q}}\right)_{\hat{\alpha}}{}^i, \qquad (Q_{\mathrm{VA}})_{\hat{\alpha}}{}^i = \left(\mathcal{P}_Q \tilde{\mathcal{Q}}\right)_{\hat{\alpha}}{}^i + \left(\mathcal{P}_S \tilde{\mathcal{Q}}\right)_{\hat{\alpha}}{}^i. \tag{4.59}$$

The corresponding algebra becomes

$$\begin{aligned} \left\{(Q_{\mathrm{SUSY}})_{\hat{\alpha}}{}^i, \left(\overline{Q_{\mathrm{SUSY}}}\right)_j{}^{\hat{\beta}}\right\} &= -\frac{1}{2}\delta_j{}^i(\mathcal{P}_Q\hat{\gamma}^{aT})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{P}_a, \\ \left\{(Q_{\mathrm{VA}})_{\hat{\alpha}}{}^i, \left(\overline{Q_{\mathrm{VA}}}\right)_j{}^{\hat{\beta}}\right\} &= -\frac{1}{2}\delta_j{}^i(\hat{\gamma}^{aT})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{P}_a - \frac{1}{2}\delta_j{}^i(\delta)_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{P}_S - \frac{1}{R}(\delta)_{\hat{\alpha}}{}^{\hat{\beta}}U_j{}^i, \\ \left\{(Q_{\mathrm{SUSY}})_{\hat{\alpha}}{}^i, \left(\overline{Q_{\mathrm{VA}}}\right)_j{}^{\hat{\beta}}\right\} &= -\frac{1}{2}\delta_j{}^i(\mathcal{P}_Q\hat{\gamma}^{aT})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{P}_a - \frac{1}{2}\delta_j{}^i(\mathcal{P}_Q)_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{P}_S \\ &\quad - \frac{1}{R}(\mathcal{P}_Q)_{\hat{\alpha}}{}^{\hat{\beta}}U_j{}^i, \end{aligned} \tag{4.60}$$

where we clearly see the appearance of translations and shift-symmetries of the scalar fields in the anti-commutators of the Volkov-Akulov-symmetry.

4.4 Conclusions

We compared the world-volume transformation rules of a D3-brane embedded in a Minkowski background with those of a D3-brane embedded in an $AdS_5 \times S^5$ background. We obtained a relation between the special supersymmetry transformations induced by the $AdS_5 \times S^5$ background and the Volkov-Akulov symmetries related to the Minkowski background. In order to relate one to the other, one needs to introduce a length scale, a result that is reaffirmed by looking at the algebra. The existence of a length scale in the algebra associated to the $AdS_5 \times S^5$ background allows for the distinction between translations and special conformal translations as an $\mathcal{O}\left(\frac{1}{R}\right)$-effect. When this effect is very small (at large R) this distinction disappears, and it is therefore no surprise that in a Minkowski background one only finds supersymmetry transformations that anti-commute into translations and shift-symmetries (i.e. the 16 supersymmetries + 16 Volkov-Akulov symmetries of [5]).
The question remains then whether we can construct higher derivative invariants coupled to supergravity in the $D = 4$, $\mathcal{N} = 4$ setting with VA-type symmetries. We will provide a tentative scheme for constructing these higher derivative invariants. Having established a relation between the conformal symmetry inherited by the $AdS_5 \times S^5$ background and the Volkov-Akulov symmetry due to the Minkowski background, we can use this relation as a tool for the construction of higher derivative invariants. The idea is to perform a construction of higher derivative invariants using superconformal methods in the theory of the brane embedded in $AdS_5 \times S^5$, followed by making the redefinitions (4.48) and (4.50), and then taking the limit necessary to obtain the Minkowski background. Our gauge choice to fix κ-symmetry is special in the sense that it has an easy limit to obtain the world-volume theory of a D3-brane in a Minkowski background. However, it might not be practical for the application of superconformal methods. Potentially there might be a different gauge choice that would give a realisation of the conformal symmetry on the brane that is easier to work with in terms of applying the superconformal methodology. The gauge choice we made in this paper, however, makes the relation with the transformation rules in the Minkowski background clear, and should be related to this unknown gauge choice by field redefinitions. If we can find such a gauge choice to simplify the construction of higher derivative invariants, we can modify the scheme by starting from this case with (as of yet) unknown κ-symmetry gauge to construct higher derivative invariants using superconformal methods. Once these are constructed field redefinitions will transform these higher derivative invariants to the gauge used in this paper, the Q-gauge. It is then only a matter of taking the large R-limit to obtain higher derivative invariants in the desired $D = 4$, $\mathcal{N} = 4$ setting with VA-type symmetries.

Chapter 5

Intersecting D4/D8 branes in massive type IIA supergravity

We saw in chapter 3 that the AdS/CFT correspondence allows one to study strongly coupled field theory systems, by instead, studying weakly coupled dual gravitational systems. A particularly interesting AdS/CFT correspondence arises in the context of 5-dimensional gauge theories. In general, 5-dimensional gauge theories are non-renormalisable and it is believed that additional degrees of freedom must be added to make the theories well defined. One may even try to argue that such theories are never well defined and this is one of the reasons we live in 4-dimensions. However, a proposed counter-example to this argument is provided by a class of 5-dimensional supersymmetric gauge theories which are conjectured to be UV-complete [62, 100, 101]. Evidence for the existence of a non-trivial UV fixed point is provided by the existence of a gravitational dual [63, 19, 113].

This gravitational dual is obtained as the near-horizon geometry of a D4-D8 brane system in massive type IIA supergravity. The near-horizon geometry is a fibration of AdS_6 over S^4. Dimensionally reducing the massive type IIA supergravity on the warped S^4 gives $F(4)$ gauged supergravity in 6 dimensions. Evidence that the 5-dimensional gauge theory captures the UV fixed point physics has been provided by finding agreement between the S^5 partition function of the gauge theory and dual gravitational system [12]. It is still an open question if the non-perturbative effects are sufficient to cure the UV-divergences or if additional degrees of freedom are necessary.

Recently, the partition function result has been extended to comparing the vacuum expectation value of the half-BPS Wilson line for totally symmetric and anti-symmetric representations [13]. In [114] this result was generalised to squashed 5-spheres. In the cases considered, the vacuum expectation value can be well approximated on the gravity side using probe branes, or, in terms of the Young tableaux discussion of section 3.5, they correspond to Young tableaux where the number of rows (or columns) is far smaller than their length. To go further (representations with arbitrary Young tableaux sizes), one must include the backreaction of the probe branes.

We have discussed in section 3.6 how these gauge theories are related to intersections of the fundamental string and D4/D8-branes (section 2.5.4) and their M-theory cousins of section 2.5.3. In this chapter we will look for solutions that correspond to the configurations of section 2.5.4, with supergroup $D(2,1;\gamma;1) \times SO(4)$. Our approach will be to make an ansatz for the (supergravity) fields based on the symmetry considerations of section 2.5.4, with the intent to look for fully localised solutions in a conformal near-horizon limit. Using this ansatz we then reduce the BPS-equations to a two-dimensional system, supplemented with algebraic expressions for the metric in terms of spinor bilinears.

In general there are two distinct cases of enhanced supersymmetry, one given by setting $\gamma = -1/2, -2$ and the second given by setting $\gamma = 1$. In the first case, we show the most general solution is given by the AdS_6 geometries of [19], which is simply the dual of the 5-dimensional gauge theory without the half-BPS Wilson line.

The second case corresponds to fundamental strings ending on D8-branes.[1] We identify three types of solutions. The first, given in section 5.3.1, we interpret as a stack of fundamental strings in the presence of D8-branes, i.e. in a background with $F_{(0)} \neq 0$. The other two, given in section 5.3.2, we interpret as fundamental strings ending on a stack of D8-branes or an O8-plane. In all three cases the geometry contains an asymptotically flat region.

We also consider solutions where $F_{(0)}$ is allowed to jump across an interface, corresponding to the presence of a stack of D8-branes. This allows for a large family of solutions, parametrised by the number of such jumps. However, we find that there is no way to glue D8-brane caps or O8-plane caps together. Consequently, we argue that there are no solutions dual to $1+0$-dimensional CFTs.

This chapter is organised as follows. In section 5.1, we present our reduction of the BPS system for $D(2,1;\gamma;1) \times SO(4)$ invariant geometries, including a summary of the reduced equations. In section 5.2, we solve the BPS system

[1] We note this system was studied in [20] without the assumption of conformal symmetry.

for the special case $\gamma = -2$ with enhanced supersymmetry. This corresponds to the case of D4/D8-branes. In section 5.3, we solve the BPS system for the special case $\gamma = 1$ with enhanced supersymmetry. This corresponds to the case of fundamental strings ending on D8-branes or an O8-plane. We also discuss the possibility of solutions with a jumping $F_{(0)}$. Finally, in section 5.4, we present a discussion of the results. Additionally, the details of some of the calculations in this chapter are presented in appendix E.

5.1 $D(2,1;\gamma;1) \times SO(4)$ invariant geometries

We consider first the most general solutions of massive IIA supergravity which preserve the symmetry $D(2,1;\gamma;1) \times SO(4)$. The bosonic subalgebra $so(1,2) \times so(4) \times so(4)$ is naturally realised on the 10-dimensional spacetime $AdS_2 \times S^3 \times S^3 \times \Sigma_2$, where Σ_2 is a 2-dimensional space. The metric takes the form

$$ds^2 = f_1^2 ds^2_{AdS_2} + f_2^2 ds^2_{S^3} + f_3^2 ds^2_{S^3} + ds^2_{\Sigma_2}. \tag{5.1}$$

The dilaton ϕ and the warp factors f_i are restricted to be functions of Σ_2. It will be convenient to introduce the frames

$$\begin{aligned} e^m &= f_1 \hat{e}^m \qquad m = 0,1, \qquad & e^i &= f_2 \hat{e}^i \qquad i = 2,3,4, \\ e^{\tilde{i}} &= f_3 \hat{e}^{\tilde{i}} \qquad \tilde{i} = 5,6,7, \qquad & e^a & \qquad\qquad a = 8,9, \end{aligned} \tag{5.2}$$

where $\hat{e}^m$ are frames on the unit AdS_2, $\hat{e}^i$ and $\hat{e}^{\tilde{i}}$ are frames on the two unit S^3's and e^a are frames on Σ_2.

The 4-form flux takes the form

$$F_{(4)} = h_1 e^{0189} + h_a e^{234a} + g_a e^{567a} \tag{5.3}$$

while the 2-form gauge potential and corresponding 3-form field strength take the form[2]

$$B_{(2)} = b_0 \hat{e}^{01} + b_1 e^{89}, \qquad H_{(3)} = dB_{(2)} = db_0 \wedge \hat{e}^{01}. \tag{5.4}$$

The coefficients h_1, h_a, g_a, b_0 and b_1 are all functions of Σ_2.

[2] In principle one could allow for an H_3-flux on either of the two three-spheres but the Bianchi identities will then fix this flux to vanish.

In order to preserve the supersymmetries, the supersymmetry variations of the fermionic fields must vanish. From (2.18) we get the following BPS equations

$$
\begin{aligned}
0 =& \Big[(D_M\phi)\Gamma^M + \frac{5}{4}F_{(0)}e^{\frac{5}{4}\phi} + \frac{1}{96}e^{\frac{\phi}{4}}(F_{MNPQ}\Gamma^{MNPQ}) \\
& - \frac{3}{8}F_{(0)}e^{\frac{3\phi}{4}}B_{MN}\Gamma^{MN}\Gamma_{11} - \frac{1}{12}e^{-\frac{\phi}{2}}H_{MNP}\Gamma^{MNP}\Gamma_{11}\Big]\epsilon \;, \\
0 =& \Big[D_M - \frac{1}{32}F_{(0)}e^{\frac{5}{4}\phi}\Gamma_M + \frac{1}{128}\frac{e^{\frac{\phi}{4}}}{2}F_{NPQR}({\Gamma_M}^{NPQR} - \frac{20}{3}{\delta_M}^N\Gamma^{PQR}) \\
& - \frac{1}{32}F_{(0)}\frac{e^{\frac{3\phi}{4}}}{2}B_{NP}({\Gamma_M}^{NP} - 14{\delta_M}^N\Gamma^P)\Gamma_{11} \\
& + \frac{1}{48}\frac{e^{-\frac{\phi}{2}}}{2}H_{NPQ}({\Gamma_M}^{NPQ} - 9{\delta_M}^N\Gamma^{PQ})\Gamma_{11}\Big]\epsilon \;. \qquad (5.5)
\end{aligned}
$$

The conventions for Γ-matrices are defined in appendix A.4. In order to obtain massless IIA supergravity we perform the replacements (2.19) and take the limit $F_{(0)} \to 0$.

Fluxes

The fluxes of our ansatz have the following interpretation in terms of the brane constructions discussed in section 2.5.4. The h_a and g_a components of $F_{(4)}$ correspond to the D4-brane and D4'-branes of table 2.3. In massless type IIA theory, the fundamental string sources the NS-NS three form $H_{(3)}$, while the D0-branes source the Ramond-Ramond two form $F_{(2)}$. In massive IIA supergravity, the gauge transformation (2.20) mixes them. A related effect occurs in the brane description. When one pulls a D0-brane through a D8-brane, a fundamental string is created which stretches between the D8-brane and the D0-brane [115]. The D8-branes can be interpreted as a magnetic source for the scalar field strength $F_{(0)}$. Since we are interested in the brane configurations of section 2.5.4 and the D4 brane sources the $F_{(4)}$ field strength only magnetically, we set h_1 to zero. Similarly, the fundamental string sources the $B_{(2)}$ gauge potential only electrically so we set b_1 to zero.

5.1.1 BPS equations

Our first step is to reduce the BPS equations (5.5) to a two-dimensional system. This is carried out as follows. First the supersymmetry parameter, ϵ, is decomposed using a basis of Killing spinors for the symmetric spaces AdS_2 and S^3. We denote by $\chi_{\eta_1,\eta_2,\eta_3}$ a basis of Killing spinors on $AdS_2\times S^3\times S^3$, where $\eta_i=\pm1$. These can be explicitly constructed following [116] and satisfy the Killing spinor equations (E.6). The basis, $\chi_{\eta_1,\eta_2,\eta_3}$, is actually overcomplete so without loss of generality, we impose the conditions $\chi_{\eta_1,\eta_2,\eta_3}=\left(B_{(1)}\otimes B_{(2)}\otimes B_{(3)}\right)\chi^*_{\eta_1,\eta_2,\eta_3}$ and $\chi_{\eta_1,\eta_2,\eta_3}=(\sigma^3\otimes I_2\otimes I_2)\chi_{-\eta_1,\eta_2,\eta_3}$. Note that these conditions are consistent with the Killing spinor equations (E.6). We then write the 10-dimensional supersymmetry parameter as

$$\epsilon=\sum_{\eta_1,\eta_2,\eta_3}\chi_{\eta_1,\eta_2,\eta_3}\otimes\left[\zeta_{\eta_1,\eta_2,\eta_3}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\hat{\zeta}_{\eta_1,\eta_2,\eta_3}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right], \tag{5.6}$$

where the coefficients of the decomposition, $\zeta_{\eta_1,\eta_2,\eta_3}$ and $\hat{\zeta}_{\eta_1,\eta_2,\eta_3}$, are collections of two-component spinors. The type IIA reality condition $\epsilon^*=\mathcal{B}\epsilon$ relates the coefficients ζ and $\hat{\zeta}$ by $\hat{\zeta}^*=i\sigma_2\zeta$. The BPS equations can then be written as equations on ζ, the coefficients of the decomposition. This is carried out explicitly for the first equation of (5.5) in appendix E.1. The final result is summarised in equations (E.5)–(E.11). In giving these expressions, we have written ζ as a single 16-component spinor with the η_i-labels corresponding to spin indices. We also introduced the notation $\tau^{(ijk)}=\sigma^i\otimes\sigma^j\otimes\sigma^k$, where the i-th Pauli matrix acts on the η_i index.

To simplify the problem, we first look for symmetries of the equations (E.5)–(E.11). Both $\tau^{(030)}$ and $\tau^{(003)}$ commute with the BPS equations. In the special case $b_1=h_1=0$, $\tau^{(300)}\sigma^3$ also commutes with the BPS equations. We are interested in the brane configurations discussed in section 2.5.4 and correspondingly set $b_1=h_1=0$ throughout the rest of the paper. The reduced BPS equations for general values of b_1 and h_1 are given in appendix E.2. As a result of the three symmetries, we can impose the following projections on ζ without loss of generality

$$\zeta=\nu_1\tau^{(300)}\sigma^3\zeta\,,\qquad \zeta=\nu_2\tau^{(030)}\zeta\,,\qquad \zeta=\nu_3\tau^{(003)}\zeta\,, \tag{5.7}$$

where the ν_i are each a sign choice for the projection. The last two projections simply project onto the components of $\zeta_{\eta_1,\eta_2,\eta_3}$ with $\eta_2=\nu_2$ and $\eta_3=\nu_3$. The first projection is solved for the case $\nu_1=+1$ by setting $\zeta_{-+}=\zeta_{+-}=0$, where the first index corresponds to η_1 and the second to the spinor index. In this case, we may group the remaining components into a variable $\xi=(\zeta_{++},\zeta_{--})$. In the case $\nu_1=-1$, we have $\zeta_{++}=\zeta_{--}=0$ and we group the remaining components into a variable $\xi=(\zeta_{-+},\zeta_{+-})$.

After imposing the above projections, the BPS equation (E.5) reduces to

$$
\begin{aligned}
0 =& \frac{1}{2\sqrt{2}} D_z\phi\sigma^3\xi^* + \frac{5}{8\sqrt{2}} F_{(0)} e^{5\phi/4} i\sigma^2\xi - \frac{1}{8\sqrt{2}} e^{\phi/4}(h_z\sigma^2\xi^* + g_z\sigma^1\xi^*) \\
&+ \frac{3}{8\sqrt{2}} F_{(0)} e^{3\phi/4} \frac{b_0}{f_1^2}\sigma^1\xi - \frac{1}{4\sqrt{2}} e^{-\phi/2}\frac{D_z b_0}{f_1^2}\xi^* \,. \qquad (5.8)
\end{aligned}
$$

The equations (E.7)–(E.9) reduce to

$$
\begin{aligned}
0 =& -\frac{\nu_1}{2f_1}\sigma^3\xi + \frac{1}{2} D_z \ln f_1 \sigma^1\xi^* - \frac{1}{32} F_{(0)} e^{5\phi/4}\xi + \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_z\xi^* + g_z\sigma^3\xi^*\right) \\
&+ \frac{7}{16} F_{(0)} \frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^3\xi + \frac{3}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2} i\sigma^2\xi^* \,, \\
0 =& -\frac{i\nu_2}{2f_2}\sigma^1\xi + \frac{1}{2} D_z \ln f_2 \sigma^1\xi^* - \frac{1}{32} F_{(0)} e^{5\phi/4}\xi + \frac{3}{16}\frac{e^{\phi/4}}{2}\left(g_z\sigma^3\xi^* - i\frac{5}{3} h_z\xi^*\right) \\
&- \frac{1}{16} F_{(0)} \frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^3\xi - \frac{1}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2} i\sigma^2\xi^* \,, \\
0 =& -\frac{i\nu_3}{2f_3}\sigma^2\xi + \frac{1}{2} D_z \ln f_3 \sigma^1\xi^* - \frac{1}{32} F_{(0)} e^{5\phi/4}\xi + \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_z\xi^* - \frac{5}{3} g_z\sigma^3\xi^*\right) \\
&- \frac{1}{16} F_{(0)} \frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^3\xi - \frac{1}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2} i\sigma^2\xi^* \,. \qquad (5.9)
\end{aligned}
$$

Note that these equations are algebraic in ξ. The remaining equations (E.10)–(E.11) reduce to

$$
\begin{aligned}
0 =& D_z\xi - \frac{1}{2} D_z(\ln\rho)\,\xi + \frac{3}{16}\frac{e^{\phi/4}}{2}\left[i\frac{8}{3} h_z\sigma^1\xi + i\frac{8}{3} g_z\sigma^2\xi\right] - \frac{1}{2}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\sigma^3\xi \,, \\
0 =& D_{\bar z}\xi + \frac{1}{2} D_{\bar z}(\ln\rho)\xi - \frac{1}{16} F_{(0)} e^{5\phi/4}\sigma^1\xi^* + \frac{3}{16}\frac{e^{\phi/4}}{2}\left[i\frac{2}{3} h_{\bar z}\sigma^1\xi + i\frac{2}{3} g_{\bar z}\sigma^2\xi\right] \\
&+ i\frac{1}{8} F_{(0)} \frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^2\xi^* - \frac{1}{4}\frac{e^{-\phi/2}}{2}\frac{D_{\bar z} b_0}{f_1^2}\sigma^3\xi \,,
\end{aligned}
\qquad (5.10)
$$

which are differential in ξ.

We now examine the number of supersymmetries present in the system. Each $\chi_{\eta_1,\eta_2,\eta_3}$ has 8 parameters and leads to 8 supersymmetries. The projections

(5.7) project onto specific values for η_2 and η_3 so that $\eta_2 = \nu_2$ and $\eta_3 = \nu_3$, while the components with different values of η_1 are related by the constraint $\chi_{\eta_1,\eta_2,\eta_3} = (\sigma^3 \otimes I_2 \otimes I_2)\chi_{-\eta_1,\eta_2,\eta_3}$. Taking the background fields fixed, we will have 8 supersymmetries for each independent solution of ξ to the above equations. Generally we expect only one such solution. To look for cases with enhanced supersymmetry, we first look for symmetries of the equations and consider sending $\xi \to M\xi$. Requiring M to commute or anti-commute with the first two terms of (5.8) leads to the possibilities $M = \sigma^1$, $i\sigma^2$ and $i\sigma^3$. As a result we find the following three symmetries

$$\begin{aligned}
&\xi \to \sigma^1 \xi \qquad && b_0 \to -b_0 \qquad && g_z \to -g_z \qquad && \nu_{1(3)} \to -\nu_{1(3)}\,, \\
&\xi \to i\sigma^2 \xi \qquad && b_0 \to -b_0 \qquad && h_z \to -h_z \qquad && \nu_{1(2)} \to -\nu_{1(2)}\,, \\
&\xi \to i\sigma^3 \xi \qquad && g_z \to -g_z \qquad && h_z \to -h_z \qquad && \nu_{2(3)} \to -\nu_{2(3)}\,.
\end{aligned} \tag{5.11}$$

Note that in general these are not symmetries of the background fields, since we are required to flip the signs of the fluxes. However, we can see that whenever two of the fluxes vanish, we will double the number of supersymmetries. This is in agreement with the brane discussion of section 2.5.4.

In appendix E.3, we carry out a further reduction of the equations. We first integrate the equations of (5.9) which are differential in the f_i. We obtain

$$f_1 = \frac{\nu_1}{c_1}\xi^\dagger \xi\,, \qquad f_2 = \frac{\nu_2}{c_2}\xi^\dagger \sigma^2 \xi\,, \qquad f_3 = \frac{\nu_3}{c_3}\xi^\dagger \sigma^1 \xi\,, \tag{5.12}$$

where the ν_i are sign choices and the c_i are constants. Using the remaining equations of (5.9), we obtain an algebraic constraint for the c_i

$$c_1 - 2c_2 + 2c_3 = 0\,, \tag{5.13}$$

and an expression for b_0

$$b_0 = \frac{8\xi^\dagger \xi}{F_{(0)} c_1^2} e^{-3\phi/4} \left(\frac{1}{2}c_1 - \frac{3}{4}c_2 + \frac{3}{4}c_3 + \frac{1}{8}F_{(0)} e^{5\phi/4} \xi^\dagger \sigma^3 \xi \right)\,. \tag{5.14}$$

We introduce the notation $b_z = D_z b_0 / f_1^2$ and treat b_z as an independent variable from b_0. We show that the BPS equations correctly enforce the differential relation between them. The final summary of reduced equations is given in section 5.1.2. There we denote the components of ξ by α and β.

Equations of motion and Bianchi identities

In general the BPS equations are not sufficient to determine a valid supergravity solution. Namely, there can be additional constraints arising from the Bianchi identities and equations of motion. With this in mind, we first look at the Bianchi identities. Since we give an anstaz directly for $B_{(2)}$, the Bianchi identity for $H_{(3)}$ is automatic. The Bianchi identity for $F_{(4)}$ takes the form

$$dF_{(4)} = 0 \qquad \Rightarrow \qquad \begin{cases} \partial_z \left(f_2^3 \rho h_{\bar{z}}\right) - \partial_{\bar{z}} \left(f_2^3 \rho h_z\right) = 0\,, \\ \partial_z \left(f_3^3 \rho g_{\bar{z}}\right) - \partial_{\bar{z}} \left(f_3^3 \rho g_z\right) = 0\,. \end{cases} \tag{5.15}$$

We solve these equations by introducing the real functions φ_1 and φ_2 as $\partial_{\bar{z}}\varphi_1 = \nu_2 f_2^3 \rho h_{\bar{z}}$ and $\partial_{\bar{z}}\varphi_2 = \nu_3 f_3^3 \rho g_{\bar{z}}$. The Bianchi identities then become integrability conditions for the fields φ_i.

In appendix E.4 we check that the Bianchi identities together with the BPS equations imply the equations of motion.

5.1.2 Summary of equations

With a partial reduction of the BPS-equations to a two-dimensional system complete, it is time to summarise the remaining variables, BPS equations and Bianchi identities. The quantities b_0 and b_z appear only algebraically and can be eliminated. The remaining variables are then given by ϕ, φ_1, φ_2, ρ and the two spinor components α and β. All of these are functions of the coordinates $(z, \bar{z})$. In total this gives 4 real variables and 2 complex variables.

The metric factors are determined in terms of the spinor components by

$$f_1 = \frac{\nu_1}{c_1}\left(|\alpha|^2 + |\beta|^2\right), \quad f_2 = \frac{i\nu_2}{c_2}\left(\alpha\beta^* - \alpha^*\beta\right), \quad f_3 = \frac{\nu_3}{c_3}\left(\alpha\beta^* + \alpha^*\beta\right). \tag{5.16}$$

The constraint for the c_i and the Bianchi identities are

$$c_1 - 2c_2 + 2c_3 = 0\,, \qquad h_z = \frac{\nu_2}{f_2^3 \rho}\partial_z \varphi_1\,, \qquad g_z = \frac{\nu_3}{f_3^3 \rho}\partial_z \varphi_2\,. \tag{5.17}$$

The BPS-equations are reduced to eight coupled equations. There are three equations algebraic in α and β

$$\begin{aligned}
0 =& -\frac{F_{(0)}}{8} e^{3\phi/4} \frac{b_0}{f_1^2}\left(|\alpha|^2 + |\beta|^2\right) + \frac{1}{2}c_1 - \frac{3}{4}c_2 + \frac{3}{4}c_3 + \frac{1}{8}F_{(0)} e^{5\phi/4}\left(|\alpha|^2 - |\beta|^2\right), \\
0 =& 2\left(c_2 + c_3\right) + e^{\phi/4} g_z \left((\alpha^*)^2 + (\beta^*)^2\right) - i e^{\phi/4} h_z \left((\alpha^*)^2 - (\beta^*)^2\right), \\
0 =& -\frac{1}{4}F_{(0)} e^{5\phi/4}\left(|\alpha|^2 - |\beta|^2\right) + \frac{3}{4}\left(2c_1 - 3c_2 + 3c_3\right) \\
& - \frac{e^{\phi/4}}{8}\left[g_z\left((\alpha^*)^2 + (\beta^*)^2\right) + i h_z \left((\alpha^*)^2 - (\beta^*)^2\right)\right] - \frac{e^{-\phi/2}}{2} b_z \alpha^* \beta^*.
\end{aligned} \tag{5.18}$$

There is an equation involving the derivative of the dilaton

$$0=\frac{1}{\rho}\partial_z\phi\alpha^*+\frac{5}{4}F_{(0)}e^{5\phi/4}\beta+\frac{i}{4}e^{\phi/4}h_z\beta^*-\frac{1}{4}e^{\phi/4}g_z\beta^*$$
$$+\frac{3}{4}F_{(0)}e^{3\phi/4}\frac{b_0}{f_1^2}\beta-\frac{1}{2}e^{-\phi/2}b_z\alpha^*\,. \quad (5.19)$$

And finally, there are four equations which are differential in α and β

$$0=\frac{1}{\rho}\partial_z\alpha-\frac{1}{2}\frac{\partial_z\rho}{\rho^2}\;\alpha+\frac{i}{4}e^{\phi/4}h_z\beta+\frac{1}{4}e^{\phi/4}g_z\beta-\frac{1}{4}e^{-\phi/2}b_z\alpha,$$
$$0=\frac{1}{\rho}\partial_z\beta-\frac{1}{2}\frac{\partial_z\rho}{\rho^2}\beta+\frac{i}{4}e^{\phi/4}h_z\alpha-\frac{1}{4}e^{\phi/4}g_z\alpha+\frac{1}{4}e^{-\phi/2}b_z\beta,$$
$$0=\frac{1}{\rho}\partial_z\alpha^*+\frac{1}{2}\frac{\partial_z\rho}{\rho^2}\alpha^*-\frac{1}{16}F_{(0)}e^{5\phi/4}\beta-\frac{i}{16}e^{\phi/4}h_z\beta^*+\frac{1}{16}e^{\phi/4}g_z\beta^*$$
$$+\frac{1}{16}F_{(0)}e^{3\phi/4}\frac{b_0}{f_1^2}\beta-\frac{1}{8}e^{-\phi/2}b_z\alpha^*,$$
$$0=\frac{1}{\rho}\partial_z\beta^*+\frac{1}{2}\frac{\partial_z\rho}{\rho^2}\beta^*-\frac{1}{16}F_{(0)}e^{5\phi/4}\alpha-\frac{i}{16}e^{\phi/4}h_z\alpha^*-\frac{1}{16}e^{\phi/4}g_z\alpha^*$$
$$-\frac{1}{16}F_{(0)}e^{3\phi/4}\frac{b_0}{f_1^2}\alpha+\frac{1}{8}e^{-\phi/2}b_z\beta^*. \quad (5.20)$$

The equations possess a conformal symmetry, with the following weighting

$$\alpha,\beta:(\frac{1}{4},-\frac{1}{4})\quad \rho:(\frac{1}{2},\frac{1}{2})\quad g_z,h_z,b_z:(1,0)\quad b_0,\phi,\varphi_1,\varphi_2:(0,0)\,. \quad (5.21)$$

We also note that the equations have a real scaling symmetry under $\alpha\to\lambda\alpha$, $\beta\to\lambda\beta$ and $c_i\to\lambda^2c_i$, where λ is an arbitrary real number. This allows us to fix one of the c_i without loss of generality, by absorbing it into the definition of α and β. There is also a symmetry under $z\to -z$, $c_i\to -c_i$ and $F_{(0)}\to -F_{(0)}$.

The massless limit

Next we consider the case $F_{(0)}=0$. From the discussion in section 2.2 we know that in order to make the massless limit $F_{(0)}=0$ well defined, we first make the replacement (2.19), in particular $B_{(2)}\to B_{(2)}-F_{(0)}^{-1}F_{(2)}$. This introduces the closed two form[3] $F_{(2)} = b_1 \hat{e}^{01}$, where b_1 is a constant. Note that $A_{(1)} \wedge F_{(2)} = 0$ and so $C_{(3)}$ remains unmodified. Sending $B_{(2)} \to B_{(2)} - F_{(2)}/F_{(0)}$ amounts to sending $b_0 \to b_0 - b_1/F_{(0)}$. The massless limit is then obtained by letting $F_{(0)} \to 0$.

5.2 Enhanced supersymmetry: $\gamma = -1/2, -2$ and D4-branes

We consider the case of supersymmetry enhancement which occurs by setting $h_z = b_0 = 0$. By turning these functions off we preserve the symmetry in the second line of (5.11) and we shall see that the supersymmetry is enhanced to $F(4;2) \times SO(3)$. Solutions that preserve the symmetry of the first line can be found in a similar way to this case, by setting $g_z = h_z = 0$. The algebraic equations (5.18) combine to give the constraint $5c_1 - 7c_2 + 8c_3 = 0$, which together with $c_1 - 2c_2 + 2c_3 = 0$, implies

$$c_1 = -c_2 = -\frac{2}{3} c_3 \,. \tag{5.22}$$

We combine α times the first equation of (5.20) with β times the second to obtain

$$\partial_z \left(\frac{\alpha^2 + \beta^2}{\rho} \right) = 0 \,, \tag{5.23}$$

which implies the existence of a holomorphic (1,0)-function κ,

$$\bar{\kappa} = \frac{\rho}{\alpha^2 + \beta^2} \,. \tag{5.24}$$

The first and second algebraic constraints in (5.18) are solved to give ϕ and g_z

$$e^{5\phi/4} = \frac{2(c_2 + c_3)}{5F_{(0)}} \frac{1}{|\alpha|^2 - |\beta|^2} \,, \qquad g_z = -2(c_2 + c_3) e^{-\phi/4} \frac{\kappa}{\rho} \,. \tag{5.25}$$

We now rewrite the four equations in (5.20) as follows. First we differentiate the conjugate of (5.24) and use the last two equations of (5.20) to eliminate derivatives of α^* and β^*. This leads to a differential equation for ρ

$$\frac{1}{\kappa} \partial_z \ln \frac{\rho^2}{|\kappa|^2} = \frac{c_2 + c_3}{20} \frac{1}{|\alpha|^2 - |\beta|^2} \frac{c_3 f_3}{\nu_3} \,. \tag{5.26}$$

[3] Recall that the D0-brane would act as a source for $F_{(2)}$. From the brane configuration in table 2.3 we know that $A_{(1)}$ would have support along the 0-direction, leaving us with $F_{(2)}$ having support along the 01-directions as the only symmetry-preserving option.

Using this, as well as the equations (5.20), we find that the $\partial_z\phi$ equation leads to the constraint $5c_1 - 7c_2 + 8c_3 = 0$ and is automatically satisfied. We rewrite the first two differential equations of (5.20) as

$$\frac{1}{\kappa}\partial_z\left(\frac{\alpha\beta}{\rho}\right) = -\frac{1}{2}(c_2+c_3)\frac{\alpha^2-\beta^2}{\rho}\,,$$
$$\frac{1}{\kappa}\partial_z\left(\frac{\alpha^2-\beta^2}{\rho}\right) = 2(c_2+c_3)\frac{\alpha\beta}{\rho}\,. \tag{5.27}$$

For the final equation we use the combination

$$\partial_z(|\alpha|^2-|\beta|^2) = \frac{5}{8}(c_2+c_3)\frac{c_3\kappa}{\nu_3}f_3\,. \tag{5.28}$$

We now move on to solving the equations as follows. The strategy will be to first introduce h and $\tilde{h}$ by $(c_2+c_3)\kappa = \partial_z\tilde{h} = i\partial_z h$, such that everything is a function of h and $\tilde{h}$ instead of z and $\bar{z}$. Then integrate the equations (5.27) and use their results to find expression for α and β. This will introduce a new holomorphic function which we will fix in terms of h and $\tilde{h}$ through equations (5.26) and (5.28). Finally, with α and β fully determined we can then write down the full solution.

The general solution to (5.27) is given by

$$\frac{\alpha\beta}{\rho} = -\frac{1}{2}\mathcal{F}_1\sin(\frac{\tilde{h}+ih}{2}) + \mathcal{F}_2\cos(\frac{\tilde{h}+ih}{2})\,,$$
$$\frac{\alpha^2-\beta^2}{\rho} = \mathcal{F}_1\cos(\frac{\tilde{h}+ih}{2}) + 2\mathcal{F}_2\sin(\frac{\tilde{h}+ih}{2})\,, \tag{5.29}$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are functions of $\bar{z}$ only. For later convenience, we redefine

$$\mathcal{F}_1 = \frac{1}{2}\left(\frac{\bar{\omega}}{\bar{\kappa}} + \frac{1}{\bar{\omega}\bar{\kappa}}\right), \qquad \mathcal{F}_2 = \frac{1}{4i}\left(\frac{\bar{\omega}}{\bar{\kappa}} - \frac{1}{\bar{\omega}\bar{\kappa}}\right)\,, \tag{5.30}$$

where ω and κ are arbitrary holomorphic functions. We will show below that this definition is consistent with (5.24). It is convenient to work with the complex combinations

$$\alpha + i\beta = \rho^{1/2}\left(\frac{\bar{\omega}}{\bar{\kappa}}\right)^{1/2} e^{\frac{h-i\tilde{h}}{4}}, \qquad \alpha - i\beta = \rho^{1/2}\left(\frac{1}{\bar{\omega}\bar{\kappa}}\right)^{1/2} e^{-\frac{h-i\tilde{h}}{4}}\,. \tag{5.31}$$

Multiplying these together reproduces (5.24). We are now ready to turn to the differential equation for ρ, we will use this equation as well as $\partial_z\bar{\omega} = 0$ to constrain the holomorphic function ω. Using the above expressions for α and β, (5.26) becomes

$$\frac{1}{\kappa}\partial_z \ln \frac{\rho^2}{|\kappa|^2} = \frac{i(c_2+c_3)}{20}\frac{\omega - \bar{\omega}e^{-i\tilde{h}}}{\omega + \bar{\omega}e^{-i\tilde{h}}}. \tag{5.32}$$

Differentiating this equation with respect to $\bar{z}$, multiplying by κ and requiring the result to be real leads to the condition

$$i\frac{1}{\bar{\kappa}}\partial_{\bar{z}}|\omega|^2 + i\frac{1}{\kappa}\partial_z|\omega|^2 = 4(c_2+c_3)i\partial_{\tilde{h}}|\omega| = 0\,. \tag{5.33}$$

The condition $\partial_{\tilde{h}}|\omega| = 0$ leads through $\partial_z\bar{\omega} = 0$ also to $\partial_h\theta_\omega = 0$, where we defined $\omega = |\omega|e^{i\theta_\omega}$.

Next we note that the differential equation for $|\alpha|^2 - |\beta|^2$, (5.28), can be combined with the ρ-equation, (5.26), to give

$$\partial_z \ln\left(\frac{\rho^2}{|\kappa|^2}(|\alpha|^2-|\beta|^2)^{-2/25}\right) = 0\,. \tag{5.34}$$

This is easily integrated to give

$$\frac{\rho^2}{|\kappa|^2} = A^{24/25}(|\alpha|-|\beta|^2)^{2/25} = A\cos^{1/12}\left(\theta_\omega + \frac{\tilde{h}}{2}\right) \tag{5.35}$$

where A is a real constant. Turning to the differential equation for ρ, we require (5.35) to be a solution. Plugging in we find that $\theta_\omega = \frac{1}{10}\tilde{h}$ and also $|\omega| = e^{-\frac{1}{10}h}$. Note that we have absorbed integration constants into the definitions of h and $\tilde{h}$.

We have now solved the system and give the final expressions for the supergravity fields. The metric factors are given by

$$\begin{aligned} f_1 &= \frac{\nu_1}{c_1}A^{1/2}\cos^{1/24}(\tfrac{3}{5}\tilde{h})\frac{1}{2}\left(e^{\frac{2}{5}h} + e^{-\frac{2}{5}h}\right), \qquad f_3 = -\frac{\nu_3}{c_3}A^{1/2}\cos^{1/24}(\tfrac{3}{5}\tilde{h})\sin(\tfrac{3}{5}\tilde{h}), \\ f_2 &= -\frac{\nu_2}{c_2}A^{1/2}\cos^{1/24}(\tfrac{3}{5}\tilde{h})\frac{1}{2}\left(e^{\frac{2}{5}h} - e^{-\frac{2}{5}h}\right), \quad \frac{\rho^2}{|\kappa|^2} = \cos^{1/24}(\tfrac{3}{5}\tilde{h}). \end{aligned} \tag{5.36}$$

The dilaton and flux are given by

$$\begin{aligned} e^{5\phi/4} &= \frac{2}{5}\frac{c_2+c_3}{F_{(0)}}A^{-1/2}\cos^{-25/24}(\tfrac{3}{5}\tilde{h}), \\ \varphi_2 &= \frac{5}{2}\frac{A^{3/2}}{c_3^3}\left(\frac{2}{5}\frac{c_2+c_3}{F_{(0)}A^{1/2}}\right)^{-1/5}\cos^{4/3}(\tfrac{3}{5}\tilde{h})\,. \end{aligned} \tag{5.37}$$

with $g_z = \nu_3(\partial_z\varphi_2)/f_3^3\rho$.

This metric reproduces exactly the AdS_6 solution of [19]. This can be seen by introducing the new coordinates x and y by $h = 5x/2$ and $\tilde{h} = 5y/3$. We also introduce the overall radius R by $A^{1/2} = \frac{2}{5}(c_2 + c_3)R$. In these coordinates, the metric is given by

$$ds^2 = R^2 \cos^{\frac{1}{12}} y \left[\left(dx^2 + \cosh^2 x\, ds^2_{AdS_2} + \sinh^2 x\, ds^2_{S^3}\right) + \frac{4}{9}\left(dy^2 + \sin^2 y\, ds^2_{S^3}\right)\right]. \tag{5.38}$$

The terms in the first set of parenthesis combine into an $AdS_2 \times S^3$ slicing of AdS_6, whereas the terms in the second set combine to make an S^4. The dilaton and flux in these coordinates are

$$e^{\phi/4} = (RF_{(0)})^{-1/5} \cos^{-5/24} y, \qquad \varphi_2 = \frac{4}{9} R^3 (RF_{(0)})^{1/5} \cos^{4/3} y. \tag{5.39}$$

The case $F_{(0)} = 0$

There are no solutions with $F_{(0)} = 0$. To see this we first take the $F_{(0)} \to 0$ limit of the BPS equations of section 5.1.2. Since we are interested in solutions with the only non-vanishing flux given by g_z, we may simply take $F_{(0)} = 0$. In this case, the algebraic equations (5.18) become

$$0 = 2c_1 - 3c_2 + 3c_3, \tag{5.40}$$

$$0 = 2(c_2 + c_3) + e^{\phi/4}\left((\alpha^*)^2 + (\beta^*)^2\right)$$

$$0 = 6(2c_1 - 3c_2 + 3c_3) - e^{\phi/4}\left((\alpha^*)^2 + (\beta^*)^2\right).$$

Along with the original constraints (5.17), we obtain a total of three constraints on the c_i

$$c_2 + c_3 = 0, \qquad 2c_1 - 3c_2 + 3c_3 = 0, \qquad c_1 - 2c_2 + 2c_3 = 0, \tag{5.41}$$

from which it follows that $c_i = 0$. This implies that the warpfactors f_i are all zero as well, and through this that our metric would be identically zero. Hence, we conclude that there are no solutions with $F_{(0)} = 0$.

5.3 Enhanced supersymmetry: $\gamma = 1$ and fundamental strings

In this section we consider the case in which supersymmetry is enhanced by setting $g_z = h_z = 0$. By turning these functions off we preserve the symmetry in the third line of (5.11) and we shall see that the supersymmetry is enhanced to $OSp(8|2,\mathbb{R})$.

The second equation of (5.18) yields the condition $c_2 = -c_3$, which together with the constraint $c_1 - 2c_2 + 2c_3 = 0$, implies $c_1 = 4c_2 = -4c_3$. Next, we combine β times the first equation of (5.20) with α times the second to yield the vanishing of a total derivative, $\partial_z \ln(\alpha\beta/\rho) = 0$. This implies the existence of a holomorphic $(1,0)$-function κ,

$$\bar{\kappa} = c_2 \frac{\rho}{\alpha\beta} \,. \tag{5.42}$$

The factor of c_2 has been chosen for convenience. We write the remaining three independent equations of (5.20) as

$$\begin{aligned}
\partial_z b_0 =& 2 f_1^2 e^{\phi/2} \partial_z \ln\left(\frac{\alpha}{\beta}\right) , \\
\partial_z \ln\left(\alpha(\beta^*)^2 \sqrt{\rho}\right) =& \frac{1}{8}\left(1 + \frac{b_0}{f_1^2} e^{-\phi/2}\right) F_{(0)} e^{5\phi/4} \frac{\rho\alpha}{\beta^*} , \\
\partial_z \ln\left(\beta(\alpha^*)^2 \sqrt{\rho}\right) =& \frac{1}{8}\left(1 - \frac{b_0}{f_1^2} e^{-\phi/2}\right) F_{(0)} e^{5\phi/4} \frac{\rho\beta}{\alpha^*} ,
\end{aligned} \tag{5.43}$$

where we have used $b_z = (\partial_z b_0)/f_1^2 \rho$. We solve these equations to give b_0 and ϕ as functions of the remaining variables α, β and ρ. The remaining equation for $\partial_z b_0$ is then automatic. After substituting in the above solutions for b_0 and ϕ and eliminating β in terms of κ, the first equation of (5.18) becomes

$$0 = |\kappa|^2 + 2\bar{\kappa}\left[\frac{|\kappa|^2|\alpha|^4}{\rho^2} \partial_z \ln\left(\frac{(\alpha^*)^2}{\alpha} \rho^{\frac{3}{2}}\right) - \frac{\rho^2}{|\kappa|^2|\alpha|^4} \partial_z \ln\left(\frac{\alpha}{\kappa^2(\alpha^*)^2} \rho^{\frac{5}{2}}\right)\right] . \tag{5.44}$$

Next we take the sum of the first equation in (5.18) and third equation in (5.18) and again eliminate b_0, ϕ and β to obtain

$$0 = \partial_z \ln\left(\frac{\alpha^4}{\kappa^2(\alpha^*)^4}\right) - 2\kappa \,. \tag{5.45}$$

The final remaining equation is given by (5.19).

We integrate (5.45) as follows. We parametrise the magnitude and phase of α by the quantities A and θ as $\alpha = \sqrt{c_2 \rho} A^{1/4} e^{i\theta}/\sqrt{\bar{\kappa}}$. We also introduce a real harmonic function h by $i\partial_z h = \kappa$. In terms of these quantities, (5.45) becomes

$$h = 4\theta\,. \tag{5.46}$$

With this we can write (5.44) as a differential equation for ρ in terms of A and κ

$$\partial_z \ln\left(\frac{\rho^2}{|\kappa|^2}\right) = -\frac{1}{4}\left(\frac{3A^2 - 2A + 3}{1 - A^2}\right)\kappa + \frac{1}{4}\left(\frac{1 + A^2}{1 - A^2}\right)\partial_z \ln A\,. \tag{5.47}$$

Using the above equation, we can eliminate ρ from the equations and cast the remaining system as a pair of first order differential equations for ϕ and A. The first is obtained the last two equations in (5.43) after eliminating b_0. The second is the differential equation (5.19). It will be slightly more convenient to introduce a new variable G for the dilaton, defined by the equation

$$e^{\frac{5\phi}{2}} = \frac{|\kappa|^2}{\rho^2}\frac{AG^2}{F_{(0)}^2}\,. \tag{5.48}$$

Note that G must have the same sign as the product $\rho\, F_{(0)}$. In terms of G and A the remaining first order system is given by

$$\begin{aligned} \partial_z \ln(G) &= -\frac{5A + 1}{1 - A^2}\partial_z \ln(A) - \frac{A^2 - 14A + 1}{1 - A^2}\kappa\,, \\ \kappa G &= -\frac{4}{1 - A}\kappa + \frac{2}{1 - A}\partial_z \ln(A)\,. \end{aligned} \tag{5.49}$$

Our approach will be to solve equations (5.49) to obtain expressions for A and G. We then integrate (5.47) to obtain ρ. The metric factors, dilaton and fluxes are then determined uniquely in terms G, A, ρ and h. The metric factors are

$$f_1^2 = \frac{\rho^2}{16|\kappa|^2} A\left(1 + A^{-1}\right)^2\,, \quad f_2^2 = \frac{4\rho^2}{|\kappa|^2}\sin^2\left(\frac{h}{2}\right)\,, \quad f_3^2 = \frac{4\rho^2}{|\kappa|^2}\cos^2\left(\frac{h}{2}\right)\,, \tag{5.50}$$

while the dilaton ϕ and flux b_0 are given by

$$e^{\frac{5\phi}{2}} = \frac{|\kappa|^2}{\rho^2}\frac{AG^2}{F_{(0)}^2}\,, \quad b_0 = \frac{e^{-\frac{3\phi}{4}}}{4F_{(0)}}\frac{\rho}{|\kappa|}\left(A^{-1/2} + A^{1/2}\right)\left(1 - \frac{1 - A}{4}G\right)\,. \tag{5.51}$$

From these expressions, we can see the presence of an $SO(8)$ symmetry as follows. By a conformal transformation, we may pick h as a coordinate. Introducing the dual coordinate $\tilde{h}$ as $\kappa = \partial_z \tilde{h}$, this corresponds to the choice $z = \tilde{h} + ih$ with $\kappa = 1/2$. With this choice of coordinates, the differential equations (5.49) imply that A and G depend only on $\tilde{h}$. Similarly, (5.47) implies ρ only depends on $\tilde{h}$. As a result, we find the metric is given by

$$ds^2 = f_1^2 ds^2_{AdS_2} + 4\rho^2 d\tilde{h}^2 + 16\rho^2 \left[\frac{dh^2}{4} + \sin^2\left(\frac{h}{2}\right) ds^2_{\mathbb{S}^3} + \cos^2\left(\frac{h}{2}\right) ds^2_{\mathbb{S}^3}\right]. \tag{5.52}$$

The metric in brackets is that of 7-sphere with unit radius, whose isometry group is $SO(8)$. As a consequence, we find that the full symmetry group is $OSp(8|2, \mathbb{R})$, as advertised.

5.3.1 Linear dilaton

We present a simple solution to the equations (5.49) obtained by taking A constant. Assuming A is constant leads to the condition $A^2 - 14A + 1 = 0$. This has two solutions $A = 7 \pm 4\sqrt{3}$ with the corresponding G given by $G = -2 \pm (4/\sqrt{3})$. For constant A, the ρ equation (5.47) becomes

$$\partial_z \ln\left(\frac{\rho^2}{|\kappa|^2}\right) = \pm\frac{5}{4\sqrt{3}}\kappa. \tag{5.53}$$

To integrate, we introduce $\tilde{h}$ so that $\kappa = \partial_z \tilde{h}$. Integrating then gives $\rho^2 = L^2|\kappa|^2 e^{\pm\frac{5}{4\sqrt{3}}\tilde{h}}$, where L is an integration constant.

We are still free to make holomorphic transformations and by a local change of coordinates, we can choose $z = \tilde{h} + ih$. This corresponds to using a conformal transformation to set $\kappa = 1/2$. Finally, it will be convenient to introduce the rescaled variables x and θ for $\tilde{h}$ and h so that $z = \pm(4\sqrt{3}/5)x + 2i\theta$.

The dilaton and flux are given by

$$e^{-5\phi/2} = \frac{3}{4}L^2 F_{(0)}^2 e^x, \qquad b_0 = 3^{3/10}\left(\frac{2L}{F_{(0)}}\right)^{2/5} e^{\frac{4}{5}x}. \tag{5.54}$$

The metric factors are given by

$$f_1^2 = L^2 e^x, \qquad f_2^2 = 4L^2\sin^2(\theta)e^x, \qquad f_3^2 = 4L^2\cos^2(\theta)e^x, \tag{5.55}$$

and the metric becomes

$$ds^2 = L^2 e^x\left(\frac{48}{25}dx^2 + ds^2_{AdS_2} + 4\, ds^2_{S^7}\right). \tag{5.56}$$

As $x \to \infty$, the geometry becomes asymptotically flat and the string coupling tends to zero. As $x \to -\infty$, the geometry becomes strongly curved and the string coupling becomes large. Note that in $(\tilde{h}, h)$-coordinates, the two solutions are simply mirrors of each other.

5.3.2 General solutions

In this section, we study general solutions to the system of equations (5.49). The differential equations have singularities at $A = 0, 1, \infty$. We start by analyzing the solution in the neighborhood of each of these points.

We start by obtaining a solution to (5.49) in the large A limit. Using the second equation of (5.49) to eliminate $\partial_z \ln A$ in the first equation and then dropping terms which are sub-leading in the large A limit, the equations reduce to

$$\partial_z \ln(A) \sim -\frac{A}{2}\kappa G + 2\kappa\,, \qquad \partial_z \ln G \sim -\frac{5}{2}\kappa G + \kappa\,. \tag{5.57}$$

The right equation can be easily integrated by again introducing $\tilde{h}$ as $\partial_z \tilde{h} = \kappa$, to give

$$G \sim \frac{2\,\mathcal{C}_1 e^{\tilde{h}}}{1 + 5\,\mathcal{C}_1 e^{\tilde{h}}}, \tag{5.58}$$

where $\mathcal{C}_1$ is a real integration constant.

Turning now to the first equation, we first assume that G is finite in the large A limit. In this case, we can neglect the second term on the left and side and the equation is easily integrated to give

$$\frac{1}{A} \sim \frac{1}{5} \ln\left(1 + 5\mathcal{C}_1 e^{\tilde{h}}\right) + \mathcal{C}_2, \tag{5.59}$$

where $\mathcal{C}_2$ is another real integration constant. We introduce a new coordinate λ by

$$\tilde{h} = \ln[\left(e^{5\lambda - 5\mathcal{C}_2} - 1\right)/5\mathcal{C}_1]. \tag{5.60}$$

Expanding around $\lambda = 0$ leads to the following asymptotic behavior

$$\begin{aligned} &\text{Case I}: && A \sim \lambda^{-1}\,, && G \sim 6\lambda\,, && (\mathcal{C}_2 = 0) \\ &\text{Case II}: && A \sim \lambda^{-1}\,, && G \sim G_0\,, && (\mathcal{C}_2 \neq 0) \end{aligned} \tag{5.61}$$

where $G_0 = 2(1 - e^{5\mathcal{C}_2})/5$. For the case $\mathcal{C}_2 = 0$, we find that AG is of order one and so we must keep the second term in the first equation of (5.57), which then yields the correct factor of 6.

By a local change of coordinates, we can again pick $z = \tilde{h} + ih$, corresponding to setting $\kappa = 1/2$. Expanding the differential equation (5.47) for ρ to leading order and integrating, we find that the metric on Σ_2 has the asymptotic form

$$\begin{aligned} \text{Case I}: \qquad & 4\rho^2(d\tilde{h}^2 + dh^2) \sim L^2\frac{d\lambda^2}{\lambda} + L^2\lambda dh^2\,, \\ \text{Case II}: \qquad & 4\rho^2(d\tilde{h}^2 + dh^2) \sim L^2\lambda^{\frac{1}{4}}\left(dh^2 + \frac{4}{G_0^2}d\lambda^2\right)\,, \end{aligned} \tag{5.62}$$

where L is an integration constant. The warp factors behave as

$$\begin{aligned} \text{Case I}: \quad & f_1^2 \sim \frac{L^2}{16}\,, \qquad f_2^2 \sim 4L^2\lambda\sin^2\left(\frac{h}{2}\right)\,, \qquad f_3^2 \sim 4L^2\lambda\cos^2\left(\frac{h}{2}\right)\,, \\ \text{Case II}: \quad & f_1^2 \sim \frac{L^2}{16\lambda^{\frac{3}{4}}}\,, \quad f_2^2 \sim 4L^2\lambda^{\frac{1}{4}}\sin^2\left(\frac{h}{2}\right)\,, \quad f_3^2 \sim 4L^2\lambda^{\frac{1}{4}}\cos^2\left(\frac{h}{2}\right)\,. \end{aligned} \tag{5.63}$$

For case I, we change coordinates to $h = 2\theta$ and $\lambda = r^2$ and for case II, we introduce $h = 2\theta$. In these coordinates, the asymptotic metrics take the form

$$\begin{aligned} \text{Case I}: \qquad & ds^2 \sim L^2\left(\frac{1}{16}ds^2_{AdS_2} + 4dr^2 + 4r^2ds^2_{\mathbb{S}^7}\right)\,, \\ \text{Case II}: \qquad & ds^2 \sim L^2\lambda^{\frac{1}{4}}\left(\frac{1}{16\lambda}ds^2_{AdS_2} + \frac{4}{G_0^2}d\lambda^2 + 4ds^2_{\mathbb{S}^7}\right)\,. \end{aligned} \tag{5.64}$$

The asymptotic values of the dilaton and flux are given by

$$\begin{aligned} \text{Case I}: \qquad & e^{\frac{5\phi}{2}} \sim \frac{36}{L^2F_{(0)}^2}\,, \qquad b_0 \sim \frac{5}{8}\left(\frac{L^8}{6^3F_{(0)}^2}\right)^{\frac{1}{5}}\,, \\ \text{Case II}: \qquad & e^{\frac{5\phi}{2}} \sim \frac{G_0^2}{L^2F_{(0)}^2\lambda^{\frac{5}{4}}}\,, \qquad b_0 \sim \frac{1}{16\lambda}\left(\frac{G_0^2L^8}{F_{(0)}^2}\right)^{\frac{1}{5}}\,. \end{aligned} \tag{5.65}$$

For case I, we see that as $r \to 0$ the geometry caps off smoothly. Furthermore, the dilaton, ϕ, and flux, b_0, both remain finite. For case II, the geometry is singular as $\lambda \to 0$, while the dilaton, ϕ, and flux, b_0, both diverge. We note that the metric is regular in string-frame, although the coupling still diverges.[4]

Next we examine the solution near $A = 0$. Assuming $A \sim 0$ and eliminating G in favor of a second order equation for A, we obtain the approximate equation

[4]This is easily seen by recalling that $ds^2_{\text{string}} = e^{\phi/2}ds^2_{\text{Einstein}}$.

$2A + \partial_h A - \partial_h^2 A = 0$, whose general solution is given by $A \sim \mathcal{C}_3 e^{-\tilde{h}} + \mathcal{C}_4 e^{2\tilde{h}}$. Since we are in the $A \sim 0$ approximation, this equation implies the following allowed behaviors: $A \sim e^{2\tilde{h}}$ as $\tilde{h} \to -\infty$, $A \sim e^{-\tilde{h}}$ as $\tilde{h} \to \infty$ or A has a first order zero in $\tilde{h}$. The first case is an exact solution with $G = 0$. This leads to a solution with $e^{\phi} = 0$ and a divergent b_0. The other two cases have the following asymptotics

$$\begin{aligned} &\text{Case III}: \qquad A \sim \lambda\,, \qquad G \sim -6\,, \qquad \left(A \sim e^{-\tilde{h}}\right) \\ &\text{Case IV}: \qquad A \sim \lambda\,, \qquad G \sim G_0 \lambda^{-1}\,, \qquad \left(A \sim \frac{G_0}{2}\tilde{h} - \frac{G_0}{2}\tilde{h}_0\right) \end{aligned} \tag{5.66}$$

where for case III, we introduced λ by $\tilde{h} = -\ln\lambda$ and for case IV, G_0 and $\tilde{h}_0$ are integration constants and we have introduced λ by $\lambda = G_0(\tilde{h} - \tilde{h}_0)/2$. It turns out that the asymptotic geometry takes the same form as in the $A \sim \infty$ cases. Namely case III leads to the same asymptotics given in (5.64) and (5.65) for case I, while case IV leads to the same asymptotics for case II.

For the special point $A = 1$, we find that A admits a series expansion as a polynomial in h. G can have one of either two behaviours. Either it admits a regular series expansion, with the value of G arbitrary at $A = 1$ or G has a linear divergence at $A = 1$ such that

$$\text{Case V}: \qquad A \sim 1\,, \qquad G \sim \frac{2}{3}\lambda^{-1}\,, \tag{5.67}$$

where $\lambda = (\tilde{h} - \tilde{h}_0)$ and $\tilde{h}_0$ is the location of $A = 1$. For this case, we change coordinates to $h = 2\theta$ and $\lambda = 2r$. In these coordinates, the asymptotic metrics take the form

$$\text{Case V}: \qquad ds^2 \sim L^2 r^{\frac{1}{12}} \left(ds^2_{AdS_2} + 16 dr^2 + 16 ds^2_{\mathbb{S}^7}\right)\,. \tag{5.68}$$

The asymptotic values of the dilaton and flux are given by

$$\text{Case V}: \qquad e^{\frac{5\phi}{2}} \sim \frac{1}{26 L^2 F_{(0)}^2 r^{\frac{25}{12}}}\,, \qquad b_0 \sim \frac{5}{4}\left(\frac{6^3 L^8}{F_{(0)}^2}\right)^{\frac{1}{5}} r^{\frac{2}{3}}\,. \tag{5.69}$$

This solution is singular as $r \to 0$. However, we observe that the singularity is of the same type as that which occurs for the AdS_6 solution given in (5.38) and (5.39). In the AdS_6 case, the singularity was attributed to the presence of an O8-plane [19]. Since we observe the same singularity structure, we interpret this solution as describing a fundamental string ending on an O8-plane. The first type of behavior corresponds to a regular interior point and yields the behavior given in figure 5.4 as we will see in the numerical analysis of the next section.

The above analysis parallels nicely with the brane picture. We associate the case I behavior with a string ending on a stack of D8-branes from the left, where we have used κ to define the orientation. The case III behavior can be associated with a fundamental string ending on a stack of D8-branes from the right. The case V behavior contains two disconnected solutions. For the first, we take $h > h_0$ with $G > 0$, this corresponds to a string ending on an O8-plane from the left and for the second we take $h < h_0$ with $G < 0$, corresponding to a string ending on an O8-plane from the right. Finally, we conjecture that the simple solution of section 5.3.1 can be associated with an infinite string in the presence of a non-zero Roman's mass.

5.3.3 Numerics

For each of the cases in the previous section, one can work out the series solution to any finite order in $e^{\tilde{h}}$. However, we find the series expansion always breaks down for some finite value of $\tilde{h}$. In order to understand the global structure of the solutions, we therefore solve the differential equations numerically. To do so, we first find approximate series solutions for each of the singular points, corresponding to cases I-V of the previous section. We use these series solutions to generate initial data for A and G away from the singular points. Finally, we use this initial data to numerically solve the pair of differential equations given in (5.49).

For case I, the differential equations admit the series solution

$$A = a_{-1}e^{-\tilde{h}} + a_0 + a_1 e^{\tilde{h}} + \ldots = \sum_{n=-1}^{\infty} a_n e^{n\tilde{h}}\,,$$

$$G = g_1 e^{\tilde{h}} + g_2 e^{2\tilde{h}} + \ldots = \sum_{n=1}^{\infty} g_n e^{n\tilde{h}}\,, \tag{5.70}$$

where the coefficients a_i and g_i are constants and by a choice of coordinates we can set $a_{-1} = 1$. The equations (5.49) can be written as

$$\begin{aligned} 0 =& A(1-A^2)\partial_{\tilde{h}}G + G(5A+1)\partial_{\tilde{h}}A + A(A^2 - 14A + 1)G\,, \\ 0 =& A(1-A)G + 4A - 2\partial_{\tilde{h}}A\,, \end{aligned} \tag{5.71}$$

where we have used $\partial_z \tilde{h} = \kappa$ and dropped an overall factor of κ from the equations. Plugging in the series expansion, one can recursively solve for all the remaining coefficients in terms of a_{-1}. Since the differential equations are non-linear, it is difficult to obtain a closed form expression for the recurrence relation. However, one can explicitly solve for the coefficients to any given order. We have used MATHEMATICA to solve for the first fifty coefficients, with the first few given by

$$\begin{aligned} &a_{-1} = 1, && a_0 = 15, && a_1 = -12, && a_2 = 144, \\ &g_1 = 6, && g_2 = -114 && g_3 = 2166 && g_4 = -41250. \end{aligned} \tag{5.72}$$

Near $\tilde{h} \sim -3$, the series expansion exhibits rapid oscillations and appears to break down. To obtain the behavior beyond this point, we solve the equations (5.71) numerically. We use the series expansion to generate the initial data. Starting at $\tilde{h}_0 = -15$, we find from the series solution that $A(\tilde{h}_0) = 3.27 \times 10^6$ and $G(\tilde{h}_0) = 1.84 \times 10^{-6}$. The result of this numerical solution is shown in figure 5.1. We see a nice agreement between the numerical solution and the series solution up to $\tilde{h} \sim -3$. Beyond that we see the numerical solution smoothly interpolates to the constant A and G solution of section 5.3.1. We conclude that case I, provides a smooth, weakly curved geometry.

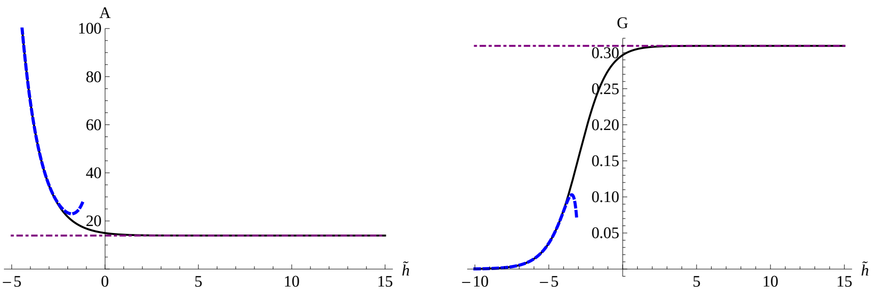


Figure 5.1: Plot of A (left) and G (right) for case I. The solid line is the numerically generated solution. The dot-dashed horizontal line is the constant solution of section 5.3.1. The dashed line is the series solution (5.70) truncated at order $n = 2$.

The remaining cases proceed in a similar manner. For case III, there is a unique numerical solution as in case I. For case V, two numerical solutions can be generated, one from taking data to the right of the $A = 1$ point and the other by taking data from the left. The results of cases I, III and V are shown together in figure 5.2. In all three cases, the solutions asymptote to the constant A and G solution of section 5.3.1. Note that case III is a reflected version of case I, while the left and right solutions of case V are reflections of each other.

For case II, there is a one parameter family of solutions, which are shown in figure 5.3 for positive values of G_0 and in figure 5.4 for negative values of G_0. Again, we observe that the solution asymptotes to the constant A and G

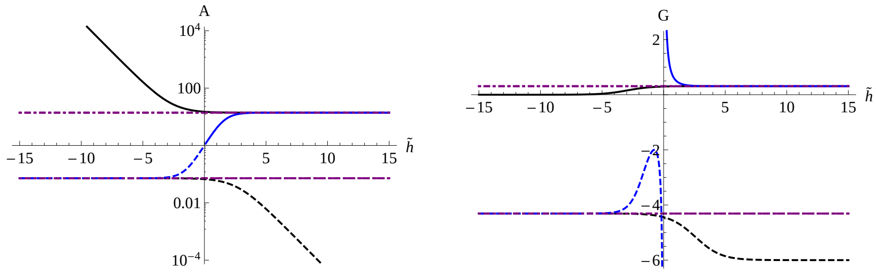


Figure 5.2: Plot of A (left) and G (right) versus $\tilde{h}$. The black solid (dashed) line corresponds to case I (III) while the blue solid (dashed) line corresponds to the right (left) extension of case V. The dot-dashed horizontal line is the constant solution of section 5.3.1.

solution. Case IV is a reflected version of case II. We note that cases II and IV are both singular and do not necessarily have a brane interpretation.

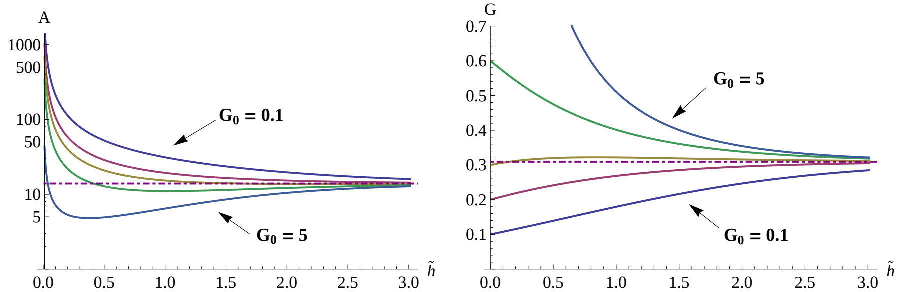


Figure 5.3: Plot of A (left) and G (right) versus $\tilde{h}$ for case II $G_0 = (0.1, 0.2, 0.3, 0.6, 5)$. The dot-dashed horizontal line is the constant G solution of section 5.3.1.

In all cases, we observe that the solutions asymptote to the constant A and G solution. This is consistent with the brane interpretation of the previous section. Namely, the fact that all the solutions asymptote to the constant A and G solution corresponds to the fact that the fundamental strings are all semi-infinite. As a result all the solutions are non-compact. From the analysis of the previous section we also conclude that cases I and III are smooth while the other cases have singularities.

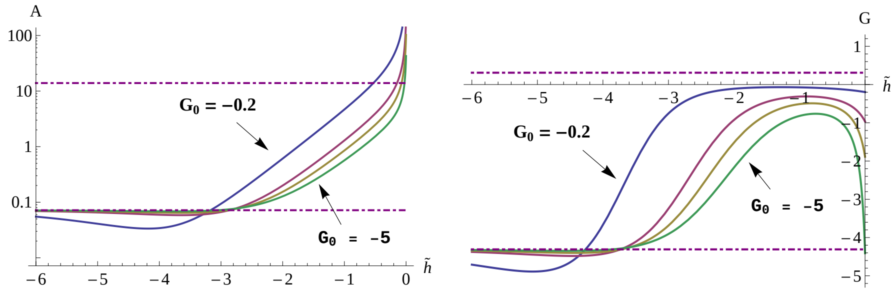


Figure 5.4: Plot of A (left) and G (right) versus $\tilde{h}$ for case II with $G_0 = (-0.2, -1, -2, -5)$. The dot-dashed horizontal lines are the constant G solution of section 5.3.1.

5.3.4 Patching solutions with jumping $F_{(0)}$

In this section, we consider D8-brane domain walls, where the value of $F_{(0)}$ jumps across the D8-brane. The physical fields are required to be continuous across the D8-brane [117]. In particular, the metric and gauge potentials are required to be continuous functions. We parametrise the jump with a real parameter λ such that $F^{-}_{(0)} = \lambda F^{+}_{(0)}$. Where $F^{-(+)}_{(0)}$ is the value of $F_{(0)}$ on the left (right) side of the D8-branes. Assuming that ρ is continuous, the expressions (5.50) for the f_i are continuous if and only if A is continuous. For ρ, we note that it is determined by the differential equation (5.47) and we may always choose the integration constant so that ρ is continuous. The expression for the dilaton in (5.51) is continuous if and only if we assume G jumps across the domain wall so that $G^{-} = \lambda G^{+}$.

At first sight, it seems that B cannot be made continuous across the interface. However, this is due to our choice of gauge for the B field. We first introduce a constant parameter b_1 and make the gauge transformation given by sending $B_{(2)} \to B_{(2)} - (b_1/F_{(0)})\hat{e}^{01}$ and $F_{(2)} \to b_1 \hat{e}^{01}$. This amounts to introducing $F_{(2)}$ and taking

$$b_0 = \frac{e^{-\frac{3\phi}{4}}}{4F_{(0)}} \frac{\rho}{|\kappa|} \left(A^{-1/2} + A^{1/2}\right) \left(1 - \frac{1-A}{4} G\right) - \frac{b_1}{F_{(0)}} \tag{5.73}$$

Note that $C_{(3)}$ does not transform since $A_{(1)} \wedge F_{(2)} = 0$. Working in the h and $\tilde{h}$ coordinates, we consider inserting the D8-brane at some value of $\tilde{h}$, which we denote by $\tilde{h}_1$. In order to obtain a solution with a continuous $B_{(2)}$ and $F_{(2)}$ we take

$$b_1 = \frac{e^{-\frac{3\phi}{4}}}{4}\frac{\rho}{|\kappa|}\left(A^{-1/2}+A^{1/2}\right)\bigg|_{\tilde{h}=\tilde{h}_i}. \tag{5.74}$$

We note that the presence of the D8-brane selects a particular gauge for $B_{(2)}$, for which the fields are continuous.

An example for a type II solution with $G_0 = 1$ is show in figure 5.5. We have chosen $\lambda > 0$ so that the value of $F_{(0)}$ is the same on either side of the D8-brane. We see that the solution always interpolates to the constant solution for any value of the jump. This just means that we are gluing together two type II solutions. In figure 5.6, we consider the same initial function but now take negative values for λ. Note that since both G and $F_{(0)}$ are negative after the jump, their product and more specifically, the sign of ρ, are positive. If ρ had flipped signs across the jump, it would be discontinuous. In this case, we again find that we are gluing together two type II solutions, now one with positive G and one with negative G. In this case, the geometry does not approach the asymptotically flat solution of section 5.3.1, but rather at each end the geometry caps off as the asymptotic type II solution of (5.64).

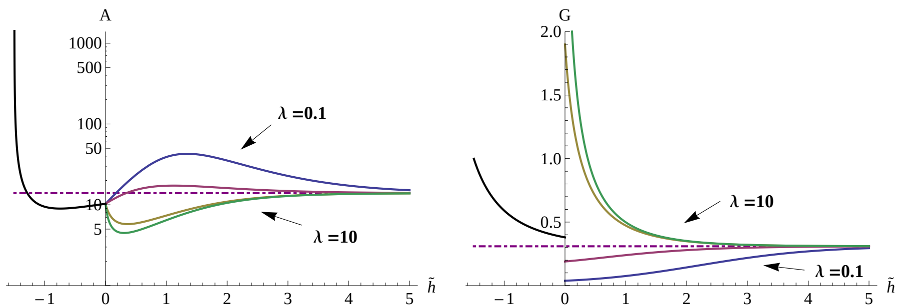


Figure 5.5: Plot of A (left) and G (right) versus $\tilde{h}$. The black line is a type II solution with $G_0 = 1$. We take $F_{(0)}$ to jump at $\tilde{h} = 0$ with jump coefficient $\lambda = (0.1, 0, 5, 5, 10)$. The dot-dashed horizontal lines are the constant G solution of section 5.3.1.

In general, one could also start with the type I solution or right type V solution and introduce a jump. We encounter similar behavior with positive values of λ yielding a geometry which asymptotes to the constant solution and negative values of λ resulting in a type II cap. The type III, IV and left V geometries yield reflected versions of the previous cases.

One may wonder whether we can construct solutions which interpolate from the smooth cap of the type I solution to the smooth cap of the type III solution

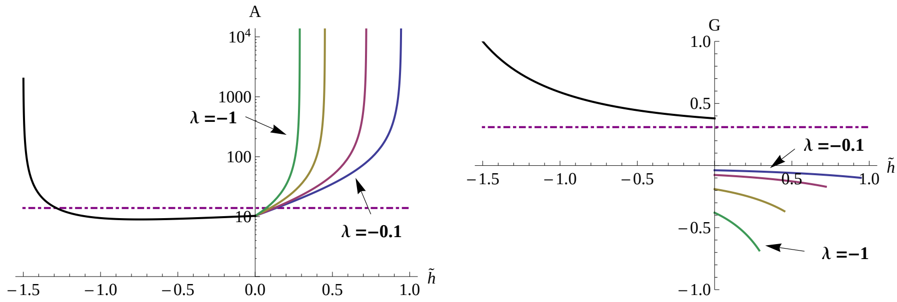


Figure 5.6: Plot of A (left) and G (right) versus $\tilde{h}$. The black line is a type II solution with $G_0 = 1$. We take $F_{(0)}$ to jump at $\tilde{h} = 0$ with jump coefficient $\lambda = (-0.1, -0.2, -0.5, -1)$. The dot-dashed horizontal lines are the constant G solution of section 5.3.1.

or to the O8-caps of the type V solutions. Unfortunately, this does not seem possible. One necessary requirement to patch together two different solutions is for their A values to overlap at some point. Examining figure 5.2, we observe that the type I, III and V solutions are disconnected, with their A values never overlapping. Thus there are no solutions which yield geometries of the form $AdS_2 \times \mathcal{M}_8$ with $\mathcal{M}_8$ compact.

5.3.5 The case $F_{(0)} = 0$

It turns out there are no solutions with $F_{(0)} = 0$. To see this we first take the $F_{(0)} \to 0$ limit of the BPS equations of section 5.1.2. As discussed at the end of the section, we send $b_0 \to b_0 - b_1/F_{(0)}$ and then take $F_{(0)} = 0$. We solve (5.18) to obtain expressions for b_1 and b_z:

$$b_1 = -2e^{-3\phi/4} \frac{f_1^2}{|\alpha|^2 + |\beta|^2} (2c_1 - 3c_2 + 3c_3) \ , \quad b_z = \frac{3}{2} \frac{e^{\phi/2}}{\alpha^* \beta^*} (2c_1 - 3c_2 + 3c_3) \ , \tag{5.75}$$

along with $c_3 = -c_2$. The differential equations (5.20) can be reduced as before to yield $\bar{\kappa} = \rho/\alpha\beta$, $b_z = (2e^{\phi/2}/\rho)\partial_z \ln(\alpha/\beta)$ and

$$\begin{aligned} \partial_z \ln\left(\alpha(\beta^*)^2\sqrt{\rho}\right) &= -\frac{1}{8}\frac{b_1}{f_1^2} e^{-\phi/2} e^{5\phi/4} \frac{\rho\alpha}{\beta^*} \ , \\ \partial_z \ln\left(\beta(\alpha^*)^2\sqrt{\rho}\right) &= \frac{1}{8}\frac{b_1}{f_1^2} e^{-\phi/2} e^{5\phi/4} \frac{\rho\beta}{\alpha^*} \ . \end{aligned} \tag{5.76}$$

We eliminate β using the definition of κ. Requiring b_1 to be constant, we can obtain algebraic expressions for $\partial_z \alpha$, $\partial_z \alpha^*$, $\partial_z \rho$ and $\partial_z \phi$. Upon plugging these expressions into (5.19), we obtain an algebraic equation which over constrains the system. To exhibit this explicitly, it is convenient to pick local coordinates so that $\kappa = \bar{\kappa} = 1/2$. Then (5.19) yields

$$4\rho^2 + |\alpha|^4 = 0\,, \tag{5.77}$$

which has no solutions other than $\rho = \alpha = 0$.

5.4 Summary

We studied $D(2,1;\gamma;1) \times SO(4)$ geometries in massive IIA supergravity and reduced the BPS-equations to a two-dimensional system. This two-dimensional system has three points of supersymmetry enhancement. Two of these lead to the AdS_6 solution of [19] which we rederived as the only solution. The remaining point of supersymmetry enhancement leads to novel solutions.

In both cases, we identify a holomorphic $(1,0)$-function, which is a ubiquitous ingredient in half-BPS solutions. In almost all known solutions, the holomorphic $(1,0)$-function is a homogenous polynomial of the spinor variables. In the current case, the degree of the polynomial is different for the two distinct cases of enhanced symmetry. This means that for the general solutions, the holomorphic $(1,0)$-function, if it exists, is not a polynomial of the spinor variables. This suggests that the resulting BPS structure is different than the analogous M-theory system.

The new solutions correspond to fundamental strings ending on D8-branes. We identify three types of solutions. The first, given in section 5.3.1, we interpret as a stack of fundamental strings in the presence of D8-branes, i.e. in a background with $F_{(0)} \neq 0$. The second two, given in section 5.3.2, we interpret as fundamental strings ending on a stack of D8-branes or an O8-plane. Case I and III correspond to fundamental strings ending on D8-branes, while case V corresponds to fundamental strings ending on an O8-plane. In all three cases the geometry contains an asymptotically flat region.

We briefly discussed glueing solutions with different values of $F_{(0)}$ together using D8-brane domain walls. In all cases there naively seems to be no decoupling limit, as the geometries contain asymptotically flat regions. However, we note that the string coupling tends to zero in these region, which may be sufficient for a valid decoupling limit.

The limiting case $F_{(0)} = 0$ (regular 'massless' IIA) admits no solutions in these points of enhanced symmetry.

Finally, we find that there are no geometries of the form $AdS_2 \times \mathcal{M}_8$ with $\mathcal{M}_8$ compact such that there are no solutions corresponding to $1 + 0$-dimensional CFTs.

Appendix A

Clifford algebras

For ease of use, we collect here a few of the relations for Clifford algebras and spinors. In the subsections of this appendix we collect some of the realisations for the Clifford algebras used in this thesis. For the two background cases in chapter 4 we need Clifford matrices tailored to their needs. We start of by constructing the algebra's needed for the $AdS_5 \times S^5$ case ($SO(2,4)$ and $SO(0,6)$) and include a compatible $SO(1,9)$ construction for the Minkowski background. We conclude with a basis for ten-dimensional gamma matrices that is suited for the $SO(2,1) \times SO(4) \times SO(4)$ isometry of chapter 5.

Gamma matrices

Gamma matrices in D dimensions satisfy the relation

$$\gamma_a \gamma_b + \gamma_b \gamma_a = \eta_{ab} I_D. \tag{A.1}$$

They are $2^{\lfloor D/2 \rfloor} \times 2^{\lfloor D/2 \rfloor}$-matrices in spinor space and an explicit basis for an Euclidean signature can be constructed in terms of the Pauli matrices

$$\begin{aligned}
\gamma_1 = \sigma^1 \otimes I_2 \otimes I_2 \otimes \dots, \quad & \gamma_2 = \sigma^2 \otimes I_2 \otimes I_2 \otimes \dots, \\
\gamma_3 = \sigma^3 \otimes \sigma^1 \otimes I_2 \otimes \dots, \quad & \gamma_4 = \sigma^3 \otimes \sigma^2 \otimes I_2 \otimes \dots, \\
\gamma_5 = \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \dots, \quad & \gamma_6 = \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes, \dots \\
& \dots
\end{aligned} \tag{A.2}$$

These matrices are all hermitian and square to the appropriate identity matrix. A construction for non-Euclidean signatures is obtained from this by multiplying the desired gamma-matrices by a factor of i. Anti-symmetric products of these gamma matrices are defined by

$$\gamma_{a_1 \dots a_r} = \gamma_{[a_1} \cdots \gamma_{a_r]}, \tag{A.3}$$

where the anti-symmetrisation is with total weight one. The identity together with the gamma matrices and their anti-symmetric products[1] constitute a basis of the D-dimensional Clifford algebra. For even dimensions ($D = 2m$) one can define

$$\gamma_* = (i)^\alpha \gamma_1 \gamma_2 \dots \gamma_D, \tag{A.4}$$

where $\alpha = \frac{1}{2}d(d-1) + s\text{mod}$, where s denotes the amount of minus signs in the signature. It squares to one and can be used to define chiral projection operators

$$P_L = \frac{1}{2}\left(I_D + \gamma_*\right), \qquad P_R = \frac{1}{2}\left(I_D - \gamma_*\right) \tag{A.5}$$

We have the following convenient relation between gamma matrices involving the Levi-Civita tensor, for even $D = 2m$

$$\gamma^{a_1 \dots a_r}\gamma_* = -(i)^\alpha (-)^{r(r-1)/2} \frac{1}{(D-r)!} \epsilon^{a_1 \dots a_r a_{(r+1)} \dots a_D} \gamma_{a_{(r+1)} \dots a_D}, \tag{A.6}$$

and for odd $D = 2m + 1$ where one uses γ_* as γ_{2m+1}

$$\gamma^{a_1 \dots a_r} = -(i)^\alpha (-)^{r(r-1)/2} \frac{1}{(D-r)!} \epsilon^{a_1 \dots a_r a_{(r+1)} \dots a_D} \gamma_{a_{(r+1)} \dots a_D}. \tag{A.7}$$

Spinors in various dimensions

A generic spinor ψ_α is a vector with $2^{\lfloor D/2 \rfloor}$ complex components, which are anti-commuting Grassmann variables.

Majorana spinors satisfy the reality condition

$$\psi = \psi^{\mathcal{C}}, \tag{A.8}$$

where $\mathcal{C}$ indicates charge conjugation. Charge conjugation is defined by an action of the charge conjugation matrix $\mathcal{B}$

$$\psi^{\mathcal{C}} = \mathcal{B}\psi, \qquad \gamma^{\mathcal{C}} = \mathcal{B}\gamma\mathcal{B}^{-1}. \tag{A.9}$$

[1] Actually not all anti-symmetric products are needed, just an independent subset. See [4]

Majorana conditions can be imposed in $D = 2, 3, 4, 8, 9, 10, 11$ and they reduce the $2^{\lfloor D/2 \rfloor}$ complex components to $2^{\lfloor D/2 \rfloor}$ real components.

Using the projectors (A.5) one can split a spinor into its chiral parts

$$\psi = \psi_L + \psi_R = P_L\psi + P_R\psi \tag{A.10}$$

defining the left and right handed spinors that satisfy

$$\psi_L = P_L\psi_L, \qquad \psi_R = P_R\psi_R. \tag{A.11}$$

These conditions are called chirality conditions or Weyl conditions.

In general dimensions D the Majorana condition is not compatible with Weyl conditions, and chiral components of a Majorana spinor are not Majorana. This compatibility is only possible in $D = 2, 10$, where the minimal spinors are Majorana-Weyl spinors and have $2^{\lfloor D/2 \rfloor - 1}$ real components.

For more details concerning Clifford algebras and spinors we refer to [4]. We now describe several of the realisations of gamma matrices used throughout the thesis.

A.1 The $SO(2,4)$ Clifford algebra

We extend the $SO(1,3)$ Clifford matrices by two more matrices as follows

$$\hat{\Gamma}_a = \gamma_a \otimes \sigma_1, \qquad \hat{\Gamma}_S = \gamma_4 \otimes \sigma_1, \qquad \hat{\Gamma}_T = I_4 \otimes (-i\sigma_2). \tag{A.12}$$

The γ_a are the $SO(1,3)$ gamma matrices and $\gamma_4 = -i\gamma_0\gamma_1\gamma_2\gamma_3$. We define

$$\hat{\Gamma}_* = -i\hat{\Gamma}_0\hat{\Gamma}_1 \ldots \hat{\Gamma}_S\hat{\Gamma}_T = I \otimes \sigma_3. \tag{A.13}$$

Since we are in 6 dimensions the minimal spinor is a Weyl spinor, the conformal spinor in 4 dimensions. We will restrict to righthanded chiral spinors, $\hat{\Gamma}_*\lambda = -\lambda$. We restrict $\hat{\Gamma}_{ab}$ to the righthanded chiral subspace[2] ($\hat{\Gamma} \to \hat{\gamma}$)

$$\hat{\gamma}_{ab} = \gamma_{ab}, \qquad \hat{\gamma}_{aS} = \gamma_a\gamma_5, \qquad \hat{\gamma}_{aT} = -\gamma_a, \qquad \hat{\gamma}_{ST} = -\gamma_5. \tag{A.14}$$

These matrices satisfy the relations

$$(\hat{\gamma}_{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\beta}}(\hat{\gamma}^{\hat{a}\hat{b}})_{\hat{\gamma}}{}^{\hat{\delta}} = 2\delta_{\hat{\alpha}}{}^{\hat{\beta}}\delta_{\hat{\gamma}}{}^{\hat{\delta}} - 8\delta_{\hat{\alpha}}{}^{\hat{\delta}}\delta_{\hat{\gamma}}{}^{\hat{\beta}}, \quad (\hat{\gamma}_{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\beta}}(\hat{\gamma}^{\hat{c}\hat{d}})_{\hat{\beta}}{}^{\hat{\alpha}} = -8\delta_{[\hat{a}}{}^{\hat{c}}\delta_{\hat{b}]}{}^{\hat{d}}, \tag{A.15}$$

where for this section $\hat{\alpha} = 1, \ldots 8$ and $\hat{a} = \{a, S, T\}$.

[2]This means that $\hat{\gamma}_{\tilde{a}\tilde{b}} = \hat{\Gamma}_{\tilde{a}\tilde{b}}\frac{1}{2}\left(1 - \hat{\Gamma}_*\right)$.

A.2 The $SO(6)$ Clifford algebra

We extend the $SO(5)$ Clifford matrices by one more matrix

$$\hat{\Gamma}'_{a'} = \gamma'_{a'} \otimes \sigma_2, \qquad \hat{\Gamma}'_{S'} = \gamma'_9 \otimes \sigma_2, \qquad \hat{\Gamma}'_{T'} = I_4 \otimes \sigma_1, \tag{A.16}$$

where $\gamma'_{a'}$ are the $SO(4)$ gamma matrices and γ'_9 is given by $\gamma'_9 = -\gamma'_5\gamma'_6\gamma'_7\gamma'_8$. We define

$$\hat{\Gamma}'_* = -i\hat{\Gamma}'_5\hat{\Gamma}'_6 \dots \hat{\Gamma}'_{S'}\hat{\Gamma}'_{T'} = I \otimes \sigma_3. \tag{A.17}$$

Like before we will restrict to righthanded chiral spinors, and identify

$$\hat{\gamma}'_{a'b'} = \gamma'_{a'b'}, \qquad \hat{\gamma}'_{a'S'} = \gamma'_{a'}\gamma'_9, \qquad \hat{\gamma}'_{a'T'} = i\gamma'_{a'}, \qquad \hat{\gamma}'_{S'T'} = i\gamma'_9. \tag{A.18}$$

These matrices satisfy a similar relation as (A.15)

A.3 The $SO(1,9)$ Clifford algebra

We will use a decomposition of 10-dimensional γ-matrices $\hat{\Gamma}^{(10D)}_M$ into the $SO(1,4)$ and $SO(5)$ matrices as follows

$$\hat{\Gamma}^{(10D)}_{\tilde{m}} = \gamma_{\tilde{m}} \otimes I_4 \otimes \sigma_1, \qquad \hat{\Gamma}^{(10D)}_{m'} = I_4 \otimes \gamma'_{m'} \otimes \sigma_2. \tag{A.19}$$

We define

$$\hat{\Gamma}^{(10D)}_* = -\hat{\Gamma}^{(10D)}_0 \dots \hat{\Gamma}^{(10D)}_9 = -I_4 \otimes I_4 \otimes \sigma_3. \tag{A.20}$$

We can write 10-dimensional spinors in this decomposition as

$$\Psi_{(10D)} = \psi \otimes \psi' \otimes \begin{pmatrix} a \\ b \end{pmatrix}, \tag{A.21}$$

however, the type IIB chirality condition $\frac{1}{2}\left(1 + \hat{\Gamma}^{(10D)}_*\right)\Psi_{(10D)} = 0$ implies that

$$0 = \frac{1}{2}\left(1 + \hat{\Gamma}^{(10D)}_*\right)\Psi_{(10D)} = \psi \otimes \psi' \otimes \begin{pmatrix} 0 \\ b \end{pmatrix} \qquad \rightarrow \qquad b = 0. \tag{A.22}$$

Our 10-dimensional chiral spinor is then

$$\Psi_\alpha{}^i = \psi_\alpha \otimes \psi'^i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{A.23}$$

where we reabsorbed the constant a into the 4-dimensional spinors. By doing this restriction to the right handed chiral subspace we can again define

$$\Gamma_{\hat{m}\hat{n}} = \hat{\Gamma}^{(10\mathrm{D})}_{\hat{m}\hat{n}} \tfrac{1}{2}\left(1 + \hat{\Gamma}^{(10\mathrm{D})}_{*}\right), \tag{A.24}$$

and identify

$$\Gamma_{\tilde{m}\tilde{n}} = \gamma_{\tilde{m}\tilde{n}} \otimes I_4, \qquad \Gamma_{\tilde{m}'\tilde{n}'} = I_4 \otimes \gamma^{'}_{\tilde{m}'\tilde{n}'}, \qquad \Gamma_{\tilde{m}\tilde{n}'} = \gamma_{\tilde{m}} \otimes \gamma^{'}_{\tilde{n}'}. \tag{A.25}$$

A.4 An $SO(2,1) \times SO(4) \times SO(4)$-adapted basis for ten-dimensional gamma matrices

We choose a basis for the Clifford algebra which is well-adapted to the $AdS_2 \times S^3 \times S^3 \times \Sigma_2$ space.

$$\begin{aligned}
\Gamma^m &= \gamma^m \otimes I_2 \otimes I_2 \otimes I_4 && m = 0,1 \\
\Gamma^i &= \sigma^3 \otimes \gamma^i \otimes I_2 \otimes \left(\sigma^1 \otimes I_2\right) && i = 2,3,4 \\
\Gamma^{\tilde{i}} &= \sigma^3 \otimes I_2 \otimes \gamma^{\tilde{i}} \otimes \left(\sigma^2 \otimes I_2\right) && \tilde{i} = 5,6,7 \\
\Gamma^a &= \sigma^3 \otimes I_2 \otimes I_2 \otimes \gamma^a && a = 8,9\,,
\end{aligned} \tag{A.26}$$

where a convenient basis for the lower Clifford algebras is as follows,

$$\begin{aligned}
i\gamma^0 &= \sigma^1 = \gamma^2 = \gamma^5 & \gamma^8 &= \sigma^3 \otimes \sigma^1 \\
\gamma^1 &= \sigma^2 = \gamma^3 = \gamma^6 & \gamma^9 &= \sigma^3 \otimes \sigma^2 \\
\sigma^3 &= \gamma^4 = \gamma^7\,.
\end{aligned} \tag{A.27}$$

The 10-dimensional chirality matrix in this basis is given by

$$\Gamma^{11} = \Gamma^{0123456789} = \sigma^3 \otimes I_2 \otimes I_2 \otimes \sigma^3 \otimes \sigma^3\,. \tag{A.28}$$

The complex conjugation matrices in each subspace are defined by

$$\begin{aligned}
(\gamma^m)^* &= +B_{(1)}\gamma^m B_{(1)}^{-1} & (B_{(1)})^* B_{(1)} &= +I_2 & B_{(1)} &= \sigma^3 \\
(\gamma^i)^* &= -B_{(2)}\gamma^i B_{(2)}^{-1} & (B_{(2)})^* B_{(2)} &= -I_2 & B_{(2)} &= \sigma^2 \\
(\gamma^{\tilde{i}})^* &= -B_{(3)}\gamma^{\tilde{i}} B_{(3)}^{-1} & (B_{(3)})^* B_{(3)} &= -I_2 & B_{(3)} &= \sigma^2 \\
(\gamma^a)^* &= +B_{(4)}\gamma^a B_{(4)}^{-1} & (B_{(4)})^* B_{(4)} &= +I_2 & B_{(4)} &= \sigma^3 \otimes \sigma^1\,,
\end{aligned} \tag{A.29}$$

where in the last column we have also listed the form of these matrices in our particular basis. The 10-dimensional complex conjugation matrix $\mathcal{B}$ is defined by $(\Gamma^M)^* = \mathcal{B}\Gamma^M\mathcal{B}^{-1}$ and $\mathcal{B}\mathcal{B}^* = 1$, and in this basis is given by

$$\mathcal{B} = B_{(1)} \otimes B_{(2)} \otimes B_{(3)} \otimes \sigma^2 \otimes \sigma^2 = \sigma^3 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \,. \tag{A.30}$$

Appendix B

The $SU(2,2|4)$ algebra in various forms

The $SU(2,2|4)$ algebra is

$$[V_{\hat{\alpha}}{}^{\hat{\beta}}, V_{\hat{\gamma}}{}^{\hat{\delta}}] = \delta_{\hat{\gamma}}{}^{\hat{\beta}} V_{\hat{\alpha}}{}^{\hat{\delta}} - \delta_{\hat{\alpha}}{}^{\hat{\delta}} V_{\hat{\gamma}}{}^{\hat{\beta}}, \qquad [U_i{}^j, U_k{}^l] = \delta_i{}^l U_k{}^j - \delta_k{}^j U_i{}^l$$

$$[V_{\hat{\alpha}}{}^{\hat{\beta}}, \mathcal{Q}_{\hat{\gamma}}{}^i] = \delta_{\hat{\gamma}}{}^{\hat{\beta}} \mathcal{Q}_{\hat{\alpha}}{}^i - \frac{1}{4}\delta_{\hat{\alpha}}{}^{\hat{\beta}} \mathcal{Q}_{\hat{\gamma}}{}^i, \qquad [U_i{}^j, \mathcal{Q}_{\hat{\alpha}}{}^k] = \delta_i{}^k \mathcal{Q}_{\hat{\alpha}}{}^j - \frac{1}{4}\delta_i{}^j \mathcal{Q}^k_{\hat{\alpha}},$$

$$\{\mathcal{Q}_{\hat{\alpha}}{}^i, \bar{\mathcal{Q}}_j{}^{\hat{\beta}}\} = \delta_j{}^i V_{\hat{\alpha}}{}^{\hat{\beta}} - \delta_{\hat{\alpha}}{}^{\hat{\beta}} U_j{}^i, \tag{B.1}$$

with all other commutators vanishing. The index $\hat{\alpha}$ runs over the values $1,\ldots 4$ in this section.

We will relate the fundamental representation of $SU(2,2)$ to the spinor of $SO(2,4)$. We rotate the generators $V_{\hat{\alpha}}{}^{\hat{\beta}}$ to $\hat{M}_{\hat{m}\hat{n}}$ by means of the $\hat{\gamma}_{\hat{a}\hat{b}}$ matrices given in (A.14)

$$V_{\hat{\alpha}}{}^{\hat{\beta}} = \frac{1}{2}(\hat{\gamma}^{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\beta}} \hat{M}_{\hat{a}\hat{b}}, \qquad \hat{M}_{\hat{a}\hat{b}} = -\frac{1}{4}(\hat{\gamma}_{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\beta}} V_{\hat{\beta}}{}^{\hat{\alpha}}. \tag{B.2}$$

This is consistent by the fact that the $V_{\hat{\alpha}}{}^{\hat{\beta}}$ are traceless and (A.15). The other bosonic subalgebra, generated by U will be considered as an internal group (an R-symmetry) for the remainder of this section. The conjugate spinor charge $\bar{\mathcal{Q}}_i{}^{\hat{\alpha}}$ is defined as the four-dimensional Dirac conjugate spinor,

$$\bar{\mathcal{Q}}_i{}^{\hat{\alpha}} = i[(\mathcal{Q}^i)^\dagger \gamma^0]^{\hat{\alpha}}. \tag{B.3}$$

With this isomorphism realised, we have a superalgebra in terms of the generators

$$\mathbf{T}_\Lambda : \qquad \hat{M}_{\hat{a}\hat{b}}, \qquad U_i{}^j, \qquad \mathcal{Q}_{\hat{\alpha}}{}^i \qquad \text{and} \qquad \bar{\mathcal{Q}}_i{}^{\hat{\alpha}}. \tag{B.4}$$

The super spacetime part of the algebra now gets the universal form

$$\begin{aligned}
[\hat{M}_{\hat{a}\hat{b}}, \hat{M}_{\hat{c}\hat{d}}] &= \hat{\eta}_{\hat{a}[\hat{c}}\hat{M}_{\hat{d}]\hat{b}} - \hat{\eta}_{\hat{b}[\hat{c}}\hat{M}_{\hat{d}]\hat{a}}, \\
[\hat{M}_{\hat{a}\hat{b}}, \mathcal{Q}_{\hat{\alpha}}{}^i] &= -\frac{1}{4}(\hat{\gamma}_{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\beta}}\mathcal{Q}_{\hat{\beta}}{}^i, \\
\{\mathcal{Q}_{\hat{\alpha}}{}^i, \mathcal{Q}_j{}^{\hat{\beta}}\} &\sim \delta_j{}^i(\hat{\gamma}^{\hat{a}\hat{b}})_{\hat{\alpha}}{}^{\hat{\beta}}\hat{M}_{\hat{a}\hat{b}} + \delta_{\hat{\alpha}}{}^{\hat{\beta}}U_i{}^j,
\end{aligned} \tag{B.5}$$

and there is the internal part which involves the generators $U_i{}^j$, which also rotate the supercharges. The metric $\hat{\eta} = \text{diag}(-++++-)$ is the (2,4) flat metric and the indices $\hat{a} = \{0,1,2,3,S,T\}$ where 0 and T are timelike directions. Remark that we chose all generators in this formula to be dimensionless. In general we define a $\mathbf{G}$ valued object A as

$$A = A^\Lambda \mathbf{T}_\Lambda. \tag{B.6}$$

For the superalgebra above we have

$$A = \hat{A}^{\hat{a}\hat{b}}\hat{M}_{\hat{a}\hat{b}} + \hat{A}_i{}^j U_j{}^i + \bar{\hat{A}}_i{}^{\hat{\alpha}}\mathcal{Q}_{\hat{\alpha}}{}^i + \bar{\mathcal{Q}}_i{}^{\hat{\alpha}}\hat{A}_{\hat{\alpha}}{}^i, \tag{B.7}$$

where these objects can be viewed as matrices. Note that $\bar{\mathcal{Q}}_i{}^{\hat{\alpha}}$ does not act on $\hat{A}_{\hat{\alpha}}{}^i$ in this notation.
We want to derive the generators of the AdS algebra and the conformal algebra in their more familiar form. Starting from the generic form of the conformal superalgebra in the $SO(2,4)$ basis (B.5), we will first decompose it into a form which is appropriate to the AdS_5 spacetime isometry algebra and then into a form which is appropriate for the conformal isometries in 4 dimensions. We call these the AdS decomposition and the conformal decomposition, respectively. We will also discuss how quantities in these decompositions are related.

B.1 The AdS decomposition

The AdS_5 space is a 5-dimensional manifold with structure group $SO(2,4)$, in order to obtain this from the algebra we split the generators into $SO(1,4)$ generators $\tilde{M}_{\tilde{m}\tilde{n}}$ and the remaining generators $\tilde{P}_{\tilde{m}}$, defined through

$$\tilde{P}_{\tilde{m}} = \frac{2}{R}\hat{M}_{\tilde{m}T}, \qquad \tilde{M}_{\tilde{m}\tilde{n}} = \hat{M}_{\tilde{m}\tilde{n}}, \tag{B.8}$$

where we have introduced the constant R, which has dimensions of a length to give the translations $\tilde{P}_{\tilde{m}}$ the canonical dimensions of L^{-1}. It will be associated with the radius of curvature of the AdS space. The S-direction will be associated with the AdS bulk direction.
The supercharges $\mathcal{Q}_{\hat{\alpha}}{}^{i}$ are rescaled to have dimensions $L^{-1/2}$,

$$\tilde{\mathcal{Q}}_{\hat{\alpha}}{}^{i} = R^{-1/2}\mathcal{Q}_{\hat{\alpha}}{}^{i}. \tag{B.9}$$

This yields a superalgebra of the form

$$[\tilde{M}_{\tilde{a}\tilde{b}}, \tilde{M}_{\tilde{c}\tilde{d}}] = \tilde{\eta}_{\tilde{a}[\tilde{c}}\tilde{M}_{\tilde{d}]\tilde{b}} - \tilde{\eta}_{\tilde{b}[\tilde{c}}\tilde{M}_{\tilde{d}]\tilde{a}},$$

$$[\tilde{P}_{\tilde{a}}, \tilde{M}_{\tilde{b}\tilde{c}}] = \tilde{\eta}_{\tilde{a}[\tilde{b}}\tilde{P}_{\tilde{c}]}, \qquad [\tilde{P}_{\tilde{a}}, \tilde{P}_{\tilde{b}}] = \frac{2}{R^2}\tilde{M}_{\tilde{a}\tilde{b}},$$

$$[\tilde{M}_{\tilde{a}\tilde{b}}, \tilde{\mathcal{Q}}_{\hat{\alpha}}{}^{i}] = -\frac{1}{4}(\hat{\gamma}_{\tilde{a}\tilde{b}})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{\mathcal{Q}}_{\hat{\beta}}{}^{i}, \qquad [\tilde{P}_{\tilde{a}}, \tilde{\mathcal{Q}}_{\hat{\alpha}}{}^{i}] = \frac{2}{R}(\hat{\gamma}_{\tilde{a}T})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{\mathcal{Q}}_{\hat{\beta}}{}^{i}$$

$$\{\tilde{\mathcal{Q}}_{\hat{\alpha}}{}^{i}, \tilde{\mathcal{Q}}_{j}{}^{\hat{\beta}}\} \sim \delta_{j}{}^{i}(\hat{\gamma}^{\tilde{a}T})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{P}_{\tilde{a}} + \frac{1}{R}\delta_{j}{}^{i}(\hat{\gamma}^{\tilde{a}\tilde{b}})_{\hat{\alpha}}{}^{\hat{\beta}}\tilde{M}_{\tilde{a}\tilde{b}} + \frac{1}{R}\delta_{\hat{\alpha}}{}^{\hat{\beta}}U_{i}{}^{j}, \tag{B.10}$$

where $\tilde{\eta} = \text{diag}(-++++)$ is the flat metric with signature (1,4).
It is interesting to note that this algebra contains the dimensionful constant R, which can not be scaled away if we want the translations to have natural dimension of a mass.
For the AdS superalgebra, we have the decomposition of a $\mathbf{G}$-valued object

$$A = \tilde{A}^{\tilde{a}}\tilde{P}_{\tilde{a}} + \tilde{A}^{\tilde{a}\tilde{b}}\tilde{M}_{\tilde{a}\tilde{b}} + \tilde{A}_{i}{}^{j}U_{j}{}^{i} + \bar{\tilde{A}}_{i}{}^{\alpha}\tilde{\mathcal{Q}}_{\hat{\alpha}}{}^{i} + \bar{\tilde{\mathcal{Q}}}_{i}{}^{\hat{\alpha}}\tilde{A}_{\hat{\alpha}}{}^{i}. \tag{B.11}$$

From this we infer that

$$\tilde{A}^{\tilde{a}} = R\hat{A}^{\tilde{a}T}, \qquad \tilde{A}^{\tilde{a}\tilde{b}} = \hat{A}^{\tilde{a}\tilde{b}}, \qquad \tilde{A}_{i}{}^{j} = \hat{A}_{i}{}^{j}, \qquad \tilde{A}_{\hat{\alpha}}{}^{i} = R^{1/2}\hat{A}_{\hat{\alpha}}{}^{i}. \tag{B.12}$$

B.2 The Conformal decomposition

We now turn to the conformal decomposition of the superalgebra. We obtain the conformal transformations (generators) in 4 dimensions, i.e. translations (P_a), Lorentz transformations (M_{ab}), dilations (D) and special conformal transformations (K_a), which also form the algebra SO(2,4), by

$$\begin{aligned}
P_a &= \frac{2}{R}(\hat{M}_{aT} + \hat{M}_{aS}) = \tilde{P}_a + \frac{2}{R}\tilde{M}_{aS}, && [P_a] = L^{-1}, \\
M_{ab} &= \hat{M}_{ab} = \tilde{M}_{ab}, && [M_{ab}] = L^0, \\
D &= 2\hat{M}_{TS} = -R\tilde{P}_S, && [D] = L^0, \\
K_a &= 2R(\hat{M}_{aT} - \hat{M}_{aS}) = R^2\tilde{P}_a - 2R\tilde{M}_{aS}, && [K_a] = L,
\end{aligned} \tag{B.13}$$

where we indicated the natural length dimensions L. The tangent directions have now been split as $\hat{a} = \{\tilde{a}, T\} = \{a, S, T\}$ and we introduce the 4-dimensional Minkowski metric $\eta_{ab} = \mathrm{diag}(-,+,+,+)$. It is natural to split the supercharge $\mathcal{Q}_{\hat{\alpha}}{}^i$ into two Lorentz supercharges, the supersymmetry $Q_\alpha{}^i$ and the conformal supersymmetry $S_\alpha{}^i$. One way to distinguish between the two is that they transform with opposite weight under the dilations. Consider the commutator

$$[D, \mathcal{Q}_{\hat{\alpha}}{}^i] = -\frac{1}{2}\hat{\gamma}_{TS}\mathcal{Q}_{\hat{\alpha}}{}^i. \tag{B.14}$$

Since $\hat{\gamma}_{TS}^2 = 1$ and $\mathrm{Tr}\,\hat{\gamma}_{TS} = 0$, we can define the projection operators

$$\mathcal{P}_{Q,S} = \frac{1}{2}(1 \pm \hat{\gamma}_{ST}). \tag{B.15}$$

We note that $\hat{\gamma}_{ST}$ commutes with $\hat{\gamma}_{ab}$ and therefore preserves the 4-dimensional Lorentz spinors as desired. This leads us to the following identification

$$\begin{aligned}
&\text{supersymmetry:} && Q_\alpha{}^i = \sqrt{2}R^{-1/2}\mathcal{P}_Q\mathcal{Q}_{\hat{\alpha}}{}^i = \sqrt{2}\mathcal{P}_Q\tilde{\mathcal{Q}}_{\hat{\alpha}}{}^i, \\
&\text{special supersymmetry:} && S_\alpha{}^i = \sqrt{2}R^{1/2}\mathcal{P}_S\mathcal{Q}_{\hat{\alpha}}{}^i = \sqrt{2}R\mathcal{P}_S\tilde{\mathcal{Q}}_{\hat{\alpha}}{}^i,
\end{aligned} \tag{B.16}$$

with $[Q_\alpha{}^i] = L^{-1/2}$ and $[S_\alpha{}^i] = L^{1/2}$. Bringing these decompositions into the algebra (B.5) we obtain

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{a[c}M_{d]b} - \eta_{b[c}M_{d]a}, & [P_a, K_b] &== 2(\eta_{ab}D + 2M_{ab}), \\
[P_a, M_{cd}] &= \eta_{a[b}P_{c]}, & [K_a, M_{cd}] &= \eta_{a[b}K_{c]}, \\
[D, P_a] &= P_a, & [D, K_a] &= -K_a, \\
[M_{ab}, Q_\alpha{}^i] &= -\frac{1}{4}(\hat{\gamma}_{ab}Q^i)_\alpha, & [M_{ab}, S_\alpha{}^i] &= -\frac{1}{4}(\hat{\gamma}_{ab}S^i)_\alpha, \\
[K_a, Q_\alpha{}^i] &= -(\hat{\gamma}_{aT}S^i)_\alpha, & [P_a, S_\alpha{}^i] &= -(\hat{\gamma}_{aT}Q^i)_\alpha,
\end{aligned} \tag{B.17}$$

$$[D, Q_\alpha{}^i] = \frac{1}{2} Q_\alpha{}^i, \quad [D, S_\alpha{}^i] = -\frac{1}{2} S_\alpha{}^i,$$

$$\{Q_\alpha{}^i, \bar{Q}_j{}^\beta\} = \delta_j{}^i (\hat{\gamma}^{aT})_\alpha{}^\beta P_a, \quad \{S_\alpha{}^i, \bar{S}_j{}^\beta\} = \delta_j{}^i (\hat{\gamma}^{aT})_\alpha{}^\beta K_a,$$

$$\{Q_\alpha{}^i, \bar{S}_j{}^\beta\} = \delta_j{}^i (\hat{\gamma}^{ab})_\alpha{}^\beta M_{ab} + \delta_j{}^i \delta_\alpha{}^\beta D - 2\delta_\alpha{}^\beta U_j{}^i.$$

By having given appropriate dimensions to the generators, this algebra contains no dimensionful constants as opposed to the *AdS*-decomposition where it was unavoidable.
A superconformal object can be decomposed as follows

$$A = A_P^a P_a + A_M^{ab} M_{ab} + A_D D + A_K^a K_a + A_i{}^j U_j{}^i + \left(\bar{A}_{Qi}{}^\alpha Q_\alpha{}^i + \bar{A}_{Si}{}^\alpha S_\alpha{}^i + \text{h.c.}\right), \tag{B.18}$$

and this yields the following relations

$$\begin{aligned}
A_P^a &= \frac{R}{2}(\hat{A}^{aT} + \hat{A}^{aS}) = \frac{1}{2}\left(\tilde{A}^a + R\tilde{A}^{aS}\right), \\
A_M^{ab} &= \hat{A}^{ab} = \tilde{A}^{ab}, \\
A_D &= \hat{A}^{TS} = -R^{-1}\tilde{A}^S, \\
A_K^a &= \frac{1}{2R}(\hat{A}^{aT} - \hat{A}^{aS}) = \frac{1}{2R^2}(\tilde{A}^a - R\tilde{A}^{aS}) \\
A_i{}^j &= \hat{A}_i{}^j = \tilde{A}_i{}^j \\
\bar{A}_{Qi}{}^\alpha &= \frac{R^{1/2}}{\sqrt{2}} \bar{\hat{A}}_i{}^{\hat{\beta}} (\mathcal{P}_Q)_{\hat{\beta}}{}^{\hat{\alpha}} = \frac{1}{\sqrt{2}} \bar{\tilde{A}}_i{}^{\hat{\beta}} (\mathcal{P}_Q)_{\hat{\beta}}{}^{\hat{\alpha}} \\
\bar{A}_{Si}{}^\alpha &= \frac{R^{-1/2}}{\sqrt{2}} \bar{\hat{A}}_i{}^{\hat{\beta}} (\mathcal{P}_S)_{\hat{\beta}}{}^{\hat{\alpha}} = \frac{1}{\sqrt{2}R} \bar{\tilde{A}}_i{}^{\hat{\beta}} (\mathcal{P}_S)_{\hat{\beta}}{}^{\hat{\alpha}}.
\end{aligned} \tag{B.19}$$

It is interesting to note that the translations and the special conformal transformations in the conformal decomposition *mix* the *AdS* translations and structure group rotations.
To conclude this section we give the *AdS* objects in terms of their conformal

counterparts.

$$\begin{aligned}
\tilde{A}^a &= A_P^a + R^2 A_K^a, \\
\tilde{A}^S &= -R A_D, \\
\tilde{A}^{aS} &= R^{-1} A_P^a - R A_K^a, \\
\tilde{A}^{ab} &= A_M^{ab}, \\
\tilde{A}_{\hat{\alpha}}{}^i &= \sqrt{2} \begin{pmatrix} A_{Q\alpha}{}^i \\ R A_{S\alpha}{}^i \end{pmatrix},
\end{aligned} \tag{B.20}$$

where just for notational reasons we have a basis in which $\hat{\gamma}_{ST}$ is diagonal.

Appendix C

$AdS_5 \times S^5$ as a coset space

Our aim in this appendix is to construct the coset space $AdS_5 \times S^5$. First we consider AdS_5 as a coset space and then we discuss the S^5 coset space. We conclude this appendix with a discussion of an appropriate choice of fermionic coordinates for the coset superspace.

C.1 AdS_5 as a coset space

The AdS_5 space is the coset

$$AdS_5 = \frac{SO(2,4)}{SO(1,4)}. \tag{C.1}$$

The algebra to be considered is the bosonic part of the algebra in appendix B.1 (ignoring the internal part). We choose horospherical coordinates,

$$ds^2 = \rho^2 dx^2 + \left(\frac{R}{\rho}\right)^2 d\rho^2, \tag{C.2}$$

where the boundary is parametrised by x^m and is at $\rho = \infty$. The coset representative for horospherical coordinates is given in the spinor representation of $SO(2,4)$. It can be derived from the supergravity Killing spinor [108] and can be written as

$$v(\tilde{x}^{\tilde{m}}) = v_{\text{conf}}(x) \left(\rho^{-1/2} \frac{1}{2} (1 - \hat{\gamma}_{ST}) + \rho^{1/2} \frac{1}{2} (1 + \hat{\gamma}_{ST})\right), \tag{C.3}$$

where the γ-matrices are defined as in A.14 and $v_{\rm conf}(x)$ is the coset representative of the 4-dimensional conformal Minkowski space

$$v_{\rm conf}(x) = 1 + \frac{x^m}{R}\hat{\gamma}_{mT}\frac{1}{2}(1+\hat{\gamma}_{ST}). \tag{C.4}$$

The flat S direction is related to the bulk direction ρ of AdS_5. Straightforward computation gives the Cartan forms (4.1)

$$v^{-1}dv \equiv L^\Lambda T_\Lambda = e^{\tilde{m}}P_{\tilde{m}} + \omega^{\tilde{m}\tilde{n}}M_{\tilde{m}\tilde{n}}, \tag{C.5}$$

with non-vanishing components

$$e^a = e^a_m dx^m = \rho\delta^a_m dx^m, \qquad e^4 = e^4_\rho d\rho = \frac{R}{\rho}d\rho, \qquad \omega^{a\rho} = \delta^a_m dx^m \frac{\rho}{R}. \tag{C.6}$$

The Killing fields Σ_0 (4.10) are determined by an $\tilde{x}$-independent $SO(2,4)$ object,

$$\Upsilon = \tilde{a}^{\tilde{m}}\tilde{P}_{\tilde{m}} + \tilde{\lambda}^{\tilde{m}\tilde{n}}_{(M)}\tilde{M}_{\tilde{m}\tilde{n}}. \tag{C.7}$$

Using the AdS-decomposition

$$\tilde{P}_{\tilde{m}} = \frac{2}{R}\hat{M}_{\tilde{m}T}, \qquad \tilde{M}_{\tilde{m}\tilde{n}} = \hat{M}_{\tilde{m}\tilde{n}}, \tag{C.8}$$

yields the AdS_5 Killing fields

$$\begin{aligned}\Sigma_0^m &= \rho\xi^m(x) + \frac{R^2}{\rho}\lambda^m_{(K)}, \qquad \Sigma_0^\rho = -\Lambda_D(x)R,\\ \Sigma_0^{m\rho} &= \frac{\rho}{R}\xi^m(x) - \frac{R}{\rho}\lambda^m_{(K)}, \qquad \Sigma_0^{mn} = \Lambda^{mn}_M(x),\end{aligned} \tag{C.9}$$

with

$$\begin{aligned}\xi^m(x) &= \xi^m_C(x) + \frac{R^2}{\rho^2}\lambda^m_{(K)},\\ \xi^m_C(x) &= a^m + \lambda^{mn}_{(M)}x_n + \lambda_D x^m + (x^2\lambda^m_{(K)} - 2x^m x\cdot\lambda_{(K)}),\\ \Lambda^{mn}_{(M)}(x) &= \lambda^{mn}_{(M)} - 4x^{[m}\lambda^{n]}_{(K)},\\ \Lambda_D(x) &= \lambda_D - 2\lambda_{(K)}\cdot x.\end{aligned} \tag{C.10}$$

We have used (B.20) to write the Killing fields in terms of the conformal parameters, where a^m, $\lambda^{mn}_{(M)}$, λ_D and $\lambda^m_{(K)}$ are the constant parameters of translations, Lorentz rotations, dilations and special conformal transformations for the conformal space in four dimensions, spanned by the coordinates x^m. We included the C as a subscript for ξ_C^m to stress that it is expressed in terms of the conformal parameters. We obtain the isometries through (4.13) as

$$\delta x^m = -\xi_C^m(x) - \frac{R^2}{\rho^2}\lambda_{(K)}^m, \qquad \delta\rho = \Lambda_D(x)\rho. \tag{C.11}$$

C.2 S^5 as a coset space

The sphere is the coset space

$$S^5 = \frac{SO(6)}{SO(5)}. \tag{C.12}$$

The algebra to be considered is the $SO(6)$ algebra

$$[\hat{M}'_{\hat{m}'\hat{n}'}, \hat{M}'_{\hat{p}'\hat{q}'}] = \delta_{\hat{m}'[\hat{p}'}\hat{M}'_{\hat{q}']\hat{n}'} - \delta_{\hat{n}'[\hat{p}'}\hat{M}'_{\hat{q}']\hat{m}'}, \tag{C.13}$$

where in the sphere decomposition

$$\tilde{P}'{}_{m'} = \frac{2}{R}\hat{M}'_{m'S'}, \qquad \tilde{M}'{}_{m'n'} = \hat{M}'_{m'n'}, \tag{C.14}$$

with m' the 5 flat tangent directions of the sphere.
We will work in stereographic coordinates $z^{m'}$

$$ds^2 = \frac{4R^2}{(1+z^2)^2}dz^2, \tag{C.15}$$

where $z^2 = z^{m'}\eta_{m'n'}z^{n'}$. The convenient coset representative for the sphere in these coordinates is

$$u(z^{m'}) = (1+z^2)^{-1/2}(1+z^{m'}\hat{\gamma}'_{m'S'}), \tag{C.16}$$

given in the spinor representation

$$\hat{M}'_{\hat{m}'\hat{n}'} = \frac{1}{4}\hat{\gamma}'_{\hat{m}'\hat{n}'}, \tag{C.17}$$

where the matrices $\hat{\gamma}'_{\hat{m}'\hat{n}'}$ are elements of the $SO(6)$ Clifford algebra. Straightforward computation gives the Cartan forms (4.1)

$$u^{-1}du = e^{m'}P_{m'} + \omega^{m'n'}M_{m'n'}, \tag{C.18}$$

with

$$e^{m'} = 2R\frac{dz^{m'}}{1+z^2}, \qquad \omega^{m'n'} = 4\frac{z^{[m'}dz^{n']}}{1+z^2}. \tag{C.19}$$

We introduce the rigid $SO(6)$-valued parameter $\Upsilon_S = \Lambda^{\hat{m}'\hat{n}'}\hat{M}'_{\hat{m}'\hat{n}'}$ and derive the Killing field (4.12),

$$\begin{aligned}
\Sigma_0^{m'} &= \frac{2R}{1+z^2}\left(\frac{1}{2}(1-z^2)\Lambda^{m'S'} + \Lambda^{m'n'}z_{n'} + z^{m'}z_{n'}\Lambda^{n'S'}\right),\\
\Sigma_0^{m'n'} &= \Lambda^{m'n'} + \frac{4}{1+z^2}\left(z^{[m'}\Lambda^{n']S'} + z^{[m'}\Lambda^{n']p'}z_{p'}\right),
\end{aligned} \tag{C.20}$$

leading to the isometries

$$-\delta z^{m'} = \xi^{m'} = \frac{1}{2}(1-z^2)\Lambda'^{m'S'} + \Lambda'^{m'n'}z_{n'} + z^{m'}z_{n'}\Lambda'^{n'S}. \tag{C.21}$$

C.3 $AdS_5 \times S^5$ and adapted fermionic coordinates

The bosonic space is of a direct product form

$$AdS_5 \times S^5. \tag{C.22}$$

The bosonic coset representative $g(X)$ then also takes the form of a direct product $g(X) = v \otimes u$, in terms of the bosonic representatives for AdS and S obtained before. We can enlarge this bosonic space to a superspace by the coset construction. We already derived the representatives for the bosonic subspaces. The only thing that is lacking is the fermionic coordinate choice, encoded in the matrix $e_{\dot{\alpha}}{}^{\alpha}$.
The conformal structure of the AdS boundary and associated isometries is most apparent in the horospherical coordinates. The coordinates x^m, which parametrise the directions parallel to the boundary $\rho \to \infty$, can then be identified with the coordinates x^m of the conformal Minkowski space. To continue this, we would like that half of the anti-commuting coordinates of the $AdS \times S$ superspace can be identified with the θ's of the conformal superspace. This can be done by appropriately considering the relation between the AdS and conformal decompositions. As a coordinate choice we will take

$$\begin{aligned}
\Theta &= \bar{\Theta}_i \mathcal{Q}^i + \bar{\mathcal{Q}}_i \Theta^i \\
&= (u^{-1})_i{}^j\bar{\theta}_j\rho^{1/2}Q^i + (u^{-1})_i{}^j\bar{\vartheta}_j\rho^{-1/2}S^i + \bar{Q}_i\rho^{1/2}\theta^j u_j{}^i + \bar{S}_i\rho^{-1/2}\vartheta^j u_j{}^i.
\end{aligned} \tag{C.23}$$

The two coordinates $\{\theta, \vartheta\}$ together build up the anti-commuting coordinate of the $AdS \times S$ superspace $\tilde{\theta}$ by

$$\tilde{\theta}^i_{\hat{\alpha}} = \sqrt{2} \begin{pmatrix} \theta^i_\alpha \\ R\vartheta^i_\alpha \end{pmatrix} . \tag{C.24}$$

We will call these coordinates the super-horospherical coordinates

$$Z^M = \{x^m, \rho, z^{m'}, \theta^i, \vartheta^i\}. \tag{C.25}$$

The parametrisation for the fermionic symmetry parameter ε will be

$$\begin{aligned} \bar{\varepsilon}^\alpha_i = & \frac{1}{\sqrt{2}} (u^{-1})_i{}^j \Big[R\rho^{-1/2} \bar{\eta}^{\hat{\beta}}_j \frac{1}{2} \left(1 - \hat{\gamma}_{ST}\right)_{\hat{\beta}}{}^\alpha \\ & + \rho^{1/2} \left(\bar{\epsilon}_j{}^{\hat{\beta}} + \bar{\eta}_j{}^{\hat{\gamma}} (\hat{\gamma}_{mT})_{\hat{\gamma}}{}^{\hat{\beta}} x^m \right) \frac{1}{2} \left(1 + \hat{\gamma}_{ST}\right)_{\hat{\beta}}{}^\alpha \Big] \\ = & \frac{1}{\sqrt{2}} (u^{-1})_i{}^j \Big[R\rho^{-1/2} \bar{\eta}^{\beta}_j \frac{1}{2} \left(1 + \gamma_5\right)_\beta{}^\alpha \\ & + \rho^{1/2} \left(\bar{\epsilon}_j{}^{\beta} - \bar{\eta}_j{}^{\gamma} (\gamma_m)_\gamma{}^\beta x^m \right) \frac{1}{2} \left(1 - \gamma_5\right)_\beta{}^\alpha \Big] . \end{aligned} \tag{C.26}$$

This is determined by the Killing spinor equation and its solution [108, 118]. We can make the same super-horospherical decomposition for the κ-symmetry parameter

$$\kappa^\alpha_+ \mathbf{K}_\alpha = \left(\rho^{1/2} u^j_i \bar{Q}_j \kappa^i_{+Q} + \text{h.c.} \right) + \left(\rho^{-1/2} u^j_i \bar{S}_j \kappa^i_{+S} + \text{h.c.} \right), \tag{C.27}$$

and the relationship between its irreducible components is modified with factors of R such that $\kappa_{+Q} = R\beta_- \kappa_{+S}$, or equivalently, $\kappa_{+S} = -\frac{1}{R}\beta_+ \kappa_{+Q}$.

Appendix D

Sample supercoset calculation - $AdS_5 \times S^5$ supersymmetry

We will calculate the supersymmetry part of δx^m using (4.13), as well as the choice for coset representative (C.23) and (C.26). We find for the fermionic part of the transformation that[1]

$$\delta x^m = -\Xi^m = -\tilde{\epsilon}_i^\beta \left(\mathcal{M}^{-1}\tanh\mathcal{M}/2\right)_\beta{}^{i\alpha}{}_j \,\Upsilon_\alpha{}^{ja} e_a^m + \text{h.c.}. \tag{D.1}$$

We know that $e_a^m = \frac{1}{\rho}\delta_a^m$ from (4.23), to lowest order we have that $\mathcal{M}^{-1}\tanh\mathcal{M}/2 \simeq \frac{1}{2}$ and finally we know that $\Upsilon_\alpha{}^{ia} = -\Theta^{j\delta} f_{j\delta\beta}{}^{i\ a}$. We have

$$\delta x^m = -\frac{1}{\rho}\left[\frac{1}{2}\tilde{\epsilon}_i^\beta \left(-\Theta^{j\delta} f_{j\delta\beta}{}^{i\ a}\right)\delta_a^m\right] + \text{h.c.} = \frac{1}{2\rho}\tilde{\epsilon}_i^\beta f_{j\delta\beta}{}^{i\ a}\Theta^{j\delta}\delta_a^m + \text{h.c.}. \tag{D.2}$$

From the algebra (B.17), we find that[2]

$$\begin{aligned} f_{j\beta}{}^{i\alpha\ a} &= \delta_j^i \left[\frac{1}{2}\left(1-\gamma_5\right)\gamma^a \frac{1}{2}\left(1+\gamma_5\right)\right]_\beta{}^\alpha + \delta_j^i \left[\frac{1}{2}\left(1+\gamma_5\right)\gamma^a \frac{1}{2}\left(1-\gamma_5\right)\right]_\beta{}^\alpha \\ &= \delta_j^i \left[\gamma^a \frac{1}{2}\left(1+\gamma_5\right)\right]_\beta{}^\alpha + \delta_j^i \left[\gamma^a \frac{1}{2}\left(1-\gamma_5\right)\right]_\beta{}^\alpha = \delta_j^i \left(\gamma^a\right)_\beta{}^\alpha , \end{aligned} \tag{D.3}$$

[1] We need to add the hermitian conjugate because we split the spinor index used in (4.12) into the indices (α, i).

[2] From the $\{Q, \bar{Q}\}$ and $\{S, \bar{S}\}$ anticommutators, reinstating the projection operators that were omitted.

such that

$$\delta x^m = \frac{1}{2\rho}\tilde{\epsilon}_i^{\beta}\left(\gamma^a\right)_{\beta}{}^{\alpha}\Theta_{\alpha i}\delta_a^m + \text{h.c.} = \frac{1}{2\rho}\bar{\tilde{\epsilon}}_i\gamma^m\Theta^i + \text{h.c.}, \tag{D.4}$$

where we have defined[3] $\gamma^m \equiv \gamma^a\delta_a^m$. We now use definitions (C.23) and (C.26) to obtain

$$\begin{aligned}
\delta x^m = {} & \frac{1}{2\rho}\frac{1}{\sqrt{2}}\left(u^{-1}\right)_i{}^j\left[R\rho^{-1/2}\bar{\eta}_j^{\beta}\frac{1}{2}\left(1+\gamma_5\right)_{\beta}{}^{\alpha}\right.\\
& \left. +\rho^{1/2}\left(\bar{\epsilon}_j{}^{\beta} - \bar{\eta}_j{}^{\gamma}(\gamma_n)_{\gamma}{}^{\beta}x^n\right)\frac{1}{2}\left(1-\gamma_5\right)_{\beta}{}^{\alpha}\right]\left(\gamma^m\right)_{\alpha}{}^{\gamma}u_k{}^i\sqrt{2}\\
& \times\left[\rho^{1/2}\frac{1}{2}\left(1+\gamma_5\right)_{\gamma}{}^{\delta}\theta_{\delta}^k + \rho^{-1/2}R\frac{1}{2}\left(1-\gamma_5\right)_{\gamma}{}^{\delta}\vartheta_{\delta}^k\right] + \text{h.c.}\\
= {} & \frac{1}{2}\left(\bar{\epsilon}_i - \bar{\eta}_i\gamma_n x^n\right)\gamma^m\theta^i + \frac{1}{2}\frac{R^2}{\rho^2}\bar{\eta}_i\gamma^m\vartheta^i + \text{h.c.}\\
= {} & \frac{1}{2}\bar{\epsilon}_i(x)\gamma^m\theta^i + \frac{1}{2}\frac{R^2}{\rho^2}\bar{\eta}_i\gamma^m\vartheta^i + \text{h.c.},
\end{aligned} \tag{D.5}$$

where we defined $\bar{\epsilon}_i(x) = \bar{\epsilon}_i - \bar{\eta}_i\gamma_n x^n$ such that $\epsilon^i(x) = \epsilon^i + x^n\gamma_n\eta^i$. Equation (D.5) gives the supersymmetry part of the transformation of x^m given in (4.26).

[3]Note that this is different from a definition using the vielbein $\gamma^m \neq e_a^m\gamma^a = \frac{1}{\rho}\delta_a^m\gamma^a = \frac{1}{\rho}\gamma^m$.

Appendix E

Reduction and partial solution of BPS-equations

In this appendix we collect the details of the partial reduction of the BPS-equations (5.5) to a two-dimensional system. We partially solve this reduced set of equations and, finally, we perform a check on this partial solution by showing that expressions we obtain for some of the fields supplemented by the remaining reduced equations solve the equations of motion and the Maxwell equations.

E.1 Reduction of BPS equations to two dimensions

We introduce complex frames e^z for the components along Σ_2, defined so that $p_z = p_8 - ip_9$ and

$$p_a \sigma^a = \begin{pmatrix} 0 & p_z \\ p_{\bar{z}} & 0 \end{pmatrix} . \qquad \text{(E.1)}$$

We begin by reducing the first BPS equation given in (5.5):

$$\begin{aligned} 0 \quad &= \Big(\tfrac{1}{2\sqrt{2}} D_M \phi \Gamma^M + \tfrac{5}{8\sqrt{2}} F_0 e^{5\phi/4} + \tfrac{1}{192\sqrt{2}} e^{\phi/4} F_{MNPQ} \Gamma^{MNPQ} \\ &- \tfrac{3}{16\sqrt{2}} F_0 e^{3\phi/4} B_{MN} \Gamma^{MN} \Gamma_{11} - \tfrac{1}{24\sqrt{2}} e^{-\phi/2} H_{MNP} \Gamma^{MNP} \Gamma_{11} \Big) \epsilon . \end{aligned} \qquad \text{(E.2)}$$

The first term is decomposed as

$$
\begin{aligned}
&D_M\phi\Gamma^M\epsilon = D_a\phi\Gamma^a\epsilon \\
&= D_a\phi(\sigma^3\otimes I_2\otimes I_2\otimes\gamma^a)\sum_{\eta_1,\eta_2,\eta_3}\chi_{\eta_1,\eta_2,\eta_3}\otimes\left[\zeta_{\eta_1,\eta_3,\eta_3}\otimes\begin{pmatrix}1\\0\end{pmatrix}+\hat{\zeta}_{\eta_1,\eta_3,\eta_3}\otimes\begin{pmatrix}0\\1\end{pmatrix}\right] \\
&= \sum_{\eta_1,\eta_2,\eta_3}\chi_{\eta_1,\eta_2,\eta_3}\otimes\left[\sigma^3\zeta_{-\eta_1,\eta_3,\eta_3}\otimes\begin{pmatrix}D_{\bar{z}}\phi\\0\end{pmatrix}+\sigma^3\hat{\zeta}_{-\eta_1,\eta_3,\eta_3}\otimes\begin{pmatrix}0\\D_z\phi\end{pmatrix}\right] \\
&= \sum_{\eta_1,\eta_2,\eta_3}\chi_{\eta_1,\eta_2,\eta_3}\otimes\left[\tau^{(100)}\sigma^3\zeta_{\eta_1,\eta_3,\eta_3}\otimes\begin{pmatrix}D_{\bar{z}}\phi\\0\end{pmatrix}+\tau^{(100)}\sigma^3\hat{\zeta}_{\eta_1,\eta_3,\eta_3}\otimes\begin{pmatrix}0\\D_z\phi\end{pmatrix}\right]
\end{aligned}
\tag{E.3}
$$

where we have written the expression so that $\chi_{\eta_1,\eta_2,\eta_3}$ is an overall coefficient. In the last line, we have used the $\tau^{(ijk)}$ notation introduced in section 5.1.1. Proceeding in a similar manner for each term and requiring (E.2) to hold independently for each $\chi_{\eta_1,\eta_2,\eta_3}$ yields two equations

$$
\begin{aligned}
0 =& \frac{1}{2\sqrt{2}}D_z\phi\tau^{(100)}\sigma^3\hat{\zeta}+\frac{5}{8\sqrt{2}}F_0e^{5\phi/4}\zeta+\frac{1}{8\sqrt{2}}e^{\phi/4}(ih_1\tau^{(100)}\zeta+h_z\sigma^2\hat{\zeta}-g_z\sigma^1\hat{\zeta}) \\
&-\frac{3}{8\sqrt{2}}F_0e^{3\phi/4}\left(\frac{b_0}{f_1^2}\sigma^3\zeta+ib_1\tau^{(100)}\sigma^3\zeta\right)+\frac{1}{4\sqrt{2}}e^{-\phi/2}\frac{D_zb_0}{f_1^2}\tau^{(100)}\hat{\zeta}\,, \\
0 =& \frac{1}{2\sqrt{2}}D_{\bar{z}}\phi\tau^{(100)}\sigma^3\zeta+\frac{5}{8\sqrt{2}}F_0e^{5\phi/4}\hat{\zeta}+\frac{1}{8\sqrt{2}}e^{\phi/4}(-ih_1\tau^{(100)}\hat{\zeta}+h_{\bar{z}}\sigma^2\zeta-g_{\bar{z}}\sigma^1\zeta) \\
&-\frac{3}{8\sqrt{2}}F_0e^{3\phi/4}\left(-\frac{b_0}{f_1^2}\sigma^3\hat{\zeta}+ib_1\tau^{(100)}\sigma^3\hat{\zeta}\right)-\frac{1}{4\sqrt{2}}e^{-\phi/2}\frac{D_{\bar{z}}b_0}{f_1^2}\tau^{(100)}\zeta\,.
\end{aligned}
\tag{E.4}
$$

Next we use the type IIA reality condition $\hat{\zeta}^*=i\sigma_2\zeta$ to eliminate $\hat{\zeta}$ from the above two equations. After doing so, we find that the equations are complex conjugates of each other, thus it is sufficient to keep only one. As a result, (E.2) is equivalent to the single equation

$$
\begin{aligned}
0 =& \frac{1}{2\sqrt{2}}D_z\phi\tau^{(100)}\sigma^3\zeta^*+\frac{5}{8\sqrt{2}}F_0e^{5\phi/4}i\sigma^2\zeta-\frac{1}{8\sqrt{2}}e^{\phi/4}(h_1\tau^{(100)}\sigma^2\zeta+h_z\sigma^2\zeta^* \\
&+g_z\sigma^1\zeta^*)+\frac{3}{8\sqrt{2}}F_0e^{3\phi/4}\left(\frac{b_0}{f_1^2}\sigma^1\zeta+ib_1\tau^{(100)}\sigma^1\zeta\right)-\frac{1}{4\sqrt{2}}e^{-\phi/2}\frac{D_zb_0}{f_1^2}\tau^{(100)}\zeta^*\,.
\end{aligned}
\tag{E.5}
$$

We proceed in a similar manner for the second equation of (5.5). Note in general that this equation is differential in the spinor parameter ϵ. First we consider when the index M is along the symmetric spaces. In this case, the equation reduces to an algebraic equation in ϵ. This is achieved by making use of the Killing spinor equations for the $\chi_{\eta_1,\eta_2,\eta_3}$

$$
\begin{aligned}
0 &= \left(\hat{e}^{\mu}_{m}\hat{\nabla}_{\mu} - \frac{\eta_1}{2}\gamma_m \otimes I_2 \otimes I_2\right)\chi_{\eta_1,\eta_2,\eta_3}\,, \\
0 &= \left(\hat{e}^{\mu}_{i}\hat{\nabla}_{\mu} - i\frac{\eta_2}{2}I_2 \otimes \gamma_i \otimes I_2\right)\chi_{\eta_1,\eta_2,\eta_3}\,, \\
0 &= \left(\hat{e}^{\mu}_{\tilde{i}}\hat{\nabla}_{\mu} - i\frac{\eta_3}{2}I_2 \otimes I_2 \otimes \gamma_{\tilde{i}}\right)\chi_{\eta_1,\eta_2,\eta_3}\,,
\end{aligned} \tag{E.6}
$$

and noting that ζ and $\hat{\zeta}$ only depend on Σ_2. We use the relation between ζ and $\hat{\zeta}$ to rewrite the equations in terms of ζ. The result is a reduction to a system of three equations, one for each symmetric space,

$$
\begin{aligned}
0 =& -\frac{1}{2f_1}\tau^{(300)}\zeta + \frac{1}{2}D_z \ln f_1 \tau^{(100)}\sigma^1\zeta^* - \frac{1}{32}F_0 e^{5\phi/4}\zeta \\
&+ \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_z\zeta^* + g_z\sigma^3\zeta^* - i\frac{5}{3}h_1\tau^{(100)}\zeta\right) \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}\left[ib_1\sigma^3\tau^{(100)}\zeta - 7\frac{b_0}{f_1^2}\sigma^3\zeta\right] + \frac{3}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\tau^{(100)}i\sigma^2\zeta^*\,,
\end{aligned} \tag{E.7}
$$

$$
\begin{aligned}
0 =& -\frac{i}{2f_2}\sigma_1\tau^{(030)}\zeta + \frac{1}{2}D_z \ln f_2 \tau^{(100)}\sigma^1\zeta^* - \frac{1}{32}F_0 e^{5\phi/4}\zeta \\
&+ \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_1\tau^{(100)}\zeta + g_z\sigma^3\zeta^* - i\frac{5}{3}h_z\zeta^*\right) \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}\left[\frac{b_0}{f_1^2}\sigma^3\zeta + ib_1\sigma^3\tau^{(100)}\zeta\right] - \frac{1}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\tau^{(100)}i\sigma^2\zeta^*\,,
\end{aligned} \tag{E.8}
$$

$$
\begin{aligned}
0 =& -\frac{i}{2f_3}\sigma_2\tau^{(003)}\zeta + \frac{1}{2}D_z \ln f_3 \tau^{(100)}\sigma^1\zeta^* - \frac{1}{32}F_0 e^{5\phi/4}\zeta \\
&+ \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_1\tau^{(100)}\zeta + ih_z\zeta^* - \frac{5}{3}g_z\sigma^3\zeta^*\right) \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}\left[\frac{b_0}{f_1^2}\sigma^3\zeta + ib_1\sigma^3\tau^{(100)}\zeta\right] - \frac{1}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\tau^{(100)}i\sigma^2\zeta^*\,.
\end{aligned} \tag{E.9}
$$

Finally, we reduce (5.5) for the components along Σ_2. In this case, the equations remain differential in ϵ and are given by

$$\begin{aligned} 0 =& D_z\zeta - \frac{1}{2}D_z(\ln\rho)\,\zeta + \frac{3}{16}\frac{e^{\phi/4}}{2}\left[i\frac{8}{3}h_z\sigma^1\tau^{(100)}\zeta + i\frac{8}{3}g_z\sigma^2\tau^{(100)}\zeta\right] \\ & - \frac{1}{2}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\sigma^3\zeta\,, \end{aligned} \tag{E.10}$$

$$\begin{aligned} 0 =& D_{\bar z}\zeta + \frac{1}{2}D_{\bar z}(\ln\rho)\zeta - \frac{1}{16}F_0 e^{5\phi/4}\sigma^1\tau^{(100)}\zeta^* + \frac{3}{16}\frac{e^{\phi/4}}{2}\left[i\frac{2}{3}h_{\bar z}\sigma^1\tau^{(100)}\zeta\right. \\ & \left.+ i\frac{2}{3}g_{\bar z}\sigma^2\tau^{(100)}\zeta + i\frac{10}{3}h_1\sigma^1\zeta^*\right] + \frac{1}{8}F_0\frac{e^{3\phi/4}}{2}\left[\frac{b_0}{f_1^2}\tau^{(100)}i\sigma^2\zeta^* - 7b_1\sigma^2\zeta^*\right] \\ & - \frac{1}{4}\frac{e^{-\phi/2}}{2}\frac{D_{\bar z}b_0}{f_1^2}\sigma^3\zeta\,. \end{aligned} \tag{E.11}$$

E.2 The general case

In the case when b_1 or h_1 no longer vanish, $\tau^{(300)}\sigma^3$ no longer commutes with the BPS equations. It is still useful to decompose the BPS equations in terms of eigenstates of $\tau^{(300)}\sigma^3$. We denote the two eigenstates by $\xi_\pm$. The equations then reduce to

$$\begin{aligned} 0 =& \frac{1}{2\sqrt{2}}D_z\phi\sigma^3\xi_\pm^* + \frac{5}{8\sqrt{2}}F_0 e^{5\phi/4}i\sigma^2\xi_\pm - \frac{1}{8\sqrt{2}}e^{\phi/4}(h_z\sigma^2\xi_\pm^* + g_z\sigma^1\xi_\pm^*) \\ & + \frac{3}{8\sqrt{2}}F_0 e^{3\phi/4}\frac{b_0}{f_1^2}\sigma^1\xi_\pm - \frac{1}{4\sqrt{2}}e^{-\phi/2}\frac{D_z b_0}{f_1^2}\xi_\pm^* - \frac{1}{8\sqrt{2}}e^{\phi/4}h_1\sigma^2\xi_\mp \\ & + \frac{3}{8\sqrt{2}}F_0 e^{3\phi/4}ib_1\sigma^1\xi_\mp\,. \end{aligned} \tag{E.12}$$

The gravitino equation reduces to the algebraic equations

$$
\begin{aligned}
0 =& -\frac{(\pm 1)}{2f_1}\sigma^3\xi_\pm + \frac{1}{2}D_z \ln f_1 \sigma^1 \xi^*_\pm - \frac{1}{32}F_0 e^{5\phi/4}\xi_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_z\xi^*_\pm + g_z\sigma^3\xi^*_\pm\right) \\
&+ \frac{7}{16}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^3\xi_\pm + \frac{3}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}i\sigma^2\xi^*_\pm - \frac{5}{16}\frac{e^{\phi/4}}{2}ih_1\xi_\mp \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}ib_1\sigma^3\xi_\mp\,, \\
0 =& -\frac{i\nu_2}{2f_2}\sigma_1\xi_\pm + \frac{1}{2}D_z \ln f_2 \sigma^1 \xi^*_\pm - \frac{1}{32}F_0 e^{5\phi/4}\xi_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}\left(g_z\sigma^3\xi^*_\pm - i\frac{5}{3}h_z\xi^*_\pm\right) \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^3\xi_\pm - \frac{1}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}i\sigma^2\xi^*_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}ih_1\xi_\mp \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}ib_1\sigma^3\xi_\mp\,, \\
0 =& -\frac{i\nu_3}{2f_3}\sigma_2\xi_\pm + \frac{1}{2}D_z \ln f_3 \sigma^1 \xi^*_\pm - \frac{1}{32}F_0 e^{5\phi/4}\xi_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}\left(ih_z\xi^*_\pm - \frac{5}{3}g_z\sigma^3\xi^*_\pm\right) \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^3\xi_\pm - \frac{1}{8}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}i\sigma^2\xi^*_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}ih_1\xi_\mp \\
&- \frac{1}{16}F_0\frac{e^{3\phi/4}}{2}ib_1\sigma^3\xi_\mp\,,
\end{aligned}
\tag{E.13}
$$

and the differential equations

$$
\begin{aligned}
0 =& D_z\xi_\pm - \frac{1}{2}D_z(\ln\rho)\,\xi_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}\left[i\frac{8}{3}h_z\sigma^1\xi_\pm + i\frac{8}{3}g_z\sigma^2\xi_\pm\right] \\
&- \frac{1}{2}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\sigma^3\xi_\pm\,,
\end{aligned}
$$

$$
\begin{aligned}
0 =& D_{\bar{z}}\xi_\pm + \frac{1}{2}D_{\bar{z}}(\ln\rho)\xi_\pm - \frac{1}{16}F_0 e^{5\phi/4}\sigma^1\xi^*_\pm + \frac{3}{16}\frac{e^{\phi/4}}{2}\left[i\frac{2}{3}h_{\bar{z}}\sigma^1\xi_\pm + i\frac{2}{3}g_{\bar{z}}\sigma^2\xi_\pm\right] \\
&+ i\frac{1}{8}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\sigma^2\xi^*_\pm - \frac{1}{4}\frac{e^{-\phi/2}}{2}\frac{D_{\bar{z}}b_0}{f_1^2}\sigma^3\xi_\pm \\
&+ \frac{5}{8}\frac{e^{\phi/4}}{2}ih_1\sigma^1\xi^*_\mp - \frac{7}{8}F_0\frac{e^{3\phi/4}}{2}b_1\sigma^2\xi^*_\mp\,.
\end{aligned}
\tag{E.14}
$$

E.3 Decoupling the BPS equations

In this appendix we further reduce the BPS equations (5.8), (5.9) and (5.10). We obtain algebraic expressions for the metric factors f_i. We obtain an algebraic expression for b_0 and show that no further constraints arise from differentiating this expression. This allows us to treat b_0 and $D_z b_0$ as independent variables, since the BPS equations will enforce the differential relation among them. We find it convenient to introduce the notation $b_z = D_z b_0/f_1^2$ and treat b_0 and b_z as independent. Finally we derive an algebraic constraint relating the c_i. The net result is the reduction of (5.8) and (5.9) to the expressions for the metric factors (E.17), the four algebraic equations (E.19), (E.22), (E.24) and (E.25) and a differential equation for the dilaton.

To solve for the metric factors, we first use (5.10) to compute the following derivatives of spinor bilinears

$$
\begin{aligned}
D_z(\xi^\dagger\xi) =& \frac{1}{16}F_0 e^{5\phi/4}\xi^t\sigma^1\xi - \frac{3}{8}\frac{e^{\phi/4}}{2}\left[ih_z\xi^\dagger\sigma^1\xi + ig_z\xi^\dagger\sigma^2\xi\right] \\
&+ \frac{3}{4}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\xi^\dagger\sigma^3\xi\,, \\
D_z(\xi^\dagger\sigma^2\xi) =& \frac{i}{16}F_0 e^{5\phi/4}\xi^t\sigma^3\xi - \frac{3}{8}\frac{e^{\phi/4}}{2}\left[\frac{5}{3}h_z\xi^\dagger\sigma^3\xi + ig_z\xi^\dagger\xi\right] \\
&+ \frac{i}{8}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\xi^t\xi + \frac{i}{4}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\xi^\dagger\sigma^1\xi\,,
\end{aligned}
$$

$$D_z(\xi^\dagger\sigma^1\xi) =\frac{1}{16}F_0 e^{5\phi/4}\xi^t\xi - \frac{3}{8}\frac{e^{\phi/4}}{2}\left[ih_z\xi^\dagger\xi - \frac{5}{3}g_z\xi^\dagger\sigma^3\xi\right] + \frac{1}{8}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\xi^t\sigma^3\xi$$
$$-\frac{i}{4}\frac{e^{-\phi/2}}{2}\frac{D_z b_0}{f_1^2}\xi^\dagger\sigma^2\xi\,. \tag{E.15}$$

To obtain equations involving only the metric factors, we multiply the three equations given in (5.9) respectively by $\xi^t\sigma^1$, $-i\xi^t\sigma^3$ and ξ^t. Combining the resulting equations with the above equations yields

$$D_z(\xi^\dagger\xi) =D_z(\ln f_1)\xi^t\xi^*\,,$$
$$D_z(\xi^\dagger\sigma^2\xi) =D_z(\ln f_2)\xi^\dagger\sigma^2\xi\,,$$
$$D_z(\xi^\dagger\sigma^1\xi) =D_z(\ln f_3)\xi^\dagger\sigma^1\xi\,. \tag{E.16}$$

These equations are integrated to give

$$f_1 = \frac{\nu_1}{c_1}\xi^\dagger\xi\,, \qquad f_2 = \frac{\nu_2}{c_2}\xi^\dagger\sigma^2\xi\,, \qquad f_3 = \frac{\nu_3}{c_3}\xi^\dagger\sigma^1\xi\,, \tag{E.17}$$

where the c_i are real constants. The factors of ν_i have been introduced for convenience.

We now move on to solving algebraically for b_0. We start by combining the first equation of (5.9) with three-halves of the second and third equations. Multiplying the resulting equation by $\xi^\dagger\sigma^3$ gives

$$0 = -\frac{\nu_1}{2f_1}\xi^\dagger\xi + \frac{3}{2}\frac{\nu_2}{2f_2}\xi^\dagger\sigma^2\xi - \frac{3}{2}\frac{\nu_3}{2f_3}\xi^\dagger\sigma^1\xi - \frac{1}{8}F_0 e^{5\phi/4}\xi^\dagger\sigma^3\xi + \frac{1}{4}F_0\frac{e^{3\phi/4}}{2}\frac{b_0}{f_1^2}\xi^\dagger\xi\,. \tag{E.18}$$

Using the above expressions for the metric factors, we solve this equation for b_0

$$b_0 = \frac{8\xi^\dagger\xi}{F_0 c_1^2}e^{-3\phi/4}\left(\frac{1}{2}c_1 - \frac{3}{4}c_2 + \frac{3}{4}c_3 + \frac{1}{8}F_0 e^{5\phi/4}\xi^\dagger\sigma^3\xi\right)\,. \tag{E.19}$$

Next we show that the derivative of b_0 is automatically reproduced by the BPS equations. To do so, we first differentiate the above expression

$$D_z b_0 =\frac{8}{F_0 c_1^2}D_z\left(\xi^\dagger\xi e^{-3\phi/4}\right)\left(\frac{1}{2}c_1 - \frac{3}{4}c_2 + \frac{3}{4}c_3 + \frac{1}{8}F_0 e^{5\phi/4}\xi^\dagger\sigma^3\xi\right)$$
$$+\frac{\xi^\dagger\xi}{c_1^2}e^{-3\phi/4}D_z\left(e^{5\phi/4}\xi^\dagger\sigma^3\xi\right)\,. \tag{E.20}$$

Next we show that the BPS equations imply this equation is automatic. We first use (5.8) and (5.10) to compute

$$D_z\left(e^{5\phi/4}\xi^\dagger\sigma^3\xi\right) = e^{3\phi/4}\frac{D_z b_0}{f_1^2}\xi^\dagger\xi - F_0 e^{2\phi}\frac{b_0}{f_1^2}\xi^t\sigma^1\xi\,,$$

$$D_z(\xi^\dagger\xi e^{-3\phi/4}) = F_0 e^{\phi/2}\xi^t\sigma^1\xi\,. \tag{E.21}$$

Using these expressions to eliminate $D_z\left(e^{5\phi/4}\xi^\dagger\sigma^3\xi\right)$ and $D_z(\xi^\dagger\xi e^{-3\phi/4})$, as well as the first equation of (E.15) and the expression (E.19) for b_0, we find that (E.20) is automatic. As a result, we may introduce b_z as $b_z = D_z b_0/f_1^2$ and treat b_z and b_0 as independent variables. The BPS equations will correctly enforce the relation between the two variables.

In total the system of equations (5.8) and (5.9) provides 7 algebraic equations and one equation differential in ϕ. Three of these equations are used to solve for the metric factors and a fourth equation gives b_0. We exhibit the remaining three equations as follows. We first obtain a simple equation relating g_z and h_z. To do so, we first take the difference of the second and third equations appearing in (5.9) and multiply the resulting expression by $\xi^\dagger\sigma^3$ to obtain

$$0 = 2\left(c_2 + c_3\right) + e^{\phi/4}g_z\xi^\dagger\xi^* - ie^{\phi/4}h_z\xi^\dagger\sigma^3\xi^*\,. \tag{E.22}$$

Next, we use the BPS equations to obtain an algebraic constraint amongst the c_i. To do so, we multiply the first equation of (5.9) by $\xi^\dagger\sigma^3$ and use (E.19) to eliminate b_0

$$\begin{aligned}0 =& -\frac{1}{2}c_1 + \frac{3}{16}F_0 e^{5\phi/4}\xi^\dagger\sigma^3\xi + \frac{3}{32}e^{\phi/4}\left(g_z\xi^\dagger\xi^* + ih_z\xi^\dagger\sigma^3\xi^*\right)\\ &+ \frac{7}{16}\left(2c_1 - 3c_2 + 3c_3\right) + \frac{3}{16}e^{-\phi/2}b_z\xi^\dagger\sigma^1\xi^*\,.\end{aligned} \tag{E.23}$$

Taking (5.8), multiplying by $\xi^\dagger\sigma^1$ and using (E.19) to eliminate b_0 gives

$$\begin{aligned}0 =& -\frac{1}{4}F_0 e^{5\phi/4}\xi^\dagger\sigma^3\xi + \frac{3}{4}\left(2c_1 - 3c_2 + 3c_3\right) - \frac{e^{\phi/4}}{8}\left(g_z\xi^\dagger\xi^* + ih_z\xi^\dagger\sigma^3\xi^*\right)\\ &- \frac{e^{-\phi/2}}{4}b_z\xi^\dagger\sigma^1\xi^*\,.\end{aligned} \tag{E.24}$$

Adding 4/3 of the first equation to the second equation then gives

$$c_1 - 2c_2 + 2c_3 = 0\,. \tag{E.25}$$

Finally, we take (E.24) for the last algebraic equation. This constraint is similar to the one encountered in [60] for M-theory, where the values of c_i controlled the $D(2,1;\gamma;1)$ group parameter γ.

E.4 Checking the equations of motion and Maxwell equations

In [119], it was shown that supersymmetry, together with the Maxwell equations and Bianchi identities imply that the dilaton equation and most of the Einstein equations are automatically satisfied. More specifically, if we denote the Einstein equations collectively by E_{MN}, one finds that $E_{MN} = 0$ provided $E_{0M} = 0$ for $M \neq 0$. Since our solution has an AdS_2 isometry, we have $E_{mn} \propto \eta_{mn}$ and $E_{mi} = E_{m\tilde{i}} = E_{ma} = 0$ and the condition is automatic. As a result, we need only to check the Maxwell equations.

The Maxwell equations are given by

$$\begin{aligned} 0 =& \nabla_P \left(e^{-\phi} H^{PMN}\right) - F_{(0)}^2 e^{3\phi/2} B^{MN} - F_{(0)} e^{\phi/4} \frac{1}{2} F^{MNPQ} B_{PQ} \\ &+ \frac{1}{24^2}\frac{1}{2}\epsilon^{MNPQRSTUVW} F_{PQRS} F_{TUVW} \, , \\ 0 =& \nabla_Q \left(e^{\phi/2} F^{QMNP}\right) + \frac{1}{72}\frac{1}{2}\epsilon^{MNPQRSTUVW} F_{QRST} H_{UVW} \, . \end{aligned} \tag{E.26}$$

Using our ansatz, these equations reduce to a set of three equations[1]

$$\begin{aligned} 0 &= D_z b_{\bar{z}} + D_{\bar{z}} b_z - D_z\phi b_{\bar{z}} - D_{\bar{z}}\phi b_z + i h_{\bar{z}} g_z e^{\phi} - i h_z g_{\bar{z}} e^{\phi} \\ &\qquad + b_z D_{\bar{z}} \ln\left(f_2^3 f_3^3 \rho\right) + b_{\bar{z}} D_z \ln\left(f_2^3 f_3^3 \rho\right) - 2F_{(0)}^2 \frac{b_0}{f_1^2} e^{5\phi/2} \, , \\ 0 &= D_z h_{\bar{z}} + D_{\bar{z}} h_z + \frac{1}{2} D_z\phi h_{\bar{z}} + \frac{1}{2} D_{\bar{z}}\phi h_z + i b_{\bar{z}} g_z e^{-\phi/2} - i b_z g_{\bar{z}} e^{-\phi/2} \\ &\qquad + h_z D_{\bar{z}} \ln\left(f_1^2 f_3^3 \rho\right) + h_{\bar{z}} D_z \ln\left(f_1^2 f_3^3 \rho\right) \, , \\ 0 &= D_z g_{\bar{z}} + D_{\bar{z}} g_z + \frac{1}{2} D_z\phi g_{\bar{z}} + \frac{1}{2} D_{\bar{z}}\phi g_z - i b_{\bar{z}} h_z e^{-\phi/2} + i b_z h_{\bar{z}} e^{-\phi/2} \\ &\qquad + g_z D_{\bar{z}} \ln\left(f_1^2 f_2^3 \rho\right) + g_{\bar{z}} D_z \ln\left(f_1^2 f_2^3 \rho\right) \, . \end{aligned} \tag{E.27}$$

To check these equations, we employ the following strategy. We use the algebraic equations to eliminate g_z, b_0 and b_z in terms of h_z, α, β and ϕ. The Bianchi identities, (5.15), are then used to obtain expressions for $\partial_z h_{\bar{z}}$ and its complex conjugate. Along with the BPS equations, (5.8) and (5.10), this allows us

[1] We are using the notation $b_z = D_z b_0 / f_1^2$.

to eliminate all derivative terms appearing in (E.27). The computations are straightforward but tedious and we do not present them here. The net result, once all derivatives have been eliminated is that the Maxwell equations are all automatically satisfied. Thus in our case, it is sufficient to keep only the BPS equations and Bianchi identities.

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1. M.M. Caldarelli, R. Emparan and B. Van Pol, *Higher-dimensional Rotating Charged Black Holes*, JHEP **1104** (2011) 013, [arXiv:1012.4517]

2. J. Estes, D. Krym and B. Van Pol, *Near Horizon Geometry of Strings Ending on Intersecting D8/D4-branes*, JHEP **1410** (2014) 103, [arXiv:1407.4305]

3. F. Coomans and B. Van Pol, *On the relation between Volkov-Akulov and special conformal supersymmetry: a D3-brane perspective*, JHEP **1412** (2014) 015, [arXiv:1409:3749]

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