On Matrix Regularisation of Supermembranes

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Abstract

I will review the theory of a Green-Schwartz supersymmetric closed membrane embedded in flat superspace in eleven dimensions. After performing a gauge-fixing, we obtain a theory which can be viewed as a limit of $SU(N)$ matrix models, where N tends to infinity. This regularisation procedure, and especially the role of the gauge groups, shall be the central topic of the thesis. Finally we discuss this procedure in the case of membranes embedded in superspaces with compactified bosonic directions.

Contents

1 Introduction

In order to consistently combine the microscopic description interacting particles, quantum field theory, with the behaviour of spacetime and matter at large scales which is ruled by general relativity, a theory of stringlike objects seems to be the most promising candidate. String theory replaces the worldline of a particle with a 2-dimensional *worldsheet* swept out by a one-dimensional object, and particle states arise from the vibrating modes of the string. Originally, this concept was introduced to explain the observed relation between mass and spin of hadrons. Although little success was achieved in this direction, the string model would soon become a very popular and promising theory for the description of nature at its most fundamental level. Its biggest advantage is the combination of diffeomorphism invariance, resulting in graviton states, with a simple harmonic oscillator spectrum which arises after an appropriate gauge fixing. The less attractive feature is the fact that its quantum mechanics is Lorentz invariant only in 26-dimensional Minkowski target space. One can include worldsheet spinors carrying a spacetime vector index in a supersymmetric fashion; this results in the so-called Ramond-Neveu-Schwarz (RNS) superstring. Again, requiring the angular momenta and Lorentz boost operators to obey the Lorentz algebra restricts the dimension of the background, to 10-dimensional Minkowski space this time.

The massless states of the RNS superstring are recognised to constitute a ten-dimensional supergravity multiplet: an irreducible representations of the super-Poincaré algebra in ten dimensions. A supergravity theory is a field theory which is locally invariant under a super-extension of the isometry group[1] of some spacetime. The resulting algebra is called supercommutative, and such algebras were all classified in [1]. Accordingly, supergravity theories can only exist in certain spaces of particular dimensions. Group-theoretical arguments show that the maximal dimension of Minkowski spacetime which allows a supergravity theory (with a finite number of fields) is not ten, but eleven. The eleven-dimensional supergravity Lagrangian was found in [2], and is believed to be the most fundamental theory because upon compactification and dimensional reduction various lower-dimensional supergravity theories can be deduced from it.

The superstring theories (there are several inequivalent ones, corresponding to different choices of boundary conditions of the spinors), not only generate the particle content of the supergravities based on ten-dimensional Minkowski space, but also generate the dynamics. The supergravity equations of motion follow from renormalisation conditions on the background fields in the action of a superstring moving in such a supergravitating target space. The connection is even more apparent in the Green-Schwartz formulation of a superstring, based on a mapping from the worldsheet into a supermanifold, whose geometry determines the supergravity background. With ten-dimensional supergravity being the classical limit of superstring theory, the rôle of the eleven-dimensional theory was not understood. The superspace formulation has a geometric constraint that implies closedness of a super four-from. This allows a Green-Schwartz coupling to a 2-dimensional extended supersymmetric object [3], called the *supermembrane*. The idea to replace fundamental particle states with the vibration modes of a membrane goes way back before string theory to the fifties, and is due to Dirac [4] in an attempt to describe the electron. The bosonic theory of membranes had little success, until the supermembrane was proposed as the eleven-dimensional alternative to superstrings. However, the Green-Schwartz supermembrane suffers several disadvantages. As opposed to string theory, which upon a suitable gauge fixing turns into a quantum system of free harmonic oscillators, the supermembrane exhibits a nonlinear higher order interaction term. The potential which governs its local self-interaction has flat directions, which cause every state of macroscopic area to be unstable, and the mass spectrum is continuous without a gap between massless and massive states [5]. These cumbersome features make a particle interpretation of the vibrating supermembrane modes difficult, if not impossible. It seems as if the description of a single membrane already incorporates multiple membrane states, because

[1] This can be generalised to 'conformal group'

cutting and gluing stringlike spikes costs no energy. At the time of these discoveries, pursuing this path seemed hopeless and supermembrane theory was declared dead. Yet some features of the ground state were deduced by rewriting the supermembrane action as gauge theory of *area-preserving diffeomorphisms* [6]. Area-preserving diffeomorphisms are smooth bijective mappings (with smooth inverse) from the membrane spacesheet to itself in such a way that a given volume element is left invariant. Roughly speaking, these mapping are generated by *Hamiltonian* or *divergence-free* vector fields on the spacesheet. 'Approximating' the infinite dimensional gauge group by $SU(N)$ as $N \longrightarrow \infty$ provides insight to the quantum mechanics, but far more importantly it clears the way to a new physical interpretation of the continuous spectrum and the rôle of the supermembrane w.r.t. superstring theory. The $SU(N)$ gauge theories which are used to approximate the membrane dynamics are called *matrix models*, and they arise by a dimensional reduction of ten-dimensional $SU(N)$ super-Yang-Mills theory to a point in space. They gained interest in the nineties, during the 'second string revolution', as it was discovered they constituted the low-energy effective theory describing an ensemble of N *Dirichlet particles* of Type IIA string theory.

Dirichlet branes [7] were acknowledged as a necessary ingredient to any string theory when T-duality was discovered. T-Duality is based on the observation that a compactified closed-string theory (as opposed to ordinary field theories) 'grows' an extra dimension as the compactification radius tends to zero (as it does if the radius tends to infinity). Even stronger, compactified string theories with a radius R and α/R are equivalent up to a bijective mapping of the Hilbert spaces. Open strings with von Neumann boundary conditions which ensure momentum conservation at the endpoints however behave differently: in the $R \longrightarrow 0$ limit their endpoints get restricted to the uncompactified sector; in the T-dual picture it corresponds to an open string whose endpoints are restricted to some extended volume, in other words, the von Neumann boundary conditions are replaced by Dirichlet boundary conditions. The existence of a static object on which strings end and transfer momentum to is in contradiction with special relativity. Hence the D-branes must be dynamical. As previously the massless states of the string governed the geometry of the background, now the massless states may be seen to govern the D-brane dynamics. More precisely, the massless modes in compactified directions become embedding coordinates of the brane, and the remaining massless states are identified as $U(N)$ gauge fields, where N is the number of branes one is considering. Requiring the branes to be charged under the R-R fields, each string theory type constitutes a particular set of such D-branes, and in particular the type IIA string model can incorporate D-particles, D-membranes and D-fourbranes and D-sixbranes.

The low-energy effective action of the branes is given by super-Yang-Mills theory reduced to the worldvolume of the branes. Here the matrix models come back in the picture: ordinary type IIA superstrings may be considered living on a very large 9-dimensional torus, whose low-energy effective action under T-duality is described by an ensemble of D0-branes, namely the matrix model. The connection between the supermembrane and the system of D-particles is only one of the fruits of dualities and D-brane physics. Eleven-dimensional supergravity came back in the picture when it was realised as the strong coupling limit of type IIA string theory [8]. Hence the eleven-dimensional world and (type IIA) ten-dimensional models are somehow connected: wrapping a supermembrane's spacesheet along a compactified dimension gives a fundamental type IIA string, and wrapping eleven-dimensional supergravity on a small circle yields type IIA supergravity in ten dimensions. Conversely in the strong coupling limit the type IIA string grows an extra dimension, and becomes effectively eleven-dimensional supergravity in the low-energy limit. Furthermore all kinds of dualities between the different types of string theory appeared, which provided arguments in favour of a unifying eleven-dimensional theory called *M-Theory*[9], producing all the string theories and supergravities by taking appropriate limits and compactifications. Motivated by the relations between type IIA string theory, eleven-dimensional supergravity and the matrix models, the authors of [10] made a bold conjecture that matrix theory captures all the degrees of freedom of M-theory in the infinite (longitudinal) momentum limit.

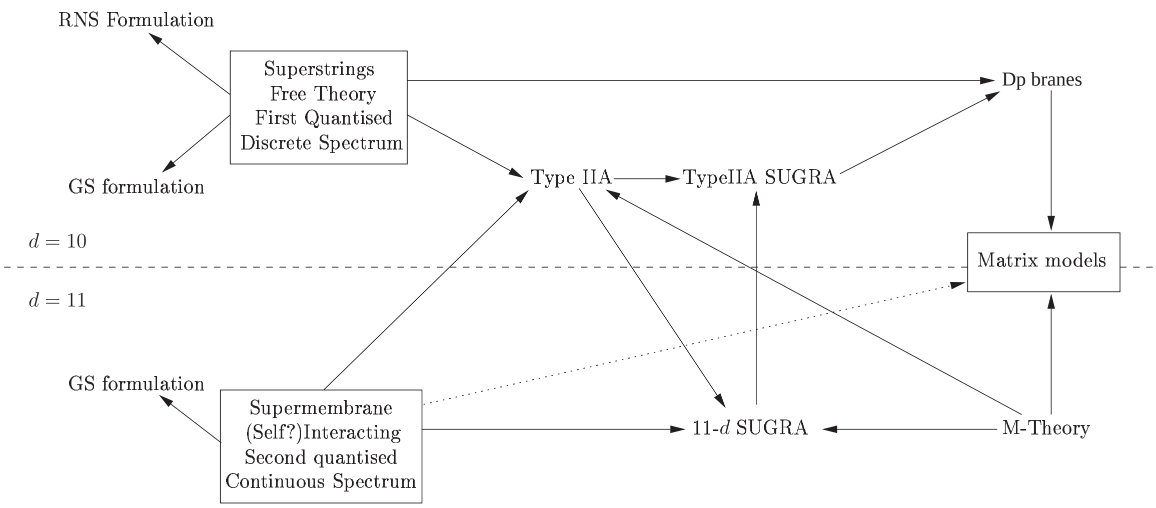


Figure 1: The framework of M-theory. The relations are discussed in the text. The dotted arrow is the subject of this thesis.

The rôle of supermembranes in this framework deserves attention, as they may be the key ingredient to a definition of M-theory. It is therefore essential to understand the matrix regularisation of a membrane, and how it should be performed on membranes moving in target spaces with compact directions. Especially the harmonic vector fields, Hamiltonian vector fields on the spacesheet which are not the gradient of a function, seem to have no place in the matrix truncation of the membrane algebra. This causes problems if one attempts to regularise wrapped membranes. The goal of this thesis is to review the membrane theory, with special attention to the harmonic diffeomorphism generators. We will include them explicitly in the gauge theory of area-preserving diffeomorphisms, and carefully study the obstructions to their matrix regularisation. Finally we will try to set up a general framework for a regularisation procedure of supermembranes in compactified target space, including harmonic vector fields. The paper is written for both physicists and mathematicians. In the first chapter some basic mathematical and physical aspects of supersymmetric field theories are treated. We start with the definition of supermanifolds and superalgebras. The second and third section are about Clifford algebras, spinor representations, and adjoints and inner products on these spinor modules. Then we put this information at work when we define superextensions of Poincaré algebras and their representations. Subsequently all this is embedded in a consistent geometrical framework and we are able to study the geometry of the supermanifolds. At the end of the first chapter we give a short summary of classical field theory. The second chapter starts with eleven-dimensional supergravity both in the ordinary formulation as well as the supermanifold formalism. These theories provide essential insight in the Lagrangian governing the supermembrane, which is studied in the second section. After a gauge-fixing, we describe the gauge theory of area-preserving diffeomorphisms in the final section. The third chapter covers the core of the paper: in its second section we introduce the matrix model and discuss its relation to the APD gauge theory. The third section is entirely devoted to the mathematics behind the regularisation. Finally, we consider winding membranes, and show how they provide the solution for the harmonic vector fields.

2 Preliminaries

2.1 Supermanifolds

2.1.1 Graded Vector Spaces

A supermembrane, the object of interest of this thesis, is an object defined by a mapping from an ordinary 3-manifold (the *worldvolume*) to an ambient curved space (the *embedding space* or *background*) which is parameterised by some ordinary, commuting ('bosonic') coordinates and a number of anticommuting ('fermionic') coordinates. In this section we shall make an attempt to rigorously define this setting by means of the theory of supermanifolds. Then this theory is applied to the *spinor* bundle over a pseudo-Riemannian manifold, collecting the anticommuting coordinates in a representation of the orthogonal group on the tangent space. A *super vector space* is a vector space V with a direct sum $V = V_0 \oplus V_1$. We shall call V_0 the *even* component and V_1 the *odd* one. A homogeneous element v, which is either in V_0 or V_1, is given a definite *parity* $\mathcal{P}(v)$, which takes the value i if v is in V_i. The super vector space V is called a $\mathbb{Z}_2$-*graded algebra* if it is equipped with a product such that $V_i V_j \subset V_{(i+j)\mathrm{mod}2}$. The resulting algebra is called *supercommutative* if for all homogeneous elements v, w in V we have $vw = (-1)^{\mathcal{P}(v)\mathcal{P}(w)} wv$.

The most basic supercommutative algebra is the *Grassmann algebra* $\bigwedge V$ over an n-dimensional vector space V over a field F. It is the linear sum of the exterior algebras of V [11],

$$\bigwedge V = \bigoplus_{p=0}^{\infty} \bigwedge^{p} V\,, \quad \bigwedge^{p} V = (\otimes^p V)/W^p\,, \tag{2.1.1}$$

where W^p is the linear space spanned by p-fold products of the form

$$v_1 \otimes \ldots \otimes v_p \in \otimes^p V \text{ with } v_i = v_j \text{ for some } 1 \le i < j \le p\,. \tag{2.1.2}$$

By definition $\bigwedge^0 V \simeq F$. The p-th exterior algebra is an F-linear vector space, rôle of the zero element is played by the elements of W^p. We define the 'wedge' product by

$$x_1 \wedge x_2 \ \wedge \ldots \wedge x_p = [x_1 \otimes x_2 \otimes \ldots \otimes x_p]\,, \tag{2.1.3}$$

where the right-hand side is the equivalence class in $\bigwedge V$ containing $x_1 \otimes \ldots \otimes x_p$. By definition, $(u+v)\wedge(u+v) = 0$ and by linearity we see $u \wedge v + v \wedge u = 0$: the wedge product is antisymmetric in all factors and contains therefore the necessary property for the construction of graded vector spaces. A basis $(e^1, \ldots e^n)$ of V provides d generators $\theta_i = [e^i]$ of the Grassmann algebra and the p-fold ordered wedge products of these classes are a basis of $\bigwedge^p V$. Hence

$$\dim(\bigwedge^{p} V) = \binom{n}{p}\,, \tag{2.1.4}$$

and consequently $\bigwedge^p V \simeq \bigwedge^{n-p} V$ and $\dim(\bigwedge V) = 2^n$. An arbitrary element of $\bigwedge V$ can thus be uniquely decomposed as

$$z = \sum_{\alpha \in (\mathbb{Z}_2)^n} c_\alpha \theta^\alpha\,, \quad \text{where } \theta^\alpha = \theta_1^{\alpha_1} \wedge \ldots \wedge \theta_n^{\alpha_n} \quad \alpha_i \in \{0,1\}\,. \tag{2.1.5}$$

The even and odd subspaces are easily identified as the subspaces spanned by the θ^α with $\alpha_1 + \ldots + \alpha_n$ respectively even and odd.

2.1.2 Supermanifolds

We proceed with the construction of superspaces with nontrivial geometries, so-called *supermanifolds*. In our definition we shall closely follow [12], which is simple and rigourous. The supermanifold formalism is founded on an ordinary smooth manifold, the *body manifold*, on which a $\mathbb{Z}_2$-graded algebra is constructed. The topology on the resulting supermanifold shall be inherited from the underlying body manifold; this is assured by the use of sheaf theory:

Definition. Let X be a topological space with topology T (the set of open subsets of X). A *sheaf of algebras* is a map $\mathcal{A} : T \longrightarrow \mathscr{C}$, where $\mathscr{C}$ is a collection of algebras over some field F, such that for each open $U \subseteq X$, $\mathcal{A}$ is an algebra and the empty set is mapped to the trivial algebra $\mathcal{A}(\emptyset) = 0$, and for all open $V \subseteq U \subseteq X$, there is an algebra morphism (the 'restriction map') $\rho_{UV} : \mathcal{A}(U) \longrightarrow \mathcal{A}(V)$ which satisfies

1. for all open $U \subseteq X$, $\rho_{UU} = \mathrm{Id}_{\mathcal{A}(U)}$,
2. for all open $W \subseteq V \subseteq U \subseteq X$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$,
3. if $\{U_\alpha\}_{\alpha \in A}$ is an open covering of an open $U \subseteq X$ and $s \in \mathcal{A}(U)$ such that $\rho_{UU_\alpha}(s) = 0$ for all $\alpha \in A$, then $s = 0$, and if $s_\alpha \in \mathcal{A}(U_\alpha)$ satisfies $\rho_{U_\alpha(U_\alpha \cap U_\beta)}(s_\alpha) = \rho_{U_\beta(U_\alpha \cap U_\beta)}(s_\beta)$ then there exists an $s \in \mathcal{A}(X)$ such that $\rho_{UU_\alpha}(s) = s_\alpha$.

Analogously one defines sheaves of groups, rings, modules etc. One may also replace 'algebras' in the definition above with $\mathbb{Z}_2$-graded algebras. An important example of a sheaf of algebras is $\mathcal{C}^\infty_M$ for a smooth manifold M, which assigns to each open $U \in M$ the algebra $C^\infty(U)$ (the algebra multiplication being multiplication of real-valued functions). A smooth vector bundle $\pi : E \longrightarrow M$ gives rise to a sheaf of $C^\infty(M)$-modules $\mathcal{E}$, which assigns to each open $U \subseteq M$ the vector space $\Gamma(U, E)$, the space of local smooth sections of the bundle. If n is the rank of the vector bundle and U suitably small, we can choose a local basis of vector fields and a local section e can then be written as $(e_1(x), \dots, e_n(x))$ where $e_i \in C^\infty(U)$. The natural multiplication by an element $f \in \mathcal{C}^\infty_M(U)$ acts uniformly on all the components, so as a module, $\Gamma(U, E)$ is a rank n free $C^\infty(U)$-module. The sheaf $\mathcal{E}$ is called a *rank n locally free sheaf of modules*. Conversely, any rank n locally free sheaf of modules over $\mathcal{C}^\infty_M$ determines a unique rank n vector bundle E over M. We can associate an exterior algebra $\bigwedge E$ to E by applying (2.1.1) to the fibers. This gives rise to a sheaf of algebras $\bigwedge \mathcal{E}$ which assigns to each open $U \subseteq M$ the space of sections $\Gamma(U, \bigwedge E)$ (such a space is an algebra under the the *pointwise* wedge product). Now let M_0 be a smooth d-dimensional manifold and $\mathcal{A}$ a sheaf of $\mathbb{Z}_2$-graded algebras,

Definition. The pair $M^{d|n} = (M_0, \mathcal{A})$ is called a (smooth) *supermanifold* of dimension $d|n$ if for each $p \in M_0$ there exists an open neighbourhood U of p and a rank n free sheaf of $\mathcal{C}^\infty_U$-modules $\mathcal{E}_U$ such that $\mathcal{A}(U) \simeq \bigwedge \mathcal{E}_U$. Local sections of $\mathcal{A}$ are called (C^∞) superfunctions on $M^{d|n}$.

The isomorphism has to be an $\mathbb{Z}_2$ graded algebra isomorphism. A *local section* of A is to be understood as an element of $\mathcal{A}(U)$. As mentioned earlier, there exist unique vector bundles E_U which give rise to $\mathcal{E}_U$ by setting $\mathcal{E}_U(V) = \Gamma(V, E_U)$ for all open $V \subseteq E_U$. Let $(\theta_1, \dots, \theta_n)$ define a local trivialisation of E_U: we have the vector space isomorphism $E_U \simeq U \times \langle \theta_1, \dots, \theta_n \rangle$, where the latter factor denotes the vector space spanned by the antisymmetrised tensor products of the generators θ_i. Moreover, let $\{x^\mu\}$ be a coordinate chart over U. We call $(x^1, \dots, x^d, \theta_1, \dots, \theta_n)$ a *local coordinate chart* of the supermanifold $M^{d|n}$. Elements of $\mathcal{A}(U)$ can be uniquely written as in (2.1.5), where the c_α are smooth real-valued functions of the coordinates $(x^1, \dots, x^d)$. The *evaluation map* $\epsilon : \mathcal{A}(U) \longrightarrow \mathcal{C}^\infty(U)$ puts all the c_α zero, except for $c_{0\dots0}$, which is left invariant by ϵ. So if we view a superfunction as a map from M_0 to the Grassmann superspace generated by the θ_i, ϵ maps a function pointwise to its body f_B. A superfunction f on U is called *homogeneous* if $f \in \mathcal{A}_0(U)$ or $f \in \mathcal{A}_1(U)$, and the algebra subscript is the degree of f.

2.1.3 The Tangent Sheaf

The *stalk* of a sheaf at a point $p \in M_0$ is the set $\mathcal{A}_p$ of equivalence classes $[U, f]$ with $U \subseteq M_0$ open and $f \in \mathcal{A}(U)$ under the equivalence relation

$$[U, f] \sim [V, g] \Leftrightarrow \exists W \subset U \cap V : W \text{ is open, } p \in W \text{ and } \rho_{UW} f = \rho_{VW} g\,. \tag{2.1.6}$$

If $\mathcal{A}(U) = \Gamma(U, E)$, then the stalk at p is the set of all classes of sections which coincide in some neighbourhood of p. The stalks of the sheaf exhibit a natural algebra structure by the multiplication

$$[U, f][V, g] = [U \cap V, \rho_{U(U\cap V)}(f)\rho_{V(U\cap V)}(g)]\,. \tag{2.1.7}$$

Note that $U \cap V \neq \emptyset$ because p belongs to both open sets. From this moment, we shall denote the elements of the stalk with their second entry, omitting the maximal neighbourhood on which they are defined (which can be made arbitrarily small and still yield a representative of the equivalence class). The evaluation map induces a surjective map $\epsilon_p : \mathcal{A}_p \longrightarrow F : \epsilon_p([U, f]) = \epsilon(f)(p)$. The *tangent space* at $p \in M^{d|n}$ is a graded vector space whose even and odd subspaces are defined as

$$T_p M_i^{d|n} = \{F\text{-linear } v : \mathcal{A}_p \longrightarrow F \mid v(fg) = v(f)\epsilon_p(g) + (-1)^{i\mathcal{P}(f)}\epsilon_p(f)v(g)\}\,, \tag{2.1.8}$$

for $i = 0, 1$. The defining equation only holds for homogeneous f but can obviously be extended to all of $\mathcal{A}_p$ by linearity. By construction, the tangent space only depends on an arbitrarily small neighbourhood of p, as is the case for ordinary manifolds. The super analog of the tangent bundle, the *tangent sheaf*, is defined as

$$TM^{d|n} = (\mathrm{Der}M^{d|n})_0 + (\mathrm{Der}M^{d|n})_1\,, \tag{2.1.9}$$

where

$$(\mathrm{Der}M^{d|n})_i = \{F\text{-linear } X : \mathcal{A}(M) \longrightarrow \mathcal{A}(M) \mid X(fg) = X(f)g + (-1)^{i\mathcal{P}(f)} f X(g)\}\,. \tag{2.1.10}$$

Its elements are called *super vector fields*. A local super coordinate system (x^μ, θ_j) gives rise to a canonical basis of vector fields $\partial/\partial x^\mu \in (\mathrm{Der}M^{d|n})_0$, $\partial/\partial\theta_j \in (\mathrm{Der}M^{d|n})_1$ which act on a section $f \in \mathcal{A}(U)$ by

$$\begin{aligned}
\frac{\partial f}{\partial x^\mu} &= \sum_{\alpha\in(\mathbb{Z}_2)^n} \frac{\partial f_\alpha}{\partial x^\mu}(x^1, \dots, x^d)\theta^\alpha \\
\frac{\partial f}{\partial \theta_j} &= \sum_{\alpha\in(\mathbb{Z}_2)^n} \alpha_j(-1)^{\alpha_1+\dots+\alpha_{j-1}} f_\alpha(x^1, \dots, x^d)\theta_1^{\alpha_1}\dots\theta_{j-1}^{\alpha_{j-1}}\theta_{j+1}^{\alpha_{j+1}}\dots\theta_n^{\alpha_n}
\end{aligned} \tag{2.1.11}$$

In physics the odd vector field operators defined above are called *left derivatives.* These fields form a local basis of $TM^{d|n}(U)$; each local vector field can be uniquely written as

$$X = \sum_{\mu=1}^{d} X^\mu(x^1, \dots, x^d, \theta_1, \dots, \theta_n)\frac{\partial}{\partial x^\mu} + \sum_{j=1}^{n} Y_j(x^1, \dots, x^d, \theta_1, \dots, \theta_n)\frac{\partial}{\partial\theta_j}\,, \tag{2.1.12}$$

where X^μ and Y_j are elements of $\mathcal{A}(U)$. The vector field (2.1.12) is even if the X^μ are even and the Y_j are odd, and it is odd if the X^μ are odd and the Y_j are even. More generally, a $(d+n)$-tuple of local vector fields $(v_1, \dots, v_{d+n}) \in \otimes^{d+n} TM^{d|n}(U)$ is called a local *frame field* if $(v_1, \dots, v_d)$ is a basis of the even subsheaf and $(v_{d+1}, \dots, v_{d+n})$ a basis of the odd subsheaf. The evaluation map may be lifted to the tangent space at $p \in M_0^d \subset M^{d|n}$ by setting $\epsilon(v)(\epsilon(f)) = v_0(f)$. This yields an isomorphism $(T_pM^{d|n})_0 \simeq T_pM_0^d$, where T_pM_0 is the ordinary tangent space to the body submanifold M_0^d. For the tangent sheaf a similar lifting is possible: $\epsilon(X)(\epsilon(f)) = \epsilon(X_0(f))$, but it does not induce an isomorphisms of sheaves because there is for example no analog of the even vector field $Y^j\partial/\partial\theta_j$ with Y_j odd in the space of sections of TM_0^d.

The *cotangent bundle* $T^*M^{d|n}$ is defined in the usual manner as the dual sheaf of $\mathcal{A}$-modules which consists of graded linear maps

$$\omega : TM^{d|n}(U) \longrightarrow \mathcal{A}(U) : \langle \omega, fX \rangle = (-1)^{\mathcal{P}(\omega)\mathcal{P}(f)} f \langle \omega, X \rangle \,, \tag{2.1.13}$$

for arbitrary $f \in \mathcal{A}_0(U) \cup \mathcal{A}_1(U)$ (extended to $\mathcal{A}(U)$ by F-linearity). In the above we have denoted the linear graded duality pairing by $\langle\ .\ ,\ .\ \rangle$. Similarly vector fields act on these *super 1-forms.* Note that the rules above imply $\langle \omega, X \rangle = (-1)^{\mathcal{P}(X)\mathcal{P}(\omega)} \langle X, \omega \rangle$. There is no natural construction of an exterior algebra on the super (co-) tangent bundle; we want to preserve the $\mathbb{Z}_2$-gradation *and* impose a $\mathbb{Z}$-gradation, which leaves two possible sign conventions: either one chooses odd elements of the super vector space to commute with the odd ones under the wedge product, or one lets them anti-commute. Our choice shall be the last one; we define

$$v \wedge w = v \otimes w - (-1)^{\mathcal{P}(v)\mathcal{P}(w)} w \otimes v \tag{2.1.14}$$

for homogeneous elements v, w of a super vector space V. The resulting Grassmann algebra $\bigwedge V$ is a $\mathbb{Z}$ graded algebra of which all the homogeneous components carry a $\mathbb{Z}_2$-gradation. For $u \in (\bigwedge^p V)_0 \cup (\bigwedge^p V)_1$ and $u \in (\bigwedge^q V)_0 \cup (\bigwedge^q V)_1$ the sign rule reads

$$u \wedge v = (-1)^{\mathcal{P}(u)\mathcal{P}(v)+pq} v \wedge u \,. \tag{2.1.15}$$

Analogously to the theory of differential forms, the (super-) wedge product gives rise to a sheaf of $\mathbb{Z}$-graded algebras

$$\bigwedge T^*M^{d|n}(U) = \bigwedge^{0} T^*M^{d|n}(U) \oplus \bigwedge^{1} T^*M^{d|n}(U) \oplus \ldots \oplus \bigwedge^{d+n} T^*M^{d|n}(U) \,, \tag{2.1.16}$$

where we have the vector space isomorphisms $\bigwedge^1 T^*M^{d|n}(U) \simeq T^*M^{d|n}(U)$, $\bigwedge^0 T^*M^{d|n}(U) \simeq \mathcal{C}^\infty(U)$ and $\bigwedge^p T^*M^{d|n}(U) \simeq \bigwedge^{d+n-p} T^*M^{d|n}(U)$. The exterior differential on this sheaf of algebras is uniquely determined by $d^2 = 0$ and

$$\begin{aligned} &d|_{\bigwedge^0 T^*M^{d|p}} : \mathcal{A}(U) \longrightarrow \bigwedge^1 T^*M^{n|p}(U) : \langle X, df \rangle = X(f) \text{ for all } X \in TM^{d|n}(U), \\ &d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \text{ for all } \alpha \in \bigwedge^p T^*M^{n|p}(U) \,. \end{aligned} \tag{2.1.17}$$

From now on we shall use the notation $\bigwedge^p T^*M^{d|n}(U) = \Omega^p_M(U)$. With these definitions one may extend the Poincaré lemma and de Rham cohomology to spaces of super forms. The *interior product* is generalised to a morphism of sheaves $\iota_X : \Omega^p_M(U) \longrightarrow \Omega^{p-1}_M(U)$ for $X \in TM^{d|n}(U)$ and $p \geq 1$ satisfying

$$\begin{aligned} &\iota_X f = 0 \quad \text{for all } f \in \Omega^0_M(U) = \mathcal{A}(U) \,, \\ &\iota_X \omega = X(\omega) \quad \text{for all } \omega \in \Omega^1_M(U) \,, \\ &\iota_X(\alpha \wedge \beta) = \iota_X(\alpha) \wedge \beta + (-1)^{p+\mathcal{P}(X)\mathcal{P}(\alpha)} \alpha \wedge \iota_X \beta \quad \text{for } \alpha \in \Omega^p_M(U) \,. \end{aligned} \tag{2.1.18}$$

2.1.4 Super-Lie Algebra Structure

Recall that for an ordinary manifold, coordinate frames in overlapping charts are related by an element of $GL(d)$. What is the superspace analogue of this group action? Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be a supercommutative $\mathbb{Z}_2$-graded algebra (the direct sum is F-linear). We define $\mathrm{Mat}(r, s, \mathcal{A})$ the vector space of $\mathcal{A}$-valued $r \times s$ matrices. This set is an $\mathcal{A}$-module by defining the (left) multiplication $(aM)_{ij} = aM_{ij}$. A partition $r = d + n, s = k + l$ gives rise to a $\mathbb{Z}_2$-graded module structure on $\mathrm{Mat}(r, s, \mathcal{A})$ by setting $\mathrm{Mat}(d|n, k|l, \mathcal{A}) = G_0 \oplus G_1$, with

$$G_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \in \mathrm{Mat}(d, k, \mathcal{A}_0), b \in \mathrm{Mat}(d, l, \mathcal{A}_1), c \in \mathrm{Mat}(n, k, \mathcal{A}_1), d \in \mathrm{Mat}(n, l, \mathcal{A}_0) \right\},$$

$$G_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \in \mathrm{Mat}(d, k, \mathcal{A}_1), b \in \mathrm{Mat}(d, l, \mathcal{A}_0), c \in \mathrm{Mat}(n, k, \mathcal{A}_0), d \in \mathrm{Mat}(n, l, \mathcal{A}_1) \right\}.$$

A $\mathbb{Z}_2$-graded module structure simply means that $\mathcal{A}_i G_j = G_{(i+j) \bmod 2}$, and it trivially follows from the graded structure of $\mathcal{A}$. On the module $\mathrm{Mat}(m|n, \mathcal{A}) \equiv \mathrm{Mat}(m|n, m|n, \mathcal{A})$ of square matrices we can consistently define a matrix multiplication. One easily sees that this turns $\mathrm{Mat}(d|n, \mathcal{A})$ into an associative supercommutative $\mathbb{Z}_2$-graded algebra with a unit element $\mathbf{1} \in \mathrm{Mat}(d|n, \mathcal{A})_0$. Using the matrix multiplication we can define a *bracket* $[X, Y] = XY - YX$ which is symmetric in the odd subspaces,

Definition. A *super Lie algebra* is a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ equipped with a *super Lie bracket* $[\,.\,,\,.\,]$. A super Lie bracket is a bilinear form $V \times V \longrightarrow V$ such that for all $X, Y, Z \in V_0 \cup V_1$ (i.e. homogeneous elements) we have

1. $\mathcal{P}([X, Y]) = (\mathcal{P}(X) + \mathcal{P}(Y)) \bmod 2$,
2. $[Y, X] = (-1)^{1+\mathcal{P}(X)\mathcal{P}(Y)}[X, Y]$,
3. $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\mathcal{P}(X)\mathcal{P}(Y)}[Y, [X, Z]]$.

The super vector space $\mathrm{Mat}(d|n, \mathcal{A})$ equipped with the bilinear bracket induced by matrix multiplication is a super Lie algebra, which we shall denote with $\mathfrak{gl}_{d|n}(\mathcal{A})$. We shall define $\mathfrak{gl}_{d|n} = \mathfrak{gl}_{d|n}(\mathbb{R})$ (its even part consists of $\mathbb{R}$-matrices with an $d \times d$ and an $n \times n$ block on the diagonal, its odd elements have $n \times d$ and $d \times n$ matrices on the off-diagonal). Another example is the tangent sheaf $TM^{d|n}(U)$, acting on $\mathcal{A}(U)$ equipped with the bracket

$$[X, Y] \in TM^{d|n}(U) : [X, Y](f) = X(Y(f)) - Y(X(f)) \quad \text{for all } f \in \mathcal{A}(U)\,. \tag{2.1.19}$$

The even part of the $\mathbb{Z}_2$-graded Lie algebra defined above exponentiates to an ordinary group. This group is a subgroup of

$$GL_{d|n}(\mathcal{A}) = (\mathrm{Mat}(d|n, \mathcal{A}))_0 \cap \mathrm{Mat}^{\times}(d|n, \mathcal{A})\,, \tag{2.1.20}$$

where $\mathrm{Mat}^{\times}(d|n, \mathcal{A})$ denotes the subspace of invertible elements of the super matrix algebra. We obtain a sheaf of groups by setting $\mathcal{G}\ell_{d|n}(U) = GL_{d|n}(\mathcal{A}(U))$. As is the case for ordinary manifolds, this sheaf of groups has a natural action on the tangent sheaf, and is the super analog of the structure group of the tangent bundle.

2.2 Clifford Algebras

2.2.1 Definition of Clifford Algebras

Let V be an d-dimensional vector space over a commutative field F (we shall only consider $\mathbb{R}$ or $\mathbb{C}$) and $q : V \to F$ a quadratic form on V. The Clifford algebra $\mathrm{C}\ell(V, q)$ over V is the vector space [13]

$$\mathrm{C}\ell(V, q) = \frac{T(V)}{W} \equiv \frac{(\bigoplus_{r=0}^{\infty} \otimes^r V)}{W}\,, \tag{2.2.1}$$

where W is the ideal generated by all elements of the form $v \otimes v - q(v)$ with $v \in V$. We define the *Clifford product* by $v \cdot w = [v \otimes w]$, the equivalence class in $\mathrm{C}\ell(V, q)$ containing $v \otimes w$, for arbitrary $v, w \in T(V)$. Then we have

$$v \cdot v = q(v) \tag{2.2.2}$$

for all $v \in V \hookrightarrow \mathrm{C}\ell(V, q)$. Note that associativity of the Clifford product is guaranteed by associativity of the tensor product and the rule $[u] \cdot [v] = [u \otimes v]$ for $u, v \in T(V)$. Equivalently, the bilinear form $g : V \otimes V \longrightarrow F : g(v, w) = q(v + w) - q(v) - q(w)$ defines an inner product on V, and the generating Clifford algebra relation is expressed in terms of this map by

$$v \cdot w + w \cdot v = 2g(v, w)\,, \tag{2.2.3}$$

for $v, w \in V \hookrightarrow \mathrm{C}\ell(V,g)$. Obviously, the Clifford algebra does not possess a $\mathbb{Z}$-grading. However, $\mathrm{C}\ell(V,g)$ exhibits a natural $\mathbb{Z}_2$-gradation, induced by the extension of the linear map $\alpha : V \to V : \alpha(v) = -v$ to an algebra automorphism of $\mathrm{C}\ell(V,g)$. We call the Clifford elements for which $\alpha(v) = v$ *even* and the ones for which $\alpha(v) = -v$ *odd*. Since α is idempotent, there is a direct sum decomposition into the even and odd Clifford algebras,

$$\mathrm{C}\ell(V,g) = \mathrm{C}\ell^0(V,g) \oplus \mathrm{C}\ell^1(V,g)\,, \qquad \mathrm{C}\ell^i(V,g) = \tfrac{1}{2}(\mathrm{Id} + (-1)^i\alpha)\mathrm{C}\ell(V,g)\,. \tag{2.2.4}$$

$\mathrm{C}\ell^0(V,g)$ is a subalgebra, $\mathrm{C}\ell^1(V,g)$ is not. The link with the first section is provided by the canonical vector space isomorphism

$$\mathrm{C}\ell(V,g) \xrightarrow{\simeq} \bigwedge V\,. \tag{2.2.5}$$

The isomorphism, call it ϕ, is uniquely determined by linearity and

$$\begin{aligned} &\phi(v) = v\,, && \text{for } v \in V \\ &\phi(v_{[1} \cdot v_2 \dots v_{p]}) = v_1 \wedge v_2 \wedge \dots \wedge v_p\,, && \text{for } v_i \in \mathrm{C}\ell(V,g)\,, \end{aligned}$$

where the brackets around indices denote antisymmetrization with unit weight,

$$v_{[1} \cdot v_2 \dots v_{p]} = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{sgn}(\sigma) v_{\sigma(1)} \cdot v_{\sigma(2)} \dots v_{\sigma(p)}\,. \tag{2.2.6}$$

Hence the Clifford algebra $\mathrm{C}\ell(V,g)$ is a vector space of dimension $2^{\dim V}$ and the completely antisymmetrized Clifford products of basis vectors of V form a basis of $\mathrm{C}\ell(V,g)$. From this point we shall no longer explicitly denote the isomorphism ϕ. Let $\mathrm{C}\ell^\times(V,g)$ denote the group of all invertible elements of the algebra w.r.t. the Clifford product (the *Clifford group*). This is a Lie group over a vector space of dimension 2^d, with Lie algebra the Clifford algebra oven V equipped with the bracket $[u,v] = u \cdot v - v \cdot u$. The adjoint representation of $\mathrm{C}\ell^\times(V,g)$ is an algebra homomorphism

$$\mathrm{Ad} : \mathrm{C}\ell^\times(V,g) \to \mathrm{Aut}(\mathrm{C}\ell(V,g)) : \mathrm{Ad}_v(x) = v \cdot x \cdot v^{-1}\,. \tag{2.2.7}$$

It can be easily shown that $g(v,v) \neq 0$ is equivalent to $\mathrm{Ad}_v(V) = V$ and then Ad_v is an algebra isomorphism. Let us denote $P(V,q)$ the subgroup of $\mathrm{C}\ell^\times(V,g)$ generated by elements satisfying this property. Furthermore denote Ad^0 the adjoint map with the 'target space' restricted to $V \hookrightarrow \mathrm{C}\ell(V,g)$:

$$\mathrm{Ad}^0 : \mathrm{C}\ell^\times(V,g) \to \mathrm{Aut}(V) : \mathrm{Ad}^0_v(x) = \mathrm{Ad}_v(x)\,. \tag{2.2.8}$$

Then there is a representation

$$P(V,q) \xrightarrow{\mathrm{Ad}^0} O(V,g) = \{\Lambda \in GL(V) : g(\Lambda u, \Lambda v) = g(u,v)\}\,, \tag{2.2.9}$$

taking values in the *orthogonal* or *Lorentz* group of V (one easily verifies that $\mathrm{Ad}_{v\cdot w} = \mathrm{Ad}_v \circ \mathrm{Ad}_w$ and $g(\mathrm{Ad}_v u, \mathrm{Ad}_v w) = g(u,w)$). The subgroup of $P(V,g)$ generated by all $v \in V$ for which $g(v,v) = \pm 1$, denoted by $\mathrm{Pin}(V,g)$, and the subgroup of $\mathrm{Pin}(V,g)$ consisting of all elements of even degree is denoted by $\mathrm{Spin}(V,g)$. Restricting the adjoint representation to $\mathrm{Spin}(V,g)$ yields a surjective homomorphism to the special orthogonal group $SO(V,g)$. This can be seen by writing Ad^0_v as $\mathrm{Ad}^0_v(w) = w - 2g(v,w)v$, which is nothing but the reflection of w across the hyperplane perpendicular to v. The Cartan-Dieudonné theorem states that every special orthogonal transformation can be written as an even number of reflections. If we restrict the adjoint map to $\mathrm{Spin}^+(V,g)$, the subalgebra of $\mathrm{Pin}(V,g)$ of vectors with (positive) unit length, we get a 2 to 1 mapping to $SO^+(V,g)$, the *orthochronous* special orthogonal transformations (v and $-v$ map to the same automorphism). They are generated by an even number of reflections in planes orthogonal to positive length and as well to negative length vectors, or in physics we say *time direction* and *parity* preserving transformations. For completeness we note that one can use the *twisted* *adjoint representation* $\widetilde{\mathrm{Ad}}_v w = \alpha(v) w v^{-1}$ to define a 2 to 1 map from $\mathrm{Pin}(V,g)$ to $O(V,g)$. Let now $V = \mathbb{R}^{r+s}$ and define the non-degenerate metric

$$g(v,v) = -v_1^2 - \ldots - v_s^2 + v_{s+1}^2 + \ldots + v_{s+r}^2 \equiv \eta_{(r,s)}(v,v)\,. \tag{2.2.10}$$

The Clifford algebra $\mathrm{C}\ell(V,g)$ shall then be denoted by $\mathrm{C}\ell_{r,s}$. Taking the field $F = \mathbb{C}$ in the definition (2.2.1) we obtain the *complexified* Clifford algebra $\mathbb{C}\ell(V,g) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{C}\ell(V,g)$. The isomorphism (2.2.5) implies $\dim_{\mathbb{R}}(\mathrm{C}\ell_{r,s}) = 2^{r+s} = \dim_{\mathbb{C}}(\mathbb{C}\ell_{r,s})$.

2.2.2 Classification

Given a complete orthogonal positively oriented basis $e_1, \ldots, e_d$ of V, the real and complex *Clifford volume elements* are defined as

$$\omega = e_1 \cdot e_2 \ldots e_d\,, \qquad \omega_{\mathbb{C}} = i^{s+[\frac{d+1}{2}]} e_1 \cdot e_2 \ldots e_d\,, \tag{2.2.11}$$

where $[n/2]$ denotes the the smallest integer larger then or equal to $n/2$. For d odd, these are *central* elements of the algebra since then they commute with the generators. The complex volume always squares to one, the real volume element only in certain dimensions,

$$\omega^2 = (-1)^{\frac{d(d+1)}{2}+s}\,, \qquad \omega_{\mathbb{C}}^2 = 1\,. \tag{2.2.12}$$

So in conclusion, ω *is central and unipotent if* $(r-s) \bmod 8 = 1$ *or* 5. Note that definition (2.2.11) is basis independent since 2 positively oriented bases of are linked by an $SO(s,r)$ transformation and a linear transformation A transforms ω to $\det(A)\omega$. If $\omega^2 = 1$, we define the real projectors

$$\pi^{\pm} = \tfrac{1}{2}(1 \pm \omega)\,, \tag{2.2.13}$$

satisfying $\pi^+ + \pi^- = 1$, $(\pi^{\pm})^2 = \pi^{\pm}$ and $\pi^+ \cdot \pi^- = 0 = \pi^- \cdot \pi^+$. Of course, in the complex case we can always define such operators. Acting with these elements on the Clifford algebra yields decompositions

$$\begin{aligned} \mathrm{C}\ell_{r,s} &= \mathrm{C}\ell_{r,s}^+ \oplus_{\mathbb{R}} \mathrm{C}\ell_{r,s}^-\,, \\ \mathbb{C}\ell_{r,s} &= \mathbb{C}\ell_{r,s}^+ \oplus_{\mathbb{C}} \mathbb{C}\ell_{r,s}^- \end{aligned} \tag{2.2.14}$$

into isomorphic subalgebras if d is *odd*. If the projectors $\pi^{\pm}$ are central, the Clifford algebra contains 2 nontrivial two-sided ideals, and is hence not simple. One can prove that if the subspaces above are ideals, they are simple and if not, $\mathrm{C}\ell_{r,s}$ contains no other nontrivial ideals. Hence

Result 2.1 *If* $(r-s) \bmod 8 = 1$ *or* 5*,* $\mathrm{C}\ell_{r,s}$ *is a direct sum of 2 simple components, otherwise it is simple. If* $(r-s) \bmod 2 = 1$*,* $\mathbb{C}\ell_{r,s}$ *is a sum of 2 simple components. In the other case it is simple.*

What about the reducibility of the even subalgebras $\mathrm{C}\ell_{r,s}^0$ and $\mathbb{C}\ell_{r,s}^0$? In odd dimensions the even subalgebra does not contain the volume element, and turns out to be simple. In even dimensions the complex volume elements are obviously central in the even subalgebras and (2.2.12) implies that ω is unipotent if $(r-s) \bmod 8 = 0, 4$, in which cases $\mathrm{C}\ell_{r,s}^0$ is a sum of 2 simple components. These properties may also be deduced from the isomorphisms

$$\mathrm{C}\ell_{r,s}^0 \simeq \mathrm{C}\ell_{r,s-1}\,, \quad \mathbb{C}\ell_{r,s}^0 \simeq \mathbb{C}\ell_{r,s-1}\,. \tag{2.2.15}$$

If $\{e^i, e^{r+1}, f^j\}$ is the set of generators of $\mathrm{C}\ell_{r+1,s}$, then it is not difficult to see that a set of generators of $\mathrm{C}\ell_{r+1,s}^0$ is $\{e^{r+1}e^i, e^{r+1}f^j\}$. Since these generators mutually anti-commute and $(e^{p+1}e^i)^2 = -1 = -(e^{p+1}f^j)$ we establish $\mathrm{C}\ell_{r+1,s}^0 \simeq \mathrm{C}\ell_{s,r}$ and tensoring both sides with $\mathbb{C}$ gives the second isomorphism in (2.2.15). An explicit calculation of the multiplication table (see [14]) establishes following isomorphisms,

$$\begin{array}{ll} \mathrm{C}\ell_{0,1} \simeq \mathbb{R}\oplus\mathbb{R}\,, & \mathrm{C}\ell_{1,0} \simeq \mathbb{C}\,, \\ \mathrm{C}\ell_{0,2} \simeq \mathbb{R}(2) \simeq \mathrm{C}\ell_{1,1}\,, & \mathrm{C}\ell_{2,0} \simeq \mathbb{H}\,, \\ \mathrm{C}\ell_{0,4} \simeq \mathbb{H}\otimes\mathbb{R}(2)\,, & \mathrm{C}\ell_{4,0} \simeq \mathbb{R}(4)\,, \end{array} \tag{2.2.16}$$

where $\mathbb{H}$ is the quaternion algebra and $\mathbb{K}(n)$ denotes the algebra of $\mathbb{K}$-valued $n\times n$ matrices. Furthermore we have the important *periodicity isomorphisms*,

$$\begin{array}{ll} \mathrm{C}\ell_{r+4,s} \simeq \mathrm{C}\ell_{r,s}\otimes\mathrm{C}\ell_{4,0}\,, & \mathrm{C}\ell_{r,s+4} \simeq \mathrm{C}\ell_{r,s}\otimes\mathrm{C}\ell_{0,4}\,, \\ \mathrm{C}\ell_{r+1,s+1} \simeq \mathrm{C}\ell_{r,s}\otimes\mathrm{C}\ell_{1,1}\,, & \mathrm{C}\ell_{r+1,s} \simeq \mathrm{C}\ell_{s+1,r}\,. \end{array} \tag{2.2.17}$$

The Clifford algebra $\mathrm{C}\ell_{r+4,s}$ is generated by $\{e^i, e^{r+1}, e^{r+2}, e^{r+3}, e^{r+4}, f^j\}$ where i runs from 1 to r and j from 1 to s and the e's square to one and the f's to minus one. A new set of generators is $\{ze^i, zf^j, e^{r+1}, e^{r+2}, e^{r+3}, e^{r+4}\}$ where $z = e^{r+1}e^{r+2}e^{r+3}e^{r+4}$. Then the last four generators commute with the first set of generators, so the algebra can be written as the tensor product of $\mathcal{A}$, generated by the first set and $\mathcal{B}$, generated by the last ones. One easily sees that $(ze^i)^2 = -(zf^j)^2 = 1$ and the last 4 generators square to one, which proves the first tensor product. The second one is proven completely analogously. Similarly, if $\mathrm{C}\ell_{r+1,s+1}$ is generated by $\{e^i, e^{r+1}, f^j, f^{s+1}\}$, then a new set of generators is $\{ze^i, zf^j, e^{r+1}, f^{s+1}\}$ with $z = e^{r+1}f^{s+1}$. One again easily calculates that the first $r+s$ generators commute with the last 2, and that the squares of the generators yield the desired tensor product. Finally, if $\{e^i, e^{r+1}, f^j\}$ is a set of generators of $\mathrm{C}\ell_{r+1,s}$, then $\{e^{r+1}, e^{r+1}e^i, e^{r+1}f^j\}$ is obviously a set of generators of too. The new ones are mutually anticommuting and we have $(e^{r+1})^2 = (e^{r+1}f^j)^2 = 1$ and $(e^{r+1}e^i)^2 = -1$, which are exactly the defining equations of the generators of $\mathrm{C}\ell_{s+1,r}$. Using the isomorphisms of algebras

$$\mathbb{K}\otimes\mathbb{R}(n)\simeq\mathbb{K}(n)\,,\quad \mathbb{R}(n)\otimes\mathbb{R}(m)\simeq\mathbb{R}(nm)\,,\quad \mathbb{C}\otimes\mathbb{C}\simeq\mathbb{C}\oplus\mathbb{C}\,,\quad \mathbb{H}\otimes\mathbb{H}\simeq\mathbb{R}(4)\,,\quad \mathbb{C}\otimes\mathbb{H}\simeq\mathbb{C}(2)\,,$$

we obtain a complete classification of Clifford algebras, given by table (2.2.1). Taking the tensor product of these algebras with $\mathbb{C}$ yields the much simpler classification of complex Clifford algebras of table (2.2.2).

$(r-s)$mod8	$\mathrm{C}\ell_{r,s}$	n	$\mathrm{C}\ell^0_{r,s}$	m
0	$\mathbb{R}(n)$	$2^{d/2}$	$\mathbb{R}(m)\oplus\mathbb{R}(m)$	$2^{(d-2)/2}$
1	$\mathbb{R}(n)\oplus\mathbb{R}(n)$	$2^{(d-1)/2}$	$\mathbb{R}(m)$	$2^{(d-1)/2}$
2	$\mathbb{R}(n)$	$2^{d/2}$	$\mathbb{C}(m)$	$2^{(d-2)/2}$
3	$\mathbb{C}(n)$	$2^{(d-1)/2}$	$\mathbb{H}(m)$	$2^{(d-3)/2}$
4	$\mathbb{H}(n)$	$2^{(d-2)/2}$	$\mathbb{H}(m)\oplus\mathbb{H}(m)$	$2^{(d-4)/2}$
5	$\mathbb{H}(n)\oplus\mathbb{H}(n)$	$2^{(d-3)/2}$	$\mathbb{H}(m)$	$2^{(d-3)/2}$
6	$\mathbb{H}(n)$	$2^{(d-2)/2}$	$\mathbb{C}(m)$	$2^{(d-2)/2}$
7	$\mathbb{C}(n)$	$2^{(d-1)/2}$	$\mathbb{R}(m)$	$2^{(d-1)/2}$

Table 2.2.1: Classification of the Clifford algebras $\mathrm{C}\ell_{r,s}$. r denotes the number of 'timelike' directions and s the number of Euclidean spatial directions, and $d = r+s$. In the third column the subalgebra of even-fold products of generators is classified

2.2.3 Representations

An irreducible representation of $\mathrm{Pin}_{r,s}$, shall be called a *pinor* representation; an irrep of $\mathrm{Spin}_{r,s}$ a *spinor* representation. We have already encountered two representations: the adjoint representations of the Pin and Spin groups, which are non-faithful and descend to orthogonal transformations

$(r-s)\mathrm{mod}2$	$\mathbb{C}\ell_{r,s}$	n	$\mathbb{C}\ell^0_{r,s}$	m
0	$\mathbb{C}(n)$	$2^{d/2}$	$\mathbb{C}(m)\oplus\mathbb{C}(m)$	$2^{(d-2)/2}$
1	$\mathbb{C}(n)\oplus\mathbb{C}(n)$	$2^{(d-1)/2}$	$\mathbb{C}(m)$	$2^{(d-1)/2}$

Table 2.2.2: Classification of the complexified Clifford algebras

on the generating vector space V. The representation theory of an algebra $\mathcal{A}$ can be studied by considering the *regular representations* of $\mathcal{A}$: left or right ideals $\mathcal{L}_i$, $\mathcal{R}_i$ together with the left action $\rho_L : \mathcal{A} \longrightarrow \mathrm{End}\mathcal{L}_i : v \mapsto L_v$ where $L_v : \mathcal{L}_i \longrightarrow \mathcal{L}_i : x \mapsto vx$ and similarly the right action defines a representation on the right ideals. Projectors play a very important rôle in this construction because they generate ideals. We call $P \in \mathcal{A}$ a *primitive projector* if $P^2 = P$ (P is idempotent) and P cannot be written as the sum of algebraically orthogonal elements. In semi-simple algebras these give us immediately minimal ideals. The basic theorem, which is proven in Appendix A of [14] is the following,

Theorem 2.2 *If P is an idempotent in a semi-simple algebra $\mathcal{A}$ then $P\mathcal{A}P$ is a division algebra iff P is primitive. Moreover, if this is the case, $\mathcal{A}P$ is a minimal left ideal and $P\mathcal{A}$ is a minimal right ideal*

Recall that a division algebra is a simple algebra of which every nonzero element has an inverse. This implies that division algebras have no nontrivial left or right ideals: for if $\mathcal{B} \subset \mathcal{A}$ is a left ideal, and $b \in \mathcal{B}$ then $b^{-1} \in \mathcal{A}$ and hence $b^{-1}\mathcal{B} \subseteq \mathcal{B}$. Consequently $b^{-1}b = e \in \mathcal{B}$. But then $\mathcal{A}e \subseteq \mathcal{B}$ and therefore $\mathcal{B} = \mathcal{A}$, and analogously for right ideals. Hence the n primitive projectors of $\mathrm{C}\ell_{r,s} \simeq \mathbb{K}(n)$ are

$$(P_i)_{mn} = \delta_{m,n}\delta_{n,i}\,, \tag{2.2.18}$$

because $P_i\mathbb{K}(n)P_i \simeq \mathbb{K}$. Multiplying a general element of the algebra with P_i from the right gives an $n \times n$ matrix with all entries zero except for the i-th column (taking values in $\mathbb{K}$); these are the minimal left ideals of the simple Clifford algebras. If $\mathrm{C}\ell_{r,s}$ is the sum of two simple components, the idempotents above are no longer primitive (since $P_i = P_i \cdot (\pi^+ + \pi^-)$ and $\pi^\pm$ are in the centre). The primitive projectors are now given by

$$P_i^+ = P_i \cdot \pi^+\,, \qquad P_i^- = P_i \cdot \pi^-\,. \tag{2.2.19}$$

One easily shows that in the simple case, the choice of the primitive projector is not important: all the representations $(L, \mathrm{C}\ell_{r,s}P_i)$ and $(R, P_i\mathrm{C}\ell_{r,s})$ are equivalent[2]. In the semi-simple case however there are 2 equivalence classes: the left and right actions on the P_i^+-projected subspaces are mutually equivalent, but inequivalent to the left and right actions on the P_i^--projected subspaces. These 2 equivalence classes are called the *chiral* or *Weyl* representations. In the following we shall not bother about the equivalence relation and we say that in the simple case there is a single representation (ρ, S), and in the semi-simple case there are 2 representations (ρ, S^+),(ρ, S^-). Let us now focus our attention to the induced representations on the Spin groups. In even dimensions, the Clifford volume element is always central in $\mathrm{C}\ell^0_{r,s}$ and so is the complex volume element in the complexified even subalgebra $\mathbb{C}\ell^0_{r,s}$. Hence, when $\omega^2 = 1$, there is always a splitting of the representation $S = S^+ \oplus S^-$. The resulting representations are irreducible *if they are induced by an irreducible representation on* $\mathrm{C}\ell_{r,s}$. The argument is as follows: suppose $\mathrm{C}\ell_{r,s}$ is simple, the volume element is unipotent and $S^\pm$, the representations of the even algebra are induced by a minimal left ideal $I \subseteq \mathrm{C}\ell_{r,s}$, that is $S^\pm = \pi^\pm I$ and $\mathrm{C}\ell^0_{r,s}S^\pm = \mathrm{C}\ell^0_{r,s}\pi^\pm I = \pi^\pm \mathrm{C}\ell^0_{r,s} I = \pi^\pm I = S^\pm$ If W is a minimal left ideal of $\mathrm{C}\ell^0_{r,s}$, either $W \subseteq S^+$ or $W \subseteq S^-$ (otherwise $W \cap S^\pm$ would provide

[2]Two representations $\rho_i : \mathcal{A} \longrightarrow \mathrm{End}(S_i)$ are *equivalent* iff there exists an intertwining isomorphism $\phi : S_1 \longrightarrow S_2$ such that $\phi \circ \rho_1(v) = \rho_2(v) \circ \phi$ for all $v \in \mathcal{A}$

us with new left ideals, contradictory to W being minimal). Since S^+ and S^- have equal dimension and they sum to I, we must have $\dim(W) \leq \frac{1}{2}\dim(I)$. On the other hand, one may verify that the space $W + vW$ with v an invertible element of $\mathrm{C}\ell^1_{r,s}$, is a left ideal of the entire Clifford algebra. Hence $\dim(W + vW) = 2\dim(W) \geq \dim(I)$[3]. Hence $\dim(W) = \frac{1}{2}\dim(I) = \dim(S^\pm)$, and by linearity we must have either $W = S^+$ or $W = S^-$. This proves the irreducibility of $S^\pm$. If the Clifford algebra is semi-simple, the projectors are central in the full algebra and hence in the even subalgebra as well, and the proof can be done analogously. The question is whether the resulting irreducible representations are equivalent or not. Observe that, for unipotent ω in odd dimensions, the automorphism $\alpha : \mathrm{C}\ell_{r,s} \to \mathrm{C}\ell_{r,s}$ interchanges the factors $\mathrm{C}\ell^+_{r,s}$ and $\mathrm{C}\ell^-_{r,s}$, defined in (2.2.14) since we have in odd dimensions $\alpha(\omega) = -\omega$ and hence $\alpha(\pi^\pm) = \pi^\mp$. Suppose now $\pi^+ \cdot v$ is even. Then $\pi^+ \cdot v = \alpha(\pi^+ \cdot v) = \alpha(\pi^+) \cdot \alpha(v) = \pi^- \cdot \alpha(v)$. The splitting of the representation $\rho(v) = \rho^+(v) + \rho^-(v)$ with $\rho^\pm(v) = \rho(\pi^\pm \cdot v)$ when restricted to the even elements yields the relation

$$\rho^\pm(v) = \rho^\mp(\alpha(v)) \tag{2.2.20}$$

for v in $\mathrm{C}\ell_{r,s}$ such that $\pi^+ \cdot v \in \mathrm{C}\ell^0_{r,s}$. This shows the equivalence of the restrictions of the irreducible representations to the even algebra in odd dimensions. The real volume element squares to one for even s if $d = 3$ or $4(\mathrm{mod} 4)$, in the first case the representations are equivalent, in the second one they can be shown not equivalent. We could have skipped the analysis above by considering the isomorphism (2.2.15).

2.2.4 The Spin Groups

Obviously, the groups $\mathrm{Spin}_{r,s}$ and $\mathrm{Spin}^+_{r,s}$ generate the even subalgebra,

$$1 + \mathrm{Spin}_{r,s} + \mathrm{Spin}_{r,s} \cdot \mathrm{Spin}_{r,s} + \ldots = \mathrm{C}\ell^0_{r,s}\,. \tag{2.2.21}$$

Since $\mathrm{Spin}_{r,s}$ is contained in the even Clifford algebra, a left ideal I of $\mathrm{C}\ell^0_{r,s}$ is also a left ideal of $\mathrm{Spin}_{r,s}$. If I is minimal in $\mathrm{C}\ell^0_{r,s}$, it is also minimal in $\mathrm{Spin}_{r,s}$, for if $W \subseteq I$ satisfies $\mathrm{Spin}_{r,s}W = W$, we can let (2.2.21) act from the left on W and get $W = \mathrm{C}\ell^0_{r,s}W$, and therefore $W = I$. The same statements hold for $\mathrm{Spin}^+_{r,s}$ and $\mathrm{Pin}_{r,s}$. So irreducible representations of the even subalgebra induce irreducible spinor representations. Along the same lines one can show that the irreducible representations of $\mathbb{C}\ell^0_{r,s}$ induce irreducible complex spinor representations, and only split up in inequivalent irreducible representations if d is even. As already mentioned, there exists a 2-to-1 correspondence between a Spin group and the orthogonal group. Hence spinor representations are also representations of $SO(r,s)$. The adjoint map $\mathrm{Ad}^0 : \mathrm{Spin}_{d-1,1} \to SO(d-1,1)$ induces a Lie algebra isomorphism

$$\mathfrak{spin}_{d-1,1} \xrightarrow{\mathrm{ad}^0} \mathfrak{so}(d-1,1) \tag{2.2.22}$$

to the algebra of linear, skew-symmetric endomorphisms of V, which in its turn isomorphic to $\bigwedge^2 \mathbb{R}^d$. In terms of a basis $\{e_\mu\}_{\mu=1,\ldots,d}$ of the base space V, $\mathfrak{so}(d-1,1)$ is spanned by the basis vectors $e_\mu \wedge e_\nu$, acting on V by

$$(e_\mu \wedge e_\nu)v = \eta(e_\mu, v)e_\nu - \eta(e_\nu, v)e_\mu = v_\mu e_\nu - v_\nu e_\mu\,, \tag{2.2.23}$$

if $v = v^\mu e_\mu$, $\eta_{\mu\nu} = \eta(e_\mu, e_\nu)$ and $v_\mu = \eta_{\mu\nu}v^\nu$. In a standard basis $(e_\mu)^\sigma = \delta^\sigma_\mu$ the rotation generators in the vector representation are given by $(e_\mu \wedge e_\nu)_{\sigma\rho} = \delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\nu\sigma}\delta_{\mu\rho}$. This matrix generates a rotation in the plane spanned by e_μ and e_ν and one easily verifies the periodicity property $\exp(i(\theta + 2\pi)(e_\mu \wedge e_\nu)) = \exp(i\theta(e_\mu \wedge e_\nu))$. The isomorphism (2.2.5) maps the Clifford algebra basis vector $\frac{1}{2}(e_\mu \cdot e_\nu - e_\nu \cdot e_\mu)$ to $2e^\mu \wedge e^\nu$ and its inverse maps $e^\mu \wedge e^\nu$ to $\frac{1}{4}[e^\mu, e^\nu]$. Acting with the derivative of the Clifford map on these algebras, it induces a representation of the Lorentz algebra $\mathfrak{so}(S)$;

$$\rho : \mathfrak{so}(d-1,1) \xrightarrow{(\mathrm{ad}^0)^{-1}} \mathfrak{spin}_{d-1,1} \xrightarrow{\Delta_*} \mathfrak{gl}(S)\,. \tag{2.2.24}$$

[3] $\dim(W) = \dim(vW)$ because if v has an inverse, the map $\phi : W \to vW : u \mapsto v \cdot u$ is a vector space isomorphism

Its basis vectors (the generators of the Lorentz group) are mapped to

$$M_{\mu\nu} = \rho(e_\mu \wedge e_\nu) = \tfrac{1}{4}[\Gamma_\mu, \Gamma_\nu] \equiv \tfrac{1}{2}\Gamma_{\mu\nu}\,, \tag{2.2.25}$$

where $\Gamma_\mu = \rho(e_\mu)$. They satisfy the Lorentz algebra relation

$$[M_{\mu\nu}, M_{\sigma\rho}] = \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho}\,. \tag{2.2.26}$$

The factor 1/2 in (2.2.25) is responsible for the *double* covering of the rotation group by the spin representation; since $(\Gamma_{\mu\nu})^2 = \mathbf{1}$ for 2 different spacelike basis vectors e_μ, e_ν the generated group element exhibits 4π-periodicity: $\exp(i(\theta + 2\pi)M_{\mu\nu}) = -\exp(i\theta M_{\mu\nu})$. Thus, a spinor (an element of the representation space S) changes sign under a full rotation. In physics there is a well known construction the representation of $\mathbb{C}\ell_{r,s}$. As the target vector space, one takes $\mathbb{C}^n$ where $n = 2^{d/2}$ in even dimensions and $2^{(d-1)/2}$ in odd dimensions. The representation morphism, fully determined by its mapping of the basis vectors is constructed iteratively in d. One starts in $d = 2$ with the matrices

$$\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ,\ \Gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,. \tag{2.2.27}$$

One easily verifies that these matrices $\mathbb{C}$-span a representation of $\mathbb{C}\ell_{0,2}$. We iteratively construct a representation of $\mathbb{C}\ell_{0,d}$ for d even as follows:

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\ \mu = 1, \dots d-2 \\ \Gamma^{d-1} = \mathbf{1}_k \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}&,\ \Gamma^d = \mathbf{1}_k \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\quad , \end{aligned} \tag{2.2.28}$$

where γ^μ are the matrices generating $\mathbb{C}\ell_{0,d-2}$ and $k = 2^{(d-2)/2}$. If d is odd one takes the $(d-1)$-dimensional basis and adds the Clifford volume element $\Gamma = i^{(d-1)/2}\Gamma^1\Gamma^1 \dots \Gamma^{d-1}$ to it. The reader may quickly verify that the obtained basis satisfies (2.2.3) in Euclidean space. Multiplying the first r matrices by i, we obtain a set of generators of $\mathbb{C}\ell_{r,s}$, which is called the *Dirac basis*.

2.3 Spinor Adjoints and Pairings

2.3.1 Pairings from Involutions

An *involution* of an algebra $\mathcal{A}$, is a *vector space* automorphism $j : \mathcal{A} \to \mathcal{A}$ such that $j \circ j = \mathrm{Id}_{\mathcal{A}}$. To emphasise the involutary property, we shall denote its image of an element v by $v^j \equiv j(v)$. Apart from the identity automorphism, there are 3 canonical involutions of the real Clifford algebras we constructed; the map

$$\xi : T^p(V) \to T^p(V) : \xi(x_1 \otimes x_2 \otimes \dots \otimes x_p) = x_p \otimes x_{p-1} \otimes \dots \otimes x_1 \tag{2.3.1}$$

is clearly an involution on $T^p(V)$. We can linearly extend this map to $T(V)$, and this induces an involution ξ on $\mathrm{C}\ell(V, g)$. The second involution is the map α (see the paragraph preceding (2.2.4)) and the third one is $\xi \circ \alpha$, which we shall denote by ζ. These are clearly also involutions of $\mathbb{C}\ell_{r,s}$, which commute with complex conjugation, which by definition only acts nontrivial on the *first* factor of $\mathbb{C} \otimes \mathrm{C}\ell_{r,s} = \mathbb{C}\ell_{r,s}$. The compositions $\xi^* = \xi \circ *$ and $\zeta^* = \zeta \circ *$ are involutions on $\mathbb{C}\ell_{r,s}$, *regarded as a real algebra*. An involution j on $\mathcal{A}$ always induces a direct sum decomposition into j-symmetric and j-skew subspaces (*not* subalgebras) because $a = \frac{1}{2}(a + a^j) + \frac{1}{2}(a - a^j)$. For $j = \alpha$ or ξ on $\mathrm{C}\ell_{r,s}$ (or $\mathbb{C}\ell_{r,s}$) this is the decomposition (2.2.4). In the definition of an involution, we stressed that it is a vector space automorphism, not necessarily satisfying $(ab)^j = a^j b^j$ under the algebra multiplication. In particular, the canonical involutions ξ, ζ, ξ^* and ζ^* are *anti-automorphisms* of the algebra: they satisfy $(ab)^j = b^j a^j$. We shall call involutions which are algebra anti-automorphisms *anti-involutions* of the algebra. Anti-involutions map the centre of a simple Clifford algebra to itself. One easily shows that the elements of the centre which are invariant under such a j form a commutative division algebra $\mathcal{E}$ (necessarily isomorphic to $\mathbb{R}$ or $\mathbb{C}$)[4], and in such a situation we call j an anti-involution *over* $\mathcal{E}$. We have the following crucial result,

Theorem 2.3 *If $\mathcal{A}$ is simple and i is an anti-involution over $\mathcal{E} \subseteq \mathcal{C}_{\mathcal{A}}$ then $j : a \mapsto a^j$ is an anti-involution over $\mathcal{E}$ iff there exists an s with $s^i = s^j = \pm s$ such that $a^j = s a^i s^{-1}$.*

One can find the proof in [14]. Observe that the choice of s is fixed up to left multiplication by an element of $\mathcal{E}$, which induces an equivalence relation on the space of involutions over $\mathcal{E}$: involutions i, j related by $a^i = \phi^{-1}(\phi(a)^j)$ for some automorphism ϕ are called *equivalent*. If $\mathcal{A}$ is simple, it is of the form $\mathbb{K}(n)$, and in these algebras we have a canonical involutary anti-automorphism, the matrix transposition T in a certain matrix basis. Note that T is an anti-involution over the centre $\mathcal{C}_{\mathcal{A}}$ of $\mathcal{A}$, which is either $\mathbb{R}\mathbf{1}_n$ or $\mathbb{C}\mathbf{1}_n$. If we are given an anti-involution j which leaves (only) central elements invariant, by the theorem above it is equivalent to T, and hence j *is* transposition in some other basis. This allows us to choose T more conveniently: let it denote transposition in a matrix basis in which the primitive idempotent P is diagonal, then by theorem 2.3 there exists a $J \in \mathcal{A}$ with $J^j = J^T = \pm J$ such that $P^T = P = J^{-1} P^j J$ from which $P^j = J P J^{-1}$. The matrix J allows us to construct pseudo-orthogonal inner products; if $\psi \in \mathcal{A}P$, then $J^{-1}\psi^j \in P\mathcal{A}$ since $P J^{-1} \psi^j = J^{-1} P^j \psi^j = J^{-1}(\psi P)^j = J^{-1}\psi^j$. Hence we define the *$j$-adjoint pairing*

$$(\ , \ )_j : \mathcal{A}P \times \mathcal{A}P \to P\mathcal{A}P : \theta, \psi \mapsto (\theta, \psi)_j = J^{-1}\theta^j \psi \,. \tag{2.3.2}$$

These inner products are nondegenerate, for if $(\theta, \psi)_j = J^{-1}\theta^j\psi = 0$ for all $\psi \in \mathcal{A}P$, then $\theta^j \mathcal{A}P = 0$, which implies $\theta^j = 0 = \theta$ because the regular representation is faithful on the minimal left ideal $\mathcal{A}P$. For all $a \in \mathcal{A}$ we have

$$(\theta, a\psi)_j = (a^j\theta, \psi)_j \,, \tag{2.3.3}$$

and for all elements $c \in \mathbb{K} = P\mathcal{A}P$ we have

$$(\theta, \psi c)_j = (\theta, \psi)_j c \ , \ \ (\theta c, \psi)_j = c^{*(j)}(\theta, \psi)_j \,, \tag{2.3.4}$$

with $c^{*(j)} = J^{-1} c^j J$ defines the *induced* involution on the division algebra $\mathbb{K}$. Acting on an inner product, the induced involution switches the the ideals up to a sign:

$$(\theta, \psi)_j = \epsilon(j)(\psi, \theta)_j^{*(j)} \,, \tag{2.3.5}$$

where $\epsilon(j) = \pm 1$ is the relative sign between J and J^j. Taking the canonical involutions ξ, ζ (possibly with conjugation) yields thus various involutions on the division algebra. Since there are only a finite number of inequivalent involutions on division algebras we can classify the canonical involutions according to their effect on the division algebra in various dimensions (see appendix **??**). The corresponding pairings are *spin invariant*: for an element $\sigma \in \mathrm{Spin}_{r,s}$ we have

$$(\sigma\theta, \sigma\psi)_\xi = A^{-1}\theta^\xi \sigma^\xi \sigma \psi = A^{-1}\theta^\xi \psi = (\theta, \psi)_\xi \,, \tag{2.3.6}$$

and similarly for $(\ , \ )_\zeta$. In fact, it is shown in [14] that these are the *only* spin-invariant pairings, up to equivalence. For the complexified simple Clifford algebras the construction may be repeated, using Hermitian conjugation (whose space of invariants is isomorphic to $\mathbb{R}$) instead of transposition. If $\mathcal{J}$ is some anti-involution which leaves the real elements of the centre invariant (such as there are ξ^* and ζ^*), it is Hermitian conjugation in some matrix basis and the adjoint involution of a pseudo-Hermitian inner product.

[4] A famous theorem by Frobenius states that all division algebras are isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Only the first two of these algebras are commutative.

Now let the algebra $\mathcal{A}$ be the sum of 2 simple components, $\mathcal{A} = \mathcal{B} \oplus \mathcal{C} = \mathcal{A}P \oplus \mathcal{A}Q$ for some central idempotents $P = 1 - Q$. If j is an involution of $\mathcal{A}$ then P^j and Q^j are central , simple orthogonal idempotents which sum to 1. Hence $\mathcal{A} = \mathcal{A}P^j \oplus \mathcal{A}Q^j$ is a decomposition into simple components. Since such a decomposition is unique up to ordering of the factors we have either $\mathcal{A}P^j = \mathcal{B}$ and $\mathcal{A}Q^j = \mathcal{C}$, or $\mathcal{A}P^j = \mathcal{C}$ and $\mathcal{A}Q^j = \mathcal{B}$. In the former case j just induces an involution on the components, and in the latter case $\mathcal{B}$ is isomorphic to $\mathcal{C}^j$ and vice versa. Applied to the primitive idempotents defined in (2.2.19) we have in the first case $(P_i^{\pm})^j = JP_i^{\pm}J^{-1}$ and in the second case $(P_i^{\pm})^j = JP_i^{\mp}J^{-1}$ for some J satisfying $J^j = \epsilon J$. It is easily verified that such involutions are unique up to equivalence. If the simple components are (real) matrix algebras over a division algebra $\mathbb{K}$, we call involutions which do not preserve the simple components $\mathbb{K}$-*swaps*. A $\mathbb{K}$-swap too induces a pairing on minimal left ideals, as is explained in [14]. However, a swap j doesn't leave the central idempotents, which span the centres in the simple components ($\pi^{\pm}$ is diagonal in $\mathbb{C}\ell^{\pm}_{r,s}$) invariant and hence will not be the adjoint of a pseudo-orthogonal (Hermitian) inner product on the simple components. In the case that j leaves the simple components invariant the theory of anti-involutions on simple algebras may be applied to the simple components.

2.3.2 Classification

We shall from now on focus our attention to the complex Clifford algebras $\mathbb{C}\ell_{r,s}$. For an algebra $\mathbb{C}(n)$, there are 3 inequivalent involutions: transposition, Hermitian conjugation, and their composition which is complex conjugation on $\mathbb{C}$ in the tensor product $\mathbb{C} \otimes \mathbb{C}\ell_{r,s}$. On the abstract level, there are ξ, ζ, ξ^*, ζ^* and the compositions $\xi \circ \xi^*$ and $\zeta \circ \zeta^*$. These are identified as follows

1. Hermitian conjugation: in even dimensions we have the equivalence of involutions $\dagger \sim \xi^* \sim \zeta^*$. If $(r, s) \bmod 2 = (0, 1)$ then ξ^* simply swaps the simple components and $\zeta^* \sim \dagger \oplus \dagger$ and vice versa if $(r, s) \bmod 2 = (1, 0)$. Restricting the involutions the even subalgebra, one obviously has $\xi^0 \sim \zeta^0$, since α is the identity on $\mathbb{C}\ell^0_{r,s}$. One may calculate that in odd dimensions the maps $(\xi^*)^0 \equiv \xi^*|_{\mathbb{C}\ell^0_{r,s}}$ and $(\zeta^*)^0 \equiv \zeta^*|_{\mathbb{C}\ell^0_{r,s}}$ are equivalent to Hermitian conjugation, if $(r, s) \bmod 2 = (0, 0)$ they are Hermitian conjugate and interchange the simple components. So if r is odd and s even, there is a $A_+ \in \mathbb{C}\ell_{r,s}$ such that $v^{\xi^*} = A_+ v^\dagger A_+^{-1}$ with

$(r, s) \bmod 2$	(0,0)	(0,1)	(1,0)	(1,1)
ξ^*	$\dagger$	S	$\dagger \oplus \dagger$	$\dagger$
ζ^*	$\dagger$	$\dagger \oplus \dagger$	S	$\dagger$
$(\xi^*)^0 = (\zeta^*)^0$	$\dagger \oplus \dagger$	$\dagger$	$\dagger$	S

Table 2.3.1: Classification of ξ^* and ζ^* on $\mathbb{C}\ell_{r,s}$ and on $\mathbb{C}\ell^0_{r,s}$ (denoted $(\xi^*)^0 = (\zeta^*)^0$). The letter S denotes a swap of the simple components.

$A_+^\dagger = A_+^{\xi^*} = \pm A_+$ and in the other cases there is an $A_- \in \mathbb{C}\ell_{r,s}$ such that $v^{\zeta^*} = A_- v^\dagger A_-^{-1}$ with $A_-^\dagger = A_-^{\xi^*} = \pm A_-$ for all $v \in \mathbb{C}\ell_{r,s}$. Corresponding to the construction of (2.3.2), we establish the Dirac product on a minimal ideal of $\mathbb{C}\ell_{r,s}$, $(\theta, \psi) \equiv (\bar{\theta})_D \psi$, where we have defined the *Dirac adjoint*

$$(\bar{\theta})_D = \begin{cases} A_+^{-1}\theta^{\xi^*} = \theta^\dagger A_+^{-1} & \text{if} \quad (r,s) \bmod 2 = (1,0) \\ A_-^{-1}\theta^{\zeta^*} = \theta^\dagger A_-^{-1} & \text{otherwise} \end{cases} . \tag{2.3.7}$$

Note that the Hermicity property of the Dirac conjugation matrix A is a matter of convention: multiplying this matrix by i gives an equivalent product with a different symmetry. Note also that if θ is a *spinor* representation, the last entry of the last row implies that its Dirac conjugate flips its chirality.

2. Matrix transposition: in [14] it is shown that ξ and ζ are for the simple Clifford algebras the adjoint involution to transposition. However, it depends on the dimension of V whether the induced pairing is symmetric or antisymmetric (in the Dirac pairing case, this was a matter of convention). In the semi-simple case, the standard involutions are either equivalent to transposition on the simple components (again with a prescribed symmetry of the associated product), or a swap. Analogously to the construction of the Dirac conjugate, we notice that

$d \bmod 8$	0	1	2	3	4	5	6	7
ξ	T_+	$T_+ \oplus T_+$	T_+	S	T_-	$T_- \oplus T_-$	T_-	S
ζ	T_+	S	T_-	$T_- \oplus T_-$	T_-	S	T_+	$T_+ \oplus T_+$
$\xi^0 = \zeta^0$	$T_+ \oplus T_+$	T_+	S	T_-	$T_- \oplus T_-$	T_-	S	T_-

Table 2.3.2: Classification of ξ and ζ on $\mathbb{C}\ell_{r,s}$ and on $\mathbb{C}\ell^0_{r,s}$ (denoted $\xi^0 = \zeta^0$). Again S denotes a swap of the simple components, and the subscripts $\pm$ denote the symmetry of the associated product.

if $d \bmod 8 = 0, 1, 2, 4, 5$ or 6 there is a C_+ in the Clifford algebra satisfying $C_+^{-1} v^T C_+ = v^\xi$, and if $d \bmod 8 = 0, 2, 3, 4, 6$ or 7 there is a C_- such that $C_-^{-1} v^T C_- = v^\xi$ for all $v \in \mathbb{C}\ell_{r,s}$. The unitary *charge conjugation matrix* $C_\pm$ satisfies $C_\pm^T = \epsilon C_\pm$, but here, as opposed to the Dirac matrix, the symmetry ϵ is not a matter of convention, but determined by table (2.3.2). The *Majorana conjugate* is defined as

$$(\bar{\theta})_M = \begin{cases} C_+^{-1}\theta^\xi = \theta^T C_+^{-1} & \text{if} \quad d \bmod 8 = 1, 5 \\ C_-^{-1}\theta^\zeta = \theta^T C_-^{-1} & \text{otherwise} \end{cases} . \tag{2.3.8}$$

Again we note that if $\bmod 8 = 2$ or 6, the associated *Majorana pairing* is between *spinors* of different chirality. The properties of the charge conjugation matrix can be directly deduced from table (2.3.2),

$d \bmod 8$	0	1	2	3	4	5	6	7
C	$C_\pm$	C_+	$C_\pm$	C_-	$C_\pm$	C_+	$C_\pm$	C_-
$C^{-1}v^T C$	$\pm v$	v	$\pm v$	$-v$	$\pm v$	v	$\pm v$	$-v$
C^T	$C_\pm$	C_+	$\pm C_\pm$	$-C_-$	$-C_\pm$	$-C_+$	$\mp C_\pm$	C_-

Table 2.3.3: Classification the charge conjugation matrices. The $\pm$-subscript denotes the sign of the action $C^{-1}v^T C$ for $v \in V$ (second row), and the last row gives the symmetry of the associated product.

3. Finally there is the composition of the previous adjoints, called *charge conjugation*, denoted $\ddagger = \dagger T = T\dagger$. This operation is *not* ordinary complex conjugation! Complex conjugation $*$ leaves $C\ell_{r,s}$ invariant (it is therefore called the *real subalgebra* of $\mathbb{C}\ell_{r,s}$), and $\ddagger$ leaves $\mathbb{R}(n)$ invariant, which is not necessarily equivalent, since the complex structure may originate from a quaternionic structure. Suppose we have a matrix basis $\{e^{ij}\}$ of $C\ell_{r,s} = \mathbb{C} \otimes \mathbb{R}(n)$. If $a = a_{ij}e^{ij}$ (where we have summation convention) for some complex numbers a_{ij}, then $a^T = a_{ji}e^{ij}$ and $a^\dagger = a^*_{ji}e^{ij}$, and hence $a^\ddagger = a^*_{ij}e^{ij}$. However, the chosen matrix basis may be complex, in which case $a^* = a^*_{ij}(e^{ij})^*$ is not equal to $a^\ddagger$. So $\ddagger$ simply conjugates matrix coefficients in some basis, and when this basis is chosen to consist only of real matrices, it coincides with complex conjugation, and hence $*$ and $\ddagger$ are equivalent anti-involutions. Since such bases always exist there is a unitary transformation matrix (not unique) $B \in \mathbb{C}(n)$ such that

$$a^* = Ba^\ddagger B^{-1} . \tag{2.3.9}$$

Of course, the involution $\ddagger$ doesn't induces a bilinear inner product (there is a double transposition in its definition, hence it maps minimal left ideals to left ideals). Moreover, it is not an anti-involution, since $(ab)^* = a^*b^*$, so taking the complex conjugate of (2.3.9) doesn't give a condition $B^* = \epsilon B$ but rather

$$B^* = \eta B^{-1}\,, \quad \eta = \pm 1\,. \tag{2.3.10}$$

The sign η is *not* a matter of convention; it can be shown that if a simple $\mathbb{C}\ell_{r,s}$ is the complexification of quaternionic algebra ($\mathbb{C}(n) \simeq \mathbb{C}\otimes\mathbb{H}(n/2)$), any matrix satisfying (2.3.9) satisfies $B^*B < 0$ (no complex rescaling of B can change this). If $\mathbb{C}\ell_{r,s}$ is semi-simple, it decomposes as $\mathbb{C}\otimes(\mathbb{R}(n)\otimes\mathbb{R}(n))$, $\mathbb{C}\otimes(\mathbb{H}(m)\otimes\mathbb{H}(m))$ or $\mathbb{C}\otimes\mathbb{C}(k)$. In the second case we again have $\eta = -1$, and in the last case complex conjugation swaps the simple components. Hence we can define an involution called *charge conjugation* on the minimal left ideals

$$\theta^c = B\theta^*\,, \tag{2.3.11}$$

if the $\mathbb{C}\ell_{r,s}$ is the complexification of a real matrix algebra or a sum thereof. Since $\ddagger$ is a composition of Hermitian conjugation and transposition, its conjugation can be expressed in terms of the conjugation matrices of the latter involutions. The reader easily verifies

$$B = C^{-1}A^*\,. \tag{2.3.12}$$

This proves the unitarity of the B-matrix, and one can also deduce the statement made about the sign of B^*B from the symmetry properties of C.

A number of remarks concerning the constructions above are to be made. The Dirac matrix representation (2.2.28) has the following Hermicity properties:

$$(\Gamma_\mu)^\dagger = -\Gamma_\mu\,, \quad \mu = 0,\ldots,r-1\,, \qquad (\Gamma_\mu)^\dagger = \Gamma_\mu\,, \quad \mu = r,\ldots,r+s-1 \tag{2.3.13}$$

Taking $A = \Gamma_0\ldots\Gamma_{r-1}$, we see that Hermitian conjugation is an automorphism conjugate to the inner automorphism associated to this matrix:

$$(\Gamma_\mu)^\dagger = (-1)^r A\Gamma_\mu A^{-1}\,, \quad A^\dagger = (-1)^r A\,. \tag{2.3.14}$$

The reader may verify that the involution $A(\mathbb{C}\ell_{r,s})^\dagger A$ is indeed conjugate to ζ^* (ξ^* if (r,s) mod $2 = (1,0)$). This comes down to showing that all the generators are purely imaginary, except if (r,s) mod $2 = (1,1)$, in which case they are real (note that we have only defined A for $r > 0$). The involutary property of the charge conjugacy is in the Dirac basis

$$B^*B = \epsilon\eta^r(-1)^{r(r-1)/2}\,, \tag{2.3.15}$$

where ϵ and η are defined by $C^T = \epsilon C$, $C^{-1}vC = \eta v$ for $v \in V \hookrightarrow \mathbb{C}\ell(V)$. A representation with the Hermicity properties above is not unique: acting adjointly with a *unitary* matrix on the generators, $\Gamma_\mu \mapsto U\Gamma_\mu U^\dagger$ leaves the algebra relation (2.2.3) and (2.3.13) invariant. Secondly, the Majorana pairing defines a *spin-invariant (complex-) bilinear product* on the spinor module, regarded as a complex vector space, i.e. a morphism $S\otimes S \longrightarrow \mathbb{C}$. This allows us to identify the spinor module S with its dual; assuming the base space is a vector space over $\mathbb{R}$ and the spinors are representations of $\mathbb{C}\ell(V) = \mathrm{C}\ell(V_{\mathbb{C}})$, left module multiplication gives a morphism $V_{\mathbb{C}}\otimes S \longrightarrow S$ and hence we get a morphism $\phi : S\otimes S^* \simeq S\otimes S \longrightarrow V_{\mathbb{C}}^* \simeq V_{\mathbb{C}}$, defined by $\langle\phi(\theta,\psi), v\rangle = (\rho(v)\theta,\psi)_M$, where $\langle\ ,\ \rangle$ denotes the inner product on $V_{\mathbb{C}}$ (the complexified inner product on V). As a generalisation, the isomorphism (2.2.6) provides a morphism $\bigwedge^k V_{\mathbb{C}}\otimes S \longrightarrow \mathbb{C}\ell(V,\eta)\otimes S \longrightarrow S$, which by self-duality of $V_{\mathbb{C}}$ and S induce morphisms $\phi^k : S\otimes S \longrightarrow \bigwedge^k V_{\mathbb{C}}$, defined by linearity and the relation $\phi^k(\theta,\psi)(e_{\mu_1},\ldots,e_{\mu_k}) = (\Gamma_{\mu_1,\ldots,\mu_k}\theta,\psi)_M$, or equivalently

$$\phi^k(\theta,\psi) = \frac{1}{k!}(\bar\theta\Gamma_{\mu_1\ldots\mu_k}\psi)\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_k} \equiv \bar\theta\wedge\Gamma^{(k)}\wedge\psi\,. \tag{2.3.16}$$

The (skew-) symmetry of these maps can be directly deduced from table (2.3.3) and we have summarised them in table (2.3.4).

d mod 8	k even		k odd	
	Morhism	Symmetry	Morphism	Symmetry
0	$S^{\pm}\otimes S^{\pm}\to\wedge^k V_{\mathbb{C}}$	$(-1)^{k(k-1)/2}$	$S^{\pm}\otimes S^{\mp}\to\wedge^k V_{\mathbb{C}}$	
1	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$(-1)^{k(k-1)/2}$	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$(-1)^{k(k-1)/2}$
2	$S^{\pm}\otimes S^{\mp}\to\wedge^k V$		$S^{\pm}\otimes S^{\pm}\to\wedge^k V_{\mathbb{C}}$	$(-1)^{k(k-1)/2}$
3	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$-(-1)^{k(k-1)/2}$	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$(-1)^{k(k-1)/2}$
4	$S^{\pm}\otimes S^{\pm}\to\wedge^k V_{\mathbb{C}}$	$-(-1)^{k(k-1)/2}$	$S^{\pm}\otimes S^{\mp}\to\wedge^k V_{\mathbb{C}}$	
5	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$-(-1)^{k(k-1)/2}$	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$-(-1)^{k(k-1)/2}$
6	$S^{\pm}\otimes S^{\mp}\to\wedge^k V_{\mathbb{C}}$		$S^{\pm}\otimes S^{\pm}\to\wedge^k V_{\mathbb{C}}$	$-(-1)^{k(k-1)/2}$
7	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$(-1)^{k(k-1)/2}$	$S\otimes S\to\wedge^k V_{\mathbb{C}}$	$-(-1)^{k(k-1)/2}$

Table 2.3.4: Classification of morphisms $S\times S\to\wedge V$

2.3.3 Majorana Spinors

When Dirac spinors carry reducible representations of the real subalgebra the irreducible subspaces are called *Majorana representations.* This occurs when the real subalgebra is a real matrix algebra or a direct sum thereof. In this situation the primitive idempotents re invariant under charge conjugation so that the eigenspaces $\mathbb{R}$-spanned by $\theta:\theta^c=\pm\theta$ contain minimal left ideals. The intersection with $+1$-eigenspace is the *Majorana subspace*, defined by

$$\theta^c=\theta\,, \tag{2.3.17}$$

and this condition can only be consistently imposed (i.e. having nontrivial solutions) if $B^*B=1$, excluding irreducible representations of quaternionic algebras. Satisfaction of the Majorana condition (2.3.17) implies that the Dirac and Majorana conjugate coincide: $\theta=B\theta^*=C^{-1}A^*\theta^*\Rightarrow \theta^TC^T=\theta^TC^{-1}=\theta^\dagger A^\dagger=\theta^\dagger A^{-1}\Rightarrow(\bar\theta)_M=(\bar\theta)_D$. As it turns out, imposing a reality condition is quite important and in dimensions and signatures where the equation above is not satisfied, one can use some tricks to introduce a notion of reality,

1. if $(r-s)$ mod $8=0$, $\mathbb{C}\ell^0_{r,s}$ is semi-simple and the real subalgebras of the components are real matrix algebras. As explained above, irreducible (chiral) representations of $\mathbb{C}\ell^0_{r,s}$ carry reducible representations of the real subalgebras (complex conjugation leaves the chiral projectors invariant); the elements in the real, chiral subspaces are *Majorana-Weyl spinors*,

2. if $(r-s)$ mod $8=1$ or 7, $\mathbb{C}\ell^0_{r,s}$ is simple and the real subalgebras are real matrix algebras, so irreducible complex spinor representations induce reducible real spinor representations; the elements in the real subspaces are *Majorana spinors*,

3. if $(r-s)$ mod $8=2$ or 6, complex conjugation swaps the simple components of $\mathbb{C}\ell^0_{r,s}$, so irreducible representations are irreducible w.r.t. ^{c}. However, an irreducible representation of $\mathbb{C}\ell_{r,s}$, which induces a pinor representation, is reducible when restricted to the real subalgebra for signature 2. Since $\mathrm{C}\ell^0_{r,s}\simeq\mathrm{C}\ell^0_{s,r}$, the pinor representation induces a reducible representation of the real even subalgebra for signature 6 as well. We stress that it is the representation of the entire Clifford algebra that contains invariant real subspaces, not the chiral representation of the even subalgebra, and both representations have equal dimension: half of the dimension of the pinor representation. We denote this situation with M/W. If we choose the Dirac representation in which $A^\dagger=(-1)^rA$ and the signature satisfies $(r-s)$ mod $8=2$, the condition $B^*B=1$ forces us to take $C=C_+$ if d mod $8=2,6$ and $C=C_-$ if d mod $8=0,4$, or in conclusion $C=C_{-(-1)^{d/2}}$. If $(r-s)$ mod $8=6$, the same argument gives us $C=C_{(-1)^{d/2}}$ (see [15] for more explicit treatment).

4. if $(r-s)$ mod $8 = 3, 4$ or 5, the real even subalgebras are quaternionic, so a Majorana condition as in (2.3.17) cannot be imposed (also the real subalgebras of the whole complexified Clifford algebra possess no reality structure). One can still define a complex structure, not on the representation S of $\mathrm{Spin}_{\mathbb{C}}(r,s)$, but on its *symplectic* extension $\tilde{S} \equiv S \otimes \mathbb{C}^2 \simeq S \oplus_{\mathbb{C}} S$ of $\mathrm{Spin}_{\mathbb{C}}(r,s) \otimes SU(2)$. We define the involution σ on this representation by $(\theta^\sigma)_i \equiv (\theta^i)^c = \Omega^{ij}\theta^c_j$ (summing over j), where
$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (2.3.18)$$
The involution σ defines a real structure on $\tilde{S}$, and the real subspace $\{\theta \in S \oplus S : \theta^i = (\theta^\sigma)_i \Leftrightarrow (\theta_i)^c = \theta^i\}$ is left invariant by $\mathrm{Spin}_{\mathbb{C}}(r,s) \otimes USP(2)$. The group $USP(2)$ is the group of unitary symplectic 2×2 matrices: $USP(2) = \{A \in \mathrm{U}(2) | A^T\Omega A = \Omega\}$. It is a well-known result of matrix algebra that $USP(2N) = SU(2N)$. The real subspace has half of the dimension of $\tilde{S}$, which has four times the (real) dimension of the original irreducible representation S; so to define a real structure on these representations one has to double the dimension. In the case $(r-s)$ mod $8 = 4$, the involution σ leaves the chiral symplectic subspaces $\tilde{S}^\pm \equiv S^\pm \otimes \mathbb{C}^2 \simeq S^\pm \oplus_{\mathbb{C}} S^\pm$ invariant; we call these real chiral subspaces *symplectic Majorana-Weyl* representations (denoted by SMW), the other cases are referred to as *symplectic Majorana* (SM). The symplectic metric Ω induces bilinear products $\tilde{S} \otimes \tilde{S} \longrightarrow \wedge^k V$ by setting $(\tilde{\theta}, \tilde{\psi}) = \Omega^{ij}(\theta_i, \psi_j)_M$. Consequently the symmetry of these products are exactly opposite to the symmetry of the products on the factors S (cf. table (2.3.4)).

$(r-s)$ mod 8	type	η	dimension	H	d_0
0	MW	$\pm$	$2^{(d-2)/2}$	$SO(N_L) \times SO(N_R)$	$2, 6$
				$SO(N)$	$0^*, 4^*$
1	M	$(-1)^{(d-1)/2}$	$2^{(d-1)/2}$	$SO(N)$	1,3
2	M/W	$(-1)^{(d+2)/2}$	$2^{d/2}$	$U(N)$	0,4
3	SM	$(-1)^{(d-1)/2}$	$2^{(d+1)/2}$	$USP(2N)$	$5, 7$
4	SMW	$\pm$	$2^{d/2}$	$USP(2N_L) \times USP(2N_R)$	6
				$SU(2N)$	$0^*, 4^*$
5	SM	$(-1)^{(d-1)/2}$	$2^{(d+1)/2}$	$USP(2N)$	$5, 7$
6	M/W	$(-1)^{d/2}$	$2^{d/2}$	$U(N)$	0,4
7	M	$(-1)^{(d-1)/2}$	$2^{(d-1)/2}$	$SO(N)$	1,3

Table 2.3.5: Classification of minimal representations in the Dirac basis of generators $\{\Gamma_\mu\}$ of $\mathbb{C}\ell_{r,s}$. The sign η is defined by $C^{-1}\Gamma^T_\mu C = \eta\Gamma_\mu$ (we classified C_η in table (2.3.3)), and it is in the symplectic cases chosen such that $B^*B = 1$ is fulfilled. The fourth column gives the dimension of minimal (chiral, or real, or both) spinor representations of the complexified Clifford algebra. The last column is the isometry group of the space spanned by N copies of the spinor module equipped with with either an orthogonal ($(r-s)$ mod $8 = 0, 1, 7$), a unitary ($(r-s)$ mod $8 = 2, 6$) or symplectic ($(r-s)$ mod $8 = 3, 4, 5$) metric. The last column represents the dimension of V_0 mod 8 in which the corresponding extension can exist, where stars indicate antichiral extensions

Restricting spinor representations to the real subspace w.r.t. charge conjugation (or some symplectic or chiral variant of this operation, as we explained above) is important because it allows us to restrict the morphisms of table (2.3.4) to the real subspace of $\Lambda^k(V_{\mathbb{C}})$. In the Dirac basis we have
$$(a\theta, \psi)^*_M = (-\eta)^r \theta^\dagger A^{-1} a^\ddagger \psi^c\,, \qquad (2.3.19)$$
for any $a \in \mathbb{C}\ell_{r,s}$. Consequently, if θ and ψ are Majorana spinors the Majorana pairing $\bar{\theta}\psi$ is a real number. If $(-\eta)^r = -1$, we may perform the unitary transformation $A \mapsto iA \Rightarrow A^{-1} \mapsto -iA^{-1}, B \mapsto -iB$ such that the right-hand side gets multiplied by $(-i)^2$, canceling the minus sign. From the Hermicity properties of this representation one easily deduces
$$\Gamma^\ddagger_\mu = \eta(-1)^r\Gamma_\mu\,, \quad \Gamma^\ddagger = (-1)^{\frac{d}{2}+r}\Gamma\,, \qquad (2.3.20)$$

which indeed indicates that the representation is Majorana-Weyl if $(r-s)$ mod $4 = 0$. Furthermore, if θ and ψ are Majorana spinors, $(\bar{\theta}\Gamma_\mu\psi)^* = (\eta)^{r+1}(\bar{\theta}\Gamma_\mu\psi)$. Hence in all spaces with $\eta = 1$ or r mod $2 = 1$ or both which allow Majorana spinors, there is a representation for which the bilinear map $S \otimes S \longrightarrow V_{\mathbb{C}}$, when restricted to the real subspace of the spinor module, maps to the real vector space.

2.4 Extensions of the Poincaré Algebra

2.4.1 The Poincaré Algebra

Let V_0 be a real vector space with metric η of signature (r,s). The *Poincaré group* associated to this space is the isometry group of (V_0,η);. It is the semi-direct product[5]

$$P(V_0) = \mathrm{Spin}(r,s) \ltimes Trans(V_0)\,, \tag{2.4.1}$$

where $Trans(V_0)$ is the *translation group* on V_0, which is of course isomorphic to V_0 itself. The tangent space at the origin (Id, t_0) of this Lie group is the *Poincaré algebra*, given by the semi-direct sum

$$\mathfrak{p}(V_0) = \mathfrak{so}(r,s) + V_0\,. \tag{2.4.2}$$

If we choose an orthonormal basis $\{P_\mu\}$ of the vector space, the Poincaré algebra is spanned by the generators $\{M_{\mu\nu}, P_\sigma\}$ which satisfy the algebra relation

$$\begin{aligned}
[P_\mu, P_\nu] &= 0\,,\\
[M_{\mu\nu}, P_\sigma] &= \eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu\,,\\
[M_{\mu\nu}, M_{\sigma\rho}] &= \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho}\,.
\end{aligned} \tag{2.4.3}$$

One easily deduces these commutators using (2.2.23) and (2.2.26). The algebra above is essential for relativistic physics: as we shall see later, it generates a subgroup of automorphism group of the classical solution space of such a (non-supersymmetric) field theory. The Lie algebra of the automorphism group will be the direct (commutative) sum of the Poincaré algebra above with some compact *internal symmetry algebra* $\mathfrak{g}$. Hence the space of classical solutions is an irreducible representation of the algebra $\mathfrak{p}(V_0) \oplus \mathfrak{g}$. The way to find these is Wightman's *little group method*. The *mass* $M^2 = P^\mu P_\mu$ is a Casimir invariant, i.e. a central element of the universal enveloping algebra, and hence its eigenspace decomposition spans the whole Hilbert space. A physical requirement is then that the space of classical solutions sits in a dense subset $\mathscr{D}$ of the total Hilbert space spanning the subspaces for which $0 \leq m < \infty$ where m is the eigenvalue of M, the so-called nonnegative energy representations. The resulting stabiliser subgroup of a particular eigenspace of $\mathscr{D}$ characterised by $P_\mu|\phi\rangle = k_\mu|\phi\rangle$ therefore depends only on the value $m^2 = k_\mu k^\mu$, and we call these one-parameter set of groups the *little groups* of the system. In d-dimensional Minkowski space these little groups split up in 2 isomorphism classes: the *massless* sector $m = 0$, where it is generated by the algebra $\mathfrak{so}(d-2) + \mathfrak{g}$ and the *massive* sector $m > 0$ where the Little algebra is $\mathfrak{so}(d-1) + \mathfrak{g}$. By the results above, this can be easily shown by choosing a particular momentum eigenvalue: for $k_\mu = \sqrt{m}\delta_{\mu,d-1}$, $(m > 0)$ the residual symmetry algebra is quickly seen to be spanned by all internal symmetry generators and the Lorentz generators M_{ij} with $0 \leq i,j < d$. For a massless state, say $k_\mu = k(\delta_{\mu,0} + \delta_{\mu,d-1})$ $(k \neq 0)$, the bosonic little group is generated by the T_a, M_{ij} with $0 < i,j < d-1$ and the Lorentz algebra elements $L_i = M_{i,d-1} - M_{i,0}$. This group is not semi-simple because the latter generators all commute among themselves; hence for the massless sector to be a finite-dimensional representation, they must act trivially on physical states. Consequently, the rotational part corresponding to the massless little group becomes isomorphic to $\mathfrak{so}(d-1)$.

[5] Group multiplication on the *semi*-direct product is given by $(A,t_u)(B,t_v) = (AB, t_u \circ t_{Av})$ for $A, B \in SO(V_0)$ and $t_u, t_v \in Trans(V_0)$ translations corresponding to the vectors $u, v \in V$

2.4.2 Representations of the Lorentz Algebra

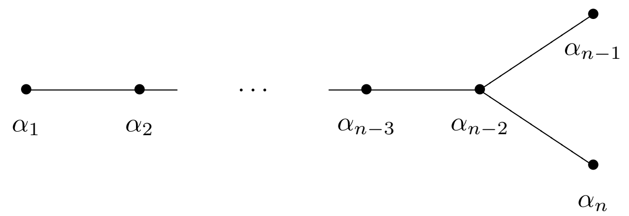
The representation theory of the orthogonal algebra $\mathfrak{so}(d-2)$ may be found in several textbooks (a comprehensive treatment may be found in [16], a more detailed exposition in [17]). It turns out that even- or oddness of $d-2$ is a crucial factor in the highest-weight representation theory. In the defining representation the basis vectors of $\mathfrak{so}(2n)$ are given by $(M_{i,j})_{k,l} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$, and therefore the Cartan generator $H_j = -iM_{2j-1,2j}$ has a 1 complex-dimensional eigenspace spanned by the eigenvectors $(|\pm e_k\rangle)_i = \delta_{j,2k-1} \pm i\delta_{j,2k}$ satisfying $H_j|\pm e_k\rangle = \pm\delta_{kj}|e_k\rangle$. Hence the e_k form a basis of $i\mathfrak{t}^*$ with components the eigenvalues w.r.t. the Cartan basis: $(e_k)_j = e_k(H_j) = H_j|e_k\rangle = \delta_{kj}$. The eigenvectors of the Cartan generators under the adjoint representation are

$$E_{\eta_1 e_j + \eta_2 e_k} = M_{2j-1,2k-1} + i\eta_1 M_{2j,2k-1} + i\eta_2 M_{2j-1,2k} - \eta_1\eta_2 M_{2j,2k}\,, \tag{2.4.4}$$

with $\eta_1, \eta_2 \in \{-1, 1\}$. These satisfy

$$[H_i, E_{\eta_1 e_j + \eta_2 e_k}] = (\eta_1\delta_{ij} + \eta_2\delta_{ik})E_{\eta_1 e_j + \eta_2 e_k} = (\eta_1 e_j + \eta_2 e_k)(H_i)E_{\eta_1 e_j + \eta_2 e_k}\,. \tag{2.4.5}$$

Consequently the roots of the algebra are of the form $\pm e_j \pm e_k$. The Killing form, a canonical inner product on the dual of the Cartan subalgebra satisfies $\langle e_i, e_j\rangle = 2\delta_{ij}$. Hence we can choose a system R^+ of positive roots associated to the fundamental Weyl chamber; we choose $e_k \pm e_j$ for $k < j$. The n simple positive roots are chosen $\alpha_i = e_i - e_{i+1}$, $i = 1, \ldots n-1$ and $\alpha_n = e_{n-1} + e_n$, yielding the Dynkin diagram



Simple root vectors which are connected with a line span an angle of $2\pi/3$, the disconnected roots are orthogonal to each other. Let us now look at the fundamental representations of this group. One easily calculates the Cartan matrix (see e.g. [18]) and applies its inverse to the set of simple positive roots given above. The resulting basis of the weight lattice are the fundamental weights

$$\omega_j = \sum_{k=1}^{j} e_k\,, \quad \omega_+ = \frac{1}{2}\Big(\sum_{k=1}^{n-1} e_k - e_n\Big)\,, \quad \omega_- = \frac{1}{2}\Big(\sum_{k=1}^{n-1} e_k + e_n\Big)\,, \tag{2.4.6}$$

where $j = 1, \ldots, n-2$. To make sense of the corresponding irreducible representations, we recall Weyl's dimension formula,

$$\dim V_\lambda = \prod_{\alpha\in R^+} \frac{\langle\lambda+\delta, \alpha\rangle}{\langle\delta, \alpha\rangle}\,, \quad \delta = \frac{1}{2}\sum_{\alpha_i\in R^+} \alpha_i\,. \tag{2.4.7}$$

One easily calculates that $\delta = \sum_{j=1}^{n-1}(n-j)e_j$. Denoting the corresponding fundamental representation vector spaces with $V_1, \ldots, V_{n-2}, V_+, V_-$, one finds the dimensions

$$\dim V_j = \binom{2n}{j}\,, \quad \dim V_+ = 2^{n-1} = \dim V_-\,. \tag{2.4.8}$$

The weights of the fundamental representations are obtained by letting the Weyl group of mirror symmetries of the weight lattice act on the fundamental weights. A Weyl reflection w.r.t. a root $\alpha = \eta_1 e_j + \eta_2 e_k$ acts on a lattice basis vector by $s_\alpha e_i = (1-\delta_{ij})(1-\delta_{ik})e_i - \eta_1\eta_2(\delta_{ij}e_k + \delta_{ik}e_j)$. After some calculations, one finds the following sets of weights:

$$W(\pi_j) = \{\pm e_{i_1} \pm \ldots \pm e_{i_j}\}\,, \quad W(\pi_\pm) = \Big\{\tfrac{1}{2}(\eta_1 e_1 + \ldots + \eta_n e_n)\Big|\prod_{i=1}^{n}\eta_i = \pm 1\Big\}\,. \tag{2.4.9}$$

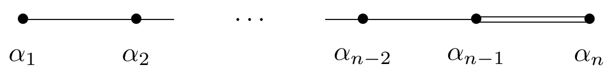
The dimensions of the associated irreducible representations again arise as the number of states $\#W(\pi)$. The first $n-2$ representations are identified as antisymmetric tensors; the last 2 are the (irreducible) chiral spinor representations. From a closer inspection of the root systems, one can derive existence and irreducibility properties of real subrepresentations, as is performed in [16]. The odd-dimensional case is simpler; the Cartan subalgebra is spanned by $-iM_{1,2}, -iM_{3,4}, \dots, -iM_{2n-1,2n}$: the maximal tori of $\mathrm{Spin}(2n)$ and $\mathrm{Spin}(2n+1)$ are isomorphic. As for $\mathrm{Spin}(2n)$, the weight vectors of the Cartan generators in the defining representation form a basis $e_1, \dots e_n$ of $\mathfrak{t}^*$, where $(e_k)_j = H_j(e_k) = \delta_{kj}$. Because the Lie algebra contains more generators than $\mathfrak{so}(2n)$, there are more co-roots:

$$\begin{aligned} E_{\eta_1 e_j + \eta_2 e_k} &= (M_{2j-1,2k-1} + i\eta_1 M_{2j,2k-1} + i\eta_2 M_{2j-1,2k} - \eta_1\eta_2 M_{2j,2k})\,, \\ F_{\eta_3 e_i} &= M_{2i-1,2n+1} + \eta_3 i M_{2i,2n+1}\,, \end{aligned} \tag{2.4.10}$$

where $\eta_1, \eta_2, \eta_3 \in \{-1, 1\}$ and $i, j, k \in \{1, \dots n\}$. Under the adjoint representation of the Cartan generators these transform as

$$\begin{aligned} [H_i, E_{\eta_1 e_j + \eta_2 e_k}] &= (\eta_1 e_j + \eta_2 e_k)(H_j) E_{\eta_1 e_j + \eta_2 e_k}\,, \\ [H_i, F_{\eta_3 e_l}] &= \eta_3 e_l(H_i) F_{\eta_3 e_l}\,. \end{aligned} \tag{2.4.11}$$

Hence the roots are $\pm e_i \pm e_j$ and $\pm e_k$ for $i, j, k = 1, \dots n$. Using $\langle e_i, e_j\rangle = 2\delta_{ij}$ we select the set of positive roots $\{e_i, e_j \pm e_k, i, j, k = 1, \dots, n, j < k\}$. It follows that the simple roots are $(\alpha_1, \dots, \alpha_n) = (e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n)$. Here we observe the structural difference with the even-dimensional case; the Dynkin diagram looks like



The double line indicates an angle of $3\pi/4$: $\cos\theta_n = \langle e_{n-1} - e_n, e_n\rangle / (|e_{n-1} - e_n||e_n|) = -1/\sqrt{2}$. The fundamental weights of this group are

$$\omega_j = \sum_{k=1}^{j} e_k\,, \quad \omega_n = \frac{1}{2}\sum_{k=1}^{n} e_k\,, \quad j = 1, \dots n-1\,. \tag{2.4.12}$$

The first $n-1$ fundamental weights are again associated to antisymmetric tensor representations. The last weight (2.4.12) is the highest weight of the single irreducible spinor representation, reflecting the fact that chiral spinors do not exist in odd dimensions (cf. table (2.3.5)). Using $\delta = \sum_{j=1}^{n}(n + \frac{1}{2} - j)e_j$ one finds via (2.4.7) that the dimension of the spinor representation is 2^n. This can also be seen by looking at the weight spaces the of the representation, which are of the form

$$W(\pi_j) = \{\pm e_{i_1} \pm \dots \pm e_{i_j}\}\,, \quad W(\pi_n) = \{\tfrac{1}{2}(e_1 \pm \dots \pm e_n)\}\,. \tag{2.4.13}$$

General irreducible representations are canonically constructed from the fundamental representations as Young symmetrised harmonic tensor spaces (cf. [17]). For the special orthogonal group, a harmonised tensor space $\mathcal{H}(\bigotimes^k V)$ is the intersection of the kernels of all *contraction operators* C_{ij},

$$C_{ij} : \bigotimes^k V \longrightarrow \bigotimes^{k-2} V : v_1 \otimes \dots \otimes v_k \mapsto g(v_i, v_j) v_1 \otimes \dots \otimes \widehat{v}_i \otimes \dots \otimes \widehat{v}_j \otimes \dots \otimes v_k\,. \tag{2.4.14}$$

Being in the harmonic subspace is therefore equivalent to satisfy all possible vanishing-trace conditions. So the antisymmetric tensor representations are harmonic tensors. The second step is to make a *partition* λ of k, a set of non-increasing integers $\lambda_1 \geq \dots \geq \lambda_l$ such that $\lambda_1 + \dots + \lambda_l = k$, and perform a *Young symmetrisation* on the harmonic tensor product. Such a partition induces a partition of the tensor product, $\bigotimes^k V = \bigotimes^{\lambda_1} V \otimes \dots \otimes \bigotimes^{\lambda_l} V$, and is visualised as a *Ferrers* *diagram*

$$
\begin{array}{r|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\lambda_1 \text{ boxes} & 1 & l+1 & & & \cdots & & \cdots & & \cdots & & & \cdots & k \\
\lambda_2 \text{ boxes} & 2 & \cdots & & & \cdots & & \cdots & & \cdots & & & & \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & & & & & & & \\
\lambda_{l-1} \text{ boxes} & & \cdots & & & \cdots & & & & & & & & \\
\lambda_l \text{ boxes} & l & \cdots & & & & & & & & & & &
\end{array}
$$

Let us denote the column lengths of the diagram above by the non-increasing sequence $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_{\lambda_1}$. A column permutation is of the form $\sigma_1 \otimes \sigma_2 \otimes \ldots \otimes \sigma_{\lambda_1}$ where each $\sigma_i \in \mathrm{Aut}(\otimes^{\tau_i} V)$ permuting the tensor factors, and analogously we define a row permutation. Let us denote the sets of these automorphisms by $\mathrm{Col}(\lambda)$ and $\mathrm{Row}(\lambda)$. The Young symmetrised representation space is the intersection

$$\bigcap_{\sigma \in \mathrm{Row}(\lambda)} \ker(\mathrm{Id} + \mathrm{sign}(\sigma)\sigma) \cap \bigcap_{\sigma \in \mathrm{Col}(\lambda)} \ker(\mathrm{Id} + \sigma)\,. \tag{2.4.15}$$

In combination with harmonisation this gives irreducible tensor representations with highest weight $\lambda_1 e_1 + \ldots + \lambda_l e_l = \tau_1 \omega_1 + \ldots + \tau_{\lambda_1} \omega_{\lambda_1}$. The fundamental antisymmetric tensor representations therefore appear as single-column diagrams. There are 2 conventional notations for the irreducible representations we constructed: either one specifies the partition and denotes the representation corresponding to the tableau above by $(\tau_1, \ldots, \tau_{\lambda_1})$, or one specified the dimension (which is calculated using the *factors over hooks rule*, cf. [16]) printed in bold as in $(3) = \mathbf{84}$ for $2n+1 = 9$. What if we include spinor representations in the tensor products above? As it turns out, the tensor product of 2 spinor representations decomposes into a sum of one-column diagrams by (2.3.16) (see [19, 16] for details); hence we may consider just tensor products of spinors with the representations constructed above. Harmonisation of a spinor times a k-form is nontrivial; the subspace

$$\mathcal{H}((k) \otimes \mathbf{2}^{n-1}) = \{A_{\mu_1 \ldots \mu_k} \theta^\alpha \mathrm{d}x^{\mu_1} \wedge \ldots \wedge x^{\mu_k} \otimes Q_\alpha \in \bigwedge^k V \otimes S(V) | A_{\mu_1 \ldots \mu_k} (\Gamma^{\mu_1 \cdots \mu_k})^\alpha{}_\beta \theta^\beta = 0\}$$

is invariant under the corresponding representation of the orthogonal group. An important application of these products is the *Rarita-Schwinger field*, which transforms as a spinor times a vector. One can easily write down the weights of such a tensor product using the following rule: if π_1 is a SO-representation with highest weight $\ell_1 e_1 + \ldots + \ell_n e_n$ and π_2 is a SO-representation with highest weight $j_1 e_1 + \ldots + j_n e_n$, then are the weights of the SO-representation $\pi_1 \otimes \pi_2$ given by $k_1 e_1 + \ldots + k_n e_n$ where $k_i \in \{-\ell_i - j_i, -\ell_i - j_i + 1, \ldots, \ell_i + j_i - 1, \ell_i + j_i\}$. This formula implies that all the weights of SO-representations have integer or half-integer coefficients. The highest eigenvalue of the set of Cartan matrices, which denoted earlier by λ_1, is called the *spin* s of the representation. Therefore spin numbers are added when taken a tensor product of representations.

2.4.3 The Super-Poincaré Algebra

A relativistic field theory is called *supersymmetric* if the automorphism group of its classical solution space is a real superextension of a Poincaré algebra,

Definition. A super algebra $\mathfrak{p} = (\mathfrak{p}_0 + C) + \mathfrak{p}_1$ is called a real *super Poincaré algebra* if the even part is a semi-direct sum of a complexified Poincaré algebra $\mathfrak{p}_0$ and a *central charge algebra* C and under the super bracket we have $[C, \mathfrak{p}_1] = [C, V_0] = [C, C] = 0$ and $[\mathfrak{p}_1, \mathfrak{p}_1] \in \mathbb{C} \otimes (V_0 \oplus C)$. Reality is provided by an involution ^{c}, which induces complex conjugation and acts as the identity on the basis vectors of the even part, and admits a nontrivial real subspace of the odd part.

Already from the Jacobi identities one derives much information on the algebra structure: define $\mathfrak{p}_1^{(0)} = \{s \in \mathfrak{p}_1 : [V_0, s] = 0\}$ and $\mathfrak{p}_1^{(i)} = \{s \in \mathfrak{p}_1 : [V_0, s] \in \mathfrak{p}_1^{(i-1)}\}$. This is an algebra filtration,

$\mathfrak{p}_1^{(0)} \subseteq \mathfrak{p}_1^{(1)} \subseteq \ldots \subseteq \mathfrak{p}_1$. It can be shown that this series is finite: at some finite k we then have $\mathfrak{p}_1^{(k)} = \mathfrak{p}_1$ Furthermore the filtration is compatible with the conjugation: $(\mathfrak{p}_1^{(i)})^c = \mathfrak{p}_1^{(i)}$, as it acts as the identity on the momentum generators. From the super Jacobi identities $[V_0, [\mathfrak{p}_1^{(0)}, \mathfrak{p}_1^{(0)}]] = 0$ and hence $[\mathfrak{p}_1^{(0)}, \mathfrak{p}_1^{(0)}] \in \mathbb{C}\otimes(V_0 \oplus C)$. Furthermore, using $[V_0, [V_0, \mathfrak{p}_1^{(1)}]] = 0$ and the Jacobi identities one finds

$$[[V_0, \mathfrak{p}_1^{(1)}], [V_0, \mathfrak{p}_1^{(1)}]] = [V_0, [V_0, \mathfrak{p}_1^{(1)}], \mathfrak{p}_1^{(1)}]] = [V_0, [V_0[\mathfrak{p}_1^{(1)}, \mathfrak{p}_1^{(1)}]]] \subseteq [V_0, [V_0, \mathfrak{p}_1^{(1)}]] = 0\,.$$

Decomposing $\mathfrak{p}_1^{(1)}$ into eigenspaces of the conjugation yields a positive definite bilinear form on $\mathfrak{p}_0^{(0)}$ on the left hand side. Hence we may conclude that $[V_0, \mathfrak{p}_1^{(1)}] = 0$, or consequently $\mathfrak{p}_1^{(1)} = \mathfrak{p}_1^{(0)}$. The same argument can be applied to all $\mathfrak{p}_1^{(i)}$, which eventually yields $\mathfrak{p}_1 = \mathfrak{p}_1^{(0)}$. The important formula we extract from this are the brackets

$$[V_0, \mathfrak{p}_1] = 0\,, \quad [\mathfrak{p}_1, \mathfrak{p}_1] \subseteq \mathbb{C} \otimes (V_0 \oplus C)\,. \tag{2.4.16}$$

By linearity of the superbracket $[\mathfrak{p}_0, \mathfrak{p}_1] \subseteq \mathfrak{p}_1$ the odd part must be a representation of the Poincaré group. From this moment, we assume the underlying vector space to be d-dimensional Minkowski space. Then $\mathfrak{p}_1$ is a representation of the Lorentz algebra $\mathfrak{so}(1, d-1) \oplus \mathfrak{g}$ with a real structure, or equivalently a tensor product of 2 representations. As for the Poincaré group, we let the representations of the (orthochronous orientation-preserving) Lorentz group be induced by the representations of its compact subgroup of rotations. From the algebra structure we conclude that the generators of the even subalgebra transform as representations with spin smaller then or equal to 1 under the rotational subalgebra $\mathfrak{so}(d-1)$. Since $[\mathfrak{p}_1, \mathfrak{p}_1]$ is a (symmetrised) tensor product of two representations of the orthogonal group, its spin two times the spin of the representation $\mathfrak{p}_1$. Now we use equality (2.4.16), in which the spin of the representations on both sides must be balanced, and the fact that the spin on the right-hand side is 1. Excluding Lorentz scalars from $\mathfrak{p}_1$ (which would lead to a trivial extension), we conclude that the spin of $\mathfrak{p}_1$ is $\frac{1}{2}$, so *the anticommuting sector is induced by a spinor representation of the rotation group*: it is therefore equivalent to a spinor representation of $\mathrm{Spin}(1, d-1)$. For $\mathfrak{g}$ trivial and Q_α (the *supersymmetry charges*) a basis of the spinor representation, we use (2.2.25) to obtain the bracket

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2} Q_\beta (\Gamma_{\mu\nu})^\beta{}_\alpha\,. \tag{2.4.17}$$

2.4.4 Superextensions

In the following V_0 is a general vector space with a metric of signature (r, s). We examine the extensions of the Poincaré algebra by a spinor module, as defined above. In general, the tensor product of 2 spinor representations decomposes into a sum of all fundamental spin-1 representations of the Lorentz algebra. Finding a superextension of the Poincaré algebra comes down to symmetrising this decomposition: if the first fundamental representation, V_0, drops out under this procedure, (non-extended) supersymmetry cannot exist (the resulting algebra would simply be a direct sum modulo central charges). Another, competing requirement of the supersymmetry algebra is reality: there must exist an involution on the spinor module which induces complex conjugation on coefficients and which commutes with the action of the spin algebra, such that the real subspace w.r.t. this involution is nonempty. When the Majorana subspace (+1 eigenspace of charge conjugation) is nonempty, the bracket of 2 Majorana spinors is real and positive definite; it looks like

$$\{Q_\alpha, Q_\beta\} = (\Gamma_\mu)_{\alpha\beta} P^\mu + \sum_{k>1} (\Gamma_{\mu_1\ldots\mu_k})_{\alpha\beta} Z^{\mu_1\cdots\mu_k}\,, \tag{2.4.18}$$

where we have denoted $Z^{\mu_1\cdots\mu_k} = e^{\mu_1} \wedge \ldots \wedge e^{\mu_k}$ and the lowered spinor indices are a shorthand notation: for $\theta = Q_\alpha \theta^\alpha$, $\bar\theta = \theta_\alpha Q^\alpha$ with $\theta_\alpha = \theta^\beta C_{\beta\alpha}$ and analogously $(\Gamma_\mu)_{\alpha\beta} = C_{\alpha\gamma}(\Gamma_\mu)^\gamma{}_\beta$ etc. The matrix C is the charge conjugation matrix $C_{\alpha\beta} = (Q_\alpha, Q_\beta)_M$ which may be symmetric or antisymmetric, but always squares to one. The bracket (2.4.18) is provided by the homomorphism

(2.3.16). The idea of a higher extension is to consistently generalise the anti-commutator above to the direct sum of several spinor representations. In the Majorana case above, we can take the tensor product of S with a vector space W carrying a positive definite bilinear pairing g. Extending the Majorana condition to this tensor product imposes V to be real. The brackets are extended to

$$\{Q_{\alpha i}, Q_{\beta j}\} = g_{ij}(\Gamma_\mu)_{\alpha\beta}P^\mu + g_{ij}\sum_k (\not{Z}^k)_{\alpha\beta}\,, \quad [M_{\mu\nu}, Q_{\alpha i}] = -\frac{1}{2}Q_{\beta i}(\Gamma_{\mu\nu})^\beta{}_\alpha\,, \tag{2.4.19}$$

where we have used the shorthand notation $\not{Z}^k$ for a rank k central charge contracted with the k-fold antisymmetrised gamma matrices. There is an extended automorphism group of the brackets above induced by the group $SO(W,g)$, which commutes with the adjoint action of the Lorentz generators. By definition such a group, if compact, is called the *R-symmetry group* of the extension. We can use it to classify the extension: a compact group is the automorphism group of a bilinear product on some vector space, and given this group the extension is recovered by taking the tensor product of a single spinor module with its vector representation. For S Majorana and nonchiral and allowing a symmetric map to V_0, g is real and symmetric and the R-symmetry group is $SO(N)$. This can happen if the $(r-s) \bmod 8 = 1$ or 7 and $(r+s) \bmod 8 = 1$ or 3. If the spinor module is chiral and charge conjugation respects the irreducible subspaces, the Majorana-Weyl case $(r-s) \bmod 8 = 0$, we separate two cases. If $d \bmod 8 = 2$, the anti-commutator of spinors with opposite chirality vanishes: we then have a *chiral superalgebra*. This allows us to tensor both irreducible real subspaces with different vector spaces W_L and W_R. The R-symmetry group turns into $SO(N_L) \times SO(N_R)$. In the cases $d \bmod 8 = 0, 4$, it is the bracket of equal-chirality spinors that vanishes (cf. (2.3.4)) and we call the algebra *anti-chiral*. In such cases we obviously must have $N_L = N_R$.

In the complex chiral case $(r-s) \bmod 8 = 2, 6$, the real subspace w.r.t. charge conjugation consists of spinors of mixed chirality. The bracket of 2 real spinors breaks up in three pieces, lying in $\{S^+, S^+\} + \{S^-, S^-\} + \{S^+, S^-\}$. Because charge conjugation switches chiral subspaces, only the latter space has a real subspace: Poincaré supersymmetry is only consistent if $S \otimes S \longrightarrow V_{\mathbb{C}}$ is nonchiral: $S^+ \vee S^- \longrightarrow V$. From table (2.3.4) we read off that this happens if $d \bmod 8 = 0$ or 4. Independent left- and right-handed extensions are therefore impossible. If we take the tensor product of the spinor module by some complex vector space W, it should be equipped with a positive-definite symmetric Hermitian form g. Hence the automorphism group of the extension becomes $U(N)$.

If $(r-s) \bmod 8 = 3, 4, 5$, the even subalgebra is quaternionic and charge conjugation is not an involution, since it squares to the map $\theta \mapsto -\theta$. Extending the module to $S \oplus S$ gives an involution $(\theta^\sigma)_i = \Omega^{ij}\theta^c_j$, with Ω defined in (2.3.18). This is an extension by a 2-dimensional vector space with a symplectic metric Ω defined on it. Replacing g by this Ω in (2.4.19) shows that Poincaré supersymmetry can then exist if $(\Gamma^\mu)_{\alpha\beta}$ is antisymmetric in the spinor indices, i.e. if $(r-s) \bmod 8 = 3, 5$ and $d \bmod 8 = 5, 7$, and for $(r-s) \bmod 8 = 4$ we obtain a chiral superalgebra if $d \bmod 8 = 6$, while for $d = 0, 4$ nonchiral superalgebras exist. Higher extensions arise by extending the two-dimensional vector space to higher even-dimensional vector spaces equipped with the symplectic metric

$$\Omega = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix}. \tag{2.4.20}$$

The new automorphisms on the extended module which leave the supercharge bracket invariant are unitary transformations L on $2N$-dimensional extension vector space W that obey $L\Omega L^T = \Omega$. This matrix group is called the *unitary symplectic group* $USP(2N)$, which is isomorphic to $SU(2N)$. Again, if the algebra is chiral we may choose different extensions for the chiral subspaces, giving us R-symmetry groups of the form $SU(2N_L) \times SU(2N_R)$. These extensions are summarised in the last column of table (2.3.5). For detailed classifications, see e.g. [20] and [21].

2.4.5 Representation Theory

Let us now include supersymmetry generators in the little group procedure to find irreducible representations of the entire super-Poincaré algebra. The reduction of the Lorentz algebra in the positive energy eigenspaces will lead to a reduction of the supersymmetry. This symmetry breaking is maximal at the massless level. From the analysis above it is clear that the representation of the Cartan generators have 2^{n-1} eigenvalues $i/2$ and 2^{n-1} eigenvalues $-i/2$ on the spinor module $\mathfrak{p}_1$ (which was already noted at the end of section (2.2)). One easily verifies that in the chiral case each irreducible subspace carries a quarter of respective eigenstates. Let $S_\pm$ be two the eigenspaces of the generator $M_{0,d-1}$, which then satisfy $\dim(S_\pm) = \frac{1}{2}\dim(S)$ and $S_+ \oplus S_- = S$, where orthogonality is provided by the spin-invariant bilinear pairing used to construct the superextension. From the super-Jacobi identities one derives that for $\theta \in S_\pm : [M_{0,d-1}, [\theta, \theta^c]] = \pm[\theta, \theta^c]$. Hence, for any state $|\phi\rangle$ in the total Hilbert space, $[\theta, \theta^c]|\phi\rangle$ is an eigenstate of $M_{0,d-1}$. But the only elements in $\mathfrak{p}_0$ (including central charges) satisfying this eigenstate condition are $c(P^0 \pm P^{d-1})$, where the sign corresponds to $\theta \in S_\pm$ and c is a positive real number. Hence, restricted to the momentum eigenspace $\mathscr{D}_0$ characterised by $P_\mu|\phi\rangle = k(\delta_{\mu 0} + \delta_{\mu(d-1)})|\phi\rangle$, all the brackets $[\theta, \theta^c]|\phi\rangle$ with $\theta \in S_-$ vanish. Because the super bracket of a spinor and its conjugate is positive definite, all the elements of S_- must be represented by zero on $\mathscr{D}_0$. Consequently, the central charges must be represented by zero on $\mathscr{D}_0$: states generated by C acting on $\mathscr{D}_0$ are in the space $[S_+, S_-]|\mathscr{D}_0\rangle = 0$, because states in $[S_\pm, S_\pm]\mathscr{D}_0$ yield linear combinations of the momenta. Also in the massive sector with $C = 0$ half of the supersymmetries are broken. However, in the massive case we look at the stabiliser of the states in $\mathscr{D}_m$, the momentum eigenspace defined by $P_\mu|\mathscr{D}_m\rangle = \sqrt{m}\delta_{\mu,d-1}|\mathscr{D}_m\rangle$ for $m > 0$. Then for $\theta \in S_\pm : [\theta, \theta^c] = \pm Cm$ on $\mathscr{D}_m$. We have therefore equal-dimensional positive and negative eigenvalue subspaces. If we denote $S = S_0 \oplus S_1$, where S_0 annihilates $\mathscr{D}_m$, we conclude $\dim S_0 \leq \frac{1}{2}\dim S$: in the massive sector less then one-half of the supersymmetries may be broken. Let us denote with S_1 the subspace of the odd sector which has a nontrivial representation on $\mathscr{D}$. The little groups are then generated by

$$\mathfrak{l} = \begin{cases} \mathfrak{so}(d-2) \oplus \mathfrak{g} \oplus S_+ (= S_1)\,, & m = 0\,, \\ \mathfrak{so}(d-1) \oplus \mathfrak{g} \oplus S_1\,, & m > 0 \end{cases}\,. \tag{2.4.21}$$

We may choose a basis Q_α of S_+ such that on $\mathscr{D}$ we have

$$[Q_\alpha, (Q^c)_\beta] = \delta_{\alpha\beta}\,. \tag{2.4.22}$$

Here we have absorbed both spinor indices and internal symmetry algebra indices in the Greek indices $\alpha, \beta, \ldots$. Let us denote the Lorentz transformation matrices of this basis with $\sigma_{\mu\nu}$: $[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2}Q_\beta(\sigma_{\mu\nu})^\beta{}_\alpha$, and define the map $T : S_1 \longrightarrow S_1^*$ given by the matrix $[Q_\alpha, Q_\beta] = T_{\alpha\beta}$, and hence $Q_\alpha = T_{\alpha\beta}C^{\beta\gamma}(Q^c)_\gamma$. The Jacobi identity $[MQQ]$ gives

$$-\tfrac{1}{2}(T_{\beta\gamma}(\sigma_{\mu\nu})^\gamma{}_\alpha + T_{\alpha\gamma}(\sigma_{\mu\nu})^\gamma{}_\beta) = 0\,. \tag{2.4.23}$$

Consequently in the universal enveloping algebra $U(\mathfrak{l})$ we have

$$\begin{aligned} \left[\tfrac{1}{2}Q_\beta(\sigma_{\mu\nu})^{\beta\gamma}(Q^c)_\gamma, Q_\alpha\right] &= -\left[\,[M_{\mu\nu}, Q_\delta]\,C^{\delta\gamma}Q^c_\gamma, Q_\alpha\right] \\ &= -\,[M_{\mu\nu}, Q_\alpha] + \tfrac{1}{2}T_{\delta\beta}(\sigma_{\mu\nu})^\beta{}_\alpha C^{\delta\gamma}(Q^c)_\gamma = -2\,[M_{\mu\nu}, Q_\delta]\,. \end{aligned} \tag{2.4.24}$$

If we denote $\bar{M}_{\mu\nu} = M_{\mu\nu} + \frac{1}{4}Q_\beta(\Gamma_{\mu\nu})^{\beta\gamma}(Q^c)_\gamma \in U(\mathfrak{l})$, then

$$[\bar{M}_{\mu\nu}, Q_\alpha] = 0\,. \tag{2.4.25}$$

Furthermore, if we expand the $[\bar{M}, \bar{M}]$ bracket in $U(\mathfrak{l})$, we find

$$[\bar{M}_{\mu\nu}, \bar{M}_{\sigma\rho}] = \eta_{\nu\sigma}\bar{M}_{\mu\rho} + \eta_{\mu\rho}\bar{M}_{\nu\sigma} - \eta_{\nu\rho}\bar{M}_{\mu\sigma} - \eta_{\mu\sigma}\bar{M}_{\nu\rho}\,. \tag{2.4.26}$$

So performing the transformation $M \mapsto M - \frac{1}{2}\,[M, Q_\alpha]\,C^{\alpha\beta}Q_\beta$ sends $U(\mathfrak{so}(d-2))$ to an algebra $\bar{U}(\mathfrak{so}(d-2))$, which is isomorphic to its original and commutes with the odd subspace. Hence

$U(\mathfrak{l}) \simeq U(\mathfrak{so}(d-2)) \otimes U(S)$ on $\mathscr{D}$. As a consequence, irreducible representations of the little group (induced by irreducible representations of the enveloping algebra) consist of tensor products of representations of the orthogonal group and representations of the spinor module, equipped with the bracket $\{Q_\alpha, Q^c_\beta\} = \delta_{\alpha\beta}$. This algebra is just a Euclidean Clifford algebra; we investigated its representation theory in the previous sections.

2.4.6 Massless Representations

The dimension of the full representation of the little group is $2^{p/2}$, where p is the dimension of the operator algebra $S|_{\mathscr{D}} \subset \mathrm{End}(\mathscr{D})$. Because the dimension of the spinor module is a power of 2, the Clifford algebra on $\mathscr{D}$ spanned by them will give rise to chiral representations. The chiral subspaces correspond to fermion and boson states, and therefore these are on-shell equal in number. If the little group stabilises a momentum vector of length zero, the integer p is exactly half of the dimension of the spinor module. Massless representations of the supersymmetry algebra which are induced by the trivial representation of the Lorentz algebra are said to form the *shortest supermultiplet.* Note that the dimension of the representation grows as an iterated exponent of the Minkowki base space V_0, as $\dim S$ grows exponentially as a function of $\dim V_0$. It is instructive to examine representations via their character of the Cartan generators $-iM_{2j-1,2j}$. Classification of representations induced by the trivial representation of $\bar{U}(\mathfrak{so}(d-2))$ is important since

$$\chi_{\rho_{SO}\otimes\rho_Q}(\exp(\zeta^{\mu\nu}M_{\mu\nu})) = \chi_{\tilde{\rho}_{SO}}(\exp(\zeta^{\mu\nu}\bar{M}_{\mu\nu})\chi_{1\otimes\rho_Q}(\exp(\zeta^{\mu\nu}M_{\mu\nu}))\,, \tag{2.4.27}$$

where $\rho_{SO} : \mathfrak{so}(d-2) \longrightarrow \mathrm{End}(\mathscr{D})$ is the representation of the homogeneous Lorentz algebra and $\tilde{\rho}_{SO}$ is its composition with the isomorphism $\bar{U}(\mathfrak{so}(d-2)) \simeq U(\mathfrak{so}(d-2))$ and $\rho_Q : \mathbb{C}\ell^0(0, \frac{1}{2}\dim(S)) \longrightarrow \mathrm{End}(\mathscr{D})$ is a spinor representation of the Clifford algebra defined by the supercharges. The representation of the super-little group induced by the trivial representation of the Lorentz group is denoted $1 \otimes \rho_Q$. In his paper [1], Nahm derives the characters of $1 \otimes \rho_Q$ recursively, starting from a minimal subalgebra $\simeq \mathfrak{u}(1)$ and then extending to the full Cartan subalgebra. Define $\chi_n(\zeta^1, \dots, \zeta^n) = \chi_{1\otimes\rho_n}(\zeta^i H_i)$ as the character of the representation of the extension of an n-dimensional commutative subalgebra $\mathfrak{t}_n$ of $\mathfrak{so}(d-2)$ by a 2^n-dimensional real spinor representation S_n. For $n = 1$, we take the Dirac basis (2.2.27) and find

$$\Gamma_{12} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_+ = \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4.28}$$

The 2 real supercharges Q_0, Q_1 are represented by the Pauli matrices (2.2.27), so we find

$$H_1 = iM_{12} = -\frac{i}{4}Q_\alpha(\Gamma_{12})^\alpha_\gamma C^{\gamma\beta}Q_\beta = \frac{1}{8}(Q_0Q_1 - Q_1Q_0) = \frac{i}{4}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.4.29}$$

Hence $\chi_1(\zeta) = \mathrm{Tr}(\exp \zeta H_1) = 2\cos(\frac{1}{4}\zeta)$. The recursion relation arises when we change the Cartan basis $\{H_1, H_2, \dots, H_n\} \mapsto \{\frac{1}{2}(H_1 + H_2), \frac{1}{2}(H_1 - H_2), H_3, \dots, H_n\}$:

$$\chi_n(\zeta^1, \zeta^2, \dots, \zeta^n) = \chi_{n-1}(\zeta_1 + \zeta_2, \zeta_3, \dots, \zeta_n)\chi_{n-1}(\zeta_1 - \zeta_2, \zeta_3, \dots, \zeta_n)\,. \tag{2.4.30}$$

Taking all the zeta's 0 except for the first one gives us the following expression:

$$\chi_{1\otimes\rho_Q}(\exp i\zeta M_{12}) = (2\cos(\tfrac{1}{4}\zeta))^{\dim S/4}\,. \tag{2.4.31}$$

We expand this formula in terms of cosines with half-integer or integer frequency and hence end up with the expression $2\cos(\frac{1}{16}\zeta \dim S)$ + lower freq. terms. We would now like to identify these terms with irreducible representations of the Lorentz groups. After a closer inspection of the Weyl group, one finds upon use of Weyl's character formula that the characters of a diagonal element $g = \mathrm{diag}(g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}, 1)$ of $\mathfrak{so}(2n+1)$ under the representation ρ^λ with highest weight $\lambda = \lambda_1 e_1 + \dots + \lambda_n e_n$ and a diagonal $h = \mathrm{diag}(h_1, h_1^{-1}, h_2, h_2^{-1}, \dots, h_n, h_n^{-1})$ of $\mathfrak{so}(2n)$

under the representations $\rho^{\lambda_\pm}$ with highest weights $\lambda_\pm = \lambda_1 e_1 + \ldots + \lambda_{n-1} e_{n-1} \pm |\lambda_n| e_n$ equal (see e.g. [22])

$$\chi_{\rho^\lambda}(g) = \frac{|S_j^{\lambda_i+n+\frac{1}{2}-i}(g)|}{|S_j^{n+\frac{1}{2}-i}(g)|}, \qquad \chi_{\rho^{\lambda_\pm}}(h) = \frac{1}{2}\left(\frac{|C_j^{\lambda_i+n-i}(h)| \pm |S_j^{\lambda_i+n-i}(h)|}{|C_j^{n-i}(h)|}\right), \tag{2.4.32}$$

where we have defined $|S_j^{k_i}(g)| = \det_{ij}(g_j^{k_i} - g_j^{-k_i})$ and $|C_j^{k_i}(g)| = \det_{ij}(g_j^{k_i} + g_j^{-k_i})$. Note that if $\lambda_n = 0$, the last column of the matrix $S_j{}^i(h)$ becomes zeros, which makes the second term in the numerator of the second character in (2.4.32) vanish (there exist no chiral representations with integer spin). Now we let g be parameterised by $\exp(i\sum_{i=1}^n \zeta_i M_{2i-1,2i}) = \mathrm{diag}(e^{i\zeta_1}, e^{-i\zeta_1}, \ldots, e^{-i\zeta_{n-1}}, e^{-i\zeta_{n-1}}, 1)$; after some algebra one finds following characters for the fundamental antisymmetric tensor representations (k) $(\lambda = e_1 + \ldots + e_k)$ and the spinor representation 2^n:

$$\chi_{(k)}(g) = \sum_{i=0}^{k} \gamma_k^i f_i(\zeta_1, \ldots, \zeta_n), \quad \chi_{2^n}(g) = 2^n \prod_{i=1}^{n} \cos(\tfrac{1}{2}\zeta_i), \tag{2.4.33}$$

where the γ_k^i are constants which have to be calculated explicitly using (2.4.32) and

$$f_i(\zeta_1, \ldots, \zeta_n) = \sum_{1 \le m_1 < \ldots < m_i \le n} \cos(\zeta_{m_1}) \ldots \cos(\zeta_{m_i}). \tag{2.4.34}$$

In the even-dimensional case the characters of the antisymmetric tensor representations have the same form. However, the chiral representations distinguish themselves in the character ring by

$$\chi_{2^n_\pm}(\exp(i\sum_{i=1}^{n} \zeta_i M_{2i-1,2i})) = \sum_{\eta_i \in \{-1,1\}} \exp\left(\tfrac{i}{2}(\eta_1\zeta_1 + \eta_2\zeta_2 + \ldots + \eta_n\zeta_n)\right), \quad \textstyle\prod_{i=1}^n \eta_i = \pm 1. \tag{2.4.35}$$

These character formulas allow us to decompose all choices of representations $\rho_{SO} \otimes \rho_Q$ into irreducible representations of the Lorentz group. Uniqueness of such a decomposition is implied by character theory. This suggests that whenever the metrised space V_0 allows a Poincaré supersymmetry algebra, it gives rise to an infinite number of massless supermultiplets, in which representations of arbitrary length and spin can occur. However, in a physical theory in Minkowski space, there is an additional restriction on such a massless supermultiplet: there exists no consistent interacting field theory built out of representations of the Lorentz group with spin bigger than 2 [23, 24]. This puts restrictions on possible extensions of the spinor module and the dimension of our Minkowski space. To determine the highest dimension in which supersymmetry is possible, we consider the shortest massless multiplet: as already mentioned, the representation $1 \otimes \rho_Q$ induces a multiplet with highest spin $\frac{1}{16}\dim S$, which gives the upper bound $\dim S \le 32$: hence from table (2.3.5) we read off that the maximal dimension is $d = 11$, and the supercharges constitute a single Majorana spinor representation of $SO(1, 10)$. Massless representations containing spin-2 particles are called *supergravity multiplets*, since they include a symmetric 2-tensor which may be identified with the metric, and hence generate gravity theories. In his article, Nahm treats these various representations in more detail, adding nontrivial representations of the Lorentz group, higher supersymmetry extensions and internal symmetries. Supersymmetry multiplets are extensively treated in [25]. For a detailed exposition of various supergravity multiplets, including the massive ones and central charges, we refer to [26].

2.5 Spinor Bundles and Supergeometry

2.5.1 Principal and Associated Bundles

Let us now implement the theory of the previous sections into a geometric framework. We call a manifold M *pseudo-Riemannian* if there exists a metric $g \in \Gamma T^*M \otimes \Gamma T^*M$ which is symmetric and nondegenerate and *Lorentzian* if it has, viewed as a bilinear form $g : T_p(M) \otimes T_p(M) \to \mathbb{R}$, signature $(- + \ldots +)$. A basis $\{e_a\}$ is called g-orthonormal if it satisfies

$$g(e_a, e_b) = g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab} \,, \tag{2.5.1}$$

where $\eta = \text{diag}(\pm 1, \ldots, \pm 1)$ is the diagonalised and normalised metric and we have in each coordinate chart frame $e_a = e_a{}^\mu \partial/\partial x^\mu$. Note that $g_{\mu\nu}$ depends on the point p of the manifold and so does the orthonormal frame, while η is at each point the same. From this moment, Latin indices shall refer to components w.r.t. an orthonormal frame basis and Greek ones to components w.r.t. the coordinate bases ∂_μ, $\mathrm{d}x^\mu$. A g-orthonormal basis of tangent vector fields defines an orthonormal set of dual co-vectors e^a. The metric and its contravariant dual can be written in terms of the new (co-) frames as follows:

$$\begin{aligned} g = g_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ab} e^a \otimes e^b \qquad &\Rightarrow \qquad g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b \,, \\ g^* = g^{\mu\nu} \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} = \eta^{ab} e_a \otimes e_b \qquad &\Rightarrow \qquad g^{\mu\nu} = \eta^{ab} e_a{}^\mu e_b{}^\nu \,. \end{aligned} \tag{2.5.2}$$

So 1-forms can be locally written as $\omega = \omega_\mu \mathrm{d}x^\mu = \omega_a e^a$ and vectors $Y = Y^\mu \partial/\partial x^\mu = Y^a e_a$. The relation (2.5.1) defining a g-orthonormal frame is at each point of the manifold unique up to an orthonormal transformation $e_a \mapsto e_b \Lambda_a{}^b$, with $\Lambda \in O(r, s)$ if the signature of η (and hence g). If we consider the fiber bundle of all g-orthonormal frames $P_O(TM)$, its fiber at each point is therefore isomorphic to $O(r, s)$. Similarly, for an orientable M we may define the bundle which assigns to each point the space of *positively oriented, orthonormal frames*, requiring $\det(e) = +1$, which we shall call the special orthogonal bundle $P_{SO}(TM)$. The transition functions in overlapping coordinate charts are Lorentz transformations, or, its *structure group is the Lorentz group.* Physical laws or quantities in which all Latin indices are contracted are invariant under *local Lorentz transformations*, and quantities in which Greek indices are contracted are invariant under *coordinate transformations* (diffeomorphisms). The orthonormal frame bundle is an example of what is called a *principal fibre bundle*:

Definition. A *principal G-bundle* is a fibre bundle $\pi : P \longrightarrow M$ with a right group action $\alpha : P \times G \longrightarrow P$ for some topological group G, which is continuous, free, fibre-preserving and transitive on each fibre.

By definition a right group action α satisfies $\alpha(x, e) = x$ if e is the unit element in G and $\alpha(\alpha(x, g), h) = \alpha(x, gh)$. Acting freely imposes the converse to the former statement: if $\alpha(x, g) = x$ for some $x \in P$, then $g = e$. Being fibre-preserving obviously means that $\pi(\alpha(x, g)) = \pi(x)$ for all group elements. A group action is transitive if there is only one G-orbit. Hence for every pair $x, y \in P$ with $\pi(x) = \pi(y)$ there exists a $g \in G$ which connects them: $x = \alpha(y, g)$. A topological space which has a continuous free transitive group action is always homeomorphic to that group, so the fibres of P are in one-to-one correspondence with G and the manifold M is homeomorphic to the quotient space P/G. We say that the P-fibres are G-*torsors*: as spaces they are homeomorphic to G, but they lack group structure because there is no preferred choice of identity element. If we require P to be a smooth manifold and the G-action to be smooth, free and proper, such that the action sends compact sets to compact sets, then P/G is diffeomorphic to M and π is smooth, as well as the local trivialisations. For a principal G-bundle the structure group is G. In this sense a principal G-bundle can be viewed as a G-bundle for which all fibres are G itself, and local transition functions correspond to left multiplication by group elements.

The easiest way to construct a principal bundle is to take globally the product of M with some Lie group G: such bundles will be called *trivial principal bundles.* A less trivial example is the *linear frame bundle* $P_{GL}(TM)$, which assigns to each point p of M the space of linearly-independent bases of T_pM. Since invertible linear transformations send bases to bases, the fibers are isomorphic to $GL(d, \mathbb{R})$. The local trivialisation assigns to each basis $X = \{X_1, \ldots, X_d\}$ of T_pM the point (p, G) of $U \times GL(d, \mathbb{R})$, where G is the matrix transforming X to the coordinate basis. A similar story holds for the orthonormal frame bundle, replacing 'linear basis' by 'orthonormal basis' and $GL(d, \mathbb{R})$ by $O(r, s)$. Analogously, an arbitrary vector bundle $\pi : E \longrightarrow M$ of rank n gives rise to a principal $GL(n, \mathbb{R})$-bundle. If E is equipped with a bilinear symmetric form $g : \Gamma(E \vee E) \longrightarrow C^\infty(M)$, we analogously define the principal bundles $P_O(E)$ and $P_{SO}(E)$, if it is orientable. The latter are examples of reductions of G-structures. Given a principal G-bundle with closed subgroup $H \subset G$, we can construct a principal H-bundle provided the transition functions take their values in H under the restriction of the fibres to H.

The inverse process to the latter constructions is called an *associated bundle construction.* Given a principal G-bundle (P, M, π) and another smooth manifold Q which allows a continuous *left* action by G under some representation $\rho : G \longrightarrow \text{Homeo}(Q)$, we define an action

$$(P \times Q) \times G \xrightarrow{\alpha \times \rho} P \times Q : ((p, q), g) \mapsto (pg, \rho(g^{-1})q)\,. \tag{2.5.3}$$

We denote $P \times_\rho Q$ the space of orbits under this action equipped with the quotient topology. There is a canonical projection $\tilde{\pi} : P \times_\rho Q \longrightarrow M : \tilde{\pi}([p, q]) = \pi(p)$ which makes $(P \times_\rho Q, M, \tilde{\pi})$ into a fiber bundle over M. The local trivialisations are constructed as follows: suppose $U \subset M$ is an open trivialisation neighbourhood for P, the homeomorphism $\phi : \pi^{-1}(U) \longrightarrow U \times G$ a local trivialisation and define the *identity section* $s : U \longrightarrow \pi^{-1}(U) : s(p) = \phi^{-1}(p, e_G)$. Then we define the homeomorphism

$$\psi : U \times Q \longrightarrow \tilde{\pi}^{-1}(U) : \psi(p, q) = [s(p), q] = [\phi^{-1}(p, e_G), q]\,. \tag{2.5.4}$$

Then $\tilde{\pi}(\psi(p, q)) = \tilde{\pi}([s(p), q]) = \tilde{\pi} \circ \rho(s(p), q) = \pi \circ \sigma(s(p), q) = p$, so that $\psi(p, q) \in \tilde{\pi}^{-1}(p)$, which makes ψ^{-1} a local trivialisation over U. The fibers of the bundle are diffeomorphic to Q since $\psi|_{p \times Q} : \{p\} \times Q \longrightarrow \tilde{\pi}^{-1}(p)$ is a diffeomorphism, and the transition functions are the ones of P, because they act from the left and therefore commute with the G-action, so the resulting fiber bundle $(P \times_\rho Q, M, \tilde{\pi})$ is a G-bundle with the trivialisation cover of P. We call it an *associated bundle* to (P, M, π). We have seen that under certain conditions one can reduce the principal G-bundle to a principal H-bundle. In [27] it shown that the number of inequivalent reductions is equal to the number of global sections of the associated G/H-bundle.

Let us take the example of the linear frame bundle $P_{GL}(TM)$. We denote the standard representation by $\rho : GL(d, \mathbb{R}) \longrightarrow \text{Aut}(\mathbb{R}^d)$. A d-tuple of real numbers a_i together with the basis X defines a unique element v of TM by $v = a_i X^i$ at each point of M. If $(\{X\}, a)$ and $(\{Y\}, b)$ are in the same equivalence class in $P_{GL}(TM) \times_\rho \mathbb{R}^d$, there is (at each point of the manifold) a linear transformation A such that $X = A^{-1}Y$ and $a = Ab$. So these couples define the same vector at each point, represented in a different basis: $a_i X^i = A_i{}^j b_j (A^{-1})^i{}_k Y^k = b_k Y^k$. Hence, different equivalent classes represent different vector fields on M: $P_{GL}(TM) \times_\rho \mathbb{R}^d = TM$. This result can be straightforwardly generalised to arbitrary rank-n vector bundles and tensor powers of representations:

$$\bigotimes^k E = P_{GL}(E) \times_{\bigotimes^k \rho} \bigotimes^k \mathbb{R}^n = P_{SO}(TM) \times_{\bigotimes^k \sigma} \bigotimes^k \mathbb{R}^n\,,$$

where $\sigma : SO(\pi^{-1}E) \longrightarrow \text{End}(\mathbb{R}^n)$ is the standard representation. The isomorphism above also holds for (anti-) symmetrised tensor powers and the dual bundle. An important example of the associated bundle construction is the *adjoint* bundle, $P \times_{\text{Ad}} \mathfrak{g}$. This is a vector bundle with fibers isomorphic to the Lie algebra $\mathfrak{g}$, whose elements on intersecting trivialisation neighbourhoods are related by $v' = g^{-1} v g$.

2.5.2 Clifford and Spinor Bundles

Now we have developed a proper machinery to put the Clifford algebras onto a manifold; recall from section (2.2) we had a 2 to 1 homomorphism $\text{Ad}^0 : \text{Spin}_{r,s} \longrightarrow SO(r, s)$. The connection with the special orthonormal frame bundle is an extremely useful tool to put the spin algebras on a bundle:

Definition. A *spin structure* on an orientable pseudo-Riemannian manifold M (signature (r, s)) is a principal $\mathrm{Spin}_{r,s}$-bundle $P_{\mathrm{Spin}}(TM)$ with a 2-sheeted covering

$$\sigma : P_{\mathrm{Spin}}(TM) \longrightarrow P_{SO}(TM) \tag{2.5.5}$$

such that $\sigma(v \cdot g) = \sigma(v) \circ \mathrm{Ad}^0(g)$ for $v \in P_{\mathrm{Spin}}(TM)$ and $g \in \mathrm{Spin}_{r,s}$. If M admits a spin structure, we call it a *spin manifold*.

Requiring $P_{\mathrm{Spin}}(TM)$ to be principal, gives a restriction on the manifold M: only spin manifolds allow a lifting of the action of $SO(r, s)$ on $P_{SO}(TM)$ to an action of $\mathrm{Spin}_{r,s}$ on $P_{\mathrm{Spin}}(TM)$. The essence of the lifting problem is to obtain transition functions for the spin bundle by 'inverting' σ and thereby choosing *in a continuous fashion* for each transition matrix one of the two elements of $\mathrm{Spin}_{r,s}$ which map to this matrix under the adjoint representation. It is beyond our scope to examine the topological conditions that arise in this procedure, but the reader may find in [13] that a pseudo-Riemannian orientable manifold admits a spin structure iff the *second Stiefel-Whitney class* $w_2(P_{SO}(TM))$ of its orthonormal oriented frame bundle vanishes. If M is a spin manifold, the *Clifford bundle* is the associated adjoint to the spin bundle,

$$\mathrm{C}\ell(TM) = P_{\mathrm{Spin}}(TM) \times_{\mathrm{Ad}} \mathrm{C}\ell_{r,s} . \tag{2.5.6}$$

And analogously we define the complexified Clifford bundle $\mathbb{C}\ell(TM)$. However, Clifford bundles can also be consistently defined on manifolds which do not allow a spin bundle (cf. [13]). Note that (s, v) and (s', v') are in the same equivalence class in $\mathrm{C}\ell(TM)$ iff $\mathrm{Ad}_s v = \mathrm{Ad}_{s'} v'$, so the Clifford product rule is conserved; given a local orthonormal oriented frame field e_a and a coordinate basis ∂_μ, we have fiberwise

$$e_a \cdot e_b + e_b \cdot e_a = 2\eta_{ab} , \qquad \partial_\mu \cdot \partial_\nu + \partial_\nu \cdot \partial_\mu = 2g_{\mu\nu} . \tag{2.5.7}$$

The Clifford bundle is thus not just a fiber bundle, it is a sheaf of algebras, like the exterior algebra. The isomorphism (2.2.5) induces a bundle isomorphism $\lambda : \wedge TM \xrightarrow{\simeq} \mathrm{C}\ell(TM)$, which preserves the even and odd subalgebra bundles. We shall from now on omit this isomorphism, regarding both algebras as algebras over the same vector space, the direct sum of tensor powers of TM. All the properties of Clifford algebras derived in previous section can be generalised to Clifford bundles. If d is odd and the signature is such that the algebra is reducible, then the central idempotents $(1 \pm \omega)/2$ with ω the volume form decompose the Clifford bundle into subbundles of simple algebras. Furthermore, we can define a *spinor bundle* $S(TM)$, which is a bundle of irreducible representations of $\mathbb{C}\ell^0(TM)$ (or a chiral/real subalgebra thereof). Again this bundle arises from an associated bundle construction

$$S(TM) = P_{\mathrm{Spin}}(TM) \times_L \mathrm{C}\ell_{r,s} P , \tag{2.5.8}$$

where P is a primitive idempotent of $\mathrm{C}\ell_{r,s}$ and L is the left multiplication representation map. Then all properties of spinors stated earlier can be directly carried over to sections of this bundle. In particular, the bilinear pairings of spinors are *globally* defined since they are spin-invariant. Charge conjugation of spinors can be locally defined by

$$\psi^c = (Q_\alpha \psi^\alpha)^c = Q_\alpha B^\alpha{}_\beta (\psi^\beta)^* . \tag{2.5.9}$$

Then some simple algebra yields that for any $a \in \Gamma\mathrm{C}\ell(TM)$, $(a\psi)^c = a^* \psi^c$. Since $s^* = s$ for $s \in \mathrm{Spin}^+_{r,s}$, charge conjugation is consistently defined on overlaps. Combining previous results yields that also the local Dirac conjugation can be extended to the whole cover of M.

2.5.3 Connections and Covariant Derivatives

Suppose (P, π, M) is a principal G-bundle over M, with $w \in P$. We can differentiate the projection map at this point to obtain a linear projection in a vector space, $(\pi_*)_w : T_wP \longrightarrow T_{\pi(w)}M$. We define the *vertical subspace* of T_wP as $T_w^V P \equiv \ker((\pi_*)_w)$, which induces the direct decomposition $T_wP = T_w^V P \oplus T_{\pi(w)}M$.

Definition. A *connection* on P is a selection of *horizontal* subspaces $T_w^H P \subset T_wP$ for each $w \in P$ such that

1. $T_wP = T_w^V P \oplus T_w^H P$,
2. $T_{w\cdot g}^H = (R_g)_* T_w^H P$,
3. $T_w^H P$ depends smoothly on w.

Here $R_g : P \longrightarrow P$ is simply the right action of the structure group and 'depending smoothly' means being the spanned bundle of a d-tuple of smooth sections of TP (by the two decompositions we have $\dim(T_w^H P) = \dim(T_{\pi(w)}M) = \dim(M) = d$). Hence restricting π_* to the horizontal subspaces yields isomorphisms $T_w^H P \simeq T_{\pi(w)}M$. Very often G will be a Lie group, isomorphic to the fibers $\pi^{-1}(p)$, acting on itself by right translation in overlaps. In these situations a connection can be formulated in terms of *Lie algebra-valued forms*. It is shown in [28] that under these conditions there exists a unique $\omega \in \Gamma(T^*P) \otimes \mathfrak{g}$ such that at each $w \in P$ we have $T_w^H P = \ker(\omega_w : T_wP \longrightarrow \mathfrak{g})$. Actually, to establish a full equivalence we need to impose following conditions on ω: (i) under the derivative of right translation, the values of ω transform under the adjoint representation: $\omega_{wg}((R_g)_* v) = \mathrm{Ad}_{g^{-1}}(\omega_w(v))$ for all $w \in P, v \in T_wP$ and $g \in G$ and (ii) under a pull-back by the embedding $R_p : G \longrightarrow P : g \mapsto p \cdot g$ for any $p \in P$ yields the *Maurer-Cartan* form: $R_p^* \omega = \omega_G$. The latter is the canonical Lie algebra-valued on the Lie group,

$$\omega_G : \Gamma TG \longrightarrow \mathfrak{g} : (\omega_G)_g(v) = (L_{g^{-1}})_* v \,. \tag{2.5.10}$$

Given a basis $\{E_a\}$ of $\mathfrak{g}$, it induces local basis of left-invariant vector fields $\{X_a\}$ on the Lie group. If $\{\omega^a\}$ is its dual basis, the Maurer-Cartan form is given by $\omega_G = \omega^a \otimes E_a$. We lift the differential and Lie algebra commutator to the complex $\Omega^*(P, \mathfrak{g})$ as follows,

$$\begin{aligned} d : \Omega^k(P, \mathfrak{g}) \longrightarrow \Omega^{k+1}(P, \mathfrak{g}) &: d(\alpha \otimes A) = d\alpha \otimes A \,, \\ [\ ,\ ]_\wedge : \Omega^k(P, \mathfrak{g}) \times \Omega^l(P, \mathfrak{g}) \longrightarrow \Omega^{k+l}(P, \mathfrak{g}) &: [\alpha \otimes A, \beta \otimes B]_\wedge = \alpha \wedge \beta \otimes [A, B]_{\mathfrak{g}} \,, \end{aligned} \tag{2.5.11}$$

these constructions are essential in the study of covariant derivatives on the principal bundle and its associated bundles.

Suppose we have a principal G-bundle (P, M, π) with a connection 1-form ω and an associated vector bundle $E = P \times_\rho V$ where $\rho : G \longrightarrow \mathrm{End}(V)$ is a representation of the structure group. Then we define a covariant derivative as a map

$$\nabla^\omega : \Omega^0(M, E) \longrightarrow \Omega^1(M, E) : \sigma = [w, v] \mapsto \nabla^\omega \sigma(X)_p = [w(p), \rho_*(\omega_{w(p)}(w_*(X)))v_p + v_*(X)_p] \,,$$

for all $p \in M$ and $X \in T_pM$. By our previous notation, $\Omega^0(M, E) = \Gamma E$ and $\Omega^1(M, E) = \Gamma(E \otimes T^*M)$. Let us explain the definition above somewhat: ∇^ω assigns to the section σ of E an E-valued one-form, i.e. an object which eats the tangent vector field $X \in \Gamma TM$ and spits out a section of E. Clearly nothing happens with the first factor in $P \times_\rho V$. The last term in the second factor defines a section of E because V is a vector space: $v_* : \Gamma TM \longrightarrow \Gamma TV = \Gamma V$. Similarly w_* assigns to the vector field X a vector field in ΓTP, which is associated to a Lie algebra value by the connection 1-form. The derivative of the representation of G correlates such values with linear transformations on V: $\rho_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$, which in its turn acts on the vector v at each point p giving us another section of V. We shall from now on omit the p- and ω-dependence in our notation and write the vector field X as a subscript: $\nabla^\omega\sigma(X) = \nabla_X\sigma$, which yields for each vector field X a map $\nabla_X : \Gamma(E) \longrightarrow \Gamma(E)$. We can extend these maps to arbitrary tensor products of the vector bundle and its dual by defining

$$\begin{gathered}\text{for } f \in C^\infty(M,\mathbb{R}) : \nabla_X f = X(f)\,,\\ \text{for } \alpha \in \Gamma E^*\,,\ \nabla_X\alpha \in \Gamma E^* : \nabla_X\alpha(Y) = -\alpha(\nabla_X Y) + X(\alpha(Y)) \text{ for all } Y \in \Gamma E\,,\\ \nabla_X : \Gamma \textstyle\bigotimes^r_s E \longrightarrow \Gamma \bigotimes^r_s E : \nabla_X(V \otimes W) = \nabla_X V \otimes W + V \otimes \nabla_X W\,,\end{gathered} \tag{2.5.12}$$

where $\bigotimes^r_s E$ is a shorthand notation for $\otimes^r E \otimes \otimes^s E^*$. The minus sign in the first term of $\nabla_X\alpha$ is there such that the Leibniz rule $\nabla_X(\alpha(Y)) = \nabla_X\alpha(Y) + \nabla_X Y(\alpha)$ is fulfilled. The space of sections of a principal fiber bundle is a $C^\infty(M)$-module by setting $(f\sigma)(p) = [w(p), f(p)v(p)]$. With the definition above, one easily derives the Leibniz rule for covariant differentiation:

$$\nabla_X(f\sigma) = f\nabla_X\sigma + df(X) \otimes \sigma\,. \tag{2.5.13}$$

Furthermore, ∇_X is $C^\infty(M,\mathbb{R})$-linear in X:

$$\nabla_{fX+gY}\sigma = f\nabla_X\sigma + g\nabla_Y\sigma\,, \tag{2.5.14}$$

because w_* and v_* are fiberwise linear maps and $\omega_p(X)$ is $C^\infty(M,\mathbb{R})$-linear in X. The product rules above are often proposed as defining properties of a covariant derivative. A connection also induces a covariant de Rham differential d_∇ on $\Gamma(\bigwedge T^*M \otimes E)$ by

$$d_\nabla([w, \beta \otimes \sigma]) = [w, d\beta \otimes \sigma + \beta \wedge \rho_* w^*\omega(\sigma)]\,. \tag{2.5.15}$$

Again we shall usually omit the factor w. Note that this differential is grading-preserving, but it has a different cohomology sequence from the ordinary de Rham complex. As an example, consider the associated bundle construction $TM = P_{GL}(TM) \times_\rho \mathbb{R}^d$ with ρ the standard representation of $GL(d,\mathbb{R})$. Suppose we have a local coordinate patch (U, x^μ) and denote its coordinate basis by $w = \{\partial_\sigma\} \in \Gamma P_{GL}(TU)$. By the linearity rules above, the covariant derivative is completely specified by the values

$$\nabla_{\partial_\mu}\partial_\nu = [\{\partial_\sigma\}, \tilde\omega(\partial_\mu))e_\nu] = [\{\partial_\sigma\}, (\tilde\omega_\mu)_\nu] \sim (\lambda_\mu)_\nu{}^\sigma\partial_\sigma\,, \tag{2.5.16}$$

where $\lambda = \rho_* w^*\omega \in \Omega^1(TM, \mathrm{End}(\mathbb{R}^d))$. We call $\Gamma^\sigma_{\nu\mu} \equiv (\lambda_\mu)_\nu{}^\sigma$ the *Christoffel symbols* of the connection, they constitute a $\mathfrak{gl}(d)$-valued one-form. The *curvature* of a connection ∇ with connection 1-form $\omega \in \Omega^1(P_G(E), \mathfrak{g})$ on an associated G-bundle is the Lie algebra-valued 2-form

$$\theta \in \Omega^2(P_G(E), \mathfrak{g}) : \theta = d_\nabla\omega = d\omega + \tfrac{1}{2}[\omega,\omega]_\wedge\,. \tag{2.5.17}$$

Again taking the pull back to the tangent bundle and the Lie algebra representation gives us a two-form $R = \rho_*(w^*\theta) : TM \otimes TM \longrightarrow \mathrm{End}(\mathbb{R}^d)$ which can be shown to satisfy

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\,. \tag{2.5.18}$$

The *torsion 1-form* associated with a connection *on the tangent bundle* is a Lie algebra-valued 1-form given in a coordinate frame by

$$T^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu}\,. \tag{2.5.19}$$

It acts on 2 tangent vector fields and a one-form by $T^\sigma_{\mu\nu}X^\mu X^\nu\eta_\sigma = T(X,Y,\eta) = (\nabla_X Y - \nabla_Y X - [X,Y])(\eta)$. The main theorem of Riemannian geometry states that there is one unique connection with vanishing torsion on a Riemannian manifold: the *Levi-Civita* connection.

Taking the orthogonal fiber bundle instead of the general linear fiber bundle in the above allows us to differentiate Clifford algebra elements and spinors. Choose continuous inverse of Ad^0 to map the special orthonormal frame bundle to the spin bundle and then act on it with the adjoint representation to obtain a map $c\ell : SO(r,s) \longrightarrow \mathrm{End}(\mathrm{C}\ell_{r,s})$. Differentiating $c\ell$ and $L|_{\mathrm{Spin}}$ give a Lie algebra representations

$$\begin{aligned} c\ell_* : \mathfrak{so}(r,s) \longrightarrow \mathrm{Der}(\mathrm{C}\ell_{r,s}) : c\ell_*(e_a \wedge e_b) = \mathrm{ad}_{\frac{1}{4}[e_a,e_b]}\,, \\ L^*|_{\mathrm{Spin}} : \mathfrak{so}(r,s) \longrightarrow \mathrm{End}(S) : L^*(e_a \wedge e_b) = L_{\frac{1}{4}[e_a,e_b]}\,. \end{aligned} \tag{2.5.20}$$

Denoting the Lie algebra-valued one form by $\omega = \omega_\mu{}^{ab} e_a \wedge e_b \otimes \mathrm{d}x^\mu$ and applying the above representations yields the familiar expressions of a covariant derivative acting on sections of the Clifford and spinor bundle,

$$\nabla_X v = X^\mu\big(\partial_\mu v - \tfrac{1}{4}\omega_{\mu a}{}^b[\Gamma^a{}_b, v]\big)\,, \quad \nabla_X\theta = X^\mu\big(\partial_\mu\theta - \tfrac{1}{4}\omega_{\mu a}{}^b\Gamma^a{}_b\theta\big)\,. \tag{2.5.21}$$

The differentials (second terms in the upper line) are taken componentwise, so given a local spinor basis $\{Q_\alpha\}$ (cf. (**??**)) in a coordinate neighbourhood (U, x^μ), the differential of $\theta = \theta^\alpha(x)Q_\alpha$ becomes $X(\theta) = X^\mu Q_\alpha \partial_\mu \theta^\alpha$, where the derivative of the $\mathbb{K}$-valued functions θ^α is taken as the derivative of a couple ($\mathbb{K} = \mathbb{H}$) or 4-tuple ($\mathbb{K} = \mathbb{C}$) of real functions. The minus signs in (**??**) are convention; recall that in the fundamental definition this sign was a matter of choice too, as long as we take the opposite sign for the duals in order to fulfill the Leibniz rule. Hence we define $\nabla_X\bar\theta = X^\mu\big(\partial_\mu\bar\theta + \frac{1}{4}\omega_{\mu a}{}^b(X)\bar\theta\Gamma^a{}_b\big)$. The representations (2.5.20) are derivations w.r.t. Clifford algebra multiplication. The reader easily verifies

$$\begin{aligned} \nabla_X(A\cdot B) = (\nabla_X A)\cdot B + A\cdot(\nabla_X B)\,, \\ \nabla_X(A\theta) = (\nabla_X A)\theta + A(\nabla_X\theta)\,. \end{aligned} \tag{2.5.22}$$

The curvature 2-form R acts on the Clifford bundle by $c\ell_*(e^*R)$ and on the spinor bundle by $L_*(e^*R)$, or equivalently

$$\begin{aligned} R(X,Y)(v) = \big(d\omega_a{}^b(X,Y) - (\omega_a{}^c \wedge \omega_c{}^b)(X,Y)\big)[\Gamma^a{}_b, v]\,, \\ R(X,Y)(\theta) = \big(d\omega_a{}^b(X,Y) - (\omega_a{}^c \wedge \omega_c{}^b)(X,Y)\big)\Gamma^a{}_b\theta\,. \end{aligned} \tag{2.5.23}$$

Note that the covariant differential commutes with the parity operator α and hence preserves the even and odd subbundles: $\nabla\mathrm{C}\ell^{0,1}(TM) = \mathrm{C}\ell^{0,1}(TM)$. Furthermore it annihilates the Clifford volume element ω (*not to be confused with the spin connection 1-form*):

$$\nabla\omega = 0\,, \tag{2.5.24}$$

and therefore it preserves the chiral subbundles: $\nabla S^\pm(TM) = S^\pm(TM)$. Given a spin invariant inner product on $S(TM)$ one easily shows that $(\nabla_X\theta,\psi) + (\theta\nabla_X\phi) = X(\theta,\psi)$ and it commutes with the charge conjugation-involution:

$$(\nabla_X\theta)^c = \nabla_X\theta^c\,, \tag{2.5.25}$$

and thus it leaves Majorana spinors real. Finally, we introduce the *Dirac operator* associated to the connection,

$$\not{D} : \Gamma(S(TM)) \longrightarrow \Gamma(S(TM)) : \not{D}\theta = \eta^{ab} e_a \cdot \nabla_{e_b}\theta\,. \tag{2.5.26}$$

2.5.4 Supergeometry

We now return to the construction of supermanifolds of the first section. The spinor bundle $S(TM)$ induces a locally free sheaf of $\mathcal{C}^\infty_M$-modules, denoted as $\mathcal{S}$, which assigns to an open subset $U \subseteq M$ the space $\mathcal{S}(U) = \Gamma(U, S(TM))$. So the spinor bundle gives rise to a supermanifold $M^{d|n} = (M, \mathcal{A} = \bigwedge\mathcal{S})$ where $n = \dim S$. The motivation for taking the spinor bundle to comprise the odd coordinates comes from the extension theory of the Poincaré algebra: if we want the isometry group of the super-tangent bundle to be a superextension of the Poincaré algebra, the odd coordinates *must* constitute a spinor representation of the rotational group. The tangent space at p can be shown to be isomorphic to $T_pM_0 \oplus S_p^*$; we have already proven in the first section that $(T_pM^{d|n})_0 \simeq T_pM_0$, and the isomorphism $(T_pM^{d|n})_1 \simeq S_p^*$ is provided by the interior multiplication map ι_X. Note that this construction doesn't need a metric structure on the spinor space. This isomorphism is not only true for tangent spaces at a certain point; there is an isomorphism of sheaves of $\mathcal{C}^\infty_{M_0}$-modules

$$(TM^{d|n})_1 \simeq \mathcal{S}^* \,. \tag{2.5.27}$$

The idea to define a notion super-pseudo-Riemannian geometry is to extent the metric g diagonally by the Majorana pairing to the odd tangent subbundle (requiring the inner product of an even and an odd vector to be zero). For higher extensions this requires the notion of a bilinear product on the **R**-symmetry group (cf. sections (2.4) and (2.3)). However, the formulation of supergeometry in terms of a supermetric (see [29, 30]) is not convenient. Recall that all geometric quantities may also be formulated in terms of a preferred frame e, a section of the orthonormal frame bundle. This approach, which is developed in [31] and adopted in practically all supergravity theories, is called the *Cartan formalism*, as it rests upon the theory of Lie algebra-valued forms. In particular, the superextension of the Poincaré algebra becomes apparent in such a formulation.

The sheaf of frames is the super analog of the linear frame bundle and its structure group $GL_{d|n}$ is the analog of the structure group $GL_n(\mathbb{R})$. We denote it by

$$\mathcal{P}_{GL}(TM^{d|n})(U) = \{(E_M) \quad \text{is a basis of } TM^{d|n}(U)\} \tag{2.5.28}$$

for all open $U \in M_0$. We may consider subsheaves with structure groups subgroups of $GL_{d|n}$, and hence extend all the principal bundles to principal sheaves on supermanifolds. The orthonormal frame bundle $\mathcal{P}_{\mathrm{Spin}}(TM^{d|n})$ is the subbundle of $\mathcal{P}_{GL}(TM^{d|n})$ with structure group

$$\mathrm{Spin}(r,s|n) = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & (\mathrm{Ad}^0)^{-1}(A) \end{array} \right) \middle| \, A \in SO(r,s) \right\} , \tag{2.5.29}$$

consisting of $E_A = (E_r, E_a)$, $r = 1, \ldots d$, $a = 1, \ldots, n$ such that $g(E_r, E_s) = \eta_{rs}$ and $(E_a, E_b)_M = C_{ab}$, where C_{ab} is the (real-valued) charge conjugation matrix of $\mathbb{C}\ell_{r,s}$. The fibers of the orthonormal frame sheaf are isomorphic to this group; again the definition of smooth sections requires M_0 to be a spin manifold. In fact, $SO(r,s|n)$ is a Lie group which is generated by the algebra

$$\mathfrak{spin}(r,s|n) = \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & (\mathrm{ad}^0)^{-1}(A) \end{array} \right) \middle| \, A \in \mathfrak{so}(r,s) \right\} . \tag{2.5.30}$$

This Lie algebra is exactly the even subalgebra of the extended Poincaré algebra $\mathfrak{p}(r,s|n)$, divided by the base space:

$$\frac{\mathfrak{p}(r,s|n)}{V_0} = \mathfrak{p}_0 + \mathfrak{p}_1 \,,$$
$$\mathfrak{p}_0 = \mathfrak{spin}(r,s|n) \,, \quad \mathfrak{p}_1 = \left\{ \left(\begin{array}{cc} 0 & 0 \\ A & 0 \end{array} \right) \middle| \, (A)_{ar} = g(E_r, \epsilon \wedge \Gamma^{(1)} \wedge E_a), \epsilon \in S(M_0) \right\} . \tag{2.5.31}$$

Given a local coordinate neighbourhood (Z^M, U) the frame is expanded as $E_A = E_A{}^M \partial_M$ and there is a corresponding dual frame $E^A = \mathrm{d}Z^N E_N{}^A \in \Gamma T^* M^{d|n}(U)$ defined by

$$E_M{}^A E_A{}^N = \delta_M{}^N \,, \quad E_A{}^M E_M{}^B = \delta_A{}^B \,. \tag{2.5.32}$$

This implies that the corresponding dual frames are of the same parity:

$$E_r = E_r{}^\mu \partial_\mu + E_r{}^\alpha \partial_\alpha \,, \quad E_a = E_a{}^\mu \partial_\mu + E_a{}^\alpha \partial_\alpha \,, \qquad \begin{cases} \mathcal{P}(E_r{}^\mu) = \mathcal{P}(E_a{}^\alpha) = 0, \\ \mathcal{P}(E_r{}^\alpha) = \mathcal{P}(E_a{}^\mu) = 1 \end{cases}$$

$$E^r = \mathrm{d}x^\mu E_\mu{}^r + \mathrm{d}\theta^\alpha E_\alpha{}^r \,, \quad E^a = \mathrm{d}x^\mu E_\mu{}^a + \mathrm{d}\theta^\alpha E_\alpha{}^a \,, \qquad \begin{cases} \mathcal{P}(E_\mu{}^r) = \mathcal{P}(E_\alpha{}^a) = 0, \\ \mathcal{P}(E_\alpha{}^r) = \mathcal{P}(E_\mu{}^a) = 1 \,. \end{cases}$$

Using the above, one easily verifies that the symmetry of the wedge product is preserved by the tangent space rotation/normalization:

$$\mathrm{d}Z^M \wedge \mathrm{d}Z^N = (-1)^{1-mn}\mathrm{d}Z^N \wedge \mathrm{d}Z^M\,, \quad E_A \wedge E_B = (-1)^{1-ab}E_B \wedge E_A\,. \tag{2.5.33}$$

Analogously to the definitions above, a connection is selection of horizontal planes on the tangent bundle of a principal fibre bundle over a supermanifold which under right translation of the base point is acted on by $(R_g)_*$ and depends smoothly on this base point. Again it can be shown that such a selection can always be written as the kernel of a super Lie algebra-valued one-form on the cotangent space to the fibre bundle satisfying certain conditions. Given a representation ρ_* of the super algebra, this *superconnection* one-form Ω then gives rise to covariant derivatives on the associated fibre bundle induced by this representation. This operator, the *supercovariant derivative* obeys the graded Leibniz property,

$$\nabla_X(fv) = X(df) \otimes v + (-1)^{\mathcal{P}(X)\mathcal{P}(f)} f\nabla_X v\,, \tag{2.5.34}$$

and linearity in the subscript: $\nabla_{fX} = f\nabla_X$. We can extend the connection to a morphism $d_\nabla : \bigwedge^p T^*M^{d|n}(U) \otimes \mathcal{E} \longrightarrow \bigwedge^{p+1} T^*M^{d|n}(U) \otimes \mathcal{E}$ by requiring

$$d_\nabla(\alpha \otimes v) = d\alpha \otimes v + (-1)^p \alpha \otimes \nabla v\,. \tag{2.5.35}$$

The curvature 2-form $\theta \in \bigwedge^2 T^*M^{d|n}(U) \otimes \mathfrak{g}_V$ given by

$$R = d\Omega - \frac{1}{2}[\Omega,\Omega] = d\Omega - \Omega \wedge \Omega\,. \tag{2.5.36}$$

As usual it acts on tangent vectors $X, Y \in TM^{d|n}(U)$ by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. As an important application, let us take $\mathcal{E} = TM^{d|n}$, $\mathcal{P}_G(\mathcal{E}) = \mathcal{P}_{\mathrm{Spin}}(TM^{d|n})$. The spin connection 1-form is expanded in terms of basis vectors of the Lie algebra (2.5.30),

$$\Omega = \mathrm{d}Z^M (\Omega_M)_A{}^B \otimes M_B{}^A = \mathrm{d}Z^M \Big((\Omega_M)_r{}^s - \frac{1}{4}(\Omega_M)_a{}^b (\Gamma_s{}^r)_b{}^a \Big) \otimes M^s{}_r\,, \tag{2.5.37}$$

where $M_s{}^r$ is a basis of $\mathfrak{so}(r,s)$, $(\Gamma_r)_a{}^b$ are the generators of the Clifford matrix-algebra and $(\Omega_M)_r{}^s$ are superfunctions. Hence the super connection 1-form is antisymmetric in its bosonic Lorentz indices r, s and either symmetric or antisymmetric (depending on the dimension and the signature) in its fermionic indices a, b. The covariant derivative of a vector field $X = X^A E_A \in TM^{d|n}(U)$ or a 1-form field $W = E^A W_A \in T^*M^{d|n}(U)$ is

$$\begin{aligned} \nabla_M X &= (\partial_M X^A + (-1)^{mb} X^B (\Omega_M)_B{}^A) E_A\,, \\ \nabla_M W &= E^A (\partial_M W_A - (\Omega_M)_A{}^B W_B)\,. \end{aligned} \tag{2.5.38}$$

Here we have used the notation $\nabla_M = \nabla_{\partial_M}$. The curvature and torsion 2-forms are defined as for ordinary manifolds,

$$R_A{}^B = \mathrm{d}\Omega_A{}^B + \Omega_A{}^C \wedge \Omega_C{}^B\,, \tag{2.5.39}$$

$$T^A = \mathrm{d}E^A + E^C \wedge \Omega_C{}^A\,. \tag{2.5.40}$$

The definition of these quantities implies the *super Bianchi identities*,

$$\mathrm{d}T^A + T^B \wedge \Omega_B{}^A - E^B \wedge R_B{}^A = 0\,, \tag{2.5.41}$$

$$\mathrm{d}R_A{}^B + R_A{}^C \wedge R_C{}^B - \Omega_A{}^C \wedge R_C{}^B = 0\,. \tag{2.5.42}$$

These play a crucial rôle if one wants to identify the supermanifold formulation of a supergravity theory with its ordinary form.

2.6 Introduction to Field Theory

2.6.1 The Double Complex

We shall end this chapter with a short summary of the Lagrangian theory of classical fields, adopting the notations and following the constructions of [25]. A field theory is constructed on a pseudo-Riemannian *spacetime manifold* M which in the standard formalism is equipped with a metric of signature $(-++\ldots+)$. A *field* is a smooth section of some fibre bundle $E \longrightarrow M$. If E is locally the product of M with some other manifold, the theory is often referred to as a nonlinear *sigma model* (the scalar fields represent coordinates on the fibres); if it is the adjoint bundle to a principal G-bundle we are doing *gauge theory*. If the theory contains more fields, E is some fiber product $\times E_i$. Clearly, we may straightforwardly extend this concept of a field to 'superfields', replacing M by some supermanifold whose bosonic sector is pseudo-Riemannian.

The classical dynamics of the fields of a theory is determined by a set of equations. These equations may be imposed by hand (constraints) or originate from minimising the integral of a density on the set (sheaf) of sections of E, the *action*. Suppose our theory is constructed from a (product of) some fibre bundle $E \longrightarrow M$ and denote the infinite-dimensional space of fields with $\mathcal{F} = \Gamma E$[6]. The idea is to set up a differential geometry on this space which allows a consistent description of the minimisation of the action. For physical applications $\Gamma\mathcal{F}$ is however too big; we shall assume that the action only depends on the fields through their derivatives, not for example on the value of a field at some point in spacetime. Fortunately, this property allows a mathematically quite rigourous framework for classical field theory. Quantities depending on derivatives of fields are phrased in terms of sections of *jet bundles*. For some bundle $\pi : E \longrightarrow M$ we define the k-jet bundle $\pi^{(k)} : J^k(E) \longrightarrow E$ as the bundle whose fiber at $v \in E$ is the vector space of equivalence classes of sections ϕ of E under the equivalence relation $\phi_1 \sim \phi_2$ if all the derivatives of ϕ_1 and ϕ_2 to order k coincide at $x = \pi(v) \in M$. A local coordinate system (x^μ, ϕ^a) on $E|_U = U \times F$ induces local coordinates on the k-jet bundle, which we denote (x^μ, ϕ^a_M) where M is a (symmetrised) multi-index of absolute value smaller or equal to k: $M = \mu_1 \ldots \mu_\ell$, $0 \leq \ell \equiv |M| \leq k$, $\mu_i = 1, \ldots, d$. Using the composition of projections, $J^k(E) \longrightarrow M$ is a fibre bundle as well, and a smooth section $\phi \in \Gamma E$ is canonically lifted to this bundle with the equivalence classes the respective derivatives of the section; we call this *jet prolongation*, and in the local coordinates introduced before we denote $j^k\phi : U \longrightarrow J^k(E)/M : x \mapsto (\phi(x), \partial_M \phi(x))$. Using jet prolongation all the quantities that shall arise on the jet bundle are pulled back to quantities defined on the infinite-dimensional space of fields $\mathcal{F}$. In practice this is substituting the ϕ^a by field components $\phi^a(x)$ and the ϕ^a_α by the field derivatives $\partial_M \phi^a(x)$. Note that $J^k(E)$ is a subbundle of $J^{k+1}(E)$. Each jet bundle is a nice finite-dimensional vector bundle of which we can look at the space of p-forms. We define the space of *local forms*

$$\Omega^{p,|-q|}_{\mathrm{loc}}(\mathcal{F} \times M) = \bigcup_k \Omega^p(J^k(E)/M, \mathbb{R}) \otimes \pi^{(k)*}(\Omega^{d-q}(M, \mathbb{R}) \wedge \mathfrak{o}_M)\,, \tag{2.6.1}$$

where $\mathfrak{o}_M$ is the oriented line bundle on M and $d = \dim(M)$. The appearance of this factor and the negative $|-q|$-notation will turn out convenient for the description of field theory. Given the local fibre bundle coordinates above, we denote the basis of $\big(\bigwedge^p J^k(E)/M\big) \wedge \big(\bigwedge^{d-q} T^*M \otimes \mathfrak{o}_M\big)$ by

$$|\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_{d-q}}| \wedge \delta\phi^{a_1}_{M_1} \wedge \delta\phi^{a_2}_{M_2} \wedge \ldots \wedge \delta\phi^{a_p}_{M_p}\,, \quad M_i = (\mu_1\mu_2 \ldots \mu_\ell)\,, \quad 0 \leq \ell \leq k\,. \tag{2.6.2}$$

Here the brackets reflect that M_i are actually symmetrised spacetime indices, as differentials commute. If one defines a Lagrangian formalism on a graded manifold, this will obviously need to get modified. So a general local $p, |-q|$-form will look like a linear combination of the above

[6] This space can be replaced by the space of *local* sections, as the jet bundle formalism is completely compatible with spacetime coordinate transformations

basis vectors, where the coefficients depend on $\phi^a, \phi^a_\mu, \phi^a_{\mu_1\mu_2}, \dots \phi^a_{\mu_1\mu_2\dots\mu_k}$ and possibly spacetime coordinates. There are two differentials that give a double complex structure to this sequence of spaces:

$$\begin{tikzcd} \Omega^{p,|-q|}_{\mathrm{loc}}(\mathcal{F}\times M) \arrow[r, "\delta"] \arrow[d, "d"] & \Omega^{p+1,|-q|}_{\mathrm{loc}}(\mathcal{F}\times M) \\ \Omega^{p,|-q+1|}_{\mathrm{loc}}(\mathcal{F}\times M) & \end{tikzcd} \quad . \tag{2.6.3}$$

By definition these differentials satisfy

$$d^2 = 0\,, \qquad \delta^2 = 0\,, \qquad d\delta + \delta d = 0\,. \tag{2.6.4}$$

However, the differentials above do not act simply on the factors of the tensor product; there is a 'mixing' which comes from the chain rule $d\phi^a_M = \sum_\mu \phi^a_{(M\mu)}\mathrm{d}x^\mu$ ($(M\mu)$ represents the $(k+1)$-index with first k indices the components of M and last index μ). The differentials act on the basis co-vectors as follows

$$d(\mathrm{d}x^\mu) = \delta(\mathrm{d}x^\mu) = \delta(\delta\phi^a_M) = 0\,, \quad d(\delta\phi^a_M) = \mathrm{d}x^\mu \wedge \delta\phi^a_{(M\mu)}\,, \tag{2.6.5}$$

where on the right hand side of the last equation summation over μ is understood (we shall from this point adopt summation convention over small Greek and Latin indices, as well as capital Latin indices). The differential calculus on the double complex is now completely determined by the rules above, the usual graded Leibnitz rule for the differential of wedge products and the actions of d and δ on a function of jet bundle variables,

$$dF = \frac{\partial F}{\partial x^\mu}\mathrm{d}x^\mu + \frac{\partial F}{\partial \phi^a_M}\mathrm{d}x^\mu \wedge \delta\phi^a_{(M\mu)}\,, \quad \delta F = \frac{\partial F}{\partial \phi^a_M}\delta\phi^a_M\,, \quad 0 \le |M| \le k\,. \tag{2.6.6}$$

A tangent vector to the k-th jet bundle at the point $(x, \phi(x))$ in E splits up in a horizontal component in T_xM and a vertical component in $T_{\phi(x)}F^k(E)$, where $F^k(E)$ is the fibre to the jet bundle associated to E. A general local vector field looks like

$$\xi = \xi^\mu(x,\phi)\frac{\partial}{\partial x^\mu} + \xi^a_M(x,\phi)\frac{\delta}{\delta\phi^a_M}\,, \quad 0 \le |M| \le k\,. \tag{2.6.7}$$

where the components again depend on the field configuration in a local fashion. Such a vector field is said to be *projectable* if the ξ^μ do not depend on ϕ or its derivatives, and if each ξ^a_M depends only on the field derivatives up to order $|M|$. Given a (local) field configuration $\phi \in \mathcal{F}$, a tangent vector $\xi_\phi \in T_\phi\mathcal{F}$ is a section of the bundle $\phi^*T(E/M)$, assigning to each point x in spacetime a vector in $T_{\phi(x)}(E/M)$: $\xi_\phi = \xi^a_\phi(x)\delta/\delta\phi^a$, where summation over a is understood. A local vector field in $T\mathcal{F}$ is therefore decomposed as $\hat{\xi} = \hat{\xi}^a(\phi, x)\delta/\delta\phi^a$ and a vector field in $\Gamma(\mathcal{F}\times M)$ is written as $\xi = \xi^a(\phi,x)\delta/\delta\phi^a + \xi^\mu(\phi,x)\partial/\partial x^\mu$. Again, we call these vector fields local if their components depend on some k-jet of the field configuration ϕ. Such a vector field can be prolonged to a vector field on the manifold $J^k(E)$ by replacing all derivatives by the appropriate jet coordinates and calculating the higher jet components in (2.6.7) recursively by the formula [32]

$$\xi^a_{(\mu M)} = \Big(\partial_\mu + \phi^b_{(N\mu)}\frac{\partial}{\partial\phi^b_N}\Big)\xi^a_M - \phi^a_{(\nu M)}\partial_\mu\xi^\nu\,, \tag{2.6.8}$$

where ℓ is some finite number denoting the maximal jet through which ξ^a_M depends on ϕ. In the formula above applying a partial derivative to the x-coordinates does not involve the chain rule: it means taking the derivative to x of the part which depends explicitly on this variable. One sees that the first term results from a chain rule of a derivative w.r.t. x. In classical field theory an infinitesimal symmetry of the system is represented by a vector field on the configuration space. Finding its k-jet prolongation is just calculating the variations of the derivatives of the fields. The prolongation of a vector field allows us to define the interior product of a local vector field on the infinite-dimensional space $\mathcal{F} \times M$ with a local $(p, |-q|)$-form, for the jet bundle $J^k(E)$ is an ordinary finite-dimensional manifold which allows an unambiguously defined differential geometry. For ξ defined as in (2.6.7), we postulate

$$\begin{gathered}\iota_\xi : \Omega^{p,|-q|}_{\mathrm{loc}}(\mathcal{F} \times M) \longrightarrow \Omega^{p-1,|-q|}_{\mathrm{loc}}(\mathcal{F} \times M) \oplus \Omega^{p,|-q+1|}_{\mathrm{loc}}(\mathcal{F} \times M) : \\ \iota_\xi(\alpha \wedge \beta) = (\iota_\xi \alpha) \wedge \beta + (-1)^{pr+(d-q)(d-s)} \alpha \wedge \beta \,, \\ \iota_\xi(f\alpha) = f \iota_\xi \alpha \,, \quad \iota_\xi(\alpha + \beta) = \iota_\xi \alpha + \iota_\xi \beta \,.\end{gathered}$$

for $\alpha \in \Omega^{p,|-q|}_{\mathrm{loc}}(\mathcal{F} \times M)$ and $\beta \in \Omega^{r,|-s|}_{\mathrm{loc}}(\mathcal{F} \times M)$ and f a function on $\mathcal{F} \times M$. To complete the definition, the interior product of this vector with the basis jet forms is

$$\iota_\xi f(\phi, x) = 0 \,, \quad \iota_\xi \mathrm{d}x^\mu = \xi^\mu(\phi, x) \,, \quad \iota_\xi \delta\phi^a_M = \xi^a_M(\phi, x) \,. \tag{2.6.9}$$

When discussing symmetries, we shall frequently use the interior product of a vector field on $\mathcal{F} \times M$ with a local form. What we then actually mean is the interior product of the k-th prolongation of this vector field with such a form, as described above.

2.6.2 Lagrangian Formalism

The *Lagrangian (density)* of a field theory is a local spacetime density:

$$L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M) \,. \tag{2.6.10}$$

Assuming L depends on the k-jet of fields, its integral defines a map $\phi \mapsto \int_M L(j^k\phi) \in \overline{\mathbb{R}}$ called the *action*. The action is typically divergent, but this is no problem since we are only interested in its (functional) derivatives. Let us now explain how the classical equations of motion arise from a given Lagrangian density. A local $(1, |-q|)$-form N is called *linear over functions* if for all $\hat{\xi} \in T_\phi\mathcal{F}$, $f \in C^\infty(M, \mathbb{R})$ and $x \in M$ one has $\iota_{j^k(f\hat{\xi})}\beta(\phi, x) = f(x)\iota_{j^k\hat{\xi}}\beta(\phi, x)$. Such a form looks like

$$\beta = \alpha_a \wedge \delta\phi^a \,, \tag{2.6.11}$$

where $\alpha_a \in \Omega^{0,|-q|}_{\mathrm{loc}}(\mathcal{F} \times M)$ is a spacetime $(d-q)$-form density depending on the k-jet of fields. We denote the space of such forms by $\Omega^{1,|-q|}_{\mathrm{lin}}(\mathcal{F} \times M)$. The main result is the following:

There exists a unique form $\eta_L \in \Omega^{1,|0|}_{\mathrm{lin}}(\mathcal{F} \times M)$ such that $\eta_L - \delta L \in \mathrm{Im}(d : \Omega^{1,|-1|}_{\mathrm{lin}}(\mathcal{F} \times M) \longrightarrow \Omega^{1,|0|}_{\mathrm{loc}}(\mathcal{F} \times M))$. A field configuration $\phi \in \mathcal{F}$ is called a classical solution if $(j^k\phi)^*\eta_L = 0$.

The physical reason behind these requirements is simply to ensure equivalence to the original method to arrive at the Euler-Lagrange equations, which involves partial integrations of $\int_M \delta L$. The existence and uniqueness of η_L is guaranteed by the decomposition [33]

$$\Omega^{1,|0|}_{\mathrm{loc}}(\mathcal{F} \times M) = \Omega^{1,|0|}_{\mathrm{lin}}(\mathcal{F} \times M) \oplus d\Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M) \,. \tag{2.6.12}$$

Denoting $\eta_L - \delta L = d\gamma$, the formula above only determines γ up to closed contributions. The exact contributions are fixed by the requirement

$$\gamma \in \Omega^{1,|-1|}_{\mathrm{lin}}(\mathcal{F} \times M) \,. \tag{2.6.13}$$

One easily shows that there exist no exact forms which are linear over functions. Now γ is unique up to cohomology cycles which are linear over functions. By the following theorem, these do not exist (see [25]):

Theorem 2.4 *For $p > 0$ the complex $(\Omega^{p,|*|}_{\mathrm{loc}}(\mathcal{F} \times M, d)$ is exact except in the top degree $|*| = 0$.*

For k-th order Lagrangian $L = \ell|\mathrm{d}^d x|$ the decomposition looks like

$$\delta L = \frac{\partial \ell}{\partial \phi^a_M}\delta\phi^a_M \wedge |d^d x| = (-1)^{|M|}\partial_M\Big(\frac{\partial \ell}{\partial \phi^a_M}\Big)\delta\phi^a \wedge |d^d x| - d\gamma\,, \quad 0 \leq |M| \leq k\,, \tag{2.6.14}$$

where the *Poincaré* one-form is given by

$$\gamma = (-1)^{|M|}\partial_M\Big(\frac{\partial \ell}{\partial \phi^a_{(M\mu)}}\Big)\delta\phi^a \wedge \iota_{\partial_\mu}|\mathrm{d}^d x|\,, \quad 0 \leq |M| \leq k-1\,, \tag{2.6.15}$$

with $(M\mu)$ the symmetrised index $(\mu_1 \dots \mu_i \mu)$, $i = 1, \dots, k-1$. The Euler-Lagrange equations are directly obtained by setting the summation on the right hand side of (2.6.14) equal to zero. Writing out the multi-indices, the equation $(j^k\phi)^*\eta_L = 0$ takes its familiar form

$$\frac{\partial \ell}{\partial \phi^a} + \sum_{j=1}^{k}(-1)^j\partial_{\mu_1}\dots\partial_{\mu_j}\big(\frac{\partial \ell}{\partial(\partial_{\mu_1}\dots\partial_{\mu_j}\phi^a)}\big) = 0\,. \tag{2.6.16}$$

The submanifold of field configurations whose jet prolongation satisfies the Euler-Lagrange equations is from this point denoted $\mathcal{M}$, and its elements are said to be *on shell.* The *local symplectic 2-form* in this situation is given by

$$\omega = \delta\gamma \quad \in \Omega^{2,|-1|}(\mathcal{F}\times M)\,. \tag{2.6.17}$$

Off-shell, this form is closed w.r.t. to the δ differential, on the shell ω is closed w.r.t. both δ and d. For the case where L depends on the first jet of sections the bundle E (i.e. the fields ϕ^a and their first-order derivatives $\partial_\mu\phi^a$), one finds

$$\omega = \frac{1}{2}\Big(\frac{\partial}{\partial\phi^a}\frac{\partial \ell}{\partial \phi^b_\mu} - \frac{\partial}{\partial\phi^b}\frac{\partial \ell}{\partial \phi^a_\mu}\Big)\delta\phi^a\wedge\delta\phi^b\wedge\iota_{\partial_\mu}|\mathrm{d}^d x|\,. \tag{2.6.18}$$

Given a hypersurface $\Sigma \subseteq M$, γ and ω respectively integrate to the *global* canonical 1-form Γ and symplectic 2-form Ω,

$$\Gamma_\Sigma = \int_\Sigma \gamma \in \Omega^{1,|-d|}_{\mathrm{loc}}(\mathcal{F}\times M)\,, \quad \Omega = \int_\Sigma \omega = \delta\Gamma_\Sigma \in \Omega^2_{\mathrm{loc}}(\mathcal{M},\mathbb{R})\,. \tag{2.6.19}$$

As it turns out, $\delta\Gamma_\Sigma$ evaluated at $\mathcal{M}$ on vectors *tangent to* $\mathcal{M}$ (*Jacobi fields*) is independent of the choice of H. These are the (Lorentz) covariant analogs of the global Poincaré-form and Poisson bracket. If we choose the hypersurface Σ to be locally perpendicular to the local coordinate ∂_0, then all the terms in the integrals above which are proportional to $\iota_{\partial_\mu}|\mathrm{d}^d x|$ with $\mu \neq 0$ give zero contribution; if one then substitutes $\pi_a = \delta L/\delta\phi^a_0$, the integrals above reduce to spacelike integrals over

$$\pi^*_\Sigma\gamma = \pi_a\delta\phi^a\wedge|\mathrm{d}^{d-1}x|\,, \quad \pi^*_\Sigma\omega = \Big(\frac{\partial\pi_b}{\partial\phi^a} - \frac{\partial\pi_a}{\partial\phi^b}\Big)\delta\phi^a\wedge\delta\phi^b\wedge|\mathrm{d}^{d-1}x|\,, \tag{2.6.20}$$

where π_Σ is the projection of a $(d-1)$-form density onto Σ and $|\mathrm{d}^{d-1}x|$ is the volume element on H. These forms will play an important rôle later, when we introduce phase space coordinates. Obviously the global Hamiltonian 2-form $\Omega \in \Omega^2_{\mathrm{loc}}(\mathcal{M})$ is closed and if it is nondegenerate it defines a symplectic structure on the space of classical solutions. However, many of the Lagrangians we consider possess *gauge symmetries*, which are a source of degeneracy of Ω, as we shall explain below.

2.6.3 Symmetries and Noether Currents

An automorphism of the space of fields $g : \mathcal{F} \longrightarrow \mathcal{F}$ is said to be *local* if all values $g(\phi)$ and $g^{-1}(\phi)$ depend on a k-jet of ϕ at $p \in M$. A *generalised symmetry* of a Lagrangian L is such a local automorphism g, together with form $\alpha \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F}\times M)$ such that

$$L(g(\phi)) - L(\phi) = d\alpha(\phi)\,, \tag{2.6.21}$$

for all fields. Adding a term $d\alpha$ to L leaves the the on-shell manifold $\mathcal{M}$ and the local and global symplectic 2-forms invariant. These automorphisms *may* be generated by infinitesimal counterparts, given by a vector field on $\mathcal{F} \times M$. A vector field $\xi \in \Gamma T(\mathcal{F} \times M)$ is said to be *decomposable* if it can be written as

$$\xi = \hat{\xi} + X\,, \quad \hat{\xi} = \hat{\xi}^a(\phi)\frac{\delta}{\delta\phi^a} \in \Gamma_{\mathrm{loc}} T\mathcal{F}\,, \quad X = X^\mu(x)\frac{\partial}{\partial x^\mu} \in \Gamma TM\,. \tag{2.6.22}$$

One easily verifies that the jet prolongation of such a vector field gives us a projectible vector field on the jet bundle. A *generalised infinitesimal symmetry* is a local decomposable vector field $\xi \in \Gamma T(\mathcal{F} \times M)$ together with a form $\alpha_\xi \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$ such that

$$\mathfrak{L}_\xi L = d\alpha_\xi\,. \tag{2.6.23}$$

Here $\mathfrak{L}_\xi$ is the Lie derivative which is $\mathfrak{L}_{\hat{\xi}} + \mathfrak{L}_X$ if the vector field is decomposed as $\xi = \hat{\xi} + X$,

$$\mathfrak{L}_\xi L = \iota_\xi(\delta + d)L + (\delta + d)\iota_\xi L = \iota_{\hat{\xi}}\delta L + d\iota_X L\,, \tag{2.6.24}$$

since $dL = 0$, $\iota_{\hat{\xi}}$ acts trivially on $\Omega^{|d|}(M)$-valued scalars on $\mathcal{F}$ and $\iota_X\delta = -\delta\iota_X$. If L depends on higher-order derivatives, δL will have nonzero components in $\delta\phi^a_\mu$. To take the interior product of ξ with such a form one should perform the first jet prolongation, as explained in the first subsection. If the form α_ξ is zero (or closed, for that matter), we call the symmetry *manifest*. From the equations above, one easily sees that if $\xi = \hat{\xi} + X$ is a symmetry, then $\hat{\xi}$ is a generalised infinitesimal symmetry by itself: the Lie derivative w.r.t. this field is the exact form $\alpha_{\hat{\xi}} = \alpha_\xi - \iota_X L$. Vice versa, if we are given a nonmanifest infinitesimal symmetry $\hat{\xi}$ acting only on the space of fields, we may seek an extension to $T(\mathcal{F} \times M)$ to turn it manifest; adding a vector field X on M which has the property that $\iota_X L + \alpha_{\hat{\xi}}$ is closed establishes this. Infinitesimal symmetries acting only on the space of field configurations in general do not integrate to generalised symmetries. The problem is that the resulting automorphism may not be a local one. For example, the vector field $\partial_0\phi\delta/\delta\phi$, corresponding to infinitesimal time translations, integrates to the automorphism $g(\phi(t,x)) = \phi(t + t_0, x)$, which definitely not local. In fact, it are only the automorphisms in a neighbourhood of $(g, \alpha) = (\mathrm{Id}_\mathcal{F}, 0)$ that are generated (when discussing invariances is physics, one uses rather the language of automorphisms depending on some infinitesimal parameter ϵ). The *Noether charge* corresponding to the infinitesimal symmetry ξ is the unique element $Q_\xi \in \Omega^0_{\mathrm{loc}}(\mathcal{M})$ with the property

$$\delta Q_\xi = -\iota_{\hat{\xi}}\Omega\,. \tag{2.6.25}$$

This equation makes sense because from its definition it is clear that an infinitesimal symmetry $\hat{\xi} \in \Gamma T\mathcal{F}$ is a vector field tangent to $\mathcal{M}$ (it generates an automorphism on the space of extremals). In particular, after selecting a spacelike hypersurface Σ it is given by

$$Q_\xi = \Gamma_\Sigma(\hat{\xi}) - \int_\Sigma \alpha_\xi = \int_\Sigma (\iota_{\hat{\xi}}\gamma - \alpha_\xi) \equiv \int_\Sigma j_\xi\,. \tag{2.6.26}$$

Here $j_\xi \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{M} \times M)$ is the *Noether current* associated with the infinitesimal symmetry (ξ, α_ξ). From the equation above it is not immediately clear that the right hand side is on-shell independent of the choice of Σ. From $\mathfrak{L}_{\hat{\xi}}\eta_L = 0$ one easily sees that $d(\mathfrak{L}_{\hat{\xi}}\gamma - \delta\alpha_{\hat{\xi}}) = 0$ on the shell. Theorem 2.4 implies that the argument of the exterior derivative is exact. Hence there exists a $\beta_\xi \in \Omega^{1,|-2|}_{\mathrm{loc}}(\mathcal{F} \times M)$ such that

$$\mathfrak{L}_\xi\gamma = \delta\alpha_\xi + d\beta_\xi \qquad \text{on } \mathcal{M} \times M. \tag{2.6.27}$$

Consequently the Noether current $j_\xi \equiv \iota_\xi\gamma - \alpha_\xi$ satisfies

$$dj_\xi = 0\,, \quad \delta j_\xi = -\iota_\xi\omega + d\beta_\xi \qquad \text{on } \mathcal{M} \times M. \tag{2.6.28}$$

The second equation above is the local counterpart of (2.6.25). The first equation implies that the definition of Q_ξ does not depend on the choice of the hypersurface: it is a *conserved* charge when evaluated at extremal fields. An important example of this phenomenon is the conservation of *energy* and *momentum*. Theories containing these constants of motion should be invariant under infinitesimal translations, linear combinations (with constant coefficients) of the vector fields

$$\xi_\mu = \frac{\partial}{\partial x^\mu} + \phi^a_\mu \frac{\delta}{\delta\phi^a}\,. \tag{2.6.29}$$

If $L = \ell|d^d x|$ is a first-order Lagrangian, one easily calculates the Noether currents $J_\mu = \iota(\hat{\xi}_\mu)\gamma + \iota(\partial_\mu)L$,

$$J_\mu = \Big(- \frac{\partial\ell}{\partial\phi^a_\nu}\phi^a_\mu + \ell\delta^\nu_\mu\Big) \wedge \iota_{\partial_\nu}|\mathrm{d}^d x|\,. \tag{2.6.30}$$

Again, if we choose Σ everywhere locally perpendicular to ∂_0, the corresponding Noether charges will all be zero except $\int_\Sigma J_0$. This quantity is called (minus) the *Hamiltonian*,

$$H = \int_\Sigma |\mathrm{d}^{d-1}x|(\pi_a\partial_0\phi^a - \ell)\,. \tag{2.6.31}$$

A description of the field theory with an explicit selection of the hypersurface, a straightforward generalisation of the choice of Σ above, will be the central idea of the *Hamiltonian formulation* of the system.

2.6.4 Hamiltonian Formalism

So far the theory has been *Lorentz covariant*: we haven't selected a preferred direction in M when discussing for instance integration over hypersurfaces. The *standard Hamiltonian formalism* breaks this covariance, it relies on a choice of hypersurfaces transverse to a coordinate direction in M assigned as *time*, parameterising the evolution of the system. Where in the Lagrangian formalism fields were sections of a bundle over spacetime and the dynamics was determined by the Euler-Lagrange equations, in the Hamiltonian formalism one works with arbitrary field configurations on a prescribed hypersurface (these configurations may be subject to constraints however) and the dynamics is formulated in terms of the evolution of these configurations w.r.t. the evolution parameter.

A *slicing* of the manifold M^d is a $(d-1)$-dimensional manifold Σ (the *spacesheet*) together with a diffeomorphism $s_M : \mathbb{R}\times\Sigma \longrightarrow M$. For $\tau\in\mathbb{R}$ we shall denote the corresponding embedded hypersurface with $\Sigma_\tau = s_M(\{\tau\}\times\Sigma)$. If M is Lorentzian there exists a class of preferred slicings of M for which all the Σ_τ are spacelike; these surfaces are called *Cauchy surfaces*, because they intersect every timelike curve once (this follows from the requirement that s_M is a diffeomorphism). The generator of the slicing is the vector field $X_s \in \Gamma TM$ satisfying

$$X_s(s_M(\tau,x)) = \frac{\partial}{\partial\tau}s_M(\tau,x)\,, \tag{2.6.32}$$

for all $\tau\in\mathbb{R}$ and $x\in\Sigma$. Given a bundle $E\longrightarrow M$, a *compatible slicing* of E is a bundle $\pi_\Sigma : E_\Sigma \longrightarrow \Sigma$ together with a bundle diffeomorphism $s_E : \mathbb{R}\times E_\Sigma \longrightarrow E$ such that the following diagram commutes:

$$\begin{tikzcd} E_\Sigma\times\mathbb{R} \arrow[r, "s_E"] \arrow[d, "\pi_\Sigma\times\mathrm{Id}"'] & E \arrow[d, "\pi"] \\ \Sigma\times\mathbb{R} \arrow[r, "s_M"] & M \end{tikzcd} \tag{2.6.33}$$

Physically this means that the slicing of the arguments of the fields should correspond to the slicing of spacetime. Again we denote $E_\tau = s_E(\{\tau\}\times E_\Sigma)$ and we associate an infinitesimal slicing

$\zeta_s \in TE$ to the bundle slicing analogously to (2.6.32). Then ζ_s generates a slicing compatible with s_M iff it projects to X_s. We note that there is always at least one slicing of E compatible to s_M.

We proceed with the construction of phase space for first-order Lagrangians. We define the sliced jet bundle $(J^1E)_\tau$ to be the restriction of the jet bundle to Σ_τ, while $J^1(E_\tau)$ is the jet bundle of the restricted bundle, whose fibres have no directions corresponding to derivatives in the X_s-direction. An infinitesimal slicing ζ of E naturally induces a slicing of the jet bundles by jet prolongation of ζ. For a field $\phi \in \Gamma E$ set $\varphi \equiv \phi|_{\Sigma_\tau} \in \Gamma E_\tau$ and $\dot{\varphi} \equiv \mathfrak{L}_\zeta \phi|_{\Sigma_\tau} \in T^V E_\tau$. Then the map $\tilde{\psi}_\zeta : \mathcal{F} \longrightarrow \Gamma(E_\tau \times T^V E_\tau) : \phi \mapsto (\varphi, \dot{\varphi})$ may be jet prolonged to a map $\psi_\zeta : \Gamma J^1(E)_\tau \longrightarrow \Gamma(J^1(E_\tau) \times T^V E_\tau) : j^1\phi(x) \mapsto (j^1\varphi(x), \dot{\varphi}(x))$, $x \in \Sigma_\tau$ which is he result of a bundle morphism, mapping the coordinates $(x^i, \phi^a, \phi^a_\mu)$ on $J^1(E)_\tau$ to $(x^i, \phi^a, \phi^a_i, \dot{\phi}^a)$ on $J^1(E_\tau) \times T^V E_\tau$, where $i = 1, \ldots, d-1$ and $\dot{\phi}^a$ is the coordinate of the derivatives along ζ. It is easy to realise that if $X_s = (\pi_E)_* \zeta$ is transverse to Σ_τ, the bundle morphism ψ_ζ is an isomorphism, and we shall call its inverse the *jet reconstructions map*. Going back to the level of sections, notice that $\dot{\varphi}$ is a vertical vector field on Σ_τ covering φ (i.e. $\dot{\varphi}(x) \in T_{\varphi(x)} E$) and that $j^1\varphi$ uniquely defines a section in $\Gamma(\Sigma_\tau, E_\tau)$. Hence if X_s is transverse to Σ_τ, ψ_ζ induces an isomorphism

$$\psi_\zeta : \Gamma_{\mathrm{hol}} J^1(E)_\tau \xrightarrow{\simeq} T\mathcal{F}_\tau \,, \tag{2.6.34}$$

where $\Gamma_{\mathrm{hol}} J^1(E)_\tau$ is the space of *holonomic sections*, i.e. sections of the first jet bundle which can be written as the jet prolongation of a field, restricted to the hypersurface Σ_τ. The instantaneous configuration space $\mathcal{F}_\tau$ is defined as $\Gamma(\Sigma_\tau, E_\tau)$.

A slicing is called *Lagrangian* if L is equivariant w.r.t. the one-parameter group of automorphisms induced by s_M; for Ψ_X the flow associated to X and $j^1\Phi_\zeta$ the flow associated to the jet prolongation of ζ, this means that $L \circ j^1\Phi_{\zeta,\tau} = (\Phi^{-1}_{X,\tau})^* \circ L$. In practice, for a given L there will be many slicings of E which satisfy this property. If the field content consists of tensors on the (co-) tangent bundle and L consists of contractions of these fields, a slicing of M naturally induces a slicing on E which is Lagrangian as long as the metric is a field variable (and therefore also sliced). Suppose now we have a spacetime slicing compatible with an infinitesimal configuration bundle slicing ζ which is Lagrangian w.r.t. L. Using jet reconstruction we can pull back the Lagrangian to section of the bundle restricted to Σ_τ; define the *instantaneous Lagrangian* $L_{\tau,\zeta} : T\mathcal{F}_\tau \longrightarrow \mathrm{Dens}(\Sigma_\tau) : L_{\tau,\zeta} \circ \psi_\zeta = L$. Then for a $(j^1\varphi, \dot{\varphi})$ which is reconstructed by $j^1\phi \circ i_\tau$ (here $\imath_\tau : \Sigma_\tau \hookrightarrow M$ is the canonical inclusion) we find using the Lagrangian property of the slicing that $L_{\tau,\zeta}(j^1\varphi, \dot{\varphi}) = i^*_\tau \iota_X L(j^1\phi)$. The *instantaneous action* $S_\zeta = \int_{\Sigma_\tau} L_{\tau,\zeta}$ defines a map $T\mathcal{F}_\tau \longrightarrow \mathbb{R}$, which induces a *Legendre transformation*. For a mapping L from some (real) vector space V to the real numbers the associated Legendre transform is a map $F_L : V \longrightarrow V^*$ such that for all $v, v' \in V$

$$\langle F_L(v), v' \rangle = L(v) + \frac{\partial}{\partial \epsilon} L(v + \epsilon(v - v'))\Big|_{\epsilon=0} \,, \tag{2.6.35}$$

where $\langle\, . \,, \,. \,\rangle$ is the duality pairing. This can be applied to the tangent space to the configuration space on Σ_τ, yielding a map to *phase space*, $F_{\tau,\zeta} : T\mathcal{F}_\tau \longrightarrow T^*\mathcal{F}_\tau$. Duality pairing on these bundles is canonically given by integration over Σ_τ. Hence an element of $T^*_\varphi \mathcal{F}_\tau$ is a section of $L(T^V E_\tau, \mathrm{Dens}(\Sigma_\tau))$, assigning to $x \in \Sigma_\tau$ a linear map $V_{\varphi(x)} E_\tau \longrightarrow \mathrm{Dens}_x(\Sigma_\tau)$. The Legendre transform maps the fibre coordinate $\dot{\varphi}$ to

$$\pi = \frac{\partial \ell_{\tau,\zeta}}{\partial \dot{\varphi}^a} \delta\varphi^a \otimes |\mathrm{d}^{d-1}x| \equiv \pi_a \wedge \delta\varphi^a \otimes |\mathrm{d}^{d-1}x| \,, \tag{2.6.36}$$

where the π_a are the *canonical momenta*: these form the new coordinates on the instantaneous phase space, replacing the derivatives in the direction of the slicing. Under the Legendre transform the (instantaneous) *Hamiltonian density* J_0 is mapped to

$$H_{\tau,\zeta} = \pi_a \dot{\varphi}^a(j^1\varphi, \pi) \otimes |\mathrm{d}^{d-1}x| - L_{\tau,\zeta}(j^1\varphi, \dot{\varphi}(j^1\varphi, \pi)) \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{P}_\tau \times \Sigma_\tau) \,. \tag{2.6.37}$$

The local symplectic 2-form ω is mapped to the phase space two-form

$$\omega_\tau = \delta\varphi^a \wedge \delta\pi_a \otimes |\mathrm{d}^{d-1}x| \in \Omega^{2,|0|}(\mathcal{F}_\tau \times \Sigma_\tau). \tag{2.6.38}$$

Note that the Hamiltonian is defined in terms of the inverse of the Legendre transform, in the expression (2.6.37) denoted by $\dot\varphi(j^1\phi, \pi)$, which does not need to exist on all of phase space. Furthermore the two-form above will in many cases not be symplectic as gauge symmetries of the Lagrangian are a source of degeneracy. The integrals over Σ_τ of these quantities will be called the total Hamiltonian $E_{\zeta,\tau} \in \Omega^{0,|-d+1|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$ and global presymplectic two-form $\Omega_\tau \in \Omega^{2,|-d+1|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$.

2.6.5 Constraints

In general the Legendre transform is not surjective. We shall call its image the *primary constraint set*, $\mathcal{P}_\tau = F_\zeta(T\mathcal{F}_\tau) \subseteq T^*\mathcal{F}_\tau$. It is proven in [34] that this set does not depend on the choice of compatible Lagrangian slicing ζ. To every phase space density $\rho \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$ the local presymplectic 2-form associates a class of vertical Hamiltonian vector fields ξ_ρ satisfying $\iota_{\xi_\rho}\omega_\tau = \delta\rho$, where all the equivalence classes are isomorphic to $\ker\omega_\tau$. Similarly, to every scalar function $Q \in \Omega^{0,|1-d|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$ on phase space the global presymplectic 2-form associates a class of Hamiltonian vector fields through $\iota_{\xi_Q}\delta\Omega_\tau = \delta Q$.

Hamiltonian dynamics lets the parameter τ above vary. We view the parameter τ as a map $\tau : \mathbb{R} \longrightarrow \mathrm{Emb}(\Sigma, M) : \tau(\lambda)(x) = s_M(x, \lambda)$ and hence $\tau(\lambda)(\Sigma) = \Sigma_\lambda$. The space in which the dynamics takes place is the one-parameter space of constraint sets $\mathcal{P}^\tau$, which is the union of all the $\mathcal{P}^{\tau(\lambda)}$ and may be viewed as a fibre bundle over $\mathbb{R}$, with a coordinate chart $(\lambda, \varphi^a, \pi_a)$. A section of this bundle defines a phase space *trajectory* $c(\lambda) = (\varphi(\lambda), \pi(\lambda))$. Viewed as a one-dimensional submanifold of $\mathcal{P}^\tau$, the tangent curve $\partial c/\partial\lambda$, which is a vector field on $\mathcal{P}^\tau$ along c, has a unique decomposition $\dot c = \partial/\partial\lambda + \xi$, where ξ is vertical on $\mathcal{P}^\tau$ and hence $\xi_\lambda \equiv \xi|_{\mathcal{P}_{\tau(\lambda)}}$ is tangent to $\mathcal{P}_{\tau(\lambda)}$,

$$\xi_\lambda = \frac{d\varphi^a}{d\lambda}\frac{\delta}{\delta\varphi^a} + \frac{d\pi_a}{d\lambda}\frac{\delta}{\delta\pi_a}\,. \tag{2.6.39}$$

The trajectory is a classical solution if it satisfies Hamilton's equations,

$$\iota_{\xi_\lambda}\omega_\lambda = \delta H_{\lambda,\zeta}\,, \tag{2.6.40}$$

where we have denoted $\omega_\lambda = \omega_{\tau(\lambda)}$ and $H_{\lambda,\zeta} = H_{\tau(\lambda),\zeta}$. Substituting the vector field ξ_λ and writing $H_{\lambda,\zeta} = h_{\lambda,\zeta} \otimes |d^{d-1}x|$ yields the familiar Hamilton equations,

$$\frac{d\varphi^a}{d\lambda} = \frac{\partial h_{\lambda,\zeta}}{\partial\varphi_a}\,, \qquad \frac{d\pi_a}{d\lambda} = -\frac{\partial h_{\lambda,\zeta}}{\partial\varphi^a}\,. \tag{2.6.41}$$

These are the Hamiltonian counterparts of the Euler-Lagrange equations: for $\phi \in \mathcal{M}$, the curve $c_\phi(\lambda) = F_\zeta(\psi_\zeta(j^1\phi \circ \iota_{\tau(\lambda)})$ satisfies the Hamilton equations, and since $F_\zeta|_{\mathcal{P}_\tau}$ and ψ_ζ are isomorphic, a c satisfying (2.6.40) corresponds to an extremal field. This however does not imply that through every point in $\mathcal{P}^\tau$ there is a classical trajectory, nor does it imply that there is a unique solution at every point. These properties are only fulfilled if ω defines a symplectic structure on $\mathcal{P}^\tau$[7]. Furthermore, a unique well-defined solution ξ of (2.6.40) may not generate a flow, which makes a physical interpretation impossible. A necessary condition for a well-defined Hamiltonian flow is the existence of a submanifold $\mathcal{C}$ of $\mathcal{P}^\tau$ such that ξ is tangent to $\mathcal{C}$, although his condition may not be sufficient if phase space is infinite dimensional.

[7] Every tangent vector in $\mathcal{P}^\tau$ can be decomposed into a piece parallel to ζ_P and a piece tangent to some $\mathcal{P}_{\tau(\lambda)}$: hence a global presymplectic form ω on $\mathcal{P}^\tau$ is uniquely defined by $\omega(\xi_1, \xi_2) = \omega_{\tau(\lambda)}(\xi_1, \xi_2)$ for $\xi_{1,2} \in T_{(\varphi,\pi)}\mathcal{P}_{\tau(\lambda)}$ and $\zeta_P \in \ker\omega$

It turns out that there is a unique maximal manifold obeying these requirements, which we shall call the *final constraint subspace* $\mathcal{C}$. It is the Hamiltonian evolution of an instantaneous constraint manifold $\mathcal{C}_\tau \subseteq \mathcal{P}_\tau$, on which finite-time propagation is well-defined on every point. There is a well-known algorithm to find $\mathcal{C}_\tau$: for an arbitrary submanifold $\mathcal{Q}_\tau \subseteq \mathcal{P}_\tau$, define the *symplectic complement* at $q \in \mathcal{Q}_\tau$ by $T_q\mathcal{Q}_\tau^\perp = \{\xi \in T_q\mathcal{P}_\tau | \forall \zeta \in T_q\mathcal{Q}_\tau : \Omega_\tau(\xi, \zeta) = 0\}$. In particular we have $(\Gamma\mathcal{P}_\tau)^\perp = \ker \omega_\tau$. The final constraint manifold is then

$$\mathcal{C}_\tau = \bigcap_\ell \mathcal{P}^\ell_{\tau,\zeta}\,, \qquad \mathcal{P}^{\ell+1}_{\tau,\zeta} = \{p \in \mathcal{P}^\ell_{\tau,\zeta} | \forall \xi \in (T_p\mathcal{P}^\ell)^\perp_{\tau,\zeta} : \iota_\xi \delta H_{\tau,\zeta} = 0\}\,. \tag{2.6.42}$$

Hence $\mathcal{C}_\tau$ is the largest submanifold of $\mathcal{P}_\tau$ such that $\iota_\xi \delta H_{\tau,\zeta}|_{\mathcal{C}_\tau} = 0$ for all $\xi \in \Gamma T\mathcal{P}_\tau^\perp$. It can be shown that it does not depend on the choice of ζ. A *constraint* is a local density-valued 0-form $F \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$ which vanishes on the final constraint set: $F|_{\mathcal{C}_\tau} = 0$. If F satisfies $\iota_\xi \delta F = 0$ for all $\xi \in \Gamma T\mathcal{C}^\perp$, it is called *first class*; otherwise we refer to F as *second class*. The final constraint manifold can always be described as the locus in $\mathcal{P}_\tau$ defined by a finite number of such constraints. This follows from the following facts: if F is a constraint then a Hamiltonian vector field $\xi_F \in \Gamma T\mathcal{P}_\tau : \iota_{\xi_F}\omega = \delta F$ exists along $\mathcal{C}_\tau$ iff for all $\zeta \in \Gamma T\mathcal{P}_\tau^\perp$ the equation $\iota_\zeta \delta F = 0$ is fulfilled on $\mathcal{C}_\tau$. If so, we also have $\xi_F \in \Gamma T\mathcal{C}_\tau^\perp$. By construction a first class constraint F therefore generates a Hamiltonian vector field tangent and symplectically polar to the final constraint manifold: $\xi_F \in \Gamma(T\mathcal{C}_\tau \cap T\mathcal{C}_\tau^\perp)$. Vice versa, at any point on the final constraint manifold $T\mathcal{C}_\tau^\perp$ is spanned by the Hamiltonian vector fields of constraints and $T\mathcal{C}_\tau \cap T\mathcal{C}_\tau^\perp$ is spanned by those of the first-class constraints. The algorithm (2.6.42) can also be carried out using constraints, one then gets a series of inclusions $\mathcal{C}_\tau \subset \mathcal{P}^\ell_{\tau,\zeta} \subset \ldots \subset \mathcal{P}_\tau \subseteq T^*\mathcal{F}_\tau$. The constraints defining P_τ (reflecting degeneracy of the Legendre morphism) are called *primary*, the ones defining $\mathcal{P}^\ell_{\tau,\zeta}$ shall be called ℓ-ary. So constraints reflect the overdetermined nature of the field equations: we have to restrict phase space to ensure all points on $\mathcal{C}$ may serve as initial value of the classical trajectory, but the primary constraints also reflect the underdetermined nature of the equations: if ξ_λ satisfies the Hamilton equations (2.6.40), then $\xi_\lambda + \zeta_\lambda$ with $\zeta_\lambda \in \Gamma(T\mathcal{C}_\lambda \cap T\mathcal{C}_\lambda^\perp)$ satisfies the Hamilton equation with $H_{\lambda,\zeta}$ substituted by $H_{\lambda,\zeta} + F$, F being the (first class) constraint corresponding to ζ_λ. Physically these equations are indistinguishable, since only the pullback to $\mathcal{C}_\lambda$ determines the dynamics. Given a set of constraints F_α generating a linearly independent complete basis of $\Gamma(T\mathcal{C}_\lambda \cap T\mathcal{C}_\lambda^\perp)$, we define the total Hamiltonian density on the extended phase space by $H_{tot,\tau,\zeta} = H_{\tau,\zeta} + \lambda^\alpha F_\alpha$ where we have added the *Lagrange multipliers* λ^α to $T^*\mathcal{F}$. Their equations of motion ensure the pullback of the total presymplectic form ω to $\mathcal{C}$, and the Hamiltonian vector field solutions of the other fields are unique up to elements in $\Gamma(T\mathcal{C} \cap T\mathcal{C}^\perp) = \ker i^*_{\mathcal{C}}\omega_\tau$, the kernel of the pullback of the presymplectic form to the final constraint manifold. If this kernel is nontrivial, it signals *gauge freedom* in the theory, and under the bracket of vector fields we call this the *gauge algebra*.

2.6.6 Gauge Symmetries

In the Lagrangian theories we shall encounter, the infinitesimal symmetries come in families, described by a set of parameters. A *global* or *rigid* infinitesimal symmetry is a map $\Xi : V \longrightarrow \Gamma_{\mathrm{loc}}(T\mathcal{F}) + \Gamma TM$ from some vector space (possibly with an algebra structure) to the space of local decomposable vector fields on $\mathcal{F} \times M$, such that each $\xi(v)$ is a generalised infinitesimal symmetry of L. The fundamental extension of this concept is allowing the $\mathbb{R}$-module V to be a $C^\infty(M, \mathbb{R})$-module, a space of sections of some fiber bundle over M. A *generalised local (gauge) infinitesimal symmetry* is determined by a vector bundle $V \longrightarrow M$ and linear maps $\Gamma V \longrightarrow \Gamma T(\mathcal{F} \times M)$: $\sigma \mapsto \xi(\sigma)$ and $\Gamma V \longrightarrow \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M) : \sigma \mapsto \alpha(\sigma)$, depending locally on σ such that for all sections σ, $(\xi(\sigma), \alpha(\sigma))$ is a generalised infinitesimal symmetry: $\mathfrak{L}_{\xi(\sigma)} L = d\alpha(\sigma)$. These gauge symmetries cause the global symplectic 2-form to be degenerate on the space of extremals: for arbitrary $\sigma \in \Gamma V$, $\phi \in \mathcal{M}$ and $\hat{\xi} \in \Gamma T\mathcal{M}$ one may show that $\Omega(\xi(\sigma), \hat{\xi})(\phi) = 0$. The key point is that $\xi(\sigma)$ for some fixed σ defines a mapping $M \longrightarrow \Gamma_{\mathrm{loc}} T\mathcal{M}$ and consequently the Noether current $j_{\xi(\sigma)}$ is actually a mapping $M \longrightarrow \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{M} \times M)$, which changes the cohomology:

Theorem 2.5 *Let $V_i \longrightarrow E$, $i = 1, \dots p$ be vector bundles over the fibre bundle E and set $V = \times_E V_i$ and $\mathcal{V}_\phi = \Gamma(\phi^* V)$. Let $\Omega^{0,|*|}_{\mathrm{Rlin}}(\mathcal{V}_\phi \times M)$ denote the subcomplex of $\Omega^{0,|*|}_{\mathrm{loc}}(\mathcal{V}_\phi \times M)$ consisting of forms $\alpha(\phi; \zeta_p, \dots, \zeta_p)$ which are $\mathbb{R}$-linear in ζ_i. Then $(\Omega^{0,|*|}_{\mathrm{Rlin}}(\mathcal{V}_\phi \times M), d)$ is exact except in the top degree $|*| = 0$.*

Note that taking all the V_i to be the vertical tangent bundle, $V_i = T^V E$ yields theorem (2.4). In the present case the gauge bundle V defines a bundle on E through the projection $V \longrightarrow TE \longrightarrow E$ induced by local infinitesimal symmetry. Since we continue to have $dj_{\xi(\sigma)} = 0$ on the shell, the theorem above states that the Noether current is exact on $\mathcal{M} \times M$. Hence, assuming sufficient decay at spatial infinity, the associated Noether charge to a local gauge symmetry vanishes.

We introduce brackets on the linear space of Noether currents, making it into a Lie algebra. A *Noether pair* is a pair (j_ξ, ξ) where ξ is a generalised infinitesimal symmetry on $\mathcal{F} \times M$ and j_ξ is its associated Noether current. We define the bracket

$$\{(j_1, \xi_1), (j_2, \xi_2)\} = (j_{[\xi_1, \xi_2]}, [\xi_1, \xi_2])\,, \tag{2.6.43}$$

where some differential calculus on $\Omega_{\mathrm{loc}}(\mathcal{F} \times M)$ yields following expression for the Noether current of the commutator of 2 infinitesimal symmetries:

$$j_{[\xi_1, \xi_2]} = \mathfrak{L}_{\xi_1} j_2 - \mathfrak{L}_{\xi_2} j_1 + (\mathfrak{L}_{\xi_2} \iota_{\xi_1} - \iota_{\xi_1} \mathfrak{L}_{\xi_2})(L + \gamma)\,. \tag{2.6.44}$$

For nonmanifest (global) symmetries the algebra defined above is typically infinite-dimensional, and becomes finite dimensional only after imposing the equations of motion. In the presence of gauge symmetries the algebra contains continuous families, and is therefore certainly infinite-dimensional. These can be modded out under the equivalence relation $(j_1, \xi_1) \sim (j_2, \xi_2)$ if $j_1 - j_2$ is d-exact on the shell. If we call $\overline{\mathcal{M}}$ the space of extremals divided by the gauge symmetries, then Ω is the pull-back of a 2-form $\overline{\Omega}$ to this space, and in the following we assume that this form defines a symplectic structure on $\overline{\mathcal{M}}$. An infinitesimal symmetry ξ projected onto an infinitesimal symmetry $\bar{\xi}$ on $\overline{\mathcal{M}}$ is then by nondegeneracy the symplectic gradient of a global charge $\overline{Q}$. The *Poisson bracket* on the space of Noether charges is defined by

$$\{\overline{Q}_1, \overline{Q}_2\} = \overline{\Omega}(\mathrm{grad}_{\overline{\Omega}} \overline{Q}_1, \mathrm{grad}_{\overline{\Omega}} \overline{Q}_1)\,. \tag{2.6.45}$$

So if we denote $\bar{\xi}_Q = \mathrm{grad}_{\overline{\Omega}} Q$, then the Poisson bracket is chosen such that $[\bar{\xi}_{Q_1}, \bar{\xi}_{Q_2}] = \bar{\xi}_{\{Q_1, Q_2\}}$. The strategy we shall follow to mod out the gauge freedom is *gauge fixing*: one then simply chooses a submanifold of $\mathcal{F}$ which yields all of $\mathcal{F}$ when translated by the gauge group; the new theory may then have residual gauge freedom, provided by the stabiliser subgroup of this submanifold. In the instantaneous Hamiltonian formalism the presymplectic structure ω_τ defines the bracket on $\Omega^{0,|0|}(\mathcal{F}_\tau \times \Sigma_\tau)$

$$\{\sigma, \rho\}_p = \omega_\tau(\xi_\sigma, \xi_\rho) = \Big(\iota(\frac{\delta}{\delta \varphi^a})\delta\sigma\Big) * \Big(\iota(\frac{\delta}{\delta \pi_a})\delta\rho\Big) - \Big(\iota(\frac{\delta}{\delta \varphi^a})\delta\rho\Big) * \Big(\iota(\frac{\delta}{\delta \pi_a})\delta\sigma\Big)\,, \tag{2.6.46}$$

where $*$ is the Hodge duality map on the spacesheet Σ_τ. The extension of this bracket to functions on $\mathcal{P}^\tau$ is straightforward: one just uses the total 2-form ω. Similarly, the global 2-form defines a bracket on $\Omega^{0,|1-d|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$, which is for 2 charges $Q_i = \int_{\Sigma_\tau} \rho_i$ equal to $\{Q_1, Q_2\}_P = \int_{\Sigma_\tau} \{\rho_1, \rho_2\}_p$. The Hamilton equations are then easily deduced from the *time evolution equation*,

$$\frac{d\rho}{d\lambda} = \frac{\partial \rho}{\partial \lambda} + \{\rho, H_{\lambda,\zeta}\}_p\,, \tag{2.6.47}$$

for any $\rho \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F}_\tau \times \Sigma_\tau)$. We then require the evolution to take place in $\mathcal{C}^\tau$. As we have seen, the constraints defining $\mathcal{C}_\lambda$ generate Hamiltonian vector fields on $\mathcal{C}_\lambda$ killing the Hamiltonian. So the constraint algorithm is quite easy using the Poisson bracket: take primary constraints, compute their time evolution and if necessary include the results in the new constraint set and so on. The primary constraints are of course found inverting the Legendre transformation. Essential is the rôle of first class constraints: their Hamiltonian vector fields are the generators of gauge transformations in the Hamiltonian formalism:

$$\iota_{\xi_F}\delta\rho(x^i,\varphi^a,\dot{\varphi}^a) = \{F,\rho\}_p\,, \tag{2.6.48}$$

for any density-valued scalar ρ on phase space. Under the Poisson bracket, their algebra is isomorphic to the on-shell algebra of gauge vector fields above. If a function on phase space commutes with all the first class constraints, we call it a *physical observable*, if a G satisfies $\{F,G\}_P = 1$ for a first-class constraint F, it is called *canonically conjugate* to the constraint. Such a quantity is then *pure gauge*, since one can multiply F by any function f on M and hence transform G to $f \otimes |d^{d-1}x|$ without changing the physics. A special case is diffeomorphism invariance of the Lagrangian theory. Then the Hamiltonian itself is a first-class constraint and time evolution is pure gauge. If the final constraint manifold is described by some complete finite independent set of constraints (F_α, S_β) where the F_α are first class and the S_α are second class, the pullback of the presymplectic form to $\mathcal{C}_\tau$, $\omega_{\mathcal{C}_\tau}$ induces a bracket on the space of density-valued functions on $\mathcal{C}_\tau$, called the *Dirac bracket*. It is given by the formula

$$\{F,G\}_d = \{F,G\}_p - *\{F,S_\alpha\}_p(C^{-1})^{\alpha\beta}\{S_\beta,G\}_p\,, \quad C_{\alpha\beta} = *\{S_\alpha,S_\beta\}_p\,, \tag{2.6.49}$$

and is crucial to the *quantisation* of the system. Note that the antisymmetry and Jacobi identity are preserved by this modification and the second class constraints satisfy $\{F,S_\alpha\}_d = 0$ by construction. The pullback of ω essentially projects all the second class constraint onto the tangent bundle $T\mathcal{C}$, making them first class w.r.t. the final constraint manifold. Analogously one defines the global Dirac brackets $\{\,.\,,\,.\,\}_D$ as the pullback of the global 2-form, and again for Noether charges this is just the integral over Σ_τ of the local brackets. A true symplectic structure is generated if we divide the set of Hamiltonian flows with initial values on $\mathcal{C}_{\tau(0)}$ by the set of gauge orbits: the flows generated by Hamiltonian vector fields of the first class constraints. One can work with this quotient as a bundle over $\mathcal{C}_\tau$, which is usually quite complicated, or work with a particular section of this bundle, which is the gauge fixing procedure mentioned earlier.

3 The Classical Supermembrane

3.1 Eleven-Dimensional Supergravity

3.1.1 Field Content

A lagrangian theory is said to exhibit n (Poincaré) supersymmetries if $\mathcal{F}$ is the space of sections of a bundle of representations of some superextension of the Poincaré algebra on the base manifold M and the Lagrangian density exhibits invariance under the infinitesimal symmetry induced by the action of this graded algebra. A special class of supersymmetric theories exhibits supersymmetry as a gauge symmetry: invariance under the linear map $\Xi : \Gamma(M, \mathfrak{p}(d|n)) \longrightarrow \Gamma T(\mathcal{F} \times M)$; these are called *supergravity* theories. Because the bracket of two supersymmetry generators is a translation, such theories necessarily should also possess translational gauge invariance, as well as gauge invariance under rotations and Lorentz boosts. At the same time a supergravity theory is required to be invariant under diffeomorphisms on the base manifold, which is allowed to have a dynamical geometry. Diffeomorphism invariance is *not* a gauge symmetry, since $\mathrm{Diff}(M)$ is not the automorphism group of some principle bundle, but the local translations will generate diffeomorphisms if a certain constraint is fulfilled. Coordinate reparameterisation invariance requires an invariant volume element on M, and hence a metric, which reduces on the shell to a traceless tensor product of 2 vector representations of the helicity group. In section 2.4 we discussed the conditions under which an irreducible representation of the super-Poincaré algebra constitutes such a tensor. In particular, the maximal dimension of Minkowski space allowing a supergravity multiplet is 11, with an $n = 1$ superextension. The goal of this and upcoming sections is to explore this maximal theory.

Let us apply the theory of previous section to construct an 11-dimensional supersymmetric field theory. Eleven-dimensional Minkowski space corresponds to $(r - s) = 7$ in table (2.3.5), we can construct a bundle of 32-component Majorana spinors in this setting. Furthermore the charge conjugation matrix C is antisymmetric and the complex conjugation matrix is symmetric (see table (2.3.3)). Table (2.3.4) indicates that bilinear morphisms to the exterior algebra bundle, $S(TM) \otimes S(TM) \longrightarrow \bigwedge^k T^*M$, are only symmetric if $k = 1, 2$ or 5. The super bracket of the supercharges (in the $N = 1$ case) therefore take the form

$$\{Q_\alpha, Q_\beta\} = (\Gamma^\mu)_{\alpha\beta} P_\mu + (\Gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (\Gamma^{\mu\nu\rho\sigma\tau})_{\alpha\beta} Z_{\mu\nu\rho\sigma\tau}\,. \tag{3.1.1}$$

We shall focus our attention to the massless shortest multiplets, which are in the previous chapter shown to transform trivially under the central charges, which will be omitted from this point. We have seen that (without imposing any restriction) the shortest massless representations are of dimension $2^{\dim S/4}$, so in 11 dimensions the shortest supermultiplet constitutes 128 bosonic and 128 fermionic states, which decompose into irreducible representations of the helicity group $SO(9)$. With some trigonometry one may calculate

$$\begin{aligned}\chi_{1\otimes F}(\exp(i\sum_{i=1}^{n}\zeta_i M_{2i-1,2i})) =& \big[4(f_1)^2 + 2f_1 - 4f_2 - 4\big] + \big[8f_3 + 4f_2 + 6f_1 + 4\big] \\ &+ \big[32 f_1 \cos(\tfrac{1}{2}\zeta_1)\cos(\tfrac{1}{2}\zeta_2)\cos(\tfrac{1}{2}\zeta_3)\cos(\tfrac{1}{2}\zeta_4)\big]\,,\end{aligned} \tag{3.1.2}$$

here the f_i are the trigonometric functions introduced in section 2.4. From the last paragraph of this section we know that the first term between brackets is the character under the harmonised $(1, 1)$ representation, which is a traceless symmetric 2-tensor, called the *graviton*. The second term is seen to be the character under the third fundamental representation (one should calculate the the coefficients γ_3^i for $n = 4$ in formula (2.4.33)), and the last bracket looks like the tensor product of a vector and a spinor, i.e. a Rarita-Schwinger representation, which is said to be the *gravitino*, the super partner of the graviton. However in the $(2n + 1)$-dimensional case such a character would look like $2^n(1 + 2\cos\zeta_1 + \ldots + 2\cos\zeta_n)\cos(\frac{1}{2}\zeta_1)\ldots\cos(\frac{1}{2}\zeta_n)$. As previously mentioned, there exists a nontrivial harmonic decomposition of the Rarita-Schwinger representation by imposing the condition

$$\gamma_i \psi^i = 0\,, \tag{3.1.3}$$

where γ_i are the $\mathbb{C}\ell_{0,2n+1}$-generators (we shall use the Dirac basis (2.2.28)). Hence we can write the 0-vector component $\Gamma_1 v^1 \theta$ in terms of the other components. This obviously fixes 2^n components. It turns out that the characters receive a term $-2^n \cos(\frac{1}{2}\zeta_1)\ldots\cos(\frac{1}{2}\zeta_n)$. Hence the character decomposition above corresponds to a decomposition $128 + 128 \to \mathcal{H}(1,1) \oplus (3) \oplus \mathcal{H}((1) \otimes 2^n)$. Putting all the ζ_i to zero gives us the dimensions of the irreducible subspaces,

$$\mathbf{128} \oplus \mathbf{128} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}\,. \tag{3.1.4}$$

The goal is to construct a field theory that exhibits diffeomorphism invariance, such that the propagating degrees of freedom on the shell furnish the representations above. The Lagrangian, built of representations of the full Lorentz group $\mathrm{Spin}(1,10)$ should exhibit gauge invariance such that the linearised equations of motion yield only (3.1.4) physical degrees of freedom. This requirement allows us to extract information about the gauge freedom and the nature of the field equations:

1. There is a symmetric 2-tensor $g \in \Gamma(T^*M \vee T^*M)$ with on-shell vanishing trace. The latter condition is established by requiring the theory to be diffeomorphism invariant. Diffeomorphisms are generated by vector fields $\xi \in \Gamma TM$, acting on a tensor field T by the Lie derivative:

$$\iota_{\Phi(\xi)}\delta T = \mathfrak{L}_\xi T\,. \tag{3.1.5}$$

   Using this gauge freedom allows us to diagonalise one row and column and fix the determinant of the field, omitting 11 degrees of freedom. We then are left with 55 components; we thus require the nature of the resulting field equation to be underdetermined, allowing us to choose another eleven components to establish irreducibility. Such additional gauge freedom is provided by an *Einstein equation*, possibly with source term.

2. A fully antisymmetric 3-tensor $A = \frac{1}{6}A_{\mu\nu\rho}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \in \Gamma(\wedge^3 TM)$. For a general p-form A, the symmetry of the action under a transformation $\Phi : \Gamma \wedge^p T^*M \longrightarrow \Gamma T(\mathcal{F} \times M)$ acting on the cotangent space as

$$\iota_{\Phi(\alpha)}\delta A = d\alpha\,, \quad \iota_{\Phi(\alpha)}\mathrm{d}x^\mu = 0\,, \tag{3.1.6}$$

   establishes the matching of the number of independent components. With the symmetry above, we may choose locally a partial trivialisation of the bundle satisfying $p^\mu A_{\mu\nu_1\ldots\nu_{p-1}} = 0$, which fixes $(d-1)(d-2)\ldots(d-p-1)$ components. What remains are

$$\binom{d}{p} - \binom{d-1}{p-1} = \binom{d-1}{p} \tag{3.1.7}$$

   components. Then the Lagrangian is prohibited to contain a *mass term* of A, i.e. a term of order A^2 or higher, which gives an additional gauge symmetry of the equations of motion. This additional symmetry allows us to fix again $\binom{d-2}{p-1}$ components, giving us 84 propagating degrees of freedom.

3. A Rarita-Schwinger field $\psi = (\psi_\mu)^\alpha \mathrm{d}x^\mu \otimes Q_\alpha \in \Gamma(T^*M \otimes S(TM))$ satisfying $\gamma_i \widetilde{\psi^i} = 0$, where $i = 1,\ldots,9$ and γ_i are the Dirac matrices of $\mathrm{Spin}(9)$. The tilded spinors are defined by the natural embedding of the spinor representation of $\mathrm{Spin}(9)$ into the one of $\mathrm{Spin}(1,10)$. Because of the supersymmetry transformations (under the action of the Q_α these transform into real bosonic quantities) the spinor components must satisfy the same reality properties as the supercharges. To establish an irreducible representation on-shell, we require the field equation to be Rarita-Schwinger-like, $\Gamma^{\mu\nu\rho}\nabla_\mu \psi_\mu = 0$ which is invariant under the gauge transformation $\Psi : \Gamma S(TM) \longrightarrow \Gamma T(\mathcal{F} \times M)$ acting on $\theta = \theta^\alpha Q_\alpha \in \Gamma S(TM)$ by

$$\iota_{\Psi(\theta)}\delta\psi = d\theta \equiv \partial_\mu \theta^\alpha \mathrm{d}x^\mu \otimes Q_\alpha\,, \tag{3.1.8}$$

where we have extended the exterior differential to the spinor bundle, $d : \Gamma S(TM) \longrightarrow \Gamma T^*M \otimes S(TM)$.

The latter gauge invariance above is extremely important because shall induce local supersymmetry.

3.1.2 Supersymmetry Gauging

The standard way to construct a Lagrangian exhibiting local infinitesimal symmetries is *gauge theory*, a field theory completely built of sections of associated fibre bundles to some principal bundle. In the present situation we have an eleven-dimensional Lorentzian spin manifold with a principal orthogonal frame bundle $P_{SO}(TM)$ on its tangent bundle. This gives rise to an adjoint bundle $\mathrm{Ad}(P_{SO})$ generating Lorentz transformations and a spinor bundle $S(TM)$. The direct sum of these bundles and the tangent bundle is the vector bundle on which the sheaf of super-Poincaré algebras $\mathfrak{p}(TM, 11|32)$ is constructed. In local coordinates, the symmetry generators consist of 110 Lorentz generators $M_{rs} \in \Gamma\mathrm{Ad}(P_{SO})$, 11 translation generators $P_r \in \Gamma TM$ and 32 supersymmetry generators $Q_\alpha \in \Gamma S(TM)$ obeying the super-algebra

$$\begin{aligned}
[M_{rs}, M_{tu}] &= \eta_{st}M_{ru} + \eta_{ru}M_{st} - \eta_{su}M_{rt} - \eta_{rt}M_{su}\,, & [P_r, P_s] &= 0\,,\\
[M_{rs}, P_t] &= \eta_{rt}P_s - \eta_{st}P_r\,, & [M_{rs}, Q_\alpha] &= \tfrac{1}{2}(\Gamma_{rs})_\alpha{}^\beta Q_\beta\,,\\
[Q_\alpha, Q_\beta] &= (\Gamma^r)_{\alpha\beta}P_r\,, & [Q_\alpha, P_r] &= 0\,.
\end{aligned} \tag{3.1.9}$$

The gauge field is a super-Poincaré algebra-valued one-form. Such a section of the total adjoint bundle $\Omega \in \mathrm{Ad}(\mathfrak{p}(TM, 11|32)$ is decomposed as

$$\Omega = (e_\mu{}^r P_r + \tfrac{1}{2}\omega_\mu{}^{rs}M_{rs} + \kappa\psi_\mu{}^\alpha Q_\alpha)\mathrm{d}x^\mu\,. \tag{3.1.10}$$

The appearance of the constant κ shall be explained later. A section $e = e_\mu{}^r P_r \otimes \mathrm{d}x^\mu$ of $P_{SO}(TM) \otimes T^*M$ is an orthonormal frame field: these degrees of freedom are equivalent to the components of the metric. A section of $\mathrm{Ad}(P_{SO}) \otimes T^*M$ is called a *spin connection*: in the determination of the on-shell multiplet no such degrees of freedom were encountered, so these components should be eliminated from the final equations of motions. Finally the gravitino field $\psi \in \Gamma(S(TM) \otimes T^*M)$ is naturally identified as the gauge field corresponding to the fermionic generators. Let $\nabla(\Omega)$ denote the covariant derivative defined by this gauge field. Following the approach of [35, 36], we define the partial field strengths of this connection in various directions of the Lie algebra:

$$[\nabla_\mu(\Omega), \nabla_\nu(\Omega)] = R^{(\Omega)}{}_{\mu\nu}(e, \omega, \psi) = R^{(P)}{}_{\mu\nu}{}^r(e, \omega, \psi)P_r + R^{(M)}{}_{\mu\nu}{}^r{}_s(\omega)M_r{}^s + R^{(Q)}{}_{\mu\nu}{}^\alpha(\omega, \psi)Q_\alpha\,.$$

Meanwhile, $\omega = \omega_\mu{}^{rs}M_{rs}$ defines a connection on $P_{SO}(TM)$, and we denote the covariant derivative induced by this section on the associated bundles to the orthonormal frame bundle by $\nabla(\omega)$. Using the Poincaré algebra commutators (3.1.9) one finds

$$\begin{aligned}
R^{(P)r}(e, \omega, \psi) &= (d_{\nabla(\omega)}e)^r + \kappa^2\bar{\psi}^\alpha(\Gamma^r)_{\alpha\beta} \wedge \psi^\beta = de^r - \omega^r{}_s \wedge e^s + \tfrac{\kappa^2}{2}\bar{\psi}_\mu\Gamma^r\psi_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,,\\
R^{(M)r}{}_s(\omega) &= (d_{\nabla(\omega)}\omega)^r{}_s = d\omega^r{}_s - \omega^r{}_t \wedge \omega^t{}_s\,,\\
R^{(Q)\alpha}(\omega, \psi) &= (d_{\nabla(\omega)}\psi)^\alpha = \big((\partial_{[\mu} - \tfrac{1}{4}\omega_{[\mu}{}^{rs}\Gamma_{rs})\psi_{\nu]}\big)^\alpha \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,.
\end{aligned} \tag{3.1.11}$$

Note that the covariant derivative defined above acts trivially on the cotangent space factor of the Rarita-Schwinger field ψ. Furthermore, if the gravitini are put zero, $R(P)$ is the torsion tensor associated to the spin connection. The first step towards a construction of a field theory which exhibits super-Poincaré symmetry is determination of the transformation rules. The fields (except for the antisymmetric 3-tensor A, which shall be included later) constitute a $\mathfrak{p}(M, 11|32)$-valued connection one-form, and hence induce a covariant on the adjoint bundle, which is postulated to be the transformation rule:

$$\Xi : \Gamma\mathrm{Ad}(P_{P(11|32)}(TM)) \longrightarrow \Gamma T(\mathcal{F} \times M) : \iota_{\Xi(\Lambda)}\delta\Omega = d\Lambda + [\Lambda, \Omega] = d_{\nabla(\Omega)}\Lambda\,. \tag{3.1.12}$$

For $\Lambda = \xi^r P_r + \frac{1}{2}\lambda_r{}^s M_r{}^s + \epsilon^\alpha Q_\alpha$ one finds the transformation rules

$$\begin{aligned}
\iota_{\Xi(\Lambda)}\delta e^r &= (d_{\nabla(\omega)}\xi)^r + \lambda^r{}_s \wedge e^s + \kappa\bar{\psi}^\alpha \wedge (\Gamma^r)_{\alpha\beta}\epsilon^\beta\,,\\
\iota_{\Xi(\Lambda)}\delta\omega^r{}_s &= (d_{\nabla(\omega)}\lambda)^r{}_s - \lambda^r{}_t \wedge \omega^t{}_s\,,\\
\iota_{\Xi(\Lambda)}\delta\psi^\alpha &= \kappa^{-1}\big(d_{\nabla(\omega)}\epsilon + \tfrac{\kappa}{4}\lambda^r{}_s(\Gamma_r{}^s)\psi\big)^\alpha\,.
\end{aligned} \tag{3.1.13}$$

Notice that the spin connection Ω is invariant under local translations and supersymmetries. Building a theory with the gauge fields above naturally raises questions on the rôle of the spin connection, which is not part of the on-shell irreducible multiplet, and the general coordinate invariance. It turns out these problems are all related and have a common solution. To give the solution requires a starting point Lagrangian, which is reasonably chosen to be the Einstein-Hilbert Lagrangian. The motivation for this choice is that upon setting the gravitini and abelian gauge field zero, the theory should produce Einstein's general relativity. Moreover, we know that this model yields an equation of motion for the spin connection which is algebraically solvable, and these degrees of freedom can therefore be eliminated. The Einstein-Hilbert Lagrangian is in the Cartan formulation equal to

$$L_{EH} = R^{(M)rs}(\omega) \wedge *(e^r \wedge e^s)\,. \tag{3.1.14}$$

The field equation of ω is solved by

$$\omega_\mu{}^r{}_s = \eta^{rt}\iota(\partial_\mu)\iota(e_t)\iota(e_s)\big(\eta_{uv}de^u \wedge e^v\big)\,, \tag{3.1.15}$$

which is the Cartan formulation of the Levi-Civita connection on TM. The field equation can be written in terms of the curvatures as $R^{(P)r}(e,\omega,\psi = 0) = 0$. Moreover, substituting the solution of the spin connection into the local translation of the vielbein gives the transformation rule $\iota_{\Xi(\xi)}\delta e^r = (\xi^\mu\partial_\mu e^r_\nu - e^r_\mu\partial_\nu\xi^\mu)\mathrm{d}x^\nu = \mathfrak{L}_\xi e^r$, which is exactly the action of an infinitesimal diffeomorphism. Actually, one can proceed on this trajectory and impose the super-Poincaré algebra (without local translations, but with coordinate transformations) to close on the shell, obtaining constraints on the curvatures which are the equations of motion of the fields. For example, the supersymmetry commutator can be calculated using the rules (3.1.13); imposing the commutator is a coordinate transformation on the vielbein field yields the constraint

$$R^{(P)r}(\omega, e, \psi) = 0\,. \tag{3.1.16}$$

Again, the spin connection can be algebraically solved of this equation:

$$\omega_\mu{}^r{}_s(e,\psi) = \eta^{rt}\iota(\partial_\mu)\iota(e_t)\iota(e_s)\big(\omega^0(e) - K(e,\psi)\big), \tag{3.1.17}$$

$$\omega^0(e) = \eta_{rs}de^r \wedge e^s\,, \quad K(e,\psi) = \tfrac{\kappa^2}{2}\bar{\psi}^\alpha \wedge \Gamma^{(1)}{}_{\alpha\beta} \wedge \psi^\beta\,. \tag{3.1.18}$$

Note that this expression yields a connection with torsion, induced by the gravitino field in the *contorsion* term $\psi \wedge \Gamma \wedge \psi$. Imposing the supersymmetry commutator on the gravitino yields the constraint

$$\Gamma^{\mu\nu\rho}R^{(Q)}_{\nu\rho}(\omega,\psi) = 0\,. \tag{3.1.19}$$

There exists an off-shell formulation yielding the same results as above: it is given by the Lagrangian

$$L = -\tfrac{1}{2\kappa^2}R^{(M)rs}(\omega) \wedge *(e_r \wedge e_s) - \tfrac{1}{2}\bar{\psi}_\alpha \wedge R^{(Q)\beta}(\psi,\omega) \wedge *(\Gamma^{(3)})^\alpha{}_\beta\,, \tag{3.1.20}$$

which is obviously diffeomorphism and Lorentz invariant. To establish super gauge symmetry, guided by the transformation rules (3.1.13), one faces the problem of the transformation law of ω. There are roughly speaking three approaches to deal with this field: the *first order formalism* (cf. [37, 38]) treats the spin connection as an independent field (whose field equation will be algebraically solvable however) with its own supersymmetry transformation. The *second order formalism* treats it in the Lagrangian above as a shorthand notation for the left-hand side of

(3.1.17), that is, ω is always considered to be on-shell. The disadvantage of this procedure is a large number of terms whose supersymmetry cancelation has to be checked. These formalisms generically will lead to different expressions for $\iota_{\Xi(\epsilon)}\delta\omega$, but on-shell (in particular, after imposing the gravitino field equation), the results will agree. The *1.5-th order formalism* (cf. [36]) essentially combines the virtues of both approaches: ω is considered to be a independent field, but the constraint (3.1.16) is imposed on the supersymmetry algebra, and therefore the variation of ω is irrelevant and may be set zero. This is the simplest description of supergravity, and we shall adopt it in what follows. Using the supersymmetry variations of e and ψ in (3.1.13) and the zero variation of ω, one may deduce that the Lagrangian above is supersymmetric up to terms cubic in the gravitino field. The expression (3.1.20) is called the *simple supergravity Lagrangian* and is the starting point of almost every supergravity theory which allows an off-shell formulation.

3.1.3 Construction of the Lagrangian

Historically, the gauging argument of the super-Poincaré algebra was not the way eleven-dimensional supergravity was constructed. Instead, the more conventional *Noether procedure* was used [2], a formalism to construct Lagrangians with a prescribed gauge symmetry [39]. Let a gauge theory Lagrangian $L^{(0)}$ of some gauge field $A \in \Gamma(\mathrm{ad}(P_G(TM)) \otimes T^*M)$ be invariant under a global infinitesimal transformation $\Phi_0 : \mathfrak{g} \longrightarrow T\mathcal{F}$ with Noether currents $j^a_\mu \mathrm{d}x^\mu \otimes E_a$ and local *abelian* gauge invariance $\Xi_0 : \Gamma\mathrm{ad}(P_G(TM)) \longrightarrow \Gamma T(\mathcal{F} \times M) : \iota_{\Xi(\Lambda)}\delta A = d\Lambda$. Assume $\mathfrak{g}$ is equipped with a symmetric bilinear pairing $\langle\, .\, ,\, .\,\rangle$ and consider the modified Lagrangian $L^{(1)} = L^{(0)} - \frac{\kappa}{2}\langle g^*(A,j)\rangle = L^{(0)} - \frac{\kappa}{2}A^a_\mu j^\mu_a$. It will be invariant *up to order* κ^0 under a gauge transformation $\Xi_1 : \Gamma\mathrm{ad}(P_G(TM)) \longrightarrow \Gamma T\mathcal{F}$ given by $\iota_{\Xi(\Lambda)}\delta A = \kappa^{-1}d\Lambda + [\Lambda, A] = d_{\nabla(A)}\Lambda$. We have converted the global invariance and the abelian gauge invariance to a *nonabelian* gauge invariance (up to first order in the *gauge coupling*): the transformation vector field is induced by a section of the adjoint bundle of $\mathfrak{g}$. This procedure may be iterated until the gauge invariance is established to all orders of the gauge coupling. Notice that we already implicitly have written down the first steps of this procedure for the supersymmetries: this explains the constant κ appearing in the fermionic transformation terms, and this also explains why we have a simple supergravity Lagrangian in terms of the curvatures. A curvature $F_\omega = d\omega + \frac{1}{2}[\omega,\omega]$ naturally exhibits a symmetry $\omega \mapsto \omega + \nabla(\omega)\lambda$. The Lagrangian (3.1.20) itself is a result in a Noether procedure (see e.g. [40, 39]), starting with a Lagrangian of the form

$$L = -\tfrac{1}{2\kappa^2}R^{(M)rs}(\omega) \wedge *(e_r \wedge e_s) - \tfrac{1}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\partial_\nu\psi_\rho\,, \tag{3.1.21}$$

which exhibits global supersymmetry and a local abelian gauge invariance $\iota_{\Xi_0(\theta)}\delta\psi_\mu = \partial_\mu\theta$ for any section θ of the spinor bundle. Accordingly the supersymmetry transformations become a gauge symmetry $\Xi_1 : \Gamma S(TM) \longrightarrow \Gamma T\mathcal{F}$, determined by

$$\iota_{\Xi_1(\epsilon)}\delta e_\mu{}^r = \kappa\psi_\mu\Gamma^r\epsilon\,, \quad \iota_{\Xi_1(\epsilon)}\delta\psi_\mu = \kappa^{-1}\nabla_\mu\epsilon\,, \tag{3.1.22}$$

and the spin connection does not transform. These results exactly agree with our gauging of the super-Poincaré algebra. However, the local supersymmetry variation above is not an exact symmetry of the simple supergravity Lagrangian, as one ends up with torsion terms quartic in the gravitino field (which are in the gauge coupling of order κ^{-1}). There is no reason to believe that pursuing the Noether procedure eventually yields a local Lagrangian with gauged supersymmetry to all orders, as we haven't included a part of the eleven-dimensional supermultiplet. Including the field A should fulfill the following properties: we require a kinetic term which exhibits local abelian gauge invariance $A \mapsto A + dC$ such that on-shell only 84 degrees of freedom are propagating and we require it to fix the supersymmetry up to all orders. The first 2 requirements are met if we include a kinetic term proportional to $F_A \wedge *F_A$, where $F_A = dA$ in the simple supergravity Lagrangian. On dimensional grounds, we choose the supersymmetry transformation $\iota_{\Xi(\Lambda)}\delta A \sim \alpha\bar{\epsilon} \wedge \Gamma^{(2)} \wedge \psi$ for some constant α. However, on-shell closure of the super-Poincaré algebra, and in particular the $\{Q, Q\}$ bracket, demands that the transformation rule for ψ is modified by a term $\sim F\epsilon$,

$$\iota_{\Xi(\epsilon)}\delta\psi^\alpha = \big(\kappa^{-1}(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{rs}\Gamma_{rs}) - \tfrac{1}{144}F_{\nu\rho\sigma\tau}(\Gamma^{\nu\rho\sigma\tau}{}_\mu - 8\Gamma^{\nu\rho\sigma}\delta_\mu{}^\tau)\big)\epsilon \wedge \mathrm{d}x^\mu\,. \tag{3.1.23}$$

To derive the transformation rule above, note that the two terms added are the only possible nontrivial couplings between F and ϵ; their prefactors may be related by closure of the Poincaré algebra up the quartic gravitino terms. With the transformation rules above, we may calculate the Noether current and establish supersymmetry up to order κ^0. Under a supersymmetry variation with $\iota_\Xi \delta\psi$ given above, we find

$$\mathfrak{L}_{\Xi(\epsilon)} L^0 = (d_{\nabla(\omega)}\bar{\epsilon})_\alpha \wedge ((J_\epsilon)^\alpha + \mathcal{O}(\psi^3)) + d\alpha\,, \tag{3.1.24}$$

where the supercurrent is calculated to be

$$(J_\epsilon)^\alpha = *(\Gamma^{(6)})^\alpha{}_\beta \wedge \psi^\beta \wedge F + \tfrac{1}{2}(\Gamma^{(2)})^\alpha{}_\beta \wedge \psi^\beta \wedge *F\,. \tag{3.1.25}$$

As prescribed by the Noether formalism we obtain a supersymmetric Lagrangian up to order κ^0 (and ignoring higher order terms in ψ) by setting $L^1 = L^0 - \kappa\bar{\psi} \wedge J_\epsilon$. With some algebra one may verify that the supersymmetry variation of this last term, which is now of the order κ, may be written as

$$\mathfrak{L}_{\Xi(\epsilon)}(\kappa\bar{\psi} \wedge J_\epsilon) = -\mathfrak{L}_{\Xi(\epsilon)}(\tfrac{\kappa}{6} F \wedge F \wedge A) + \mathcal{O}(\psi^3)\,. \tag{3.1.26}$$

The Chern-Simons-like term $F \wedge F \wedge A$ is invariant under the tensorial gauge transformation $A \to A + d\alpha$ up to total derivative, as well as local Lorentz transformations and diffeomorphisms. Including it in the action establishes local supersymmetry up to all orders in κ. To fix the cubic terms in the Lagrangian and the $\psi^2\epsilon$-terms in the transformation laws, we again investigate the transformation of the gravitino field. For the $[Q, Q]\psi$ bracket to result into a coordinate reparameterisation, the derivatives of the supersymmetry parameter coming from the F-terms should be canceled. This is established by replacing it with the *supercovariant* quantity

$$\widehat{F} = F + 6\kappa\bar{\psi}_\alpha \wedge (\Gamma^{(2)})^\alpha{}_\beta \wedge \psi^\beta\,. \tag{3.1.27}$$

A quantity is called supercovariant if its supersymmetry variation depends on the zeroth jet of the supersymmetry parameter. We shall denote the variation (3.1.23) with F replaced by $\widehat{F}$ by the supercovariant derivative $\hat{\nabla}(\omega)$. Finally, the quartic terms in the Lagrangian are found by exploiting the 1.5-th order formalism. If ω is a solution of its field equation, i.e. an extremum of the action, adding a expression of the order ψ^2 will modify the action by terms of the order ψ^4 or higher. Another source of quartic terms will be the Noether coupling. These are modified such that the supersymmetry is established. The easiest way is to use a trick: demanding that the field equation of ψ is the supercovariant Rarita-Schwinger equation $\Gamma^{\mu\nu\rho}\widehat{\nabla}_\nu(\widehat{\omega})\psi_\rho = 0$, where $\hat{\omega}(e, \psi)$ is the supercovariant solution of the connection, given by (3.1.17). This is effectively done by replacing ω by $\frac{1}{2}(\omega + \widehat{\omega})$ in the kinetic gravitino term and F by $\frac{1}{2}(F + \widehat{F})$ in the Noether coupling term:

$$\begin{aligned} L \;=\;& -\tfrac{1}{2\kappa^2} R^{rs}(\Omega) \wedge *(e_r \wedge e_s) - \tfrac{1}{2}\bar{\psi}_\alpha \wedge R^{(Q)\beta}\big(\tfrac{1}{2}(\omega + \widehat{\omega})\big) \wedge *(\Gamma^{(3)})^\alpha{}_\beta - \tfrac{1}{2} F \wedge *F \\ & -\kappa\bar{\psi}_\alpha \wedge \Big(*\, (\Gamma^{(6)})^\alpha{}_\beta \wedge \psi^\beta \wedge \big(\tfrac{1}{2}(F + \widehat{F})\big) + \tfrac{1}{2}(\Gamma^{(2)})^\alpha{}_\beta \wedge \psi^\beta \wedge *\big(\tfrac{1}{2}(F + \widehat{F})\big)\Big) \\ & +\frac{\kappa}{6} F \wedge F \wedge A\,. \end{aligned} \tag{3.1.28}$$

Expanding the differentials, wedge products and bilinear spinor maps (suppressing spinor indices however), one finds the Lagrangian in its familiar form [2, 39, 41]

$$\begin{aligned} L \;=\;& \frac{e}{4\kappa^2} R(\Omega, e) - \frac{e}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \nabla_\nu \big(\frac{\omega + \widehat{\omega}}{2}\big)\psi_\rho - \frac{e}{48} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \\ & - \frac{e\kappa}{192}(\bar{\psi}_\mu \Gamma^{\mu\nu\rho\sigma\tau\lambda}\psi_\nu + 12\bar{\psi}^\rho \Gamma^{\sigma\tau}\psi^\lambda)(F_{\rho\sigma\tau\lambda} + \widehat{F}_{\rho\sigma\tau\lambda}) \\ & + \frac{2\kappa}{(12)^4} \epsilon^{\mu_1\mu_2\cdots\mu_{11}} F_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_5\mu_6\mu_7\mu_8} A_{\mu_9\mu_{10}\mu_{11}}\,. \end{aligned} \tag{3.1.29}$$

with

$$F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]}\,, \quad \widehat{F}_{\mu\nu\rho\sigma} = F_{\mu\nu\rho\sigma} + 3\kappa\bar{\psi}_\mu\Gamma_{\nu\rho}\psi_\sigma\,. \tag{3.1.30}$$

The connection is solved by its equation of motion

$$\omega = \frac{1}{6}(\widehat{\omega}_\mu{}^{rs} + \frac{\kappa^2}{4}\bar{\psi}^\nu\Gamma_{\nu\rho\mu}{}^{rs}\psi^\rho)\mathrm{d}x^\mu \wedge e_a \wedge e_b\,, \tag{3.1.31}$$

Note that the gauge coupling constant κ, which is related to Newton's constant in eleven dimensions, can be absorbed in the fields; a redefinition $\psi \mapsto \kappa^{-1}\psi$, $A \mapsto \kappa^{-1}A$ turns κ into an overall (and hence irrelevant) factor which makes the Lagrangian dimensionless. In much of the literature this factor is therefore omitted.

3.1.4 Field equations and Symmetries

The equations of motion associated to the action above are

$$G(\widehat{\omega}) = \tfrac{1}{2}e^r \wedge *(\widehat{F} \wedge \iota(e_r)(*\widehat{F}) - \iota(e_r)(\widehat{F} \wedge *\widehat{F}))\,, \tag{3.1.32}$$

$$d_{\widehat{\nabla}} * \widehat{F} - \widehat{F} \wedge \widehat{F} = 0\,, \tag{3.1.33}$$

$$(d_{\widehat{\nabla}}\psi)^\alpha \wedge (\Gamma^{(3)})_\alpha{}^\beta = 0\,, \tag{3.1.34}$$

where G is the Einstein tensor and $d_{\widehat{\nabla}}$ is the covariant exterior derivative associated to $\widehat{\nabla}(\widehat{\omega})$. In the absence of the gravitino equation (3.1.33) reduces to $d_\nabla H = 0$, where H is the *dual field strength*: $H = *F - F \wedge A$ and d_∇ is the de Rham differential associated with the Levi-Civita connection. The symmetries of the action are

1. *General diffeomorphism invariance*: a linear map $\Psi : \Gamma TM \longrightarrow \Gamma T(\mathcal{F} \times M)$ such that for a vector field $\xi = \xi^\mu(x)\partial_\mu$ on M, the corresponding infinitesimal generalized transformation acts as
$$\iota_{\Psi(\xi)}\delta\phi = \mathfrak{L}_\xi\phi\,, \quad \iota_{\Psi(\xi)}\mathrm{d}x^\mu = \mathfrak{L}_\xi\mathrm{d}x^\mu = \partial_\nu\xi^\mu\mathrm{d}x^\nu\,, \tag{3.1.35}$$
where we have canonically lifted the Lie derivative to the gauge bundle $\mathrm{ad}(\mathfrak{p}(TM, 11|32)) \otimes T^*M$ by letting it act only on the second factor. As usual, this symmetry is manifest, as all scalar terms transform covariantly, canceling contributions from the volume element.

2. *Local supersymmetry and Lorentz invariance:* the theory is constructed as a gauge theory of the super-extension of the Poincaré algebra. The symmetry is a linear map $\Xi : \Gamma\mathrm{ad}(\mathfrak{p}(TM, 11|32)) \longrightarrow \Gamma T(\mathcal{F} \times M)$ such that for $\Lambda = \frac{1}{2}\lambda_r{}^s(x)M^r{}_s + \epsilon^\alpha(x)Q_\alpha$ we have the modified gauge transformations laws
$$\iota_{\Xi(\Lambda)}\delta e^r = \lambda^r{}_s \wedge e^s + \kappa\bar{\psi}_\alpha \wedge (\Gamma^r)^\alpha{}_\beta\epsilon^\beta\,,$$
$$\iota_{\Xi(\Lambda)}\delta\omega^r{}_s = (d_{\nabla(\omega)}\lambda)^r{}_s - \lambda^s{}_t \wedge \omega^t{}_s\,,$$
$$\iota_{\Xi(\Lambda)}\delta\psi^\alpha = \frac{1}{4}\lambda^r{}_s(\Gamma_r{}^s)^\alpha{}_\beta\psi^\beta + \kappa^{-1}(d_{\widehat{\nabla}}\epsilon)^\alpha\,,$$
$$\iota_{\Xi(\Lambda)}\delta A = 3\bar{\epsilon}_\alpha(\Gamma^{(2)})^\alpha{}_\beta \wedge \psi^\beta\,. \tag{3.1.36}$$
Although these symmetries are local, they act trivially on the $\mathrm{d}x^\mu$. We shall see that the Lorentz invariance is manifest, but the supersymmetry is not (this also follows from the fact that we used the 1.5-th order formalism).

3. *Tensor gauge invariance:* Already from the construction of the supermultiplet it was emphasised that the antisymmetric tensor field A contains 84 degrees of freedom on shell if the Lagrangian is forced to be invariant under (3.1.6). The reader may easily verify that this is indeed the case. The symmetry is nonmanifest due to the Chern-Simons-like term $F \wedge F \wedge A$, giving a surface contribution $d(F \wedge F \wedge \alpha)$. The other fields as well as the basis forms $\mathrm{d}x^\mu$ transform trivially under this symmetry.

4. An odd number of coordinate reflections combined with $A \mapsto -A$, which is not an infinitesimal symmetry, but generates disconnected components of the automorphism group of $\mathcal{M}$.

As already pointed out, the supersymmetry anti-commutators may give terms proportional to the field equations and generators of gauge symmetries. On the vielbein field e^r and the antisymmetric tensor gauge field A, one finds that the supersymmetry anticommutator $\{\iota_{\Xi(\epsilon_1)}, \iota_{\Xi(\epsilon_2)}\}$ is just a sum of a diffeomorphism, a Lorentz transformation, a supersymmetry and a tensor gauge transformation. It is the gravitino field which must be a classical solution in order for the symmetry algebra to close:

$$\begin{aligned}[\iota_{\Xi(\epsilon_1)}, \iota_{\Xi(\epsilon_2)}]\delta\psi^\alpha =& \iota_{\xi_{12}} d\psi^\alpha + d\iota_{\xi_{12}}\psi^\alpha + \tfrac{1}{4}(\lambda_{12})^r{}_s (\Gamma_r{}^s)^\alpha{}_\beta \psi^\beta + \kappa^{-1}(d_{\widehat{\nabla}}\epsilon_{12})^\alpha \\ &+ c_{12}(d_{\widehat{\nabla}}\psi)^\beta \wedge (\Gamma^{(3)})_\beta{}^\alpha\,,\end{aligned} \tag{3.1.37}$$

for suitably chosen $\xi_{12} \in \Gamma TM$, $\lambda_{12} \in \Gamma(TM \otimes \mathfrak{so}(TM))$, $\epsilon_{12} \in \Gamma S(TM)$ and $c_{12} \in C^\infty(M, \mathbb{R})$. The property that only the extremal configurations form a representation of the symmetry algebra (and in particular, supersymmetry on the gravitino field spoiling off-shell closure of the algebra) is typical for supergravity theories. Often this phenomenon is an indication of a complicated quantisation procedure, and much time has been spent on seeking a set of auxiliary fields and transformation laws such that the gauge symmetries become manifest. In the eleven-dimensional case however, the algebra structure does not give rise to severe quantisation problems (see [42]).

Viewing eleven-dimensional supergravity as a gauge theory of the super-Poincaré algebra raises the natural question what the rôle of the antisymmetric tensor field A is. Already in the original paper [2], Cremmer, Julia and Scherk suggested that this representation could be related to the gauge fields corresponding to the central charges in (3.1.1), so that the theory is a true gauge theory of the *M-algebra*, the super-Poincaré algebra with central charges in eleven dimensions (often denoted $\mathfrak{osp}(1|32)$). Various attempts were made to make the dependence explicit: Bars [43] interpreted A as a projection of the Lorentz connection onto the invariant subspace of a gauge transformation, after introducing a new gauge principle. In the action obtained, A was pure gauge and only after adding the Chern-Simons-like term $F \wedge F \wedge A$, the original strong gauge symmetry was broken to the correct invariance (3.1.6). The kinetic term $F \wedge *F$ however had to be added by hand. Another approach is relating A to a gauge field $B_\mu{}^{abcde}$ associated to the 5-charge Z_{abcde} by $dA = *dB$, directly carrying the gauge principle (3.1.6) to the B-formulation of the action. However, due to the Chern-Simons-like term such formulation cannot exist. D'Auria and Fré came in 1984 up with perhaps the most satisfactory construction [44]: a Goldstone mechanism where the A-field becomes a composite field made out of gauge fields of certain central extensions of the M-algebra. They found 2 possible extensions of the super-Poincaré algebra. Recently [45] it was discovered that there exists an entire one-parameter group of M-algebra extensions with gauge theories all yielding the CJS Lagrangian with composite A.

3.2 Superspace Formulation

3.2.1 Superfield Content

We already mentioned that there exists a formulation of eleven-dimensional supergravity as a constrained geometry on a supermanifold. Since this is a priori not evident, there is no fixed scheme to derive this formulation, so we shall proceed by making use of our common sense rather than axiomatic procedures. We shall associate the components of the various supergeometric quantities with the fields in the eleven-dimensional supergravity multiplet (e, ψ, A). This procedure, called *gauge completion* comes down to matching the super-Bianchi identities with field equations and superdiffeomorphisms and super tensor gauge transformations with the symmetries of the CJS Lagrangian, and this order by order in the odd coordinates. First of all, since the superextension of the Poincaré algebra in eleven dimensional supergravity is by a single spinor module $S(TM)$, the superspace formulation should be based upon a supermanifold $M^{11|32}$ where the 32 odd coordinates are all captured in a single Majorana spinor, so we set

$$Z^M = (x^\mu, \theta^\alpha)\,, \quad \mu = 0, \ldots, 10 \quad \alpha = 1, \ldots, 32\,. \tag{3.2.1}$$

One argument in favour of a superspace formulation is the *flat* solution of simple supergravity. Putting the curvatures (3.1.11) equal to zero yields the Maurer-Cartan equations

$$\begin{aligned} de^r &= {\omega^r}_s \wedge e^s - \tfrac{\kappa^2}{2}\bar{\psi}^\alpha (\Gamma^r)_\alpha{}^\beta \wedge \psi_\beta\,, \\ d{\omega^r}_s &= {\omega^r}_t \wedge {\omega^t}_s\,, \\ d\psi^\alpha &= {\omega^r}_s ({\Gamma_r}^s)^\alpha{}_\beta \psi^\beta\,. \end{aligned} \tag{3.2.2}$$

The solutions of these equations (on a body manifolds M_0 with trivial topology) are given by

$${\omega^r}_s = 0\,, \quad \psi^\alpha = d\theta^\alpha\,, \quad e^r = {\delta_\mu}^r \mathrm{d}x^\mu - \tfrac{\kappa^2}{2} d\bar{\theta}^\alpha (\Gamma^r)^\alpha{}_\beta \theta^\beta\,, \tag{3.2.3}$$

for arbitrary $\theta \in S(M)$. The idea is that the general solutions can be expressed in terms of this parameter, and the first terms of the polynomial expansions shall always be equal to the solutions above (this expectation is based upon the analogy with general relativity, where solutions of the Einstein equation are locally flat). So we see naturally a fermionic coordinate arise, parameterising $\mathcal{M}$ and vice versa a superspace formulation with 32 odd coordinates which are components of a single spinor is expected to generate $\mathcal{M}$ after making suitable field identifications. This also shows that the superspace formulation is an *on-shell* formulation, since only the equations of motion (or the Maurer-Cartan equations for flat solutions) are generated.

What should our superfield content be? At the end of the previous section, we mentioned that there is no standard geometrical meaning of the antisymmetric tensor gauge field (neglecting central charges). Let us use a completely antisymmetric rank 3 super tensor field to represent these degrees of freedom. Obviously such an object has by far more components than an ordinary three-form; the remaining components will also depend in a local fashion on the the graviton and gravitino fields and the spin connection (off-shell). Furthermore, the superspace geometry shall be described in the well-defined Maurer-Cartan formalism instead of (pseudo-) Riemannian supergeometry. The field content is therefore

1. A frame of super vector fields [8] (super vielbein): $E^A = \mathrm{d}Z^M {E_M}^A$,
2. A spin connection super one-form: ${\Omega_A}^B = \mathrm{d}Z^M {\Omega_{MA}}^B$,
3. An antisymmetric rank three super tensor field: $B = \frac{1}{6} E^C \wedge E^B \wedge E^A B_{ABC}$.

As already mentioned, ordinary diffeomorphisms and local supersymmetries are in supergeometry two special cases of a single transformation, a *superdiffeomorphism*. Let us denote the manifold of superfields configurations above by $\mathcal{G}$,

$$\mathcal{G} = \Gamma P_{SO}(T^* M^{11|32}) \times \Gamma(T^* M^{11|32} \otimes \mathfrak{spin}(1, 10|32)) \times \Gamma \wedge^3 T^* M^{11|32}\,. \tag{3.2.4}$$

The first factor contains the field configurations of the supervielbein, the second factor those of the spin connection and the last one are the sections of the super three-form bundle. An infinitesimal superdiffeomorphism is a map $\Xi : \Gamma T M^{11|32} \longrightarrow \Gamma T(\mathcal{G} \times M^{11|32})$ such that for a super vector field $X = X^M \partial_M$ and some (algebra-valued) super p-form A,

$$\iota_{\Xi(X)} \mathrm{d}Z^M = \iota_X \mathrm{d}Z^M = X^M\,, \quad \iota_{\Xi(X)} \delta A = \iota_X dA + d\iota_X A\,. \tag{3.2.5}$$

[8] Components w.r.t. the coordinate basis $\mathrm{d}Z^M$ shall be denoted with $M, N, \ldots = (\mu, \alpha), (\nu, \beta), \ldots$ and components w.r.t. the super vielbein basis E^A shall be denoted by the letters $A, B, \ldots = (r, a), (s, b), \ldots$

Hence superdiffeomorphisms act on a basis of $\Gamma T\mathcal{G}$ according to

$$\begin{aligned}
\iota_{\Xi(X)}\delta E^A &= \mathrm{d}Z^M(X^N\partial_N E_M{}^A + (\partial_M X^N)E_N{}^A)\,,\\
\iota_{\Xi(X)}\Omega_{MA}{}^B &= \mathrm{d}Z^M(X^N\partial_N\Omega_{MA}{}^B + (\partial_M X^N)\Omega_{NA}{}^B)\,,\\
\iota_{\Xi(X)}\delta B &= \tfrac{1}{6}\mathrm{d}Z^M\wedge\mathrm{d}Z^N\wedge\mathrm{d}Z^O(X^P\partial_P B_{MNO} + 3(\partial_{[M}X^P)B_{|P|NO]})\,,
\end{aligned} \tag{3.2.6}$$

where the $|\,.\,|$ around an index denotes exclusion from antisymmetrisation. Requiring invariance of the final field equations under the infinitesimal transformations above automatically produces diffeomorphism invariance and local supersymmetry of eleven-dimensional supergravity. An infinitesimal local Lorentz transformation is a map $\Psi : C^\infty(M^{11|32}, \mathfrak{spin}(1,10|32)) \longrightarrow \Gamma TM^{11|32}$ so that for $\Lambda(Z) = \frac{1}{2}\Lambda_A{}^B(Z)M^A{}_B = \frac{1}{2}(\lambda_r{}^s(Z) + \lambda_\alpha{}^\beta(Z)(\Gamma_r{}^s)^\alpha{}_\beta)e^r\wedge e_s$, we have

$$\begin{aligned}
\iota_{\Psi(\Lambda)}\delta E^B &= E^A\Lambda_A{}^B\,,\\
\iota_{\Psi(\Lambda)}\delta\Omega_A{}^B &= -d\Lambda_A{}^B - \Lambda_A{}^C\Omega_C{}^B + \Omega_A{}^C\Lambda_C{}^B\,,
\end{aligned} \tag{3.2.7}$$

and B is invariant under a Lorentz transformation. Finally we demand super tensor gauge invariance from our final equations of motion. This is a linear map $\Phi : \wedge^2T^*M^{11|32} \longrightarrow T(\mathcal{G}\times M)$, which acts trivially on the coordinates, the orthonormal superframe and the spin connection, but acts on B by

$$\iota_{\Psi(C)}\delta B = dC\,. \tag{3.2.8}$$

3.2.2 Gauge Completion

Since e^a and ψ^α are super partners, we expect them to be the components of the single supergravity gauge field E^A, while Ω and C are straightforward superextensions of the corresponding ordinary quantities. This determines the first steps of the gauge completion,

$$\begin{aligned}
E_\mu{}^r(x,\theta=0) &= e_\mu{}^r(x)\,, & E_\mu{}^a(x,\theta=0) &= \psi_\mu{}^a(x)\,,\\
\Omega_{\mu r}{}^s(x,\theta=0) &= \widehat{\omega}_{\mu r}{}^s(x)\,, & B_{\mu\nu\rho}(x,\theta=0) &= A_{\mu\nu\rho}(x)\,.
\end{aligned} \tag{3.2.9}$$

Using these zeroth-order identifications and comparing the transformations above with the symmetries (3.1.35), (3.1.36) and (3.1.6), we can make following zeroth-order identifications of the gauge parameters,

$$\begin{aligned}
X^\mu(x,\theta=0) &= \xi^\mu(x)\,, & X^\alpha(x,\theta=0) &= \epsilon^\alpha(x)\,,\\
\Lambda_r{}^s(x,\theta=0) &= \lambda_r{}^s(x)\,, & C_{\mu\nu}(x,\theta=0) &= \alpha_{\mu\nu}(x)\,.
\end{aligned} \tag{3.2.10}$$

Recall that in the above we have not considered the mixed components or the fermionic ones of the Lorentz connection and the Lorentz parameter. These are to all orders in θ determined by the requirement that the odd components are spinor representations of the Lorentz group,

$$\begin{aligned}
(\Omega_M)_r{}^a(x,\theta) = (\Omega_M)_b{}^s(x,\theta) &= 0\,, & (\Omega_M)_a{}^b(x,\theta) &= \tfrac{1}{4}(\Omega_M)_r{}^s(x,\theta)(\Gamma_r{}^s)_a{}^b\,,\\
\Lambda_r{}^a(x,\theta) = \Lambda_b{}^s(x,\theta) &= 0\,, & \Lambda_a{}^b(x,\theta) &= \tfrac{1}{4}\Lambda_r{}^s(x,\theta)(\Gamma_r{}^s)_a{}^b\,.
\end{aligned} \tag{3.2.11}$$

Computing higher order terms is quite a job. One of the difficulties is the dependence of the symmetry parameters on the component fields. This gives rise to extra terms in the symmetry algebra,

$$[\mathfrak{L}_{\Xi(X_1)} + \mathfrak{L}_{\Psi(\Lambda_1)} + \mathfrak{L}_{\Phi(C_1)}, \mathfrak{L}_{\Xi(X_2)} + \mathfrak{L}_{\Psi(\Lambda_2)} + \mathfrak{L}_{\Phi(C_2)}] = \mathfrak{L}_{\Xi(X_3)} + \mathfrak{L}_{\Psi(\Lambda_3)} + \mathfrak{L}_{\Phi(C_3)}\,, \tag{3.2.12}$$

where

$$\begin{aligned}
X_3 &= \iota_{X_2}dX_1 + \mathfrak{L}_{\Xi_2}X_1 - (1\leftrightarrow 2)\,,\\
\Lambda_3 &= \iota_{X_2}d\Lambda_1 + \mathfrak{L}_{\Xi_2}\Lambda_1 + \Lambda_2\Lambda_1 - (1\leftrightarrow 2)\,,\\
C_3 &= \iota_{X_2}dC_1 + \mathfrak{L}_{\Xi_2}C_1 - (1\leftrightarrow 2)\,.
\end{aligned} \tag{3.2.13}$$

Each second term in the above represents a transformation of the super algebra parameter itself, i.e. Ξ_i are the vector fields on $\mathcal{F}$

$$\Xi_i = \Psi(\xi_i) + \Xi(\tfrac{1}{2}(\lambda_i)_r{}^a M^r{}_a + \epsilon_i{}^\alpha Q_\alpha) + \Phi(\alpha_i)\,, \tag{3.2.14}$$

with ξ_i, λ_i and ϵ_i defined in terms of the super transformation parameters by (3.2.10) and Ψ, Ξ and Φ are resp. the diffeomorphism map, the super-Poincaré transformation map and the tensor gauge transformation map of ordinary supergravity (cf. (3.1.35), (3.1.36) and (3.1.6)). The calculation of the components of the superspace quantities is now a matter of matching the transformation rules[9] order-by-order in θ with those of ordinary supergravity (see previous section). This is a rather delicate and exhaustive procedure, and we shall merely write down the results, which were up to first order calculated in [46], the second order terms (which are relevant for the coupling to branes, as we shall see later) can be found in [47]. The super-diffeomorphism parameters in $X = X^M\partial_M$ are given by

$$\begin{aligned}
X^\mu &= \xi^\mu + \bar\theta\Gamma^\mu\epsilon - (\bar\theta\Gamma^\nu\epsilon)\bar\theta\Gamma^\mu\psi_\nu + \mathcal{O}(\theta^3)\,,\\
X^\alpha &= \epsilon^\alpha - \tfrac{1}{4}\Big(\lambda^{rs} - (\bar\theta\Gamma^\mu\epsilon)\widehat\omega_\mu{}^{rs}\Big)(\Gamma_{rs}\theta)^\alpha - \Big((\bar\theta\Gamma^\mu\epsilon) - (\bar\theta\Gamma^\nu\epsilon)(\bar\theta\Gamma^\mu\psi_\nu)\Big)\psi_\mu{}^\alpha + \mathcal{O}(\widehat F\theta^2) + \mathcal{O}(\theta^3)\,.
\end{aligned}$$

As it turns out, the superspace formulation of supergravity only depends on the first-order terms, while the coupling to branes depends on second-order terms (and even third-order terms in the three-tensor), but as it turns out we shall not need the explicit form of the superdiffeomorphism parameter component proportional to $\widehat F\theta^2$. The other generators of the symmetry algebra are the Lorentz transformations and the tensor gauge transformations. The former are determined by (3.2.11) and

$$\Lambda^{rs} = \lambda^{rs} - (\bar\theta\Gamma^\mu\epsilon)\widehat\omega_\mu{}^{rs} + \tfrac{1}{144}\bar\theta(\Gamma^{rs\mu\nu\rho\sigma}\widehat F_{\mu\nu\rho\sigma} + 24\Gamma_{\mu\nu}\widehat F^{rs\mu\nu})\epsilon + \mathcal{O}(\theta^2)\,, \tag{3.2.15}$$

and the abelian tensor gauge transformation is identified with

$$\begin{aligned}
C_{\mu\nu} &= \big(\bar\epsilon + (\bar\theta\Gamma^\rho\epsilon)\bar\psi_\rho\big)(A_{\mu\nu\rho}\Gamma^\rho + \Gamma_{\mu\nu})\theta + \tfrac{4}{3}\bar\theta\Gamma^\rho\psi_{[\mu}\bar\theta\Gamma_{\nu]\rho}\epsilon + \tfrac{4}{3}\bar\theta\Gamma^\rho\epsilon\bar\theta\Gamma_{\rho[\mu}\psi_{\nu]} + \mathcal{O}(\theta^3)\,,\\
C_{\mu\alpha} &= \tfrac{1}{6}\bar\theta\Gamma^\nu\epsilon(\bar\theta\Gamma_\nu)_\alpha + \tfrac{1}{6}\bar\theta\Gamma_{\mu\nu}\epsilon(\bar\theta\Gamma^\nu)_\alpha + \mathcal{O}(\theta^3)\,,\\
C_{\alpha\beta} &= \mathcal{O}(\theta^3)\,,
\end{aligned} \tag{3.2.16}$$

The θ-expansion of the fields can now be determined by looking their transformation laws. Again we note that we are not interested in the $\widehat F\theta^2$-terms in the supervielbein,

$$\begin{aligned}
E_\mu{}^r &= e_\mu{}^r + 2\bar\theta\Gamma^r\psi_\mu + \bar\theta\Gamma^r\widehat\nabla_\mu(\widehat\omega)\theta + \mathcal{O}(\theta^3)\,,\\
E_\mu{}^a &= \psi_\mu{}^a + (\widehat\nabla_\mu(\widehat\omega)\theta)^a + \mathcal{O}(\theta^2)\,,\\
E_\alpha{}^r &= -(\bar\theta\Gamma^r)_\alpha + \mathcal{O}(\theta^3)\,,\\
E_\alpha{}^a &= \delta_\alpha{}^a + \mathcal{O}(\theta^3)\,.
\end{aligned} \tag{3.2.17}$$

The Lorentz superconnection 1-form is more conveniently expressed in differential form language:

$$\begin{aligned}
\Omega_\alpha{}^{rs} &= \tfrac{1}{6}*\Big[\big(*(\Gamma^{(6)}\theta)_\alpha\wedge\widehat F + 2(\Gamma^{(2)}\theta)_\alpha\wedge *\widehat F\big)\wedge e^r\wedge e^s\Big] + \mathcal{O}(\theta^2)\,,\\
\Omega_\mu{}^{rs} &= \widehat\omega_\mu{}^{rs} + 2\iota(\partial_\mu)*\big[*(\bar\theta\Gamma^{(1)}\wedge d_{\widehat\nabla}\psi)\wedge e^r\wedge e^s\big] + \mathcal{O}(\theta^2)\,,
\end{aligned} \tag{3.2.18}$$

where again the other components can be immediately obtained using (3.2.11). Finally, we give the expansion of the B-field up to third order in θ,

$$\begin{aligned}
B_{\mu\nu\rho} &= A_{\mu\nu\rho} - 6\bar\theta\Gamma_{[\mu\nu}\psi_{\rho]} - 3\bar\theta\Gamma_{[\mu\nu}\widehat\nabla_{\rho]}(\widehat\omega)\theta - 12\bar\theta\Gamma_{\sigma[\mu}\psi_\nu\bar\theta\Gamma^\sigma\psi_{\rho]} + \mathcal{O}(\theta^3)\,,\\
B_{\mu\nu\alpha} &= (\bar\theta\Gamma_{\mu\nu})_\alpha - \tfrac{8}{3}\bar\theta\Gamma^\rho\psi_{[\mu}(\bar\theta\Gamma_{\nu]\rho})_\alpha - \tfrac{4}{3}\bar\theta\Gamma_{\rho[\mu}\psi_{\nu]}(\bar\theta\Gamma^\rho)_\alpha + \mathcal{O}(\theta^3)\,,\\
B_{\mu\alpha\beta} &= (\bar\theta\Gamma_{\mu\nu})_{(\alpha}(\bar\theta\Gamma^\nu)_{\beta)} + \mathcal{O}(\theta^3)\,,\\
B_{\alpha\beta\gamma} &= (\bar\theta\Gamma_{\mu\nu})_{(\alpha}(\bar\theta\Gamma^\mu)_\beta(\bar\theta\Gamma^\nu)_{\gamma)}\,,
\end{aligned} \tag{3.2.19}$$

[9]Not entirely: one uses the assumption that the fermionic and mixed components of the tensor gauge transformation parameter is of the order θ^2: $C_{\alpha M} = \mathcal{O}(\theta^2)$. Furthermore one will encounter ambiguities which can be solved by making suitable higher-order coordinate redefinitions and gauge choices.

where the brackets (. , .) around indices denote symmetrisation with unit weight. In the last line we explicitly wrote down the third-order term which survives when imposing a flat superspace geometry. This finalises the gauge completion.

3.2.3 Torsion Constraints

Let us now try to reproduce the field equations of supergravity from the supergeometry with these identifications. The torsion components of the spin connection are up to zeroth order

$$\begin{aligned} &T_{rs}{}^{t}(x,\theta=0)=T_{ab}{}^{c}(x,\theta=0)=T_{ar}{}^{s}(x,\theta=0)=0\,, && T_{ab}{}^{r}(x,\theta=0)=\tfrac{1}{2}(\Gamma^{r})_{ab}\,,\\ &T_{ar}{}^{b}(x,\theta=0)=\widehat{F}_{stuv}(T^{stuv}{}_{r})_{a}{}^{b}\,, && T_{rs}{}^{a}(x,\theta=0)=\psi_{rs}{}^{a}\,, \end{aligned} \tag{3.2.20}$$

where $T^{stuv}{}_{r}=\frac{1}{288}(\Gamma^{stuv}{}_{r}+\Gamma^{stu}\delta_{r}{}^{v})$ and $\psi_{rs}{}^{a}=\nabla_{[r}(\widehat{\omega})\psi_{s]}{}^{a}$ is the supercovariant gravitino field strength. The curvature obeys in its second 2 entries the same G-structure as the spin connection 1-form (3.2.11), $R_{AB}{}^{cd}=\frac{1}{4}R_{AB}{}^{rs}(\Gamma_{rs})^{cd}$, $R_{AB}{}^{cr}=R_{AB}{}^{sd}=0$, and is therefore completely determined by the following components,

$$\begin{aligned} R_{rs}{}^{tu}(x,\theta=0)&=2(\nabla_{[r}(\widehat{\omega})\widehat{\omega}_{s]}{}^{tu}+\widehat{\omega}_{[r}{}^{tv}\widehat{\omega}_{s]}{}^{u}{}_{t})\,,\\ R_{ab}{}^{rs}(x,\theta=0)&=-\frac{1}{72}(\Gamma^{rstuvw}+24\eta^{r[t}\eta^{|s|u}\Gamma^{vw]})_{ab}\widehat{F}_{tuvw}\,,\\ R_{arst}(x,\theta=0)&=3(\bar{\psi}_{[sr}\Gamma_{t]})_{a}\,. \end{aligned} \tag{3.2.21}$$

Let us now take a glance at the super Bianchi identities governing these geometries. First note that for a Lorentzian connection (the structure group being (2.5.30)), the second identity (2.5.42) follows from the first one (2.5.41). Written out in components, the relevant equation is thus

$$Z_{ABC}{}^{D}\equiv\sum_{[ABC]}\Big(R_{ABC}{}^{D}-\nabla_{A}T_{BC}{}^{D}-T_{AB}{}^{F}T_{FC}{}^{D}\Big)=0\,, \tag{3.2.22}$$

where the sum runs over graded antisymmetrized combinations with unit weight. At first sight, $Z_{ABC}{}^{D}=0$ yields a large number of equations (each component and each order in θ gives an equation). However, the *body* ($\theta=0$ part) of the equation contains all the information because of supercovariance; namely, if a supercovariant tensor is zero up to order θ, then it is zero to all orders, since applying an infinitesimal superdiffeomorphism gives an equation $\epsilon^{\alpha}\partial Z_{ABC}{}^{D}/\partial\theta^{\alpha}(x,\theta=0)=0$, and hence inductively one proves that all derivatives vanish up to order θ, or equivalently the tensor vanishes up to all orders. Note that in order to obtain the component version of (3.2.22) up to order θ^{0}, we need to now the torsion components up to order θ. This can be done by explicit computation using (3.2.18) and (3.2.17), or with the trick of the superdiffeomorphisms: since $T_{MN}{}^{A}$ is a (super-Lie algebra valued) *tensor*, its components transform under a superdiffeomorphism generated by $X^{M}\partial_{M}$ as $T_{MN}{}^{A}\mapsto X^{O}\partial_{O}T_{MN}{}^{A}+(\partial_{M}X^{O})T_{ON}{}^{A}-(-1)^{mn}(\partial_{N}X^{O})T_{OM}{}^{A}$. Taking for X a rigid supersymmetry that is independent of θ one easily derives that only $T_{\mu\nu}{}^{a}$ and $T_{\alpha\mu}{}^{a}$ constitute higher-order terms; the first-order term being exactly (minus) their supersymmetry variation with supersymmetry parameter θ,

$$\begin{aligned} T_{ar}{}^{b}(\theta,x)&=T_{ar}{}^{b}(\theta=0,x)-\tfrac{1}{48}(\bar{\theta}\Gamma_{[st}\nabla_{u}(\widehat{\omega})\psi_{v]})(T^{stuv}{}_{r})_{a}{}^{b}+\mathcal{O}(\theta^{2})\,,\\ T_{rs}{}^{a}(\theta,x)&=T_{rs}{}^{a}(\theta=0,x)+\big((\tfrac{-1}{8}R_{rstu}\Gamma^{tu}+\tfrac{1}{2}[S_{r},S_{s}]+\nabla_{[r}S_{s]})\theta\big)^{a}+\mathcal{O}(\theta^{2})\,, \end{aligned} \tag{3.2.23}$$

where $S_{r}=\widehat{F}_{stuv}T^{stuv}{}_{r}$. Inserting these results in the Bianchi identities exactly reproduces the equations of motion and the Bianchi identities of supergravity. In particular, $Z_{rst}{}^{u}=0$ and $Z_{rst}{}^{a}=0$ lead respectively to the Bianchi identities

$$R_{[rst]}{}^{u}(\widehat{\omega})=0\,,\quad \nabla_{[r}\psi_{st]}{}^{a}+\widehat{F}_{vwxy}T_{[t}{}^{vwxy}\psi_{rs]}{}^{a}=0\,, \tag{3.2.24}$$

which we did not mention in the treatment of supergravity previous chapter. These identities just arise from the definition of the field strengths $R_{rst}{}^{u}$ and $\psi_{rs}{}^{a}$. Furthermore the equations $Z_{abr}{}^{s} = 0$, $Z_{abc}{}^{r} = 0$ and $Z_{abc}{}^{d} = 0$ may be shown to vanish algebraically. The gravitino field equation (??) is produced by both $Z_{ars}{}^{t} = 0$ and $Z_{abr}{}^{c} = 0$ (the latter in an unconventional form however, cf. [46]), and the other field equations follow from $Z_{rsa}{}^{b} = 0$, which leads to

$$R_{rsa}{}^{b} = \nabla_r T_{sa}{}^{b} - \nabla_s T_{ra}{}^{b} + \nabla_a T_{rs}{}^{b} + T_{ra}{}^{c} T_{cs}{}^{b} - T_{sa}{}^{c} T_{cr}{}^{b} \,. \tag{3.2.25}$$

Contracting this equation with $(\Gamma_{tu})^{a}{}_{b}$ yields the supercovariant Einstein equation (3.1.32), contracting it with $(\Gamma_{u})^{a}{}_{b}$ gives us the equation of motion of the antisymmetric tensor field (3.1.33). Notice that we have not used the superfields B_{ABC} at all. It may however be observed that all torsion and curvature components can be written in terms of the exterior derivative (curl) H of this super three-form. It was observed in [48] that all the field equations and Bianchi identities also arise from the supercovariant equation

$$(\Gamma^{rst})_{a} H_{rstu}(x,\theta) = 0 \,, \quad H = d_{\nabla(\widehat{\omega})} B \,. \tag{3.2.26}$$

Using the gauge choice (3.2.19) one may verify that the ψ field equation is given by setting the zeroth order component in θ of the left-hand side equal to zero, while the $e_{\mu}{}^{a}$ and $A_{\mu\nu\rho}$ equations and their Bianchi identities arise from the part of the equation above proportional to θ and the θ^2 part yields the Bianchi identity for the gravitino curvature. Including the H field provides a way to obtain the field equations without using the gauge completion; the input of this theory is the equation (3.2.22) and the super Bianchi identity of the H field, $d_{\nabla(\widehat{\omega})} H = 0$, which reads in component language

$$\sum_{(ABCDE)} \Big(\nabla_A H_{BCDE} + (1 - (-1)^{ce+cb+ed}) T_{AB}{}^{F} H_{FCDE} \Big) = 0 \,. \tag{3.2.27}$$

This yields a coupled system of differential equations with too many degrees of freedom; one needs to impose a minimal set of constraints to obtain the supergeometries described above. These are the *torsion constraints* governing eleven-dimensional supergravity in supermanifold formulation,

$$\begin{aligned} &T_{rs}{}^{t} = T_{ab}{}^{c} = 0 \,, && H_{abcr} = H_{abcd} = 0 \,, \\ &T_{ab}{}^{r} = \tfrac{1}{2}(\Gamma^{r})_{ab} \,, && H_{abrs} = -\tfrac{1}{6}(\Gamma_{rs})_{ab} \,. \end{aligned} \tag{3.2.28}$$

The constraints above are required to hold up to all orders in θ. The first set of equations has already been shown to be obeyed by the supergeometry constructed above by gauge completion; the constraints on the H-field may be verified using the same methods (i.e. direct calculation up to first order in θ and extension of the result to all orders using a supersymmetry variation). Vice versa, considering the two coupled super Bianchi identities in combination with these requirements yields the supergravity equations of motion and their Bianchi identities, where the fields are identified as the $\theta = 0$ components of the respective superfields: $e_{\mu}{}^{a} = E_{\mu}{}^{a}|_{\theta=0}$, $\psi_{\mu}{}^{\alpha} = E_{\mu}{}^{\alpha}|_{\theta=0}$ and $\widehat{F}_{\mu\nu\rho\sigma} = H_{\mu\nu\rho\sigma}|_{\theta=0}$. The constraints above and their equivalence to on-shell eleven-dimensional supergravity were derived in [48] (simultaneously with [46]).

3.2.4 Flat Superspace

Finally let us consider the flat superspace solution of the geometrical systems described above. Setting the supercurvature equal to zero is equivalent to setting the gravitino and antisymmetric tensor field strengths equal to zero. Hence locally these fields are (in trivial topologies) derivatives of respectively a spinor field and a 2-form (in correspondence with (3.2.3). This means that they become pure gauge: by performing a supersymmetry and a tensor gauge transformation they may be put equal to zero. Furthermore the Lorentz connection vanishes because then $R_{rst}{}^{u}(\omega) = 0$ and the bosonic vielbein becomes equal to the coordinate basis, $e_\mu{}^r = \delta_\mu{}^r$. However, the constraints above should still be satisfied. The torsion constraints allow the solution

$$\begin{aligned} E_\mu{}^r(x,\theta) &= \delta_\mu{}^r\,, & E_\mu{}^a(x,\theta) &= 0\,, \\ E_\alpha{}^r(x,\theta) &= -(\bar\theta\Gamma^r)_\alpha\,, & E_\alpha{}^a(x,\theta) &= \delta_\alpha{}^a\,, \end{aligned} \tag{3.2.29}$$

which may be derived from the supercovariance trick on the first-order flat expressions of the supervielbeins. The constraints on H together with the Bianchi identities may be integrated to B, obtaining the unique solution

$$\begin{aligned} B_{\mu\nu\rho}(x,\theta) &= 0\,, \\ B_{\mu\nu\alpha}(x,\theta) &= (\bar\theta\Gamma_{\mu\nu})_\alpha\,, \\ B_{\mu\alpha\beta}(x,\theta) &= (\bar\theta\Gamma_{\mu\nu})_{(\alpha}(\bar\theta\Gamma^\nu)_{\beta)}\,, \\ B_{\alpha\beta\gamma}(x,\theta) &= (\bar\theta\Gamma_{\mu\nu})_{(\alpha}(\bar\theta\Gamma^\mu)_\beta(\bar\theta\Gamma^\nu)_{\gamma)}\,, \end{aligned} \tag{3.2.30}$$

in correspondence with the gauge completion (3.2.19). Because the torsion constraints are obeyed by all supergeometries, flat superspace has zero curvature but a nonzero torsion component $T_{ab}{}^r = (\Gamma^r)_{ab}$. This property is due to the structure group of the connection, which is not the Lorentz group as for ordinary theories, but the super-Poincaré group divided by translations, which has its representations on the tangent sheaf (2.5.31). In fact, it is the odd sector of the superalgebra, corresponding to the lower left block nonzero in its representation in $GL_{d|n}$, which produces an obstruction for a torsion-free geometry, as is explained in [49].

3.3 The Supermembrane

3.3.1 Green-Schwartz p-Brane Action

A *super p-brane* is a $(p+1)$-dimensional Lorentzian manifold V^{p+1} equipped with a mapping $Z : V^{p+1} \longrightarrow M^{d|n}$ into a supermanifold and a Lagrangian $L \in \Omega^{0|0|}_{\mathrm{loc}}(\mathcal{F} \times V^{p+1})$ which is supersymmetric. The configuration space $\mathcal{F}$ consists of the components of Z and the components of the background superfields, which appear in the GS Lagrangian pulled back to the worldvolume. Conceptually, it is important to keep in mind that V^{p+1} is nondynamical, depends on the choice of the physicist and should be regarded as a generalised coordinate space. The image of the worldvolume on the other hand *is* dynamical, and does not have to be homeomorphic to V^{p+1}: we only require smoothness from Z. In fact, this image does not even have to be a manifold: later we shall consider membranes with stringlike spikes for instance. In fibre bundle language, the base manifold is the worldvolume, and the embedding coordinate fields are sections of a fibre bundle which looks locally like the product of the worldvolume and the target superspace $M^{d|n}$. We assume the brane worldvolume is Lorentzian and equipped with a metric $g \in \Gamma(T^*V^{p+1} \vee T^*V^{p+1})$. Let $*$ denote the Hodge star operator on the worldvolume induced by the volume element $\sqrt{-g}$. The Green-Schwartz action of a p-brane coupled to a supergravity super-$(p+1)$-form B is

$$S_{GS}[Z,E,B,g] = \int_{V^{p+1}} \left[*\Big(\frac{p-1}{2}\Big) - \frac{1}{2}\eta_{rs}(Z^*(E^r) \wedge *Z^*(E^s)) + Z^*(B) \right]\,. \tag{3.3.1}$$

Let us write this out in components w.r.t. a local coordinate basis $\{\sigma^i\}$ on V^{p+1}; the pull-backs are easily seen to be

$$\begin{aligned} Z^*(E^A) &= E_i{}^A(Z)\mathrm{d}\sigma^i\,, \\ Z^*(B) &= \frac{1}{(p+1)!} E_{i_1}{}^{A_1} \dots E_{i_{p+1}}{}^{A_{p+1}} B_{A_1 \dots A_{p+1}}(Z(\sigma))\mathrm{d}\sigma^{i_1} \wedge \dots \wedge \mathrm{d}\sigma^{i_{p+1}}\,, \end{aligned} \tag{3.3.2}$$

where $E_i{}^A = \partial_i Z^M E_M{}^A$. So in components we have

$$S_{GS}[Z,E,B,g] = \int_{V^{p+1}} \mathrm{d}^{p+1}\sigma \Big(\tfrac{1}{2}\sqrt{-g}(p-1) - \tfrac{1}{2}\sqrt{-g}g^{ij}E_i{}^r E_j{}^s \eta_{rs}\,, \\ + \frac{\varepsilon^{i_1 \ldots i_{p+1}}}{(p+1)!} E_{i_1}{}^{A_1} \ldots E_{i_{p+1}}{}^{A_{p+1}} B_{A_1 \ldots A_{p+1}} \Big)\,. \tag{3.3.3}$$

Note at this point that the Lagrangian contains no derivatives in g: this is an *auxiliary field*. The equation of motion of this variable can be algebraically solved; its solution defines g to be the pull-back of the metric on the body of $M^{d|n}$ induced by the vielbeins E^r:

$$g_{ij} - \eta_{rs} E_i{}^r E_j{}^s = 0\,. \tag{3.3.4}$$

Suppose now we want the action (3.3.3) to possess worldvolume supersymmetry. For this symmetry the worldvolume boson and fermion degrees of freedom should match on-shell. One easily verifies that the Lagrangian has a worldvolume diffeomorphism invariance, $\Xi : \Gamma TV^{p+1} \longrightarrow \Gamma(T\mathcal{F} \times TV^{p+1})$ which acts on a pull-backed form by $\iota_{\Xi(\xi)}\delta Z^*(\beta) = \mathfrak{L}_\xi Z^*\beta$. Then we may choose a gauge in which $p+1$ of the embedding coordinates are fixed. Hence there are $d-p-1$ on-shell bosonic degrees of freedom. At first sight, there are $n/2$ fermionic degrees of freedom, since the $(Z^*E)^\alpha$ equation of motion contains a Dirac-like projection operator with half of the eigenvalues vanishing. If this would be the end of the story, superbranes would not exist because there are too many anticommuting propagating degrees of freedom. However, a closer inspection of the action (3.3.3) reveals a local fermionic symmetry $K : \Gamma(V^{p+1}, S(TM_0)) \longrightarrow T\mathcal{F}$ called *kappa symmetry*, acting by

$$\iota_{K(\kappa)}\delta Z^* E^r = 0\,, \quad \iota_{K(\kappa)}\delta Z^* E^\alpha = (\mathbf{1} - \Gamma)^\alpha{}_\beta \kappa^\beta\,, \tag{3.3.5}$$

where

$$\Gamma = *(\wedge^{p+1} Z^*\Gamma^{(1)}) = \frac{q}{(p+1)!\sqrt{-g}} \epsilon^{i_1 \ldots i_{p+1}} E_{i_1}{}^{r_1} \ldots E_{i_{p+1}}{}^{r_{p+1}} \Gamma_{r_1 \ldots r_{p+1}}\,, \tag{3.3.6}$$

with $Z^*\Gamma^{(1)} = (Z^*E^r)\Gamma^r = E_i{}^r \Gamma_r \mathrm{d}\sigma^i$ and $q = (-1)^{(p+1)(p-3)/4}$. If the gauge symmetry above holds, half of the remaining fermionic degrees of freedom are unphysical. This is because Γ squares to the identity and is traceless, and consequently half of its eigenvalues are zero. It was shown in [50] that kappa symmetry can always be achieved by choosing the appropriate transformation law of g, upon implementing certain constraints on the background supergeometry. These are given by

$$\begin{aligned} T^r{}_{\alpha\beta} &= (\Gamma^r)_{\alpha\beta}\,, & H_{\alpha r_{p+1} \ldots r_1} &= \frac{q}{p!} \zeta_\beta (\Gamma_{r_1 \ldots r_{p+1}})^\beta{}_\alpha\,, \\ T_{(rs)\alpha} &= \eta_{rs}\zeta_\alpha\,, & H_{\alpha\beta r_p \ldots r_1} &= \frac{q(-1)^p}{(p+1)!} (\Gamma_{r_1 \ldots r_p})_{\alpha\beta}\,, \\ H_{\alpha\beta\gamma A_1 \ldots A_{p-1}} &= 0\,. \end{aligned} \tag{3.3.7}$$

where ζ is an arbitrary target space Majorana spinor. Choosing it to be zero yields for $p=2$ and $d=11$ (the supermembrane) exactly the superspace constraints (3.2.28) governing on-shell supergravity. It turns out that supergeometries with nonzero ζ are gauge equivalent, so on-shell eleven-dimensional supergravity is consistent with the supermembrane. If $\zeta = 0$, the Bianchi identity $dH = 0$ with the constraints above imposed on H yields a Fierz identity

$$d\bar\theta_\alpha \wedge (\Gamma_s d\theta)^\alpha \wedge d\bar\theta_\beta \wedge (\Gamma^{s r_1 \ldots r_{p-2}} d\theta)^\beta = 0\,, \tag{3.3.8}$$

which only holds for for specific values of p and d. Now we get a realistic requirement for the existence of unbroken supersymmetry; given an N-extended embedding supermanifold, the equation $d-p-1 = \frac{1}{4}N \dim S$ must be satisfied, as well as condition (3.3.8). One obtains that N has to be one or two, $d \leq 11$ and $p \leq 6$. Imposing these restrictions yields the so-called *brane scan* (table (2), see e.g. [51]), a diagram containing all the superbranes in various dimensions. The values (d,p) naturally form four sequences: the real, complex quaternionic and octonionic groups[10].

[10] Satisfaction of equation (3.3.8) provides a *triality* when written down in the light-cone gauge, and it is shown in [52] that the objects in a triality must be (not necessarily associative) division algebras.

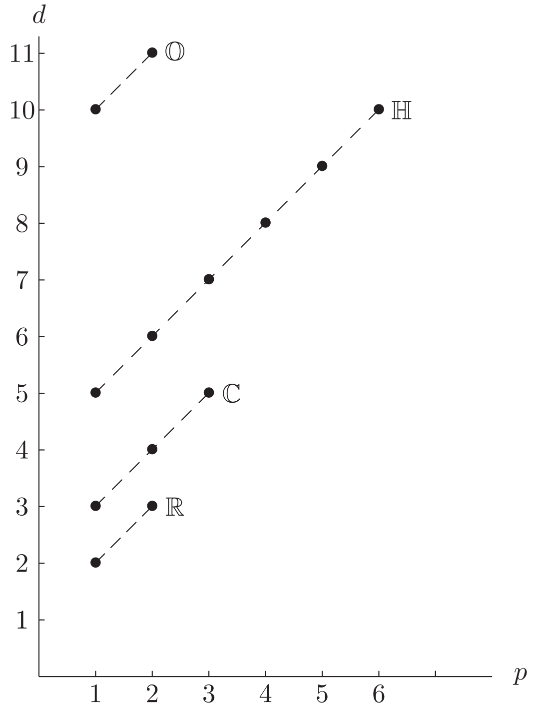


Figure 2: The superbrane scan. The four sequences are indicated by dotted lines

3.3.2 The Supermembrane Action

The $p = 2$ version of (3.3.3), the supermembrane action, looks like

$$S[Z(\sigma), g(\sigma)] = \int d^3\sigma \left(-\tfrac{1}{2}\sqrt{-g}g^{ij}E_i{}^rE_j{}^s\eta_{rs} + \tfrac{1}{2}\sqrt{-g} - \tfrac{1}{6}\varepsilon^{ijk}E_i{}^AE_j{}^BE_k{}^CB_{CBA}\right)\,, \qquad (3.3.9)$$

where g_{ij} represents the world volume metric. This Lagrangian exhibits superdiffeomorphism and super tensor gauge invariance (cf. (3.2.6) and (3.2.8)) on the background field configuration space $T\mathcal{F}_0$ and rigid super-Poincaré invariance on the worldvolume as well as kappa symmetry, provided $H = dB$ fulfills the constraints above. Let us consider the Green-Schwartz supermembrane Lagrangian in flat superspace. Splitting the supercoordinates into even and odd sectors, $Z^M(\sigma) = (X^\mu(\sigma), \theta^\alpha(\sigma))$ and substituting the flat superspace gauge completions (3.2.29) and (3.2.30) yields the scalar Lagrangian

$$L = -\sqrt{-g} - \varepsilon^{ijk}\left(\tfrac{1}{2}\partial_iX^\mu(\partial_jX^\nu + \bar\theta\Gamma^\nu\partial_j\theta) + \tfrac{1}{6}(\bar\theta\Gamma^\mu\partial_i\theta)(\bar\theta\Gamma^\nu\partial_j\theta)\right)(\bar\theta\Gamma_{\mu\nu}\partial_k\theta)\,, \qquad (3.3.10)$$

where the metric is taken on-shell: $g_{ij} = \eta_{\mu\nu}\partial_iX^\mu\partial_jX^\nu$. The space of field configurations $\mathcal{F}$ is reduced to $\Gamma(V, M_0) \times \Gamma(V, S(TM_0))$ where M_0 is the bosonic target-space manifold and V is the worldvolume. We shall make following assumptions about this base space: it is locally of the form $\mathbb{R} \times \Sigma$, where Σ, the *spacesheet*, is a compact smooth 2-manifold without boundary with fixed de Rham cohomology. This action exhibits the following worldvolume symmetries:

1. Diffeomorphism invariance: a linear map $\Xi : \Gamma TV \longrightarrow \Gamma T\mathcal{F}$ acting with $\xi = \xi^i\frac{\partial}{\partial\sigma^i}$ on $T^*\mathcal{F}$ by
$$\iota_{\Xi(\xi)}\delta X^\mu = \xi^i\partial_iX^\mu\,, \quad \iota_{\Xi(\xi)}\delta\theta^\alpha = \xi^i\partial_i\theta^\alpha\,, \quad \iota_{\Xi(\xi)}\mathrm{d}\sigma^i = \xi^i\,. \qquad (3.3.11)$$
This invariance is essential for a globally defined action on the worldvolume. Since it is manifest, it is also exhibited by membranes modeled on worldvolumes with boundaries (which we will not consider). As already mentioned, we can fully consume this symmetry to locally

fix a gauge $X(\sigma^0,\sigma^1,\sigma^2) = (\sigma^0,\sigma^1,\sigma^2, f_1(\sigma^1,\sigma^2,\sigma^3),\dots,f_8(\sigma^1,\sigma^2,\sigma^3))$, leaving 8 degrees of freedom in the bosonic sector.

2. Kappa gauge symmetry: a linear map $K : \Gamma(V, S(TM_0)) \longrightarrow \Gamma T\mathcal{F}$ defined by

$$\iota_{K(\kappa)}\delta X^\mu = \bar{\kappa}(1-\Gamma)\Gamma^\mu\theta\,, \quad \iota_{K(\kappa)}\delta\theta^\alpha = (1-\Gamma)^\alpha{}_\beta\kappa^\beta\,, \tag{3.3.12}$$

where Γ is the $p=2$ version of (3.3.6). As we already have mentioned, this gauge freedom is essential for matching the boson and fermion degrees of freedom, making supersymmetry possible. We also note that if the membrane has a boundary, restrictions on κ are required. The reader also may verify that the Lie derivative of L contains apart from this surface term a term proportional to

$$\bar{\theta}_{[1}\Gamma^\mu\theta_2\bar{\theta}_3\Gamma_{|\mu\nu|}\theta_{4]} = 0\,, \tag{3.3.13}$$

for spinors $\theta_1,\dots,\theta_4$ related to θ. This property of $C\ell_{1,10}$ is again the 2-brane version of the requirement (3.3.8) for the existence of superbranes.

3. Global super-Poincaré invariance, $\Psi : \mathfrak{p}(1,10|32) \longrightarrow \Gamma T\mathcal{F}$, acting on $\Lambda = a^\mu P_\mu + \frac{1}{2}\lambda^{\mu\nu}M_{\mu\nu} + \epsilon^\alpha Q_\alpha$ by

$$\iota_{\Psi(\Lambda)}\delta X^\mu = a^\mu + \lambda^\mu{}_\nu X^\nu + \bar{\theta}\Gamma^\mu\epsilon\,, \quad \iota_{\Psi(\Lambda)}\delta\theta^\alpha = \tfrac{1}{4}\lambda^\mu{}_\nu(\Gamma_\mu{}^\nu)^\alpha{}_\beta\theta^\beta + \epsilon^\alpha\,. \tag{3.3.14}$$

Since this is a global symmetry, it gives rise to a Noether current $j_\Lambda = (a^\mu K_\mu{}^i + \frac{1}{2}\lambda^{\mu\nu}L_{\mu\nu}{}^i + \epsilon^\alpha J_\alpha{}^i)\iota(\partial_i)|d^3\sigma|$ where

$$\begin{aligned}
J_\alpha{}^i =& -2\sqrt{-g}(\bar{\theta}\Gamma^i)_\alpha \\
&+\varepsilon^{ijk}\Big((\bar{\theta}\Gamma_{jk})_\alpha - \tfrac{4}{3}\Big[E_k{}^\mu - \tfrac{2}{5}\bar{\theta}\Gamma^\mu\partial_k\theta\Big]\Big[(\bar{\theta}\Gamma_{\mu\nu}\partial_j\theta)(\bar{\theta}\Gamma^\nu)_\alpha - (\bar{\theta}\Gamma^\nu\partial_j\theta)(\bar{\theta}\Gamma_{\mu\nu})_\alpha\Big]\Big)\,, \\
K_\mu{}^i =& \sqrt{-g}g^{ij}\eta_{\mu\nu}E_j{}^\nu - \varepsilon^{ijk}\Big(E_j{}^\nu - \tfrac{1}{2}\bar{\theta}\Gamma^\nu\partial_j\theta\Big)\bar{\theta}\Gamma_{\mu\nu}\partial_k\theta\,, \\
L_{\mu\nu}{}^i =& 2X_{[\mu}K_{\nu]}{}^i - \tfrac{1}{2}\sqrt{-g}g^{ij}\bar{\theta}\Gamma_j\Gamma_{\mu\nu}\theta + \tfrac{1}{4}\varepsilon^{ijk}\bar{\theta}\Gamma_{jk}\Gamma_{\mu\nu}\theta \\
&+\frac{1}{4}\varepsilon^{ijk}\Big(E_j{}^\rho - \frac{1}{3}\bar{\theta}\Gamma^\rho\partial_j\theta\Big)\Big((\bar{\theta}\Gamma^\sigma\Gamma_{\mu\nu}\theta)(\bar{\theta}\Gamma_{\rho\sigma}\partial_k\theta) + (\bar{\theta}\Gamma_{\rho\sigma}\Gamma_{\mu\nu}\theta)(\bar{\theta}\Gamma^\sigma\partial_k\theta)\Big)\,,
\end{aligned}$$

where we have used the notation $\Gamma_i = E_i{}^\mu\Gamma_\mu$ and $\Gamma^i = g^{ij}\Gamma_j$. integrating these quantities over an arbitrary spacelike surface in the worldvolume yields the corresponding conserved charges.

The field equations corresponding to (3.3.10) are easily calculated to be

$$\partial_i\left(\sqrt{-g}g^{ij}E_j{}^\mu\right) - \varepsilon^{ijk}E_i{}^\nu\partial_j\bar{\theta}\Gamma^\mu{}_\nu\partial_k\theta = 0\,, \tag{3.3.15}$$

$$g^{ij}(\mathbf{1}+\Gamma)\Gamma_\mu E_i{}^\mu\partial_j\theta = 0\,. \tag{3.3.16}$$

Observe the matching of fermion and boson degrees of freedom: the θ-equation of motion is of the form $P\theta = 0$, with P a projector with half of the eigenvalues vanishing, making half of the θ-components pure gauge. Together with kappa symmetry (also observe that $(\mathbf{1}-\Gamma)P = 0$, allowing half of the surviving coordinates to be gauged away) a quarter of the 32 spinor components are propagating degrees of freedom, which matches the eight bosons. It was noted in [3] that the equations above may be written as $\partial_i K^{\mu i} = 0$ and $F = P\theta = 0$ and can be algebraically related by $E_j{}^\mu\partial_i K_\mu{}^i = 2\partial_j\bar{\theta}F$, making the on-shell matching of the number of components even more clear. Bergshoeff et al. also showed that under the global super-Poincaré transformations these quantities transformed into themselves, while a kappa transformation makes them transform into each other, and the flat superspace Lagrangian simply becomes

$$L = K^\mu \wedge *dX_\mu - \bar{\theta}(1+\Gamma)\Gamma^{(1)} \wedge *d\theta\,, \tag{3.3.17}$$

where $K^\mu = K_i{}^\mu d\sigma^i$ is the momentum 1-form.

3.4 Gauge Fixing

3.4.1 Lightcone Gauge

Recall the the embedding of a membrane in flat superspace yields a sigma model field theory on three-manifold V, the fundamental fields (X^μ, θ^α) being sections of a fibre bundle $P^{11|32} \longrightarrow V$, where the fibres are isomorphic to flat 11|32-dimensional Minkowski superspace. The action exhibits various gauge symmetries, which define an equivalence relation $\sim$ on the space of field configurations $\mathcal{F}$. Gauge fixing is essentially the restriction of the configuration space to the space of representatives of $\mathcal{F}/\sim$. The goal of this procedure is to establish a symplectic structure on the manifold of classical solutions, as was explained in section 2.6. In practice however, we shall perform only a partial gauge fixing, which results in a reduction of $\mathcal{F}$ with residual gauge symmetry. If we denote by G the group of gauge transformations on J^kE and H is some subgroup, we define the equivalence relation $\phi_1 \sim \phi_2$ iff $j^k\phi_1 = g(j^k\phi_2)$ for some $g \in G/H$. Then H is the residual gauge symmetry group: we have chosen representatives which still may be related by some gauge transformation in H.

Roughly speaking, gauge fixing the theory shall take place in 2 stages, following the approach of [53], and at the same time we shall introduce the Hamiltonian formalism. Let us begin by introducing so-called *lightcone coordinates* on the fibres of $P^{11|32}$. This is a simple linear redefinition of coordinates in such a way that the world line of a light ray in the $\mu = 10$-direction becomes an axis along the new basis, in other words, a rotation of the (X^0, X^{10}) by 45 degrees,

$$X^\pm = \frac{X^{10} \pm X^0}{\sqrt{2}} \,. \tag{3.4.1}$$

Accordingly, we can write all objects with components in the tangent space to the target space in this basis. The other *transverse* tangent space directions will be denoted with small Latin letters $a, b, \ldots = 1, 2, \ldots 9$. For the lightcone gamma matrices, we have the identities $\{\Gamma^+, \Gamma^-\} = 2\mathbf{1}$, $(\Gamma^+)^2 = (\Gamma^-)^2 = 0$ and $\{\Gamma^\pm, \Gamma^a\} = 0$. The idea of the lightcone gauge is to restrict the local trivialisations of $P^{11|32}$ to those which are the identity on overlaps in the X^+ coordinate. As a consequence, X^+ should only depend on σ^0, and we take this dependence linear, with proportionality factor 1. For the fermionic sector, we note that the nilpotent matrix Γ^+ has half of its eigenvalues zero, and we reduce the field space to those configurations whose spinor part is annihilated by this matrix

$$X^+(\tau, \sigma^1, \sigma^2) = \tau + X^+(0)\,, \quad \Gamma^+\theta = 0\,, \tag{3.4.2}$$

where we have denoted $\tau = \sigma^0$. The residual symmetry is found by performing a total variation on the conditions above and putting these equal to zero on the gauge-fixed subspace, which fixes a number of gauge parameters. The variation of the variable X^+ by a kappa gauge transformation $\kappa : V \longrightarrow S(TM^{11})$, a coordinate transformation $\xi \in \Gamma TV$ and a super-Poincaré algebra element $\Lambda = a^\mu P_\mu + \epsilon^\alpha Q_\alpha + \frac{1}{2}\lambda^{\mu\nu}M_{\mu\nu} \in \mathfrak{p}(1, 10|32)$ yields

$$(\iota_{K(\kappa)} + \iota_{\Xi(\xi)} + \iota_{\Psi(\Lambda)})\delta X^+ = \tfrac{1}{2}\lambda^{+-}\tau + \tfrac{1}{2}\lambda^{+a} + \xi^0(\sigma) + a^+ \,. \tag{3.4.3}$$

Setting this equal to zero fixes the time reparameterisation parameter ξ^0. Looking at the fermionic gauge fixing, which sets half of the components of θ equal to zero, one might naively conclude that only half of the supersymmetries preserve this configuration space. This would be true if the kappa symmetry was fully consumed by the constraint above. However, it turns out that there are still supersymmetry transformations which do not obey $\Gamma^+\epsilon = 0$, but can be compensated by a kappa gauge transformation as to preserve the gauge condition. This can in principle be verified at the present stage, but to avoid lengthy calculations we shall do this after the next bosonic gauge fixing. It is convenient to split off the time components of the metric in our notation:

$$g = \begin{pmatrix} g_{00} & u \\ u^T & \bar{g} \end{pmatrix} . \tag{3.4.4}$$

The embedding equation yields following expressions for these variables

$$
\begin{aligned}
g_{00} &= 2\partial_0 X^- + \partial_0 X^a \partial_0 X_a + 2\bar{\theta}\Gamma^- \partial_0 \theta \,, \\
u_r = g_{0r} &= \partial_r X^- + \partial_0 X^a \partial_r X_a + \bar{\theta}\Gamma^- \partial_r \theta \,, \\
\bar{g}_{rs} = g_{rs} &= \partial_r X^a \partial_s X_a \,.
\end{aligned} \tag{3.4.5}
$$

When calculating these components, one should keep in mind that $\bar{\theta}\Gamma^\mu \partial_i \theta$ vanishes except for $\mu = -$. From this moment, we label the residual spatial worldvolume components with $r, s = 1, 2$ and the transverse lightcone components with $a, b = 1, \ldots, 9$. If we define $\bar{g}$ as the determinant of $\bar{g}_{rs}$ and $\bar{g}^{rs}$ as its inverse matrix we have the following properties:

$$
g = -\Delta \bar{g} \,, \qquad g^{00} = -\Delta^{-1} \,, \qquad g^{0r} = \Delta^{-1} \bar{g}^{rs} u_s \tag{3.4.6}
$$

with $\Delta = -g_{00} + u_r \bar{g}^{rs} u_s$. Since the original pull-backed metric was assumed to be negative definite and τ is identified with a Minkowskian time parameter, we shall require from now on that $\bar{g}_{rs}$ is positive definite and $\Delta, -g_{00} > 0$. Using these identifications the Lagrangian in the light-cone gauge becomes

$$
L = \Big(- \sqrt{\bar{g}\Delta} + \varepsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma_a \partial_s \theta \Big) \mathrm{d}\tau \wedge |\mathrm{d}^2 \sigma| \,. \tag{3.4.7}
$$

3.4.2 Hamiltonian Formulation

Our gauge fixing procedure automatically clears the way for an introduction of the Hamiltonian formalism, since we have made a coordinate choice which induces a preferred slicing direction: we choose Σ_τ the τ level set w.r.t. the local coordinates $(\tau, \sigma^1, \sigma^2)$ on the worldvolume. Here we assume that V is such that it allows a slicing

$$
\phi : \mathbb{R} \times \Sigma \xrightarrow{\simeq} V \tag{3.4.8}
$$

such that each $\Sigma_\tau = \phi(\tau, \Sigma) \subset V$ is a smooth, compact, orientable 2-manifold without boundary. We have started working with a first jet bundle with coordinates $(\sigma^i, X^\mu, \theta^\alpha, \partial_i X^\mu, \partial_i \theta^\alpha)$. Subsequently we have performed a gauge fixing, reducing this system to a system containing only 10 bosonic coordinates and derivatives of these, and only 16 fermionic coordinates and partial derivatives. Hence there are 10 bosonic canonical momenta and and 16 fermionic modes,

$$
P_a = \sqrt{\frac{\bar{g}}{\Delta}} \left(\partial_0 X_a - u_r \bar{g}^{rs} \partial_s X_a \right) \,, \quad P^+ = \sqrt{\frac{\bar{g}}{\Delta}} \,, \quad S^\alpha = -\sqrt{\frac{\bar{g}}{\Delta}} (\Gamma^- \theta)^\alpha \,. \tag{3.4.9}
$$

One easily shows that the spinor S satisfies $\Gamma^+ S = S$, and since half of the eigenvalues of Γ^+ are zero, there are only 16 nontrivial components of S. The Legendre transform is not surjective; the equations (3.4.9) are not invertible to expressions of the τ-derivatives of the fields into phase space variables because there are linear relations between the momenta. These primary constraints are

$$
\psi_r = \big(P_a \partial_r X^a + P^+ \partial_r X^- + \bar{S} \partial_r \theta \big) \otimes |\mathrm{d}^2 \sigma| = 0 \,, \tag{3.4.10}
$$

$$
\chi^\alpha = \big(S^\alpha + P^+ (\Gamma^- \theta)^\alpha \big) \otimes |\mathrm{d}^2 \sigma| = 0 \,. \tag{3.4.11}
$$

and we denote the locus in $T^* \mathcal{F}_\tau$ defined by these equations by $\mathcal{P}_\tau$. So instantaneous phase space $T^* \mathcal{F}_\tau$ is coordinated by $(X^-, X^a, \theta^\alpha, \partial_r X^-, \partial_r \theta^\alpha, P_a, P^+, S_\alpha)$ where the fermionic coordinates satisfy the respective projection conditions. The Hamiltonian is the spacesheet density-valued function $(P_a \partial_0 X^a + P^+ \partial_0 X^- + S_\alpha \partial_0 \theta^\alpha) \otimes |\mathrm{d}^2 \sigma| - L_\tau$ on $\mathcal{P}_\tau$. A simple calculation leads to

$$
H_\tau = \left(\frac{P_a P^a + \bar{g}}{2P^+} - \varepsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma_a \partial_s \theta \right) \otimes |\mathrm{d}^2 \sigma| \,, \tag{3.4.12}
$$

which is a constraint too: $H_\tau|_{\mathcal{P}_\tau} = 0$, as original theory exhibited reparameterisation invariance. It is minus the momentum corresponding to the gauge-fixed coordinate X^+, as it generates time translations. The global (time evolution) Hamiltonian incorporating these constraints is

$$\mathscr{H}_{\tau,tot} = \int_{\Sigma_\tau} \left(H_\tau + c^r\psi_r + \bar{d}\chi \right) , \tag{3.4.13}$$

where we have introduced the Lagrange multipliers c^r and d^α. We quickly verify there are no secondary constraints:

$$\{\psi_r, H_\tau\}_p = d(\iota_{\partial_r} H_\tau) , \qquad \{\chi^\alpha, H_\tau\}_p = d\beta^\alpha , \qquad \beta^\alpha = -2(\Gamma^-\Gamma_a\theta)^\alpha \wedge dX^a ,$$

where d is the exterior differential on the spacesheet: $dX^a = \partial_r X^a \mathrm{d}\sigma^r$. The Poisson brackets of constraints are given by

$$\{\psi_r, \psi_s\}_p = d(\iota_{\partial_s}\psi_r - \iota_{\partial_r}\psi_s) , \qquad \{\psi_r, \chi^\alpha\}_p = -d\iota_{\partial_r}\chi^\alpha , \qquad \{\chi^\alpha, \chi^\beta\}_p = 2P^+(\Gamma^-)^{\alpha\beta} \otimes |\mathrm{d}^2\sigma| .$$

So the ψ_r are first class constraints and the χ^α are second class. The latter are usual in theories containing a Dirac-like kinetic term for fermionic fields (linear in the time derivative of the spinor). They ensure the matching of the number of bosonic and fermionic phase-space variables on the mass-shell: since the momenta are linearly related to the spinor components, only 16 degrees of freedom are propagating, in accordance with the 8 bosonic coordinates and their momenta. The former constraints are first class, so these must be generators of the residual gauge invariance. This is of course the residual diffeomorphism invariance, generated by time-dependent spacesheet reparameterisations,

$$\sigma^r \to \sigma^r + \xi^r(\tau, \sigma) . \tag{3.4.14}$$

We have not yet examined the fermionic residual gauge symmetry. This shall be done in the upcoming paragraph, after some further bosonic gauge fixing.

3.4.3 More Gauge Fixing

Under this the residual reparameterisations, $u^r \equiv \bar{g}^{rs}u_s$ transforms as

$$\mathfrak{L}_{\Xi(\xi)} u^r = -\partial_0\xi^r + \partial_s\xi^r u^s - \xi^s\partial_s u^r . \tag{3.4.15}$$

In particular, $u^r + \mathfrak{L}_\xi \delta u^r = 0$ defines a coupled system of partial differential equations on a compact 2-manifold (where the unknown functions are the ξ^r). It can be decoupled and solved by putting the matrix $\partial_r u^s$ in its Jordan-normal form. Hence we may use the residual reparameterisation invariance to set locally

$$u^r = 0 , \tag{3.4.16}$$

provided this is a consistent set of equations, which we shall check below, when examining the residual gauge symmetry. Hence we have $u_r = 0$, $g = -g_{00}\bar{g}$ and the transverse canonical momenta reduce to $P^a = P^+\partial_0 X^a$. The Hamilton equation for the transverse bosonic degrees of freedom reduces to

$$\partial_0 X^a = \frac{P^a}{P^+} + c^r\partial_r X^a , \tag{3.4.17}$$

which yields $c^r\bar{g}_{rs}c^s = 0$ and therefore, this gauge choice sets the lagrange multiplier c^r to zero. Consequently, on-shell we have

$$\partial_0 P^+ = 0 . \tag{3.4.18}$$

Since P^+ transforms as a density (cf. (3.4.9)), we make the final gauge choice

$$P^+ = P_0^+ \sqrt{w(\sigma)} . \tag{3.4.19}$$

Here P_0^+ is a constant, w is a time-independent density on the membrane space sheet which is normalised to unity at all times τ,

$$\int_{\Sigma_\tau} d^2\sigma \sqrt{w(\sigma)} = 1\,. \tag{3.4.20}$$

Here we have denoted $d^2\sigma = |\mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2|$. The density w can be viewed as the normalised determinant of the membrane space sheet metric $w_{rs}(\sigma)$. Such a Riemannian structure can always be constructed on Σ_τ under the assumptions it is closed, smooth and orientable (w is nowhere zero, as w_{rs} is nowhere degenerate). Hence w_{rs} should not be confused with the induced metric g_{ij}, nor with its 'space sheet square' $\bar{g}_{rs}$, which are dynamical quantities, reflecting the way Σ is mapped into target space. Their relevant relation is $w = -\bar{g}/((P_0^+)^2 g_{00})$ upon use of the field equations. The equations (3.4.16) and (3.4.19) complete the gauge choice. The next step is to examine the residual gauge symmetry in the Lagrangian formalism. We already verified that the condition $X^+ = \tau$ fixes the time component of diffeomorphism-generating vector fields. The spinorial gauge-fixed object $\Gamma^+\theta$ transforms under a general transformation as

$$(\mathfrak{L}_{K(\kappa)} + \mathfrak{L}_{\Xi(\xi)} + \mathfrak{L}_{\Psi(\Lambda)})\Gamma^+\theta = \Gamma^+(\mathbf{1} - \Gamma)\kappa(\sigma) + \Gamma^+\epsilon + \tfrac{1}{2}\lambda^{a+}\Gamma_a\theta\,. \tag{3.4.21}$$

We proceed by substituting the $SO(1,10)$ spinors by two-component vectors consisting of $SO(9)$ spinors: $\theta = (\theta_1, \theta_2)$. Let γ_a, $a = 1, \dots 8$ denote the 16×16 matrices generating $C\ell_{0,8}$. Following the construction (2.2.28), we obtain the generators of $C\ell_{0,10}$ as follows:

$$\Gamma^a = \gamma^a \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^9 = \mathbf{1}_{16} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^{10} = \mathbf{1}_{16} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.4.22}$$

Since $\gamma^1 \dots \gamma^8 = \mathbf{1}_{16}$, we shall denote Γ^9 as $\mathrm{diag}(\gamma^9, -\gamma^9)$ to make the embedding of $SO(9)$ in $C\ell_{0,10}$ explicit. To obtain the Dirac matrices of eleven-dimensional Minkowski space, we define Γ^0 as i times the volume element of $C\ell_{0,10}$, generated by the matrices above:

$$\Gamma^0 = i\Gamma^{11} = i(i\Gamma^1 \dots \Gamma^{10}) = -\Gamma^9\Gamma^{10} = \mathbf{1}_{16} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{3.4.23}$$

In eleven dimensional Minkowski space, the charge conjugation matrix is antisymmetric; we therefore take $C = \Gamma^{10}$:

$$\theta \mapsto \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad \bar{\theta} \mapsto \begin{pmatrix} -i\theta_2^T & i\theta_1^T \end{pmatrix}. \tag{3.4.24}$$

The lightcone gamma matrices are given by

$$\Gamma^+ = \sqrt{2}\mathbf{1}_{16} \otimes \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad \Gamma^- = \sqrt{2}\mathbf{1}_{16} \otimes \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \tag{3.4.25}$$

so that a spinor θ fulfilling the gauge choice has $\theta_1 = 0$. For such spinors we have $\bar{\theta}\Gamma^-\partial_r\theta = \sqrt{2}\theta_2^T\theta_2$. We can expand the fermionic variation in this $SO(9)$ spinor module, obtaining for instance

$$\Gamma^+\Gamma\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \tag{3.4.26}$$

where a straightforward calculation shows that

$$\begin{aligned}\chi &= \frac{\varepsilon^{rs}}{2\sqrt{-g}}\Big[\sqrt{2}\partial_0 X^a\partial_r X^b\partial_s X^c\gamma_{abc}\kappa_1 + 2\partial_r X^b\partial_s X^c\gamma_{bc}\kappa_2 - 2\sqrt{2}(\partial_r X^- + \bar{\theta}\Gamma^-\partial_r\theta)\partial_s X^c\gamma_c\kappa_1\Big] \\ &= \frac{1}{\sqrt{-g}}\Big(\frac{1}{\sqrt{2}}W(\partial_0 X^a\gamma_a)\kappa_1 + W\kappa_2\Big),\end{aligned}$$

where $\varepsilon^{rs} = -\varepsilon^{0rs}$ and $W = \varepsilon^{rs}\partial_r X^a\partial_s X^b\gamma_{ab}$. Here we used the gauge fixing condition $u^r = 0$ to rewrite the last term of the first line, which cancels against a part of the first term upon substituting

$\gamma_{abc} = \gamma_{bc}\gamma_a + \delta_{ab}\gamma_c - \delta_{ac}\gamma_b$. Substituting this expression in the total variation (3.4.21) of the fermionic coordinates yields a relation between the kappa gauge symmetry parameter and the nontrivial residual supersymmetry parameter:

$$\kappa_2 = \frac{1}{\sqrt{2}P_0^+\sqrt{w}}\Big[W(\epsilon_1 - \frac{1}{\sqrt{2}}\lambda^{a+}\gamma_a\theta_2) + (W + P_0^+\sqrt{w}\partial_o X^a\gamma_a)\kappa_1\Big]\,, \tag{3.4.27}$$

where we have used the identity

$$W^2 = 2\overline{g}\mathbf{1} = 2P_0^+\sqrt{w}\sqrt{-g}\mathbf{1}\,. \tag{3.4.28}$$

At first sight one might think half of the kappa symmetry (parameterised by the spinor field κ_1) preserves the gauge choice. Substituting the right hand side of (3.4.27) into a kappa transformation law will show that all the κ_1-dependence drops out. Hence this parameter is physically irrelevant, and may be set to zero. What remains is a global kappa symmetry consisting of a Lorentz transformation and a supersymmetry. Hence there are 32 residual supersymmetries, half of which are actually compensating a kappa symmetry. To find these supersymmetries we put the Lorentz variation equal to zero and substitute the result in the kappa transformation law; this gives us

$$(\mathbf{1}-\Gamma)\begin{pmatrix} 0 \\ W\epsilon_1/P_0^+\sqrt{2w}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon_1 \\ ((2P_0^+\sqrt{w})^{-1}W + \partial_0 X^a\gamma_a)\epsilon_1\end{pmatrix}. \tag{3.4.29}$$

Together with the 16 residual supersymmetries satisfying $\Gamma^+\epsilon = 0$ (i.e. supersymmetries with $\epsilon_1 = 0$), the fermionic transformation above constitute the supersymmetry variation

$$\iota_{\Xi_0(\epsilon)}\delta\theta = \left[\tfrac{1}{2}\Gamma^+(\partial_0 X^a\Gamma_a + \Gamma^-) + \tfrac{1}{4}(P_0^+\sqrt{w})^{-1}\partial_r X^a\partial_s X^b\Gamma^+\Gamma_{ab}\right]\epsilon\,. \tag{3.4.30}$$

The kappa symmetries given by (3.4.27) annihilate the transverse bosonic degrees of freedom on the gauge-fixed configuration subspace. To see this, write $\iota_{K(\kappa)}\delta X^a = \frac{1}{2}\overline{\kappa}\Gamma\Gamma^a\Gamma^+\Gamma^-\theta$ and observe that no terms containing a Γ^+ in Γ contribute. The only nontrivial term is proportional to $\Gamma^-_{ab}\Gamma^+\Gamma^-$, but using $\Gamma^+\kappa = 0$ for κ given by (3.4.27), one immediately concludes also these terms vanish. Hence there are 16 supersymmetries acting on the transverse bosonic variables as

$$\iota_{\Xi(\epsilon)}\delta X^a = \overline{\epsilon}\Gamma^a\theta\,. \tag{3.4.31}$$

The subalgebra of infinitesimal symmetries stabilising condition (3.4.16) on the gauge-fixed mass shell are determined by the following differential equations [3]:

$$\partial_0\xi^r = \lambda^{+a}\varepsilon^{rs}\overline{\theta}\Gamma^-\Gamma_a\partial_s\theta - (P_0^+\sqrt{w})^{-2}\overline{g}g^{rs}\lambda^{+a}\partial_s X_a + \sqrt{2}\overline{\epsilon}\Gamma^-\partial_s\theta\,. \tag{3.4.32}$$

To obtain these, one should also make use of the fermionic equations of motion in the lightcone gauge. The equation above only fixes ξ up to time-independent contributions $\xi^r \mapsto \xi^r + \eta^r(\sigma)$. These contributions are further restricted by the invariance of the last gauge-fixing requirement:

$$P_0^+\partial_r(\sqrt{w}\xi^r) = -\lambda^{a+}\partial_0 X_a + \lambda^{+-}\,, \tag{3.4.33}$$

which fixes the vector field ξ up to spatial contributions satisfying $\partial_r(\eta^r\sqrt{w}) = 0$. These constitute the residual gauge invariance, and generate the so-called group of *area-preserving diffeomorphisms*. This residual invariance shall be extensively investigated in the upcoming section.

3.4.4 The Spectrum

We start our brief discussion of the gauge-fixed membrane mass spectrum with a study of the rôle of the centre-of-mass phase space variables. These are defined by

$$\begin{aligned} X_0^a(\tau) &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|\sqrt{w(\sigma)}X^a(\tau,\sigma)\,, & \theta_0^\alpha(\tau) &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|\sqrt{w(\sigma)}\theta^\alpha(\tau,\sigma)\,, \\ P_0^a(\tau) &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|P^a(\tau,\sigma)\,, & S_0^\alpha(\tau) &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|S^\alpha(\sigma,\tau)\,. \end{aligned} \tag{3.4.34}$$

The Hamilton equation of P^- (or the observation that P^- is the generator of X^+, i.e. time) shows that this quantity is equal to minus the Hamiltonian density. Hence its corresponding CM mode is

$$P_0^-(\tau) = -\int_{\Sigma_\tau} H_\tau \,. \tag{3.4.35}$$

The mass-squared of the membrane is defined by $\mathscr{M}^2 + P_0^\mu P_{0\mu} = 0$ and yields

$$\mathscr{M}^2(\tau) = \int_{\Sigma_\tau} d^2\sigma \Big(\frac{P^a P_a + \bar{g}}{\sqrt{w(\sigma)}} - 2P_0^+ \varepsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma^a \partial_s \theta \Big) - (P_0)^a (P_0)_a \,. \tag{3.4.36}$$

Note that the mass does not depend on the CM coordinates and momenta defined in (3.4.34), if the membrane does not wind around a certain direction and has no boundary. Writing the integrand above as $M^2 \in \mathrm{Dens}(\Sigma_\tau)$, we find with the chain rule that for a phase space variable W^M with normalised zero mode $(W_0)^M$ as above,

$$\iota\Big(\frac{\delta}{\delta W_0^M}\Big)\delta\mathscr{M}^2 = \int_{\Sigma_\tau} \iota\Big(\frac{\delta}{\delta W^M}\Big)\delta M^2 - 2(P_0)^a \int_{\Sigma_\tau} \iota\Big(\frac{\delta}{\delta W^M}\Big)\delta P_a \,. \tag{3.4.37}$$

Hence differentiating the mass squared to the zero modes gives

$$\begin{aligned} \iota\Big(\frac{\delta}{\delta X_0^a}\Big)\delta\mathscr{M}^2 &= 2\int_{\Sigma_\tau} d\gamma_a \,, \quad \gamma_a = -\frac{\varepsilon^{rs}}{\sqrt{w}} \partial_r X_a \partial_s X_b \wedge dX^b + P_0^+ \bar{\theta} \Gamma^- \Gamma_a \wedge d\theta \,, \\ \iota\Big(\frac{\delta}{\delta(\theta_0)_\alpha}\Big)\delta\mathscr{M}^2 &= 4P_0^+ \int_{\Sigma_\tau} d\beta^\alpha \,, \qquad \beta^\alpha = X^a (\Gamma^- \Gamma_a)^\alpha{}_\beta \wedge d\theta^\beta \,. \end{aligned} \tag{3.4.38}$$

The independence of the $\mathbf{X}_0$ mode is reasonable from the physical point of view. The decoupling of the fermionic zero mode turns out to be very important, as it can be used to show the existence of a massless $d = 11$ supergravity multiplet in the membrane spectrum. Furthermore we observe that the transverse CM-momenta are decoupled as well; the eigenstates of the Hamiltonian therefore factorise into a relativistic particle wave function of the CM modes and a wave function describing the higher modes. In a first attempt to analyse the stability of this nontrivial part of the spectrum we set the spinors equal to zero and observe that the remaining transverse bosonic Hamiltonian density is of the standard form $T + V$, where the potential density is given by

$$V(\mathbf{X}) = \bar{g} = (\epsilon^{rs} \partial_r X^a \partial_s X^b)^2 \,. \tag{3.4.39}$$

This quantity (which we have assumed to be positive definite) becomes zero in regions where the coordinates X^a only depend on a linear combination of σ^1 and σ^2, say $X^a(\tau, \sigma^1, \sigma^2) = X^a(\tau, z)$ where $z = a_r \sigma^r$. We then have

$$\bar{g}_{rs} = \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_2 a_1 & a_2^2 \end{pmatrix} (\partial_z X^a(\tau, z))^2 \,, \tag{3.4.40}$$

so that $\bar{g} = 0$. Geometrically, these are regions where the bosonic image of the spacesheet of the membrane degenerates into a one-dimensional, stringlike manifold. The important difference with string theory in this regime is that the energy of the stretched membrane is not proportional to, but independent of the length of the object. Hence, as the brane moves through a potential valley, certain components can escape to infinity at a finite cost of energy and which mass can become arbitrarily small. All bosonic p-branes with a Nambu-based action suffer from the potential instability mentioned above, since all positive $(p-1)$-metric determinants of the form (3.4.39) contain valleys corresponding to $p - 1$ brane configurations, except for the $p = 1$ case, string theory, where the quadratic potential fully confines all the modes. Another way of looking at the instability is from the second-quantisation point of view: since 2 membranes connected by an infinitely thin tube is physically equivalent to a disconnected configuration, there is no conservation of 'membrane number', nor a physical meaning of membrane topology. Here it should be noted that these properties reflect the nature of the embedded bosonic submanifold of superspace, not the worldvolume which is of course nondynamical and required to be smooth and connected.

At the classical level the bosonic membrane spectrum thus allows no interpretation in terms of elementary particle states. At the quantum level however, this behaviour changes drastically. It turns out that the ground state energy induces an effective potential which prevents stringlike states to escape through the potential valleys. Another interpretation of this phenomenon comes from the uncertainty principle, suppressing stringlike configurations because of the deviation in the momentum wave function. The bosonic membrane Hamiltonian exhibits qualitative similarities with the two-dimensional quantum mechanical Hamiltonian

$$H = p_x^2 + p_y^2 + x^2 y^2 \tag{3.4.41}$$

as the potential has zero-energy valleys along the x and y axes. However, this Hamiltonian can be written as the sum of a free particle Hamiltonian and two harmonic-oscillator Hamiltonians with variable frequencies,

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + (\tfrac{1}{2}p_x^2 + \tfrac{1}{2}y^2x^2) + (\tfrac{1}{2}p_y^2 + \tfrac{1}{2}x^2y^2)\,. \tag{3.4.42}$$

The last two Hamiltonians are viewed as operators associated to a particle moving on a line. Their eigenvalues are therefore always bigger then the ground state energy of a harmonic oscillator (here we have used the convention $\hbar = 1$),

$$H \geq \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(|x| + |y|)\,. \tag{3.4.43}$$

The inequality above should be read as an inequality of all eigenvalues. The Hamiltonian is now bounded from below by an effective Hamiltonian incorporating the harmonic oscillator zero-point fluctuations, which looks like an inverted pyramid and has definitely a discrete spectrum. Later on, we shall see that the finite-N approximation of the bosonic membrane has a nonlinear potential which is essentially a linear combination of these potentials, and therefore it has a discrete spectrum.

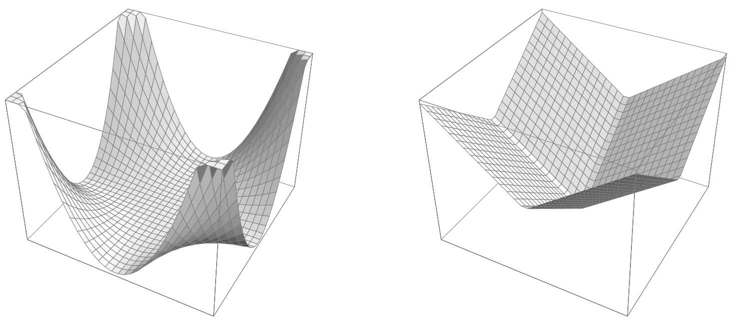


Figure 3: On the left hand side, the potential in the Hamiltonian of the toy model describing the bosonic membrane, containing 2 flat valleys. On the right hand side the effective, uncoupled confining potential obtained by including ground state energies of the harmonic oscillators; the flat directions have disappeared.

Turning supersymmetry on changes the spectrum drastically. This is essentially due to the fact that a supersymmetric harmonic oscillator has zero ground state energy. Hence the spectrum of a supermembrane in flat superspace is continuous. A rigourous proof of this statement was constructed in [5]. Due to this fact it is not clear whether (normalisable) massless states even exist. After the discovery of the instability of the supermembrane the theory was declared dead for a while. This changed with the second string revolution and the discovery of D-branes. In this new perspective the continuous spectrum and second-quantised nature reflect the interpretation of a supermembrane as a supersymmetric quantummechanical system of Dirichlet particles in ten-dimensional Minkowski space. This interpretation requires however a new formulation of the membrane action as a gauge theory, which shall be the topic of the upcoming section.

3.5 APD Gauge Theory

3.5.1 The Residual Constraint

In the previous section we introduced the lightcone gauge, keeping only 16 fermionic degrees of freedom and 10 bosonic degrees of freedom. The momentum corresponding to the fixed X^+ coordinate became a constraint, namely the Hamiltonian. Subsequently we diagonalised the timelike components of the pulled-back metric to obtain a gauge-fixed transversal momentum $P^+ = P_0^+\sqrt{w}$. As we shall see, the X^- coordinate corresponding to this momentum degree of freedom, which no longer appears in the Hamiltonian or Lagrangian, gives rise to a constraint as well. This constraint generates the residual gauge freedom, consisting of area-preserving diffeomorphisms, (time-independent) spacesheet diffeomorphisms preserving the density $\sqrt{w}$. The next step will be to interpret the resulting theory as a gauge theory of the infinite-dimensional algebra on the time axis. The space-dependency of the remaining variables will be replaced by Lie-algebra valuedness and we will introduce a connection whose gauge fields generate the supermembrane constraints.

An until this point unexplained feature of the Hamiltonian (3.4.12) is its independence of the X^- coordinate. This degree of freedom is determined by the gauge condition (3.4.16). Putting the constraints ψ_r equal to zero is equivalent to the requirement

$$dX^- = -\partial_0 X_a dX^a - \bar{\theta}\Gamma^- d\theta \equiv \gamma\,. \tag{3.5.1}$$

So γ is an exact one-form. This gives rise to a set of constraints using Hodge theory on the spacesheet exterior algebra. We assume that the density $\sqrt{w(\sigma)}$ belongs to a time-independent Riemannian structure $w_{rs}(\sigma)$, but we will have to make sure that no physical observables depend on this tensor other then through its determinant, as it is introduced by hand. Every compact pseudo-Riemannian manifold M^n has a nondegenerate bilinear symmetric inner product on its space of sections of homogeneous k-form bundles. For $\alpha, \beta \in \Omega^k(M)$ it is given by

$$\langle \alpha, \beta\rangle = \int_M \alpha \wedge *\beta\,, \tag{3.5.2}$$

where $* : \Omega^k(M) \longrightarrow \Omega^{k-n}(M)$ is the Hodge isomorphism, induced by the Riemannian duality between vectors and one-forms. It gives rise to the *adjoint* operator of the de Rham differential, $\delta = *d* : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ which satisfies $\langle \alpha, d\beta\rangle = \langle \delta\alpha, \beta\rangle$ for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k-1}(M)$. The *Laplace-Beltrami* operator is the grading-preserving differential operator

$$\Delta = d\delta + \delta d : \Omega^k(M) \longrightarrow \Omega^k(M)\,. \tag{3.5.3}$$

A differential form α is called *harmonic* if it satisfies $\Delta\alpha = 0$ (which is equivalent to $d\alpha = \delta\alpha = 0$), and we shall denote the space of such k-forms with $H^k_\Delta(M)$. One can show that this is a finite-dimensional vector space which is isomorphic to the de Rham cohomology vector space $H^k(M)$. At the basis of our proceedings lies the following theorem:

$$\Delta(\Omega^k(M)) = (H^k_\Delta(M))^\perp\,. \tag{3.5.4}$$

The image of Δ being perpendicular to the space of harmonic forms immediately follows from self-adjointness of Δ. The inclusion backwards, the statement that every α satisfying $\langle \alpha, H_\Delta(M)\rangle = 0$ is the Laplacian of some form is not trivial to show, and requires ellepticity properties of the Laplacian. Composing both sides of (3.5.4) with the space of harmonic forms gives the *Hodge decompostion*

$$d\Omega^{k-1}(M) \oplus \delta\Omega^{k+1}(M) \oplus H^k_\Delta(M) = \Omega^k(M)\,. \tag{3.5.5}$$

By orthogonality of this decomposition and $d^2 = 0$ and the fact that harmonic forms are closed, we have

$$\ker(d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)) = d(\Omega^{k-1}(M)) \oplus H^k_\Delta(M)\,. \tag{3.5.6}$$

The constraint (3.5.1) states that γ is exact. By the orthogonal decomposition above this is equivalent to the requirements $d\gamma = 0$ and $\langle \gamma, H^1_\Delta(\Sigma_\tau)\rangle = 0$. Given a complete basis φ_λ of harmonic forms, (3.5.1) corresponds to

$$\psi \equiv d(\partial_0 X_a) \wedge dX^a + d\bar{\theta}\Gamma^- \wedge d\theta = 0\,, \tag{3.5.7}$$

$$\psi_\lambda \equiv \int_{\Sigma_\tau} \varphi_\lambda \wedge *(\partial_0 X_a dX^a + \bar{\theta}\Gamma^- d\theta) = 0\,. \tag{3.5.8}$$

We mention that this construction gets modified in compact target spaces, as there exist exact $\mathbb{R}/\mathbb{Z}$-valued forms which do not integrate to zero along a noncontractible closed loop on Σ_τ. These theories shall be treated in the next chapter. Note also that the number of independent harmonic forms is directly related to the topology of the spacesheet. A standard result from algebraic topology of closed 2-dimensional manifolds is

$$\dim(H^0(\Sigma_\tau)) - \dim(H^1(\Sigma_\tau)) + \dim(H^2(\Sigma_\tau)) = \chi(\Sigma_\tau) = 2 - 2g(\Sigma_\tau)\,. \tag{3.5.9}$$

Here $\chi(\Sigma_\tau)$ is the Euler characteristic and $g(\Sigma_\tau)$ is the genus of the manifold. The zeroth cohomology vector space is the space of constant functions on Σ_τ, and therefore one-dimensional for connected spacesheets. Furthermore, if Σ_τ is oriented, by Hodge duality $H^2(\Sigma_\tau)$ is isomorphic to $H^0(\Sigma_\tau)^*$, which is again one dimensional. We can then solve (3.5.9), which gives

$$\dim(H^1(\Sigma_\tau)) = 2g(\Sigma_\tau)\,. \tag{3.5.10}$$

Using $H^1_\Delta(\Sigma_\tau) \cong H^1(\Sigma_\tau)$, we find there are $2g(\Sigma_\tau)$ linearly independent harmonic one-forms.

3.5.2 Area-Preserving Diffeomorphisms

The symmetry group generated by the residual (first class) constraints is a subgroup of the world-volume reparameterisation group, consisting of time-independent invertible and differentiable mappings which leave the density $\sqrt{w}$ invariant:

$$\sigma^r \to \sigma^r + \xi^r(\sigma)\,, \quad \text{with } \partial_r(\sqrt{w(\sigma)}\xi^r(\sigma)) = \nabla_r \xi^r(\sigma) = 0\,, \tag{3.5.11}$$

where ∇_r is the Levi-Civita covariant derivative with respect to the metric $w_{rs}(\sigma)$. This transformation leaves the constraint (3.5.1) invariant for spacesheets without boundary, since $\iota_\xi \delta\gamma = \partial_r(\xi^r\gamma)$, and furthermore it leaves the gauge choice (3.4.18) by definition invariant. The equation above defines the algebra of area-preserving diffeomorphisms. Furthermore, the Riemannian metric defines a smooth, closed, nondegenerate 2-form

$$\nu = \sqrt{w(\sigma)}\mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2\,, \tag{3.5.12}$$

defining the spacesheet to be a *symplectic* manifold. The criterium (3.5.11) for the vector field $\xi = \xi^r\partial_r$ is equivalent to

$$\mathfrak{L}_\xi \nu = 0\,. \tag{3.5.13}$$

Vector fields satisfying this condition shall be referred to as locally Hamiltonian vector fields, and in mathematics the group generated by these vectors is the group of *symplectomorphisms* $\mathrm{SDiff}_\lambda(\Sigma_\tau)$, diffeomorphisms $\varphi : \Sigma_\tau \longrightarrow \Sigma_\tau$ restricted to satisfy

$$\varphi^*\nu = \nu\,. \tag{3.5.14}$$

There have been various proposals to construct a topology and a differential calculus on this group and establish some kind of Lie theory. The major problems with such constructions are the absence of a complete Banach norm on the spaces underlying diffeomorphism groups, which makes the definition of smoothness of translation along the group (composition of maps) cumbersome. Omori [54] circumvented this problem by considering the diffeomorphism group as a limit of groups modeled on Banach spaces. It is well-known that the space of smooth functions on Σ_τ is not Banach, but its completion to $C^k(\Sigma_\tau)$ for $0 \leq k < \infty$ possesses a complete norm (the uniform C^k topology) and also the completion to the Sobolev space $W^k(\Sigma_\tau)$ is Banach. Hence the diffeomorphism group is a limit of 'nice' infinite-dimensional Lie groups, a so-called *ILH (inverse limit of Hilbert) Lie group*. For a review of this approach and for applications in plasma physics, the theory of incompressible fluids and general relativity, we refer to [55, 56]. A more modern approach is to modify the vector space on which the group manifold should be modeled on. In [57], an extensive examination of the properties these spaces should satisfy is performed, resulting in the definition of a *convenient vector space*. We shall not go into detail about this, since we shall only work in the Lie algebra of Hamiltonian vector fields. It is however important to realise that only diffeomorphisms in some neighbourhood of the identity are generated by such vector fields. For example, the membrane modeled on a sphere has a disconnected component in its gauge group, obtained by composing an 'ordinary' local diffeomorphism with the antipodal map. Even stronger, an artefact of infinite-dimensionality is the property of the exponent (flow mapping in this case) being not surjective in *any* neighbourhood of the identity. Since we work with the Lie algebra generating these diffeomorphisms, we shall ignore entire classes of diffeomorphisms, and the inclusion of these would correspond to an infinite but discrete set of residual symmetries of the membrane. The identification and inclusion of these transformations is an open question.

The area-preserving diffeomorphisms form a so-called *regular* subgroup of the total diffeomorphisms, because it does have the property that the exponent maps smooth curves in the algebra to smooth curves in the group. Among the regular Lie subgroups of $\mathrm{Diff}(\Sigma)$ are the group of orientation-preserving diffeomorphisms $\mathrm{Diff}_+(\Sigma)$ (provided Σ is orientable), the group of analytic diffeomorphisms $\mathrm{Diff}^\omega(\Sigma)$ (provided Σ is analytic), the group of volume-preserving diffeomorphisms $\mathrm{Diff}_w(\Sigma)$ (provided a nonsingular volume form w on Σ is given) and the group of symplectomorphisms $\mathrm{SDiff}_\nu(\Sigma)$, preserving a symplectic form ν on Σ (provided Σ is symplectic). The 2 last ones coincide in our 2-dimensional case, and since Σ_τ is postulated to be compact, we may omit the c-subscript. The Lie algebra of the symplectic diffeomorphism group may be shown to be the space $\mathfrak{X}_\nu(\Sigma_\tau)$ of vector fields satisfying (3.5.13), equipped with the *negative* of the usual Lie bracket of vector fields[11]. The trivial solutions to (3.5.13) are the so-called *globally Hamiltonian* vector fields,

$$\xi(\sigma) = \frac{\varepsilon^{rs}}{\sqrt{w(\sigma)}}\partial_s f(\sigma)\frac{\partial}{\partial\sigma^r} \equiv \mathrm{grad}_\nu f\,, \tag{3.5.15}$$

for some $f \in C^\infty(\Sigma, \mathbb{R})$. However, for nontrivial spacesheet topologies, there are more solutions; the linear space of these solutions is isomorphic to the harmonic space $H^1_\Delta(\Sigma_\tau)$ of the membrane spacesheet. Because ν is symplectic the mapping

$$g : \Gamma T\Sigma_\tau \longrightarrow \Omega^1(\Sigma_\tau) : \xi \mapsto \iota_\xi \nu \tag{3.5.16}$$

is an isomorphism. It associates to the vector field $\xi^r\partial_r$ the one form $\beta = g(\xi)$ satisfying

$$\sqrt{w(\sigma)}\xi^r(\sigma) = \varepsilon^{rs}\beta_s(\sigma)\,. \tag{3.5.17}$$

If ξ^r is area-preserving, then β is closed and vice versa:

$$d\beta = \partial_r(\sqrt{w(\sigma)}\xi^r(\sigma))d\sigma^1 \wedge d\sigma^2 = 0\,. \tag{3.5.18}$$

The trivial solutions (3.5.15) correspond to β being exact: $\beta = df$. Invoking the decomposition theorem (3.5.6) with $k = 1$ and $M = \Sigma_\tau$, the general solution of a closed form β looks like $\beta = df + a^\lambda\varphi_\lambda$ where $a_1, \ldots, a_{2g(\Sigma)}$ are constants. Hence the general solution of a Hamiltonian vector field ξ is

$$\xi = \mathrm{grad}_\nu f + a^\lambda\phi_\lambda\,, \quad \lambda = 1, \ldots, 2g(\Sigma_\tau)\,, \tag{3.5.19}$$

where the ϕ_λ are *harmonic vector fields* on the manifold Σ_τ: their divergence vanishes but they cannot be written as a gradient of a scalar function. They are related to the cycles φ_λ by the

[11] because the adjoint action exponentiates to the adjoint transformation by the inverse diffeomorphism

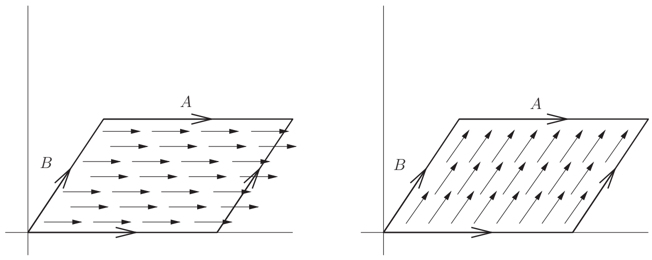


Figure 4: Two independent harmonic vector fields on a flat torus (parallelogram with opposite sides identified); these fields are divergence-free, but can not be written as the gradient of a continuous scalar function on the torus.

inverse of g,

$$\phi_\lambda(\sigma) = \frac{\varepsilon^{rs}}{\sqrt{w(\sigma)}}\varphi_{\lambda s}(\sigma)\partial_r \,. \tag{3.5.20}$$

We can pull back the inner product structure on Ω^1 to the space of vector fields with the Hodge and Riemannian dual. One then obtains the inner product of vector fields ξ, ζ,

$$\langle \xi, \zeta\rangle = \int_{\Sigma_\tau} \iota_\xi \nu \wedge *\iota_\zeta \nu = \int_{\Sigma_\tau} *w(\xi,\zeta) = \int_{\Sigma_\tau} |\mathrm{d}^2\sigma|\sqrt{w(\sigma)}w_{rs}(\sigma)\xi^r(\sigma)\zeta^s(\sigma)\,. \tag{3.5.21}$$

With respect to this inner product the space of Hamiltonian vector fields has an orthogonal decomposition

$$\mathfrak{X}_\nu(\Sigma_\tau) = \mathfrak{X}^G_\nu(\Sigma_\tau) \oplus \mathfrak{X}^\Delta_\nu(\Sigma_\tau)\,, \quad \mathfrak{X}^G_\nu(\Sigma_\tau) = \mathrm{grad}_\nu(C^\infty(\Sigma_\tau))\,, \quad \mathfrak{X}^\Delta_\nu(\Sigma_\tau) = g^{-1}(H^1_\Delta(\Sigma_\tau))\,.$$

The space of Hamiltonian vector fields is given a Lie algebra structure by the minus the Lie bracket of vector fields. This gives rise to the exact sequence of Lie algebras

$$0 \longrightarrow H^0(\Sigma) \longrightarrow C^\infty(\Sigma_\tau, \mathbb{R}) \stackrel{\mathrm{grad}_\nu}{\longrightarrow} \mathfrak{X}_\nu(\Sigma_\tau) \stackrel{\gamma}{\longrightarrow} H^1(\Sigma_\tau) \longrightarrow 0\,, \tag{3.5.22}$$

where $\gamma(\xi) = [\iota_\xi \nu]$. For these maps to be Lie algebra homomorphisms, the vector spaces $H^0(\Sigma_\tau)$ and $H^1(\Sigma_\tau)$ should be equipped with the zero bracket, $\mathfrak{X}_\nu(\Sigma_\tau)$ is equipped with minus the Lie bracket of vector fields and $C^\infty(\Sigma_\tau, \mathbb{R})$ is given a Lie algebra structure by the *spacesheet Poisson bracket*,

$$\{f, g\} = \nu(\mathrm{grad}_\nu f, \mathrm{grad}_\nu g) = \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}}\partial_r f(\sigma)\partial_s g(\sigma)\,, \tag{3.5.23}$$

which can easily be seen to satisfy the Jacobi identity. We shall denote the resulting Poisson algebra on the spacesheet by $P(\Sigma_\tau)$. Up to (integration) constants, grad_ν maps injectively to $\mathfrak{X}^G_\nu(\Sigma_\tau)$; since constant functions are central under the Poisson bracket, $P(\Sigma_\tau)$ is isomorphic to a central extension of $\mathfrak{X}^G_\nu(\Sigma_\tau)$. Moreover, the exact sequence implies that $\mathfrak{X}^G_\nu(\Sigma_\tau)$ is an ideal of the symplectic diffeomorphism algebra,

$$[\mathfrak{X}_\nu(\Sigma_\tau), \mathfrak{X}_\nu(\Sigma_\tau)] \subseteq \mathfrak{X}^G_\nu(\Sigma_\tau)\,. \tag{3.5.24}$$

One may calculate the commutators explicitly:

$$\begin{aligned}[][\mathrm{grad}_\nu f, \mathrm{grad}_\nu g] &= \mathrm{grad}_\nu(\{f, g\})\,,\\ [\phi_\lambda, \mathrm{grad}_\nu f] &= -\mathrm{grad}_\nu(\phi_\lambda(f)) = -\mathrm{grad}_\nu({\phi_\lambda}^r\partial_r f)\,,\\ [\phi_\lambda, \phi_{\lambda'}] &= -\mathrm{grad}_\nu(\sqrt{w(\sigma)}\epsilon_{rs}{\phi_\lambda}^r{\phi_{\lambda'}}^s)\,.\end{aligned} \tag{3.5.25}$$

3.5.3 Gauging Area-Preserving Diffeomorphisms

The next step is to *gauge the symmetry*, i.e. to construct a theory with an extra gauge field corresponding the APD invariance such that it coincides with the original theory upon fixing a gauge for the gauge field and implementing its equation of motion. Recall that the ingredients of a gauge theory are a Lie group G which is the structure group of a principal fibre bundle P_G and a gauge field that defines a connection on the associated bundles to P_G. Is it possible to construct a gauge theory associated to the APD group? The following theorem by Michor [57] answers affirmative:

Theorem 3.1 *For N and M smooth finite-dimensional closed connected manifolds with $\dim(N) \leq \dim(M)$, the set $\mathrm{Emb}(N,M)$ of all smooth embeddings $N \longrightarrow M$ is an open submanifold of $C^\infty(N,M)$, the infinite-dimensional manifold of smooth mappings from N to M. $\mathrm{Emb}(N,M)$ is the total space of a smooth principal fibre bundle with structure group $\mathrm{Diff}(N)$ on the space $B(N,M) = \mathrm{Emb}(N,M)/\mathrm{Diff}(N)$ of all smooth submanifolds of M of type N.*

One straightforwardly generalises this theorem for M a supermanifold. Moreover, restricting the diffeomorphism group to $\mathrm{Diff}_\nu(M)$, we restrict the space of embeddings to $\mathrm{Emb}_\nu(N,M)$, the space of embedding preserving the symplectic form ν under pullback. Because we assume the existence of a slicing diffeomorphism, there is a projection $\pi : B_\nu(\Sigma_\tau, M^{11|32}) \longrightarrow \mathbb{R}$. We can view $\mathrm{Emb}_\nu(\Sigma_\tau, M^{11|32})$ as a principal $\mathrm{Diff}_\nu(\Sigma_\tau)$-bundle over $\mathbb{R}$ if on coordinate overlaps the fibres are related by the composition of an area-preserving diffeomorphism. This imposes a restriction on the structure of the worldvolume: we had already assumed that the two-manifolds Σ_τ are compact and without boundary and we will now assume that they are all diffeomorphic. In particular, this implies that $\dim(H^p(\Sigma_\tau))$ is independent of τ, and hence the Lie algebras of Hamiltonian vector fields are isomorphic (acted upon by some Ad_φ), and the gauge symmetry may be described by curves in a fixed algebra. Recall that this is a reasonable requirement for a Hamiltonian treatment of diffeomorphism-invariant theories on topologically nontrivial base manifolds, since we do not want to deal with time-dependent gauge groups.

We assume thus that all the spacesheets are diffeomorphic to some fixed Σ, and we are therefore dealing with a principal $\mathrm{Diff}_\nu(\Sigma)$-bundle $\mathrm{Emb}_\nu(\Sigma, M^{32|11}) \longrightarrow \mathbb{R}$. The components of Z, namely X^μ and θ^α correspond to Hamiltonian vector fields when acted on with the symplectic gradient and as such transform under the adjoint representation of the algebra $\mathfrak{X}_\nu(\Sigma)$,

$$Z \mapsto \mathrm{grad}_\nu(Z) \in \Gamma(\oplus^{11+32}\mathrm{Emb}_\nu(\Sigma, M^{11|32}) \times_{\mathrm{Ad}} \mathfrak{X}_\nu(\Sigma), \mathbb{R})\,. \tag{3.5.26}$$

This representation has some important physical consequences: as the constant functions are mapped to the zero element in the representation space, all the transformation laws will be determined up to constants (integration constants arising from inverting the gradient). These may be set zero, as they are absorbed in the invariance of the physical laws under global translation in the target space. On the other hand, there are Lie algebra elements which cannot be reached by taking the gradient of a scalar function: the harmonic vector fields shall only play a role in the connection on the adjoint bundle. However, in the next chapter we will consider membranes moving in compact target spaces, where we will see that the gradient of circle-valued functions may contain nonzero contributions of harmonic vector fields.

From the representation of the embedding coordinates above, we immediately obtain their transformation law,

$$\iota_\xi \delta(\mathrm{grad}_\nu Z^M(\tau,\sigma)) = [\xi(\tau), \mathrm{grad}_\nu Z^M(\tau,\sigma)]\,, \tag{3.5.27}$$

where the brackets are the Lie brackets in $\mathfrak{X}_\nu(\Sigma)$, which is minus the Lie bracket of vector fields. By the algebra relations (3.5.25) the right hand side is a gradient and moreover grad_ν commutes with ι_ξ since these are operations in orthogonal directions of the double complex. We may therefore derive a transformation rule for the scalar fields up to a time-varying constant, which is set to zero and shall manifest as zero-mode translation invariance. We obtain for $\xi(\tau,\sigma) = \mathrm{grad}_\nu f(\tau,\sigma) + \chi^\lambda(\tau)\phi_\lambda(\sigma)$

$$\begin{aligned}\iota_\xi \delta X^a(\tau,\sigma) &= \{f(\tau,\sigma), X^a(\tau,\sigma)\} - \chi^\lambda(\tau)\phi^r_\lambda(\sigma)\partial_r X^a(\tau,\sigma)\,,\\ \iota_\xi \delta\theta^\alpha(\tau,\sigma) &= \{f(\tau,\sigma), \theta^\alpha(\tau,\sigma)\} - \chi^\lambda(\tau)\phi^r_\lambda(\sigma)\partial_r \theta^\alpha(\tau,\sigma)\,. \end{aligned} \tag{3.5.28}$$

Of course these APD diffeomorphism generating vector fields act on the spacesheet volume form too:

$$\iota_\xi|\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2| = \varepsilon_{rs}\xi^r\mathrm{d}\sigma^s = \frac{1}{\sqrt{w(\sigma)}}df(\tau,\sigma) + \varepsilon_{rs}\chi^\lambda(\tau)\phi^r_\lambda(\sigma)\mathrm{d}\sigma^s\,. \tag{3.5.29}$$

Let us now introduce a connection one-form $\Omega\in\Gamma(T^*\mathbb{R}\otimes\mathfrak{X}_\nu(\Sigma))$:

$$\Omega = (\mathrm{grad}_\nu\omega + A^\lambda\phi_\lambda)\otimes\mathrm{d}\tau\,. \tag{3.5.30}$$

This Lie algebra-valued form on the real line acts on the embedding coordinates through the adjoint representation: $\Omega(Z) = [\Omega, \mathrm{grad}_\nu Z]$, or equivalently

$$\Omega(X^a) = (\{X^a,\omega\} + A^\lambda\phi_\lambda(X^a))\mathrm{d}\tau\,,$$

and similarly it acts on the fermionic coordinates. We obtain a covariant time derivative on the sections of the fibre bundle by setting

$$\nabla_0 Z^M = \partial_0 Z^M + \Omega(Z^M)\,, \tag{3.5.31}$$

and it gives rise to a covariant de Rham differential $d_\nabla : C^\infty(\mathbb{R},\mathbb{R})\otimes\mathfrak{X}_\lambda(\Sigma)\longrightarrow \Gamma T^*\mathbb{R}\otimes\mathfrak{X}_\lambda(\Sigma)$ by the rule $d_\nabla(f\otimes X) = df\otimes X - f\mathrm{d}\tau\otimes[\Omega, X]$. As usual, this provides the transformation rule of the gauge field under a time-dependent area-preserving diffeomorphism ξ,

$$\iota_\xi\Omega = d_\nabla\xi = (\partial_0\xi - [\Omega,\xi])\otimes\mathrm{d}\tau\,. \tag{3.5.32}$$

which reads in components

$$\iota_\xi\delta\omega = \partial_0 f - \{\omega, f\} - \chi^\lambda\phi_\lambda(\omega) + A^\lambda\phi_\lambda(f) + A^\lambda\chi^{\lambda'}\Phi_{\lambda\lambda'}\,,\quad \iota_\xi\delta A^\lambda = \partial_0\chi^\lambda\,, \tag{3.5.33}$$

where $\Phi_{\lambda\lambda'} = \sqrt{w}\epsilon_{rs}\phi^r_\lambda\phi^s_{\lambda'}$. The Lagrangian we claim to be gauge-invariant is the one proposed in [53],

$$\begin{aligned} L = \mathrm{d}\tau\wedge|\mathrm{d}^2\sigma|P_0^+\sqrt{w}\Big[&\tfrac{1}{2}\nabla_0 X^a\nabla_0 X_a + \bar\theta\Gamma_-\nabla_0\theta - \tfrac{1}{4}(P_0^+)^{-2}\{X^a,X^b\}\{X_a,X_b\}\\ &+ (P_0^+)^{-1}\bar\theta\Gamma^-\Gamma_a\{X^a,\theta\}\Big]\,, \end{aligned} \tag{3.5.34}$$

where $\mathrm{d}\tau$ is assumed to be positively oriented and and we have rescaled the spacesheet volume element by the constant factor P_0^+, but the covariant derivative is not affected by this rescaling; this causes no problems for APD invariance, since all terms will transform covariantly and therefore yield all separately total derivatives. So to check APD invariance we only have to verify that the $\nabla_0 Z^M$ transform covariantly. By construction this is the case:

$$\mathfrak{L}_\xi\nabla_0 X^a = \{f, \nabla_0 X^a\} - \chi^\lambda\phi_\lambda(\nabla_0 X^a)\,. \tag{3.5.35}$$

Similarly the covariant derivative of the fermionic coordinates transform. To calculate these transformation rules explicitly, one should use the following property,

$$\phi_\lambda(\{f,g\}) = \{\phi_\lambda(f), g\} + \{f, \phi_\lambda(g)\}\,, \tag{3.5.36}$$

which may be derived directly or from the Jacobi identities of $\mathfrak{X}_\lambda(\Sigma)$. This property will turn out to be important in next chapter: it defines the harmonic vector fields as *outer derivations* on the Poisson algebra. Writing $L = \ell P_0^+\sqrt{w}\mathrm{d}\tau\wedge|\mathrm{d}^2\sigma|$, we see that ℓ is a scalar polynomial in the fields, their covariant time derivatives and spacesheet Poisson brackets between them and therefore transforms as $\mathfrak{L}_\xi\ell = \{f,\ell\} - \chi^\lambda\phi_\lambda(\ell)$. Hence the area-preserving diffeomorphisms are a manifest symmetry of our gauge theory,

$$\mathfrak{L}_\xi L = d(\ell P_0^+\sqrt{w}\iota_\xi|\mathrm{d}^2\sigma|) + (\mathfrak{L}_\xi\ell)P_0^+\sqrt{w}|\mathrm{d}^2\sigma| = 0\,. \tag{3.5.37}$$

3.5.4 Supersymmetry Transformations

The theory above also exhibits global supersymmetry; it was already established in [53] that for $\chi_\lambda = 0$ the Lagrangian is invariant under the supersymmetry transformation $\Xi : S_M(\mathbb{R}^{11}) \longrightarrow \Gamma T\mathcal{F}$, the domain being the Majorana-fermionic sector of a fiber, the resulting vector field being

$$\begin{aligned}\iota_{\Xi_0(\epsilon)}\delta X^a &= -2\bar{\epsilon}\Gamma^a\theta\,,\\ \iota_{\Xi_0(\epsilon)}\delta\theta &= \big(\tfrac{1}{2}\Gamma^+(D_0X^a\Gamma_a+\Gamma^-)+\tfrac{1}{4}(P_0^+)^{-1}\{X^a,X^b\}\Gamma^+\Gamma_{ab}\big)\epsilon\,,\\ \iota_{\Xi_0(\epsilon)}\omega &= -2(P_0^+)^{-1}\bar{\epsilon}\theta\,.\end{aligned} \tag{3.5.38}$$

where $D_0 = \nabla_0|_{\chi=0} = \partial_0 - \{\omega,\quad\}$. Notice that this transformation preserves the gauge choices, and for supersymmetry parameters satisfying the gauge condition $\Gamma_+\epsilon = 0$, the variation in θ is just a translation by ϵ. Note also that ω comes with an 'extra' factor $(P_0^+)^{-1}$, this to compensate the fact that we haven't rescaled the volume element in the covariant derivative; for the diffeomorphism invariance this did not cause problems since all the terms themselves transformed into total derivatives, however for supersymmetry different terms have to cancel each other. The obvious way to include harmonic vector fields is to replace D_0 with ∇_0 in the transformation rule above. This corresponds to adding a vector field $\Xi_1(\epsilon) \in \Gamma T\mathcal{F}$, defined by

$$\iota_{\Xi_1(\epsilon)}\delta X^a = \iota_{\Xi_1(\epsilon)}\delta\omega = 0\,,\quad \iota_{\Xi_1(\epsilon)}\delta\theta = \frac{1}{2}A^\lambda\phi_\lambda(X^a)\Gamma^+\Gamma_a\epsilon \tag{3.5.39}$$

and accordingly we split the Lagrangian $L = L_0 + L_1$ with $L_0 = L|_{A=0}$ and

$$L_1 = \mathrm{d}\tau\wedge|\mathrm{d}^2\sigma|P_0^+\sqrt{w}\Big[A^\lambda\phi_\lambda(X^a)D_0X_a+\frac{1}{2}A^\lambda A^{\lambda'}\phi_\lambda(X^a)\phi_{\lambda'}(X_a)+A^\lambda\bar{\theta}\Gamma_-\phi_\lambda(\theta)\Big]\,.$$

The gauge theory of $\mathfrak{X}_\nu^G(\Sigma)$, given by L_0, can be shown (nonmanifestly) supersymmetric under Ξ_0. The remaining terms, to which the gauge field A^λ contributes, are coming from $\mathfrak{L}_{\Xi_1(\epsilon)}L_0 + \mathfrak{L}_{(\Xi_0+\Xi_1)(\epsilon)}L_1$, which may be written as

$$\partial_0\zeta_1\mathrm{d}\tau\wedge|\mathrm{d}^2\sigma|P_0^+\sqrt{w}+\mathrm{d}\tau\wedge d\eta_1+\mathrm{d}\tau\wedge|\mathrm{d}^2\sigma|P_0^+\sqrt{w}\big(\phi_\lambda(X_a)\nabla_0X^a+\bar{\theta}\Gamma^-\phi_\lambda(\theta)\big)\iota_{(\Xi_0+\Xi_1)(\epsilon)}\delta A^\lambda\,,$$

where the boundary terms are given by

$$\begin{aligned}\zeta_1 &= A^\lambda\phi_\lambda(X^a)\bar{\theta}\Gamma_a\epsilon\,,\\ \eta_1 &= P_0^+A^\lambda\phi_\lambda(X^b)\big((\bar{\theta}\Gamma_b\epsilon)\wedge d\omega+(\bar{\theta}\Gamma_b\Gamma_a\epsilon)\wedge dX^a\big)\\ &\quad+\sqrt{w}\epsilon_{rs}\phi_{\lambda r}\bar{\theta}\big(P_0^+(A^{\lambda'}\phi_{\lambda'}(X^a)+\nabla_0X^a)\Gamma_a-\frac{1}{2}\{X^a,X^b\}\Gamma_{ab}\big)\epsilon\wedge\mathrm{d}\sigma^s\,.\end{aligned}$$

Hence adding harmonic vector field components to the theory gives (up to total derivatives) an additional term in the supersymmetry variation which is proportional to the variation in the A-field. We may therefore choose

$$\iota_{\Xi(\epsilon)}\delta A^\lambda = 0\,, \tag{3.5.40}$$

if this transformation obeys the super-Poincaré algebra. This is the case up to an area-preserving diffeomorphism, in analogy with the supersymmetry algebra of eleven-dimensional supergravity. Note that the trivial transformation law (3.5.40) is just a different notation of the supersymmetry transformation of the gauge field given in [47, 58]: in these papers the gauge field is written in terms of its (spacesheet) Hodge dual, $\omega_r\mathrm{d}\sigma^r$ and transforms as $\iota_{\Xi(\epsilon)}\delta\omega_r = -2(P_0^+)^{-1}\bar{\epsilon}\partial_r\theta$. Taking the Hodge dual and subsequently the dual vector by contracting with w^{rs} on both sides yields that Ω_0 transforms as $\mathrm{grad}_\nu(-2(P_0^+)^{-1}\bar{\epsilon}\theta)$, which captures exactly the transformation laws of ω and A^λ in our context.

3.5.5 Field equations and Noether Charges

The commutator of two supersymmetry transformations can be calculated and yields terms which cannot be identified as rigid translations. This signals nonmanifest supersymmetry (as usual in GS superstring theory and supergravity): the field equations (namely those of the APD gauge fields) should be imposed to make the symmetry algebra close. The equations of motion of respectively the X^a-, θ-, ω- and A^λ-fields are:

$$\begin{aligned} P_0^+\nabla_0^2 X^a - (P_0^+)^{-1}\{\{X^a,X^b\},X_b\} + \{\bar\theta,\Gamma^-\Gamma^a\theta\} &= 0\,,\\ \nabla_0\theta + (P_0^+)^{-1}\Gamma_a\{X^a,\theta\} &= 0\,,\\ \{\nabla_0 X^a, X_a\} + \{\bar\theta\Gamma^-,\theta\} &= 0\,,\\ \int_\Sigma |\mathrm{d}^2\sigma|\phi^r_\lambda(\nabla_0 X^a\partial_r X_a + \bar\theta\Gamma^-\partial_r\theta) &= 0\,. \end{aligned} \tag{3.5.41}$$

The right hand side of the last equation is a spacesheet integral because A^λ only depends on τ. Observe that the gauge field equations are exactly the constraints (3.5.7) and (3.5.8) upon choosing the gauge $\omega = A^\lambda = 0$. The X^- coordinate, which does not appear in the Lagrangian is, up to exact contributions, defined along closed loops by integrating these field equations over a surface with such a loop as its boundary. In the presence of winding around the X^--direction, this procedure works up to winding numbers, as we shall explain later. We have the nontrivial supersymmetry brackets

$$\begin{aligned} \big[\iota_{\Xi(\epsilon_1)}\delta,\iota_{\Xi(\epsilon_2)}\delta\big]\,X^a &= -2\bar\epsilon_2\Gamma^+\epsilon_1\nabla_0X^a - 2\bar\epsilon_2\Gamma^a\epsilon_1 + \{2\bar\epsilon_2\Gamma_b\Gamma^+\epsilon_1X^b,X^a\}\,,\\ \big[\iota_{\Xi(\epsilon_1)}\delta,\iota_{\Xi(\epsilon_2)}\delta\big]\,\theta^\alpha &= \varepsilon^{rs}(\bar\epsilon_r\Gamma_a\chi)\Gamma^{+a}\epsilon_s + 2(P_0^+)^{-1}\{X_a\big((\bar\epsilon_2\Gamma^+\epsilon_1)\Gamma_a + (\bar\epsilon_2\Gamma^{+a}\epsilon_2\big),\theta\}\,,\\ \big[\iota_{\Xi(\epsilon_1)}\delta,\iota_{\Xi(\epsilon_2)}\delta\big]\,\omega &= -2(P_0^+)^{-1}\bar\epsilon_2(\nabla_0X_a\Gamma^{+a} + \Gamma^+\Gamma^-)\epsilon_1\,, \end{aligned}$$

In the supersymmetry commutator acting on θ we have denoted the spinor field equation (left hand side of the 3rd equation in (3.5.41)) with χ. Observe that modulo area-preserving gauge transformations (possibly with field-dependent coefficients) and field equations, these commutators all yield translations. The momenta corresponding to the theory (3.5.34) are easily seen to be $P^a = P_0^+\sqrt{w}\nabla_0X^a$, the transverse bosonic modes and $S^\alpha = -P_0^+\sqrt{w}(\Gamma^-\theta)^\alpha$ the fermionic modes, while the momenta corresponding to the gauge fields are zero. We see once again that these quantities coincide with the gauge-fixed theory (3.4.9). The full correspondence becomes apparent in the Hamiltonian theory: the Hamiltonian density of our gauge theory is

$$H = \frac{1}{P_0^+}|\mathrm{d}^2\sigma|\sqrt{w}\Big[\frac{P^aP_a}{2w} + \frac14\{X^a,X^b\}\{X_a,X_b\} - P_0^+\bar\theta\Gamma^-\Gamma_a\{X^a,\theta\}\Big]\,. \tag{3.5.42}$$

This expression coincides with (3.4.12) upon using that $\{X^a,X^b\}\{X_a,X_b\} = 2w^{-1}\bar g$. Let us now investigate the algebra of the various Noether charges on the final constraint manifold. The APD invariance is a gauge symmetry, and hence the integral of the constraint (3.5.7) multiplied with $\sqrt{w}$, is zero. The rigid symmetries are translational invariance, with Noether charges the momentum zero modes and the total Hamiltonian, Lorentz invariance, with lightcone charges

$$\begin{aligned} M^{ab} &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|\Big(X^aP^b - P^aX^b + \tfrac12P_0^+\bar\theta\Gamma^-\Gamma^{ab}\theta\Big)\,,\\ M^{+-} &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|(-\sqrt{w}P_0^+X^- - *H_\tau\tau)\,,\\ M^{+a} &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|(-\sqrt{w}P_0^+X^a + P^a\tau)\,,\\ M^{-a} &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|\Big(P^aX^- + *H_\tau X^a + \tfrac12P_b\bar\theta\Gamma^-\Gamma^{ab}\theta + \sqrt{w}\{X_b,X_c\}\bar\theta\Gamma^-\Gamma^{abc}\theta\Big)\,. \end{aligned} \tag{3.5.43}$$

Finally there is supersymmetry, whose Noether current reads

$$J^0 = |\mathrm{d}^2\sigma|\Big(2(P_a\Gamma^a + \sqrt{w}P_0^+\Gamma^-) + \sqrt{w}\{X_a, X_b\}\Gamma^{ab}\Big)\theta\,. \tag{3.5.44}$$

To recover the 9-dimensional Euclidean super-Poincaré algebra corresponding the gauge-fixed theory, we extract the $SO(9)$-blocks in $Q = \int_{\Sigma_\tau} J^0$ as $Q = Q^+ + Q^-$ with $Q^\pm = \frac{1}{2}\Gamma^\pm\Gamma^\mp Q$. Then

$$Q^+ = \int_{\Sigma_\tau} |\mathrm{d}^2\sigma|(2P^a\Gamma_a + \sqrt{w}P_0^+\{X^a, X^b\}\Gamma_{ab})\theta\,, \qquad Q^- = 2P_0^+\Gamma^-\theta_0\,. \tag{3.5.45}$$

Note that Q^- acts only on the CM variables, where Q^+ gives us the supersymmetry transformations on the higher fluctuations. Recall formula (2.6.49), which is the local version of this pulled back Poisson algebra; in our case the second class constraints, χ^α yield for 2 Noether charges $Q_i = \int_{\Sigma_\tau} J_i$, $i = 1, 2$,

$$\{Q_1, Q_2\}_D = \int_{\Sigma_\tau} \big(\{J_1, J_2\}_p - (4P_0^+\sqrt{w})^{-1}\{J_1, \chi^\alpha\}_p(\Gamma^+)_{\alpha\beta} * \{\chi^\beta, J_2\}_p\big)\,, \tag{3.5.46}$$

and the Dirac brackets of Noether currents are given by the integrand in the equation above. We find for instance following nonzero Dirac brackets of the phase space variables

$$\{X^a, P^b\}_d = \delta^{ab}\,, \qquad \{\theta_\alpha, \theta_\beta\}_d = \frac{1}{4\sqrt{w}}(P_0^+)^{-1}(\Gamma^+)_{\alpha\beta}\,, \tag{3.5.47}$$

Of course, the momenta commute with the Hamiltonian under the Dirac bracket. However, we want to keep track of the surface terms: later, if we compactify the bosonic target space, these surface terms will give winding contributions. The translational sector of the super-Poincaré algebra (spanned by the *bosonic momenta*) reads

$$\{P_0^a, \mathscr{H}_\tau\}_D = (P_0^+)^{-1}\int_{\Sigma_\tau} d\gamma^a\,, \qquad \gamma^a = \{X_b, X^a\} \wedge dX^b + \bar{\theta}\Gamma^-\Gamma^a \wedge d\theta\,, \tag{3.5.48}$$

while the brackets of the other momenta are manifestly zero. The algebra of the supercurrents reads

$$\begin{aligned}
\{Q_\alpha^-, Q_\beta^-\}_D =&\ -2(\Gamma^-)_{\alpha\beta}P_0^+\,,\\
\{Q_\alpha^+, Q_\beta^-\}_D =&\ -(\Gamma_a\Gamma^+\Gamma^-)_{\alpha\beta}P_0^a - (\Gamma^{ab}\Gamma^+\Gamma^-)_{\alpha\beta}\int_{\Sigma_\tau} d\beta_{ab}\,,\\
\{Q_\alpha^+, Q_\beta^+\}_D =&\ 2(\Gamma^+)_{\alpha\beta}\mathscr{H}_\tau + 2(\Gamma_a\Gamma^+)_{\alpha\beta}\int_{\Sigma_\tau}\sqrt{w}dX^a \wedge \psi\\
&+ (\Gamma^a\Gamma^+)_{\alpha\beta}\int_{\Sigma_\tau} d\beta_a + (\Gamma^{abcd}\Gamma^+)_{\alpha\beta}\int_{\Sigma_\tau} d\beta_{abcd}\,,
\end{aligned} \tag{3.5.49}$$

where

$$\begin{aligned}
\beta_a &= \frac{2}{\sqrt{w}}X_aP_b \wedge dX^b + \bar{\theta}\Gamma^-(2X_a - \frac{3}{4}X^b\Gamma_{ab}) \wedge d\theta\,,\\
\beta_{ab} &= -\frac{1}{4}(X_a \wedge dX_b - X_b \wedge dX_a)\,,\\
\beta_{abcd} &= \frac{1}{12}\bar{\theta}\Gamma^-(X_{[a}\Gamma_{bcd]}) \wedge d\theta\,.
\end{aligned} \tag{3.5.50}$$

Hence we see that if the surface terms drop out, the supercharge algebra on the constraint manifold indeed coincides with the anticommuting sector of the super-Poincaré algebra in eleven dimensions. However, if the target space is compact, these surface terms may give contributions. At this point it is not at all clear whether the Lorentz generators, the momentum zero modes and the supercharges constitute the full eleven-dimensional super-Poincaré algebra. In the following section, we shall perform an abstract mode decomposition and give the full algebra of these Noether charges.

4 Matrix Regularisation

4.1 More APD Gauge Theory

4.1.1 A Basis for $C^\infty(\Sigma_\tau, \mathbb{R})$

The final step for the gauge-fixed supermembrane to be seen as a gauge theory on the time axis is by expanding the area-preserving diffeomorphisms in a complete basis and define a bilinear product on the space of Hamiltonian vector fields. This procedure also nicely connects the membranes with the theory of matrices, as we shall see in upcoming section. Recall that the majority of the area-preserving diffeomorphisms arises from scalar functions on Σ_τ, under the symplectic gradient. We therefore start by considering the usual symmetric, bilinear product of real-valued functions on Σ_τ,

$$\langle f, g\rangle = \int_{\Sigma_\tau} f \wedge *g = \int_{\Sigma_\tau} |d^2\sigma| \sqrt{w(\sigma)} f(\sigma) g(\sigma)\,. \qquad (4.1.1)$$

It is well-known that the space of smooth functions on Σ_τ is not complete w.r.t. the norm $||f|| = \sqrt{\langle f, f\rangle}$. Consequently we shall consider its completion to the Sobolev space $W^{0,2}(\Sigma_\tau)$ of functions bounded w.r.t. this norm, which is a Hilbert space. The pairing above induces a morphism $\eta^0 : W^{0,2}(\Sigma_\tau) \longrightarrow (W^{0,2}(\Sigma_\tau))^*$, which is by degeneracy of the pairing not injective and by infinite-dimensionality not surjective. For instance, the Dirac delta distribution associated to a point $\sigma \in \Sigma_\tau$, defined by $\delta_\sigma \in (W^{0,2}(\Sigma_\tau))^* : \delta_\sigma[f] = f(\sigma)$ for all $f \in W^{0,2}(\Sigma_\tau)$ has no counterpart in the Sobolev space. However, restricted to $C^\infty(\Sigma_\tau)$, η^0 is injective, and we shall denote its image with $\mathcal{D}_\infty(\Sigma_\tau)$: it is the space of linear functionals given by integration of the product with some smooth density. Suppose now we are given a basis $Y_I(\sigma)$ of $W^{0,2}(\Sigma_\tau)$, consisting of *smooth* functions. Then by the triangle inequality the metric components $\eta_{IJ} = \langle Y_I, Y_J\rangle$ are all finite numbers, and a function $f = f^I Y_I$ is part of the Hilbert space if $\eta_{IJ} f^I f^J$ is finite. We shall denote the corresponding dual basis by Y^I, and require it be normalised by $Y^I[Y_J] = \delta^I_J$. We emphasise that although we write $Y_I(\sigma)$ with $\sigma = \sigma^1, \sigma^2 \in U$, an open compact subset of $\mathbb{R}^2$, the functions depend on these coordinates *through the coordinate functions* of Σ_τ, so the basis should be compatible with the topology of the spacesheet. Furthermore, the basis is constructed for uncompactified membranes: if the brane winds around a certain dimension, the component in this direction should be expanded in a function basis of $W^{0,2}(\Sigma_\tau, \mathbb{R}/\mathbb{Z})$. We shall comment on this later.

The index I is in practice a double integer-valued object, but this is of minor importance, as long as we can sum over it. What is of importance is the subtraction of the zero modes: the linear morphism $f \mapsto f - \langle f, 1\rangle$ is a surjective map to $W^{0,2}_1(\Sigma_\tau, \mathbb{R})$, the space of bounded functions modulo constants under addition. Hence we may construct from any basis Y_I a basis Y_A, Y_0 where $\sqrt{w} Y_A$ integrates to zero and $Y_0 = 1$ spans the kernel of the morphism. The dual basis vectors Y^A satisfy a *completeness* relation in $W^{0,2}_1(\Sigma_\tau, \mathbb{R})$, which is easily derived from the relation $Y^I[Y_J] = \delta^I_J$ in the total space. For a fixed $\sigma \in \Sigma_\tau$,

$$Y_A(\sigma) Y^A = \frac{1}{\sqrt{w(\sigma)}} \delta_\sigma - \langle 1, \,.\,\rangle\,. \qquad (4.1.2)$$

There is a canonical basis of smooth bounded functions on Σ_τ given by the eigenvectors of the Laplace-Beltrami operator (4.1.3), which acts on zero-forms by

$$\Delta f = \delta df = *d * df = \frac{1}{\sqrt{w}} \partial_r (\sqrt{w} w^{rs} \partial_s f)\,. \qquad (4.1.3)$$

We set $\Delta Y_A = -\lambda_A Y_A$ (no summation), $\lambda_A > 0$. Using the elliptic properties of Δ, one can show that the the (smooth) eigenvectors of the Laplacian indeed form a complete basis of our Sobolev space and that its eigenvalues are negative definite. Choosing the basis to consist of these eigenvectors causes no loss of generality, since all the formulas will be basis-independent; some quantities depend on explicitly on the eigenvalues, and the easiest way to deduce these is by a convenient choice of Y_A. One of the simplifications is that the metric η becomes diagonal, because Δ is self-adjoint with respect to (4.1.1).

4.1.2 Green's Function Associated to X^-

One object depending explicitly on the eigenvalues λ_A is Green's functional associated to the differential equation defining X^-. This function is constructed from Green's function associated to (4.1.3). Recall that for a linear invertible operator A on some bounded measure space C of functions, the operator equation $Af = g$ is solved by $f(y) = \delta_y[f] = \delta_y[A^{-1}g] = (A^{-1})^*\delta_y[g]$, and the operator $G_y = (A^{-1})^*\delta_y$ is called Green's functional. In our case the function vector space is $W_1^{0,2}(\Sigma_\tau)$ and the pairing is given by (4.1.1). Green's function associated to the Laplacian operator is

$$G_\sigma = -\sum_A \frac{1}{\lambda_A} Y_A(\sigma) Y^A \,. \tag{4.1.4}$$

Letting σ vary, it defines an element in $W_1^{0,2}(\Sigma_\tau) \otimes (W_1^{0,2}(\Sigma_\tau))^*$, on which the Laplacian acts as

$$\Delta_\sigma G_\sigma = \frac{1}{\sqrt{w(\sigma)}}\delta_\sigma - \langle 1, \,.\, \rangle \,. \tag{4.1.5}$$

The differential equation $\Delta f = g$ is then solved by $f(\sigma) = G_\sigma[g]$. From this function we deduce Green's functional associated to the system of PDE's defining X^-, $dX^- = \gamma$ (cf. 3.5.1). For this we need to construct a basis on $\Omega^1(\Sigma_\tau)$ with the Y_A. This can be done straightforwardly, expanding the one-form components in this basis, or it can be done with the Hodge decomposition theorem (3.5.5):

$$\Omega^1(\Sigma_\tau) = d(\Omega^0(\Sigma_\tau)) \oplus *d(\Omega^0(\Sigma_\tau)) \oplus H^1_\Delta(\Sigma_\tau) \,. \tag{4.1.6}$$

For the second factor we used that 2-dimensional manifolds exhibit the isomorphism $*\Omega^2(\Sigma) = \Omega^0(\Sigma)$. Hence we obtain the basis

$$\alpha_A^{(1)} = \frac{1}{\sqrt{\lambda^A}} dY_A \,, \quad \alpha_A^{(2)} = *\alpha_A^{(1)} = \frac{1}{\sqrt{\lambda^A}} \sqrt{w} \epsilon_{rs} w^{st} \partial_t Y_A \wedge \mathrm{d}\sigma^r \,, \tag{4.1.7}$$

In the formula above there is of course no summation over the A-index. These are again eigenvectors of the Laplace operator in $\Omega^1(\Sigma_\tau)$: $\Delta\alpha_A^{(i)} = -\lambda_A \alpha_A^{(i)}$. Observe the double degeneracy of the nonzero eigenvalues. The factor $(\lambda^A)^{-1/2}$ normalises the inner products between these forms:

$$\int_{\Sigma_\tau} \alpha_A^{(i)} \wedge *\alpha_B^{(j)} = \delta^{ij}\eta_{AB} \,. \tag{4.1.8}$$

The pairing of one forms induces a duality morphism $\eta^1 : \Omega^1(\Sigma_\tau) \longrightarrow (\Omega^1(\sigma_\tau))^*$. Since $\Omega^1(\Sigma_\tau) = C^\infty(\Sigma_\tau)$, there exists a de Rham differential $d^* : \mathcal{D}_\infty(\Sigma_\tau) \longrightarrow (\Omega^1(\sigma_\tau))^*$, given by $d^* = (\eta^0)^{-1} \circ d \circ \eta^1$. In the usual way η^1 pulls the Hodge star isomorphism back to the dual space $\Omega^1(\Sigma_\tau)^*$. By construction, the element $Y^0 = \langle \,.\,, 1\rangle$ is in the kernel of d^*; hence the basis Y^A induces basis vectors on the space of smooth functional-valued one forms, we which we denote by $\alpha^{(1)A} = d^*Y_A/\sqrt{\lambda^A}$ and $\alpha^{(2)A} = *d^*Y_A/\sqrt{\lambda^A}$. The remaining basis vectors are the duals of the harmonic one-forms φ_λ. We shall assume these to be properly normalised:

$$\int_{\Sigma_\tau} \varphi_\lambda \wedge *\alpha_A^{(i)} = 0 \,, \qquad \int_{\Sigma_\tau} \varphi_\lambda \wedge *\varphi_{\lambda'} = \delta_{\lambda\lambda'} \,. \tag{4.1.9}$$

There is a completeness relation between the components of the basis one-forms and their duals: for every $\sigma \in \Sigma_\tau$ we have

$$\sum_{i,A} \alpha_{Ar}^{(i)}(\sigma)\alpha_s^{(i)A} = \frac{w_{rs}(\sigma)}{\sqrt{w(\sigma)}}\delta_\sigma - \sum_{\lambda=1,\ldots,2g} \varphi_{\lambda r}(\sigma)\varphi_s^\lambda \,. \tag{4.1.10}$$

This should be read as an equation of linear functionals: the objects φ_s^λ are the components of the harmonic one-forms in $\Omega^1(\Sigma_\tau)^*$, which are again the linear functionals defined by $f \mapsto \int_{\Sigma_\tau} \varphi_s^\lambda \wedge * f$. Green's function associated to $dX^- = \gamma$ is a map $\sigma \mapsto \bar{G}_\sigma \otimes X \in W_1^{0,2}(\Sigma_\tau)^* \otimes \Gamma T\Sigma_\tau$ from the spacesheet to the space of functional-valued vector fields such that

$$df = \beta \Rightarrow f(\sigma) = \bar{G}_\sigma[\iota_X \beta] + \text{const.}\,. \tag{4.1.11}$$

Hence, for every fixed σ this functional should satisfy

$$\delta_\sigma[\operatorname{div}(X)]\bar{G}_\sigma = -\frac{1}{\sqrt{w(\sigma)}}\delta_\sigma + \langle 1,\,.\,\rangle\,. \tag{4.1.12}$$

The explicit expression of Green's function follows from the completeness relation (4.1.10). Contracting this formula with $w^{rs}(\sigma)$ and using an orthonormal homology basis yields on the right hand side -2 times the right hand side of (4.1.12) and the left hand side may be written as a covariant derivative of a vector field. A quick calculation yields

$$\bar{G}_\sigma \otimes X = \sum_A \frac{1}{\lambda_A} w^{rs}(\sigma)\partial_r Y_A(\sigma) Y^A \otimes \partial_s\,, \tag{4.1.13}$$

so the X^- coordinate has an integral representation

$$X^-(\sigma) = (P_0^+)^{-1} \sum_{A,B} \eta^{AB} \Big[\int_{\Sigma_\tau} *w^*(\frac{1}{\lambda_B} dY_B, \gamma)\Big] Y_A(\sigma) + \text{constant}\,, \tag{4.1.14}$$

where γ is defined in (3.5.1) and w^* is the inner product on $\Omega^1(\Sigma_\tau)$ induced by w. The constant is the centre-of-mass coordinate q^-, the variable canonically conjugate to P_0^+.

4.1.3 A Basis of Hamiltonian Vector Fields

We can now straightforwardly formulate the gauge theory of area-preserving diffeomorphisms from previous chapter in terms of the basis Y_A. The symplectic gradient maps the basis of functions to a basis of globally Hamiltonian vector fields,

$$\zeta_A = \frac{\varepsilon^{rs}}{\sqrt{w}}(\partial_s Y_A)\partial_r\,. \tag{4.1.15}$$

Together with the additional harmonic vector fields ϕ_λ this gives a real vector space structure to $\mathfrak{X}_\nu(\Sigma_\tau)$. Moreover, there is a nondegenerate bilinear pairing $\mathfrak{X}_\nu(\Sigma_\tau) \otimes \mathfrak{X}_\nu(\Sigma_\tau) \longrightarrow \mathbb{R}$ constructed by combining the Riemannian metric on Σ_τ and the pairing of bounded smooth functions on Σ_τ; this is just the product (3.5.21). Lifting the duality morphism η to $\mathfrak{X}_\nu(\Sigma_\tau)$ yields a basis of the space $\mathfrak{X}_\nu^*(\Sigma_\tau)$ of functional-valued Hamiltonian vector fields, contracting with w^{rs} and taking the Hodge dual yields the space $\Omega_C^1(\Sigma_\tau) = d\Omega^0(\Sigma_\tau) \oplus H_\Delta^1(\Sigma_\tau)$ of closed one-forms. The duality morphism is the combination of both morphisms to the space $\Omega_C^1(\Sigma_\tau)^*$ of functional-valued closed one-forms. Its induced basis vectors are then $\sqrt{\lambda_A}\alpha^{(1)A}$, which we have introduced earlier. The advantage of the normalisation (4.1.15) is the relative simple expressions of the structure constants; the disadvantage is the fact that the orthonormality relations between the ζ_A are not normalised:

$$\int_{\Sigma_\tau} *w(\zeta_A, \zeta_B) = \lambda_A \eta_{AB}\,. \tag{4.1.16}$$

A simple modification of the Riemannian inner product yields a normalised basis (ζ_A, ϕ_λ) of the APD algebra. This is the procedure of Gram-Schmidt: the linear map $g : \mathfrak{X}_\nu(\Sigma_\tau) \longrightarrow \mathfrak{X}_\nu(\Sigma_\tau) : g(\xi) = \sum_A w(\xi, \zeta_A)\zeta_A + \sum_\lambda w(\xi, \phi_{(\lambda)})\phi_\lambda$ obviously has zero kernel by completeness of the basis. Hence it is an automorphism of the APD algebra; its inverse normalises the basis through the modified product $\tilde{w}(\xi, \zeta) = w(g^{-1}(\xi), \zeta)$. This inner product is still symmetric because g and its inverse are self-adjoint w.r.t. the Riemannian pairing.

The Poisson algebra on the spacesheet is isomorphic to a central extension of the ideal of globally Hamiltonian vector fields. In particular, the structure constants defined by $[\zeta_A, \zeta_B] = f_{AB}{}^C \zeta_C$ are deduced from this extension,

$$\{Y_A, Y_B\} = f_{AB}{}^C Y_C\,, \quad f_{AB}{}^C = \eta^{CD} \int_{\Sigma_\tau} |\mathrm{d}^2\sigma| \sqrt{w(\sigma)} Y_D(\sigma) \{Y_A(\sigma), Y_B(\sigma)\}\,. \tag{4.1.17}$$

Together these structure constants define a tensor F_p in $(\mathcal{D}_\infty \wedge \mathcal{D}_\infty \otimes C^\infty)(\Sigma_\tau)$ defined by $F_p[f,g] = \{f,g\}$ for all smooth functions f and g on the space sheet. Moreover, the tensor $(\mathrm{Id}\otimes\mathrm{Id}\otimes\eta)\circ F_p$ is an element of $\bigwedge^3 \mathcal{D}_\infty(\Sigma_\tau)$, as one sees by partial integration that its components $f_{ABC} = \eta_{CD} f_{AB}{}^D$ are totally antisymmetric. For nontrivial spacesheet topologies the APD algebra has the additional harmonic generators yielding the structure constants

$$\begin{aligned} f_{\lambda B}{}^C &= -\eta^{CD} \int_{\Sigma_\tau} |\mathrm{d}^2\sigma| \sqrt{w(\sigma)}\,, Y_D(\sigma) \phi_\lambda{}^r \partial_r Y_B(\sigma)\,, \\ f_{\lambda\lambda'}{}^C &= -\eta^{CD} \int_{\Sigma_\tau} |\mathrm{d}^2\sigma| w(\sigma) \epsilon_{rs} \phi_\lambda{}^r \phi_{\lambda'}{}^s Y_D(\sigma)\,, \end{aligned} \tag{4.1.18}$$

which of course satisfy the Jacobi identities too and are by definition antisymmetric in the lower indices. Furthermore, partial integration yields $f_{\lambda BC} = -f_{\lambda CB}$. Together these structure constants constitute the tensor $F \in \bigwedge^2 \Omega^*_C(\Sigma_\tau) \otimes \mathfrak{X}^G_\nu(\Sigma_\tau)$. Note that the upper index takes no λ-values because $\mathfrak{X}^G_\nu(\Sigma_\tau)$ is an ideal of the full algebra. We also see that the zero modes play a trivial rôle in this algebra: the structure constants $f_{0A}{}^B$, $f_{00}{}^B$ and $f_{\lambda 0}{}^B$ are all zero. This corresponds to the observation in the previous section that the the Poisson algebra including constant functions is just a central extension of $\mathfrak{X}^G_\nu(\Sigma_\tau)$.

4.1.4 APD Gauge Theory Revisited

Now let us expand the action and its Hamiltonian in this function basis. We write $X^a = X^a_0 + X^{aA} Y_A$ and $\theta^\alpha = \theta^\alpha_0 + \theta^{\alpha A} Y_A$ (adopting summation convention), assuming that the embedding is square integrable under pullback to the world volume. The spacesheet integral of the Lagrangian (3.5.34) is expanded according to

$$\begin{aligned} \mathscr{L} = \;& \mathscr{L}_0 + \tfrac{1}{2} P_0^+ \nabla_0 X_a{}^A \nabla_0 X^a{}_A + P_0^+ \bar\theta_A \Gamma^- \nabla_0 \theta^A - \tfrac{1}{4} (P_0^+)^{-1} f_{AB}{}^E f_{CDE} X_a{}^A X_b{}^B X^{aC} X^{bD} \\ & - f_{AB}{}^C X_a{}^A \bar\theta^B \Gamma^- \Gamma^a \theta_C\,, \end{aligned} \tag{4.1.19}$$

where we have split off the CM Lagrangian

$$\mathscr{L}_0 = P_0^+ (\tfrac{1}{2} \partial_0 X_{a0} X^a_0 + \bar\theta_0 \Gamma^- \partial_0 \theta_0)\,. \tag{4.1.20}$$

Note that we require sufficient differentiability of the field configurations to make the Lagrangian well-defined; this corresponds to the requirement that sums like $f^C_{AB} X^{aA} X^{bB}$ converge. The APD gauge field can be written as $\Omega = (\omega^A \zeta_A + A^\lambda \phi_\lambda) \otimes \mathrm{d}\tau$ and the covariant derivative induced by this connection acts on the bosonic components of a section by

$$\begin{aligned} \nabla_0 X^{aA} = \;& \partial_0 X^{aA} - \Omega_C{}^A X^{aC}\,, \\ = \;& \partial_0 X^{aA} - f_{BC}{}^A \omega^B X^{aC} - f_{\lambda C}{}^A A^\lambda X^{aC}\,, \end{aligned} \tag{4.1.21}$$

and in the same way it acts on the components $\theta^{\alpha A}$. A Hamiltonian vector field $\xi = \xi^A \zeta_A + \chi^\lambda \phi_\lambda$ induces a transformation on the embedding fields by

$$\iota_\xi \delta X^{aA} = \big(f_{BC}{}^A \xi^B + f_{\lambda C}{}^A \chi^\lambda \big) X^{aC}\,,$$

and similarly for the fermions. Therefore The transverse components and nonzero spinor components all transform under the adjoint representation of $\mathrm{Diff}_\nu(\Sigma_\tau)$. The gauge fields transform as

$$\begin{aligned}\iota_\xi\delta\omega^A &= \nabla_0\omega^A + f_{\lambda C}{}^A\chi^\lambda\omega^C + f_{\lambda\lambda'}{}^A\chi^\lambda A^{\lambda'}\,,\\ \iota_\xi\delta A^\lambda &= \partial_0\chi^\lambda\,.\end{aligned} \tag{4.1.22}$$

The action of the Hamiltonian vector field on local spacesheet coordinates is irrelevant as we have integrated out all space-dependence (it has been replaced with an unbounded index). The symmetry obviously remains manifest, but the cancelation of total derivative terms by the volume element transformation is replaced by certain identities between the so-called f-, c- and d-tensors, reflecting the vanishing of the integral of a total derivative, which we shall discuss below. The zero modes are obviously invariant under area-preserving transformations. The nontrivial supersymmetry transformations of the components become $\iota_\epsilon\delta X^{aA} = -2\bar\epsilon\Gamma^a\theta^A$, $\iota_\epsilon\delta\omega^A = -2(P_0^+)^{-1}\bar\epsilon\theta^A$, and

$$\iota_\epsilon\delta\theta^A = \frac{1}{2}(\nabla_0X^{aA}\Gamma_a + \Gamma^-)\epsilon + \frac{1}{4}(P_0^+)^{-1}f_{BC}{}^AX^{aB}X^{bC}\Gamma^-\Gamma_{ab}\epsilon\,. \tag{4.1.23}$$

Again, the zero modes transform among themselves as $\iota_\epsilon\delta X^{a0} = -2\bar\epsilon\Gamma^a\theta^0$ and $\iota_\epsilon\delta\theta^0 = \frac{1}{2}\partial_0X^{a0}\Gamma_a\epsilon$. The equations of motion become

$$\begin{aligned}P_0^+\Big(\delta_C{}^A(\partial_0)^2 - 2\Omega_C{}^A\partial_0 + W_C{}^A - (P_0^+)^{-2}f_{BD}{}^Af_{CE}{}^BX^{bE}X_b{}^D\Big)X^{aC} + f_{BC}{}^A\bar\theta^B\Gamma^{-a}\theta^C &= 0\,,\\ \Gamma^-\Big(\partial_0 - \Omega_C{}^A\Big)\theta^C + (P_0^+)^{-1}\Gamma_af_{BC}{}^AX^{aA}\theta^B &= 0\,,\\ f_{BC}{}^A\Big((\partial_0X^{aB} - \Omega_D{}^BX^{aD})X_a{}^C + \bar\theta^B\Gamma^-\theta^C\Big) &= 0\,,\\ f_{\lambda B}{}^C\eta_{AC}\Big((\partial_0X^{aA} - \Omega_D{}^AX^{aD})X_a{}^B + \bar\theta^A\Gamma^-\theta^B\Big) &= 0\,,\end{aligned} \tag{4.1.24}$$

where $W_C{}^A = -\partial_0\Omega_C{}^A + \Omega_C{}^B\Omega_B{}^A$, which reads in terms of the component gauge fields

$$\begin{aligned}W_C{}^A = \;& (f_{BD}{}^Af_{\lambda C}{}^D + f_{BC}{}^Df_{\lambda D}{}^A)\omega^DA^\lambda + f_{BE}{}^Af_{DC}{}^E\omega^B\omega^D\\ &+ f_{\lambda B}{}^Af_{\lambda' D}{}^AA^\lambda A^{\lambda'} - f_{BC}{}^A\partial_0\omega^B - f_{\lambda C}{}^A\partial_0A^\lambda\,.\end{aligned} \tag{4.1.25}$$

The CM modes are completely decoupled and they form a system of differential equations among themselves:

$$\partial_0^2X^{a0} = 0\,,\quad \Gamma^-\partial_0\theta_0 = 0\,. \tag{4.1.26}$$

Note that the expansion in the function basis has converted a theory of fields, sections of a bundle over a manifold N, to a theory of an infinite number of interacting 'particles', sections of a bundle over the time axis. Passing to the Hamiltonian formalism, we define the canonical momentum densities $P_a{}^A = P_0^+(\partial_0X_a{}^A - \Omega_C{}^AX_a{}^C)$ and $S^{\alpha A} = -P_0^+(\Gamma^-\theta^A)^\alpha$. The Hamiltonian becomes

$$\mathscr{H}_\tau = \frac{1}{2P_0^+}\Big[P_a{}^0P^{a0} + P_a{}^AP^a{}_A + \tfrac{1}{2}f_{AB}{}^Cf_{DEC}X_a{}^AX_b{}^BX^{aD}X^{bE} - 2P_0^+f_{ABC}\bar\theta^C\Gamma^-\Gamma_aX^{aA}\theta^B\Big]\,. \tag{4.1.27}$$

The factors P_0^+ inside the brackets may be canceled by a rescaling of the fermionic fields by $(P_0^+)^{-1/2}$. Note that the prefactor $(P_0^+)^{-1}$ vanishes in the physically relevant quantity $\mathscr{M}^2 = 2P_0^+\mathscr{H}_\tau - P_0^aP_{0a}$. Although we shall not perform this procedure, we recommend this trick to the reader who wishes to calculate results himself: the reason is that in our formulation the zero mode of the X^- coordinate q^-, which is canonically conjugate to P_0^+, has nonzero Dirac brackets with the fermionic modes because the second-class constraints χ^α involve the P_0^+ coordinate, when not rescaled. Evaluating Dirac commutators, this causes terms which cancel the terms that arise by the P_0^+-factors in front of the fermionic terms. Once again the equations of motion of the gauge fields (multiplied by a factor P_0^+) become the usual first-class constraints on phase space,

$$\begin{aligned}\psi^A &= f_{BC}{}^A(X^{aB}P_a{}^C + P_0^+\bar\theta^B\Gamma^-\theta^C)\,,\\ \psi_\lambda &= f_{\lambda BC}(X^{aB}P_a{}^C + P_0^+\bar\theta^B\Gamma^-\theta^C)\,.\end{aligned} \tag{4.1.28}$$

These are the generators of the area-preserving transformation gauge symmetry in the Hamiltonian formalism,

$$\{\psi_A,\psi_B\}_D = f_{AB}{}^C\psi_C\,,\quad \{\psi_A\psi_\lambda\}_D = f_{A\lambda}{}^C\psi_C\,,\quad \{\psi_\lambda,\psi_{\lambda'}\}_D = f_{\lambda\lambda'}{}^C\psi_C\,. \tag{4.1.29}$$

4.1.5 Super-Poincaré Algebra Structure

Finally, we would like to mention the super-Poincaré algebra structure of the other Noether charges. For this it is convenient to introduce two more tensors. The first one captures the pointwise product of smooth functions in $W_1^{0,2}(\Sigma_\tau)$; it is the element $d\in\bigotimes^3\mathcal{D}_\infty(\Sigma_\tau)$ such that $\delta_\sigma\eta^{-1}(d[f,g]) = f(\sigma)g(\sigma)$. Its components read

$$d_{ABC} = \int_{\Sigma_\tau} |\mathrm{d}^2\sigma|\sqrt{w(\sigma)}Y_A(\sigma)Y_B(\sigma)Y_C(\sigma)\,, \tag{4.1.30}$$

and is obviously totally symmetric. The zeroth components are fixed by $d_{AB0}=\eta_{AB}$ and $d_{A00}=0$. The second tensor is associated to the normalised bilinear product $\tilde{w}$ introduced earlier. We define $c\in\mathfrak{X}_\nu(\Sigma_\tau)^*\otimes\mathfrak{X}_\nu(\Sigma_\tau)^*\otimes\mathcal{D}_\infty(\Sigma_\tau)$ by $c(\xi_1,\xi_2)[f] = \langle\tilde{w}(\xi_1,\xi_2),f\rangle$. Hence its components are

$$\begin{aligned}
c_{ABC} = c(\zeta_A,\zeta_B)[Y_C] &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|\frac{\sqrt{w(\sigma)}}{\lambda^A}w^{rs}(\sigma)\partial_r Y_A(\sigma)\partial_s Y_B(\sigma)Y_C(\sigma)\,,\\
c_{\lambda BC} = c(\phi_\lambda,\zeta_B)[Y_C] &= \int_{\Sigma_\tau}|\mathrm{d}^2\sigma|\epsilon^{st}w_{rs}(\sigma)\phi^r_\lambda(\sigma)\partial_t Y_B(\sigma)Y_C(\sigma)\,,\\
c_{\lambda\lambda' C} = c(\phi_\lambda,\phi_\lambda)[Y_C] &= \int_\sigma |\mathrm{d}^2\sigma|\sqrt{w(\sigma)}w_{rs}(\sigma)\phi^r_\lambda(\sigma)\phi^s_{\lambda'}(\sigma)Y_C(\sigma)\,.
\end{aligned}\tag{4.1.31}$$

This quantity also determines the function expansion of Green's function (4.1.13):

$$c_{ABC} = \langle g_{AB},Y_C\rangle\,,\qquad g_{AB}(\sigma) = \bar{G}_\sigma[Y_A]\delta_\sigma[X(Y_B)]\,. \tag{4.1.32}$$

Hence this quantity arises in the decomposition of the transverse component X^-:

$$X^{-A} = -\frac{1}{P_0^+}\Big(X^{aA}P_{a0} + P_0^+\bar{\theta}^A\Gamma^-\theta_0 + c^A{}_{BC}(X^{aB}P_a{}^C + P_0^+\bar{\theta}^B\Gamma^-\theta^C)\Big)\,. \tag{4.1.33}$$

Note that our definition slightly differs from the original paper [6]: the relation is $(c_{ABC})_{\mathrm{dWMN}} = -2c_{ACB}$ and similarly the harmonic components are related. There various identities involving the quantities η, f, c and d:

$$\begin{gathered}
f_{[AB}{}^C f_{D]C}{}^E = f_{[\lambda B}{}^C f_{D]C}{}^E = f_{[\lambda\lambda'}{}^C f_{D]C}{}^E = f_{[\lambda\lambda'}{}^C f_{\lambda'']C}{}^E = 0\,,\\
f_{A(B}{}^E d_{CD)E} = f_{[AB}{}^C c^D{}_{E]C} = d_{ABC}f^A{}_{[DE}f^B{}_{F]G} = 0\,,\\
c_{ABC} + c_{ACB} = d_{ABC}\,,\\
d_{EA[B}d_{C]D}{}^E = -\eta_{A[B}\eta_{C]D}\,,\\
f_{AB}{}^E c_{ECD} = c_{EAB}f^E{}_{AB} + f_{CA}{}^E d_{BDE} + c_{\lambda AB}f^\lambda{}_{CD}\,.
\end{gathered}\tag{4.1.34}$$

One recognises the Jacobi identity of the APD algebra in the first line. Because c depends explicitly on the metric w, its APD transformation rule is nontrivial (opposed to the APD-covariant quantities η, f and d); hence the X^- coordinate transforms noncovariantly. However, a calculation learns that the additional terms vanish on the final constraint manifold:

$$\begin{aligned}
\iota_\xi\delta X^{-A} = {} & (f_{BC}{}^A\xi^B + f_{\lambda C}{}^A\chi^\lambda)X^{-C} - \frac{\xi^B}{P_0^+}\Big((c^A{}_{BC}+c_{CB}{}^A)\psi^C + c_{\lambda B}{}^A\psi^\lambda\Big)\\
& - \frac{\chi^\lambda}{P_0^+}\Big((c^A{}_{\lambda C}+c_{C\lambda}{}^A)\psi^C + c_{\lambda'\lambda}{}^A\psi^{\lambda'}\Big)\,.
\end{aligned}\tag{4.1.35}$$

The generators of the lightcone super-Poincaré algebra are the zero modes P_0^+ and P_{0a}, the Hamiltonian (4.1.27), the lightcone supercharges and the Lorentz generators. We split off the zero modes and write for the generators $G = G^{(0)} + \widetilde{G} + \text{rest}$, where $G^{(0)}$ only involves zero modes and $\widetilde{G}$ only nonzero modes (except the constant P_0^+). Then

$$
\begin{aligned}
&Q^+ = (Q^+)^{(0)} + \widetilde{Q}^+ \,, \quad (Q^+)^{(0)} = 2P_{a0}\Gamma^a\theta_0 \,, \quad \widetilde{Q}^+ = (2P_a{}^A\Gamma^a + f_{BC}{}^A X^{aB} X^{bC} \Gamma_{ab})\theta_A \,, \\
&Q^- = (Q^-)^{(0)} = P_0^+ \Gamma^- \theta_0 \,. \qquad (4.1.36)
\end{aligned}
$$

The first three Lorentz generators (3.5.43) take the relatively simple form because there are no mixed terms in their expressions; we have $M^{ab} = (M^{ab})^{(0)} + \widetilde{M}^{ab}$, $M^{+-} = (M^{+-})^{(0)} + \widetilde{M}^{+-}$ and $M^{+a} = (M^{+a})^{(0)}$, with

$$
\begin{aligned}
(M^{ab})^{(0)} &= X_0^a P_0^b - X_0^b P_0^a + \tfrac{1}{2} P_0^+ \bar{\theta}_0 \Gamma^- \Gamma^{ab} \theta_0 \,, \\
\widetilde{M}^{ab} &= X^{aA} P^b{}_A - X^{bA} P^a{}_A + \tfrac{1}{2} P_0^+ \bar{\theta}^A \Gamma^- \Gamma^{ab} \theta_A \,, \\
(M^{+-})^{(0)} &= -P_0^+ q^- \,, \\
\widetilde{M}^{+-} &= -\mathscr{H}_\tau \tau \,, \\
(M^{+a})^{(0)} &= -P_0^+ X_0^a + P_0^a \tau \,, \qquad (4.1.37)
\end{aligned}
$$

where q^- is the zero mode of X^-, a quantity canonically conjugate to P_0^+. The remaining generator has a more complicated decomposition:

$$
M^{-a} = (M^{-a})^{(0)} + X_0^a \mathscr{H}_\tau + \frac{1}{P_0^+} P_{b0} \widetilde{M}^{ab} + \bar{\theta}_0 \Gamma^- \Gamma^a \widetilde{Q}^+ + \widetilde{M}^{-a} \,, \qquad (4.1.38)
$$

where $(M^{-a})^{(0)} = q^- P_0^a + \frac{1}{2} P_{b0} \bar{\theta}_0 \Gamma^- \Gamma^{ab} \theta_0$ and

$$
\begin{aligned}
\widetilde{M}^{-a} = \; & \frac{d_{ABC}}{P_0^+} \Big[X^{aA} \big(P_b{}^B P^{bC} + \tfrac{1}{4} f_{DE}{}^B f_{FG}{}^C X_b{}^D X_c{}^E X^{bF} X^{cG} - P_0^+ f_{DE}{}^B \bar{\theta}^C \Gamma^- \Gamma_b X^{bD} \theta^E \big) \\
& + \tfrac{1}{2} P_0^+ P_b{}^A \bar{\theta}^B \Gamma^- \Gamma^{ab} \theta^C + P_0^+ f_{DE}{}^A X_b{}^D X_c{}^E \bar{\theta}^B \Gamma^- \Gamma^{abc} \theta^C \Big] \\
& - \frac{c_{ABC}}{P_0^+} P^{aA} (X^{bB} P_b{}^C + P_0^+ \bar{\theta}^B \Gamma^- \theta^C) \,. \qquad (4.1.39)
\end{aligned}
$$

An explicit verification of supersymmetry and Lorentz invariance of the theory is the cancelation of all Dirac brackets of the generators with the invariant mass-squared (3.4.36), which in the mode decomposition reads

$$
\mathscr{M}^2 = P_a{}^A P^a{}_A + \tfrac{1}{2} f_{AB}{}^C f_{DEC} X_a{}^A X_b{}^B X^{aD} X^{bE} - 2P_0^+ f_{ABC} \bar{\theta}^C \Gamma^- \Gamma_a X^{aA} \theta^B \,, \qquad (4.1.40)
$$

and is independent of the zero modes. Hence all its Dirac brackets with zero-mode coordinates in phase space vanish, as is required by translational invariance, and for a super-Poincaré generator G we have $\{(G)^{(0)}, \mathscr{M}^2\}_D = 0$. Furthermore the expression (4.1.40) is quickly seen to be manifestly invariant under transverse rotation of the field variables, which gives $\{\widetilde{M}^{ab}, \mathscr{M}^2\}_D = 0$. Hence the only nontrivial commutators are provided by $\widetilde{Q}^+$ and $\widetilde{M}^{-a}$. A tedious calculation, performed in [6] yields

$$
\begin{aligned}
&\{\widetilde{Q}^+, \mathscr{M}^2\}_D = 2\psi_A \theta^A \,, \\
&\{\widetilde{M}^{-a}, \mathscr{M}^2\}_D = -\frac{2}{P_0^+} \Big(f_{DE}{}^B X^{bC} X^{aD} X_b{}^E + P_0^+ \bar{\theta}^B \Gamma^- \theta^C \Big) (c_{ABC} \psi^A + c_{\lambda BC} \psi^\lambda) \,. \qquad (4.1.41)
\end{aligned}
$$

Hence on the final constraint space the invariant mass squared commutes with all the generators of the super-Poincaré algebra. The closure of the super-Poincaré algebra is quickly verified, except for two commutators involving M^{-a}. The commutators that manifestly satisfy the algebra (2.4.3) are

$$\begin{aligned} \{M^{ab}, M^{cd}\}_D &= \delta^{ac}M^{bd} + \delta^{bd}M^{ac} - \delta^{ad}M^{bc} - \delta^{bc}M^{ad} & \{M^{+-}, M^{+a}\}_D &= -M^{+a}\,, \\ \{M^{+a}, M^{-b}\}_D &= M^{ab} + \delta^{ab}M^{+-} & \{M^{+a}, M^{+b}\}_D &= 0\,, \\ \{M^{ab}, M^{\pm c}\}_D &= \delta^{ac}M^{\pm b} - \delta^{bc}M^{\pm a} & \{M^{ab}, M^{+-}\}_D &= 0\,. \end{aligned} \tag{4.1.42}$$

The remaining commutators were calculated in [59], and miraculously enough they satisfy the Lorentz algebra on the final constraint manifold:

$$\begin{aligned} \{M^{+-}, M^{-a}\}_D &= M^{-a} + (P_0^+)^{-1}\bar\theta_0\Gamma^-\Gamma^a\theta^A\psi_A\,, \\ &\quad -(P_0^+)^{-2}(f_{DE}{}^B X^{bC}X^{aD}X_b{}^E + P_0^+\bar\theta^B\Gamma^-\theta^C)(c_{ABC}\psi^A + c_{\lambda BC}\psi^\lambda)\,, \\ \{M^{-a}, M^{-b}\}_D &= \frac{-2}{(P_0^+)^2}\Big[P_0^+\Big(\Phi\big(X^{[a}\otimes(\bar\theta\Gamma^{b]}\Gamma^-)_\alpha\otimes\theta^\alpha\big) - \tfrac12\Phi\big((\bar\theta\Gamma^-\Gamma^{abc})_\alpha\otimes\theta^\alpha\otimes X_c\big)\Big)\,, \\ &\quad +\frac{1}{\lambda_A}\Big(\frac{1}{\lambda_C}f_C{}^{AB}\psi^C + f_\lambda{}^{AB}\psi^\lambda - \frac{1}{2\lambda_B}f_C{}^{AB}\psi^C\Big)P^{[a}{}_A P^{b]}{}_B\,, \\ &\quad +\Phi\big(X^{[a}\otimes\{X^{b]}, X^c\}\otimes X_c\big) + \tfrac12 P_0^+\bar\theta_0\Gamma^-\Gamma^{[a}X^{b]D}\theta^C d_{CDE}\psi^E\,, \\ &\quad -P_0^+\bar\theta_0\Gamma_-\Gamma^{[a}\Gamma^{b]c}\theta^C X_c{}^D(c_{EDC}\psi^E + c_{\lambda DC}\psi^\lambda)\Big]\,, \end{aligned} \tag{4.1.43}$$

where $\Phi : C^\infty(\Sigma_\tau)\otimes C^\infty(\Sigma_\tau)\otimes C^\infty(\Sigma_\tau)\longrightarrow \mathbb{R}$ is the trilinear map

$$\Phi(V^{ABC}Y_A\otimes Y_B\otimes Y_C) = V_A{}^A{}_C\psi^C + V^{ABC}d_{AB}{}^D(c_{ECD}\psi^E + c_{\lambda CD}\psi^\lambda)\,. \tag{4.1.44}$$

So we observe that $\{M^{-a}, M^{-b}\}_D$ vanishes on the physical subspace, which is required by the lightcone Lorentz algebra. The fermionic sector was already shown in the previous chapter to close on the final constraint manifold:

$$\begin{aligned} \{Q^-{}_\alpha, Q^-{}_\beta\}_D = -2(\Gamma^-)_{\alpha\beta}P_0^+\,, &\quad \{Q^+{}_\alpha, Q^-{}_\beta\}_D = -(\Gamma_a\Gamma^+\Gamma^-)P_0^a\,, \\ \{Q^+{}_\alpha, Q^+{}_\beta\}_D &= 2(\Gamma^+)_{\alpha\beta}\mathscr{H}_\tau - 2(\Gamma_a\Gamma^+)_{\alpha\beta}X^a{}_A\psi^A\,. \end{aligned} \tag{4.1.45}$$

With the results above and some additional calculations it is easy to verify the remaining commutators. The (off-shell) nonzero results are

$$\begin{aligned} \{M^{+-}, P_0^+\}_D &= -P_0^+\,, & \{M^{+-}, \mathscr{H}_\tau\}_D &= \mathscr{H}_\tau\,, \\ \{M^{+a}, P_0^c\}_D &= -\delta^{ac}P_0^+\,, & \{M^{+a}, \mathscr{H}_\tau\}_D &= P_0^a\,, \\ \{M^{-a}, P_0^c\}_D &= \delta^{ac}\mathscr{H}_\tau\,, & \{M^{-a}, P_0^+\}_D &= P^a\,, \\ \{M^{ab}, P_0^c\}_D &= \delta^{ac}P_0^b - \delta^{bc}P_0^a\,, & \{M^{ab}, Q^\pm\}_D &= \tfrac12\Gamma^{ab}Q^\pm\,, \\ \{M^{+-}, Q^+\}_D &= \tfrac12 Q^+ + 2(P_0^+)^{-1}\theta^A\psi_A\tau\,, & \{M^{+-}, Q^-\}_D &= \tfrac12 Q^-\,, \\ \{M^{+a}, Q^+\}_D &= \tfrac12\Gamma^{+a}Q^-\,, & \{M^{-a}, Q^-\}_D &= \tfrac12\Gamma^{-a}Q^+\,, \end{aligned} \tag{4.1.46}$$

and

$$\{M^{-a}, Q^+\}_D = (P_0^+)^{-1}\Big[(X^{aA}\theta^B d_{ABC} - X_0^a\theta_C)\psi^C - X_b^B\Gamma^{ab}\theta^C(c_{ABC}\psi^A + c_{\lambda BC}\psi^{(\lambda)})\Big]\,.$$

It is important to realise that the classical on-shell closure of the symmetry algebra does not imply this property at the quantum mechanical level. In superstring theory, the Hilbert space is defined by means of Virasoro constraints and the Lorentz algebra acting on it is well-defined by virtue of normal ordering prescriptions. This procedure seems to be inapplicable for the supermembrane model. However, there is a way to 'relate' this theory to a well-defined quantum theory with a finite number of degrees of freedom, as we shall see below.

4.2 Matrix Regularisation

In this section we shall interpret the one-dimensional gauge theory of area-preserving diffeomorphisms as a limiting case of a well-known class of models called the matrix models. We shall therefore begin with a quick review of the matrix model, without going into details on its relation to superstring theory and M-theory.

4.2.1 Super-Yang-Mills Theory

The standard way to understand the matrix model is by dimensional reduction of a minimally supersymmetric Yang-Mills theory (SYM). The data used to construct such a theory are

1. A principal compact-G-bundle $P_G(M)$ on a manifold M^d with Minkowski metric η.
2. A bi-invariant scalar product $\langle .,. \rangle$ on the Lie algebra $\mathfrak{g}$.
3. A spinor bundle $S(M)$, possibly with a graded product structure.

The field content of the theory is determined by a gauge field $A \in \Omega^1(M, \mathfrak{g})$ and an adjoint representation-valued spinor $\theta \in \Omega^0(S \otimes \mathrm{ad}P)$. Note that the bilinear pairing on $\mathfrak{g}$ is trivially extended to the adjoint bundle. As usual, the gauge field A gives rise to the curvature $F_A \in \Omega^2(M, \mathrm{ad}P)$ given by $F_A = dA + \frac{1}{2}qA \wedge A$, where the de Rham differential and wedge product are extended to the principal bundle as in (2.5.11) and a covariant derivative on $\Omega^0(S \otimes \mathrm{ad}P)$ by $\nabla_X \theta = X(\theta) - q[\eta(X, A), \theta]$, where q is the gauge coupling constant. The SYM Lagrangian is then given by

$$L = -\frac{q^{-1}}{2}\left(\langle F_A \wedge *F_A \rangle - \langle \bar{\theta} \wedge *\Gamma^\mu \nabla_\mu \theta \rangle\right). \tag{4.2.1}$$

Here we have defined for $A_1 = \alpha_1 \otimes g_1$ and $A_2 = \alpha_2 \otimes g_2 \in \Omega^k(M \otimes \mathfrak{g})$ the bilinear pairing $\langle A_1 \wedge *A_2 \rangle = (\alpha_1 \wedge *\alpha_2)\langle g_1, g_2 \rangle$, and similarly the second term is defined by $\theta_1 = s_1 \otimes g_1$, $\theta_2 = s_2 \otimes h \in \Omega^0(S \otimes \mathrm{ad}P)$: $\langle \theta_1 \wedge *\theta_2 \rangle = |\mathrm{d}^d x| \bar{s_1} s_2 \langle g_1, g_2 \rangle$. Hence, for the second term to be nontrivial (not a total derivative), $\bar{s}_1 \wedge \Gamma^{(1)} \wedge s_2$ should be antisymmetric under $1 \leftrightarrow 2$; if the charge conjugation matrix and the Dirac matrices are symmetric, we choose the spinor components to constitute the first-order homogeneous sector of the supermanifold $M^{0|n}$ with $n = \dim(S)$. The action is manifestly gauge invariant: the map $\Phi : \Omega^0(M, G) \longrightarrow \Gamma T\mathcal{F}$ given by

$$\iota_\xi \delta A = g^{-1} A g + g dg\,, \quad \iota_\xi \delta \theta = g^{-1} \theta g\,, \tag{4.2.2}$$

for $\xi = \Phi(g)$ satisfies $\mathfrak{L}_\xi L = 0$. In certain dimensions, the Lagrangian is globally supersymmetric: we postulate the supersymmetry transformation $\Xi : S \longrightarrow \Gamma T\mathcal{F}$ defined by

$$\iota_\xi \delta A_\mu = \bar{\epsilon} \Gamma_\mu \theta\,, \quad \iota_\xi \delta \theta = \tfrac{1}{2} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon\,, \tag{4.2.3}$$

for $\xi = \Xi(\epsilon)$. These transformation rules may be derived by power counting and on-shell closure of the supersymmetry algebra. Some calculations yields the variation

$$\mathfrak{L}_\xi L = \tfrac{1}{2}(\Gamma^\mu)_{\alpha\beta} (\bar{\epsilon}\Gamma_\mu)_\gamma \langle \theta^\alpha \wedge *[\theta^\beta, \theta^\gamma] \rangle + d\alpha\,, \tag{4.2.4}$$

where $\alpha = \frac{1}{4} q^{-1} \bar{\epsilon} \Gamma^{\mu\nu\rho} \langle F_{\nu\rho}, \theta \rangle \iota_{\partial_\mu} |\mathrm{d}^d x|$. Hence if G is abelian the theory is supersymmetric. In the more interesting case where G is not abelian, there are restrictions on spacetime and spinor type which make the SYM Lagrangian supersymmetric. In particular, the first term in the variation above vanishes provided the Fierz identity

$$(\Gamma^\mu)_{\alpha\beta} (\Gamma_\mu)_{\gamma\delta} + (\Gamma^\mu)_{\alpha\delta} (\Gamma_\mu)_{\beta\gamma} + (\Gamma^\mu)_{\alpha\gamma} (\Gamma_\mu)_{\delta\beta} = 0 \tag{4.2.5}$$

is satisfied. In [60] it is shown that this is only the case for $d = 3, 4, 6$ and 10 with minimal supersymmetry. This is exactly the sequence for which bosonic and fermionic degrees of freedom can match on the shell: by gauge invariance, the field A has $d-2$ physical degrees of freedom, and the Dirac-like equation of motion of the fermion field projects out half of its components. Hence there is a matching if $d = 2 + n/2$, yielding the sequence $(d, n) = (3, 2(\mathrm{M})), (4, 4(\mathrm{M})), (6, 8(\mathrm{SMW})), (10, 16(\mathrm{MW}))$, where we have noted the spinor type as in table (2.3.5). One verifies that the associated super current takes the form

$$J_\alpha = q^{-1}\eta^{\mu\rho}(\Gamma^\nu)_{\alpha\beta}\langle F_{\mu\nu}, \theta^\beta\rangle \frac{\partial}{\partial x^\rho} \otimes |\mathrm{d}^d x| \,. \tag{4.2.6}$$

Furthermore one easily verifies the usual Poincaré invariance. For completeness we include the equations of motion:

$$d_A * F_A - \frac{q}{2}[\bar\theta \wedge *\Gamma^{(1)}\theta] = 0\,, \quad \not{D}_A\theta = 0\,, \tag{4.2.7}$$

where $[\bar\theta \wedge *\Gamma^{(1)}\theta] = (\Gamma^\mu)_{\alpha\beta}[\theta^\alpha, \theta^\beta]\iota_{\partial_\mu}|\mathrm{d}^d x|$ and $\not{D}_A = \Gamma^\mu\nabla_\mu$. Note that if we use a superspace formalism the commutator $[\theta^\alpha, \theta^\beta]$ is symmetric.

4.2.2 $U(N)$ Matrix Mechanics

We are now interested in a *dimensional reduction* of these supersymmetric gauge theories, since we would like to make the connection with gauge theory of area-preserving diffeomorphisms on the time axis constructed in the previous chapter. This process goes as follows: let $x^0, \ldots, x^{d-1}$ be a coordinate chart on d-dimensional Minkowski space M^d and $L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M^d)$ and $\gamma \in \Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M^d)$ define a Poincaré-invariant theory. Such a theory 'contains' lower dimensional theories which may be found by considering only fields which are constant in certain directions. Let $x^0, \ldots x^{d-2}$ denote the induced coordinates on $(d-1)$-dimensional Minkowski space $M^{d-1} = M^d/\exp\mathbb{R}\partial_{d-1}$ obtained by dividing by translations in the x^{d-1} direction. Denote $\hat\xi_{d-1}$ denote the vector field on $\mathcal{F}$ induced by ∂_{d-1}, the generator of translations in this direction. Then we define

$$\mathcal{F}_{d-1} = \{\phi \in \mathcal{F} : \mathfrak{L}(\hat\xi_{d-1})\phi = 0\}\,, \tag{4.2.8}$$

which is just the space of fields constant in the x^{d-1}-direction. Then the dimensionally reduced theory is given by

$$L_{d-1} = \iota(\partial_{d-1})L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F}_{d-1} \times M^{d-1})\,, \tag{4.2.9}$$

and the reduced γ_{d-1} may be derived from this Lagrangian. Obviously, a scalar field on M^d reduces to a scalar field on M^{d-1}. For higher spin fields, the reduction is more complicated. A k-form field reduces to a k-form field plus a $k-1$-form field corresponding to the terms in the original field containing $\mathrm{d}x^{d-1}$. To reduce a spinor field, one should take a closer look at the decomposition of spinor representations of $SO(1, d-1)$. This is not necessary however, since we are only interested in particular simple reduction of the theory, namely the iterative procedure to M^1, the affine time line. The resulting decomposition of a spinor yields a direct sum of representations of $S(M^1)$, complex numbers, which constitute an $SO(1, d-1)$ spinor of the original type, in other words, the reduction is decomposition in components which depend only on x^0. Let us now focus on the 10-dimensional theory with $G = U(N)$ and its Lie algebra $\mathfrak{u}(N)$ equipped with the bi-invariant positive definite scalar product $\langle U, V\rangle = -\mathrm{Tr}(UV)$ (the motivation of this choice will be discussed below). The spinor bundle S is by supersymmetry a Majorana-Weyl representation of the Lorentz group $SO(1, 9)$ equipped with an anticommuting algebra structure, since the gamma matrices and charge conjugation matrix are symmetric (for a superspace construction of super Yang-Mills theory, see [25]). We then reduce this theory to the time axis, by subsequently reducing the 8 spatial coordinates. The gauge field A reduces to the sum of a one-form field and 9 scalars, which are just its components: $A \mapsto A_0(x^0)\mathrm{d}x^0 + X_1(x^0) + \ldots + X_9(x^0)$. The $N \times N$ Hermitian matrices X_i together transform under the vector representation of $SO(8)$. The field θ reduces to $\mathfrak{u}(N)$-valued scalars $\theta_\alpha(x^0)$ which constitute a real spinor representation of $SO(1, 9)$. The Weyl condition on the Majorana spinor θ we choose is

$$(\mathbf{1} - \Gamma)\theta = 0\,, \tag{4.2.10}$$

where Γ is the volume element in $C\ell_{0,10}$, which we in the eleven-dimensional theory of the previous chapter have denoted by $-i\Gamma^0$. A straightforward calculation of the chirality condition above yields $\theta_1 = \theta^2$ for $\theta = (\theta_1, \theta_2)$. Under an invertible linear transformation on the spinor module (multiplication by $(\mathbf{1} - i\Gamma^{10})$), the chirality condition above becomes the restriction $\theta_1 = 0$, the gauge-fixing condition on spinors in the eleven-dimensional supermembrane theory. In ten dimensions, the charge conjugation matrix is antisymmetric and can be chosen equal to Γ^0, so that $\bar{\theta}\Gamma^0\nabla_0\theta$ becomes $-\theta_2^T\nabla_0\theta_2$ and $\bar{\theta}\Gamma^{0a}\theta$ turns into $-\theta_2^T\Gamma^a\theta_2$. After a rescaling of the spinor components by a factor i, the dimensional reduction of (4.2.1) reads

$$L = \frac{1}{2q}\mathrm{Tr}\Big[\nabla_0 X^a \nabla_0 X_a - \frac{q^2}{2}[X_a, X_b][X^a, X^b] + \theta^T\nabla_0\theta + q\theta^T\Gamma^a[X_a, \theta]\Big]\mathrm{d}x^0\,, \tag{4.2.11}$$

where the transpose T is with respect to the spinor module, while the trace applies to the gauge algebra. The A_0-field is contained in the covariant derivative: $\nabla_0 X_a = \partial_0 X_a - q[A_0, X_a]$. Under the reduction the residual infinitesimal gauge symmetry is as expected given by $\Phi : \Omega^0(M^1, \mathfrak{u}(N))\Gamma T \longrightarrow \mathcal{F}_1$:

$$\iota_\xi\delta X^a = [X^a, g]\,, \quad \iota_\xi\delta\theta^\alpha = [\theta^\alpha, g]\,, \quad \iota_\xi\delta A_0 = \partial_0 g + [A_0, g]\,, \tag{4.2.12}$$

for $\xi = \Phi(g) \in \Gamma T\mathcal{F}_1$. Furthermore there are the supersymmetry transformations

$$\iota_\xi\delta X^a = \bar{\epsilon}\Gamma^a\theta\,, \quad \iota_\xi\delta A_0 = \epsilon^T\theta\,, \quad \iota_\xi\delta\theta = \nabla_0 X_a\Gamma^0\Gamma^a\epsilon + \frac{q}{2}[X_a, X_b]\Gamma^{ab}\epsilon\,, \tag{4.2.13}$$

for $\xi = \Xi(\epsilon) \in \Gamma T\mathcal{F}$. However, since the gauge group $U(N)$ contains an abelian factor $U(1)$, there are *additional* supersymmetry transformations

$$\iota_\xi\delta X^a = \iota_\xi\delta A_0 = 0\,, \quad \iota_\xi\delta\theta = \epsilon\,. \tag{4.2.14}$$

These are just constant translations of the spinor, proportional a central element of the gauge algebra generating the $U(1)$ factor (namely, i times the unit matrix). This does also apply to the bosonic fields, resulting in translations $X^a \mapsto X^a + c^a$, where c^a are proportional to the unit matrix. We shall denote the generators of the two types of supersymmetries suggestively Q^+ and Q^-. These are both 16-component $\mathfrak{u}(N)$-valued Majorana-Weyl spinors. They are given by

$$Q^+ = -q^{-1}\mathrm{Tr}\big[(\nabla_0 X_a\Gamma^a + \tfrac{1}{2}[X_a, X_b]\Gamma^{ab})\theta\big]\,, \quad Q^- = -q^{-1}\mathrm{Tr}\,\theta\,. \tag{4.2.15}$$

Passing to the Hamiltonian formalism, instantaneous phase space is spanned by $9+16+1$ unitary matrices X^a, θ^α and A_0 and the momenta $P_a = q^{-1}\nabla_0 X_a$ (as usual, fermionic momentum is a linear transformation of θ). The gauge symmetry of the Lagrangian gives rise to a set of first-class (Gauss) constraints

$$\psi = [X^a, P_a] + \frac{q^{-1}}{2}[\theta^T, \theta] = 0\,, \tag{4.2.16}$$

which give rise to the $\mathfrak{su}(N)$-algebra in the standard representation,

$$[\psi_{ij}, \psi_{k\ell}]_D = i(\delta_{i\ell}\psi_{jk} - \delta_{jk}\psi_{i\ell})\,. \tag{4.2.17}$$

The Hamiltonian of the system is given by

$$H = q\mathrm{Tr}\Big[\frac{1}{2}P_a P^a + \frac{1}{4}[X_a, X_b][X^a, X^b] - \frac{q^{-1}}{2}\theta^T\Gamma^a[X_a, \theta]\Big]\,. \tag{4.2.18}$$

4.2.3 Matrix Models in String Theory

The dimensional reduction performed above leads to a gauge theory of time-dependent matrices, and is an example of a *matrix model.* These theories have been known for quite some time, and their spectrum was analysed in [61]. In the nineties these models were found to play a key rôle in the description of the low-energy effective behaviour of Dirichlet branes in superstring theory. When considering a Nambu-Goto-based theory containing open strings, one always faces the problem of imposing boundary conditions at the endpoints of the string. To prevent momentum flow in and out of the string through these points, one may impose *von Neumann* boundary conditions, requiring the derivative of the embedding coordinates along the spatial worldsheet directions to vanish:

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,\sigma_0) = \frac{\partial X^\mu}{\partial\sigma}(\tau,\sigma_1) = 0\,. \tag{4.2.19}$$

Suppose now one takes p embedding space directions to be compact. Because the mass of a string is proportional to its length, the closed strings winding around the compactified dimensions become very light as the radii tend to zero, but on the other hand configurations with a nonzero number of momentum quanta become infinitely heavy. For example, if $p = 1$ the mass-squared is given in terms of the Kaluza-Klein momentum quantum number n along the circle of radius R and the winding number w by

$$\mathscr{M}^2 = \Big(\frac{n}{R} + \frac{wR}{\alpha}\Big)^2 + \frac{4}{\alpha}(N-1)\,, \tag{4.2.20}$$

where α is the string coupling constant and N is the number of left-moving modes in the uncompactified directions. Observe that interchanging winding number and momentum quantum number is physically equivalent[12] to interchanging the compactification radius and its inverse (multiplied by the string coupling constant). This phenomenon is called *T-duality.* So for closed strings, letting the compactification radius go to zero does not produce a dimensionally reduced theory, as would be the case for particle theories, but it 'grows' extra dimensions and the full, uncompactified theory emerges. Open strings however behave like particles and seem to get dimensionally reduced when the radii become very small, which seems to produce a paradox in theories containing both open and closed strings. This seeming contradiction is resolved by taking into account the transformation of the von Neumann boundary conditions under the T-duality. A simple calculation shows that under a linear change of embedding coordinates $X \mapsto X'$, the duality transformation changes p of the boundary conditions (4.2.19) into conditions of the form $\partial_\tau X'^i(\tau,\sigma_0) = \partial_\tau X'^i(\tau,\sigma_1) = 0$, which restricts only the endpoints of the open string to a hyperplane, perpendicular to the compactified dimensions. These sort of boundary conditions are called *Dirichlet* boundary conditions and the $(D-p)$-dimensional hyperplane that is defined by them is called a *D-brane.* Integrating the compactified embedding coordinates in the T-dualised theory along the spacesheet yields an integer times the length of the compactification circles, so both endpoints actually lay on the same hypersurface, and the winding number is a nondynamical integer, namely the Kaluza-Klein momentum number:

$$[X'^i(\tau,\sigma_0)] = [X'^i(\tau,\sigma_1)] = \theta_i R'_i\,, \quad \theta_i \in [0,2\pi]\,, \quad X'^i(\tau,\sigma_0) - X'^i(\tau,\sigma_1) = 2\pi n_i R_i\,, \tag{4.2.21}$$

where the brackets denote the equivalence class under the compactification identification $X \sim Y \Leftrightarrow X^i - Y^i \in 2\pi R_i\mathbb{Z}$. The angles θ_i determine the position of the Dirichlet brane within the toroidally compactified subspace of the target space. Now we can consider systems with multiple D-branes; for simplicity assume only one target space direction is compactified on a circle. The open strings stretched between the N branes are then characterised not by a single winding number (the T-dual of KK momentum), but by 2 integers labeling the branes. Hence a general state in the Hilbert space of a theory with N $(d-2)$-branes is decomposed as

$$|k\rangle = \sum_{i,j=1}^{N} |k,(i,j)\rangle\lambda_{ij}\,. \tag{4.2.22}$$

The mass of a string stretched between 2 different branes has a minimum since it is proportional to the length of the string, which cannot become smaller then the distance between the branes:

$$\mathscr{M}^2|k,ij\rangle = \Big(\frac{(2\pi n + (\theta_i-\theta_j)R'}{2\pi\alpha}\Big)^2|k,ij\rangle + \frac{1}{\alpha}(N-1)|k,ij\rangle\,. \tag{4.2.23}$$

[12]Not only at the level of the mass-spectrum, but for all interaction diagrams

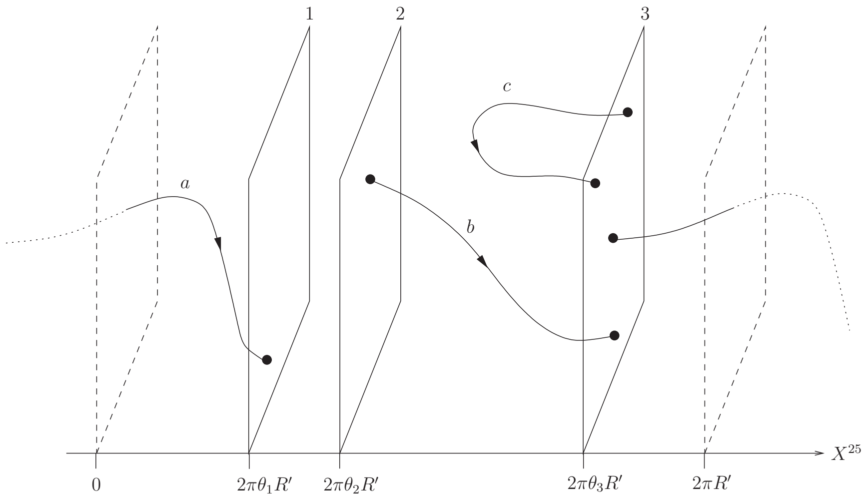


Figure 5: A three brane configuration with three strings stretched between them: string a is in the eigenstate $|31\rangle$, b is in the state $|23\rangle$ and string c is in $|33\rangle$. The latter string configuration can become arbitrarily light. The dashed planes are not D-branes: they are the identified hyperspaces perpendicular to the compactified direction X^{25}.

Now consider an interaction between string 1 and 2, yielding strings 3 and 4, by joining and then splitting. Assuming the stretching labels (Chan-Paton factors) are nondynamical, the interaction vertex receives a factor (see [7] for a more detailed exposition),

$$\lambda^1_{ij}\lambda^2_{jk}\lambda^3_{k\ell}\lambda^4_{\ell m} = \mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4)\,. \tag{4.2.24}$$

This quantity is invariant under

$$\lambda^a \mapsto U\lambda^a U^{-1}\,, \tag{4.2.25}$$

where $U \in U(N)$ (to preserve the norm of the state). This is a global *worldsheet* symmetry of the theory. By an ingenious mechanism it can be promoted to a *spacetime* gauge symmetry, where the associated gauge field lives on the worldvolume of the Dp-brane by dimensional reduction of super-Yang-Mills theory to the brane worldvolume [8]. Hence the matrix model, introduced in the previous paragraph can be interpreted as low-energy effective action of type IIA string theory which is in every spacelike direction compactified and hence the endpoints of the strings are confined to the worldvolume of zero-dimensional D-branes, called *Dirichlet particles*. However, a physical interpretation of the degrees of freedom in terms of positions of particles only makes sense at the classical vacuum: for the bosonic model this vacuum is characterised by the vanishing potential $\frac{1}{2}[X^a, X^b][X_a, X_b] = 0$, which means that all the matrices X^a are simultaneously diagonalisable. The eigenvalues $\theta^a_1, \ldots, \theta^a_N$ represent the positions of the N D0-branes on the X^a-axis. The degeneracy of these ground states comes from the action of $U(N)$, permuting the eigenvalues and reflecting the fact that the Dirichlet particles are identical bosons. This breaks into the N-fold product $U(1) \times \ldots \times U(1)$ if the eigenvalues coincide, which is exactly the symmetry group of N string states beginning and ending at the same brane.

In [10, 62] an even more fundamental rôle is assigned to the matrix model in the context of *M-theory*. M-theory is the conjectured eleven-dimensional supersymmetric Poincaré-invariant quantum theory unifying all string theories and eleven-dimensional supermembrane theory and supergravity. That is, in the low-energy limit it should generate the 11D supergravity multiplet and its interactions, and under compactification along a spacelike circle it should generate the ten-dimensional superstring. M-theory is believed to yield the matrix model (4.2.11) upon a *discrete lightcone quantisation* (DLCQ). This procedure rests upon the observation that compactifying a theory along a lightlike circle can be approximated by wrapping the theory around a family of spacelike circles [63]. Since M-theory is postulated to produce the Type IIA Green-Schwarz superstring spectrum under spacelike compactification, the lightlike compactification induces a series of string theory Lagrangians with vanishing coupling and string length scales. Careful analysis of the limit of these theories leads to the conclusion that the sector of the spectrum with longitudinal momentum $P^+ = R/N$ is exactly described by nonrelativistic $U(N)$ matrix theory.

4.2.4 Spherical Membranes from Matrices

The correspondence between the $U(N)$ matrix model Lagrangian (4.2.11) and the APD gauge theory Lagrangian (3.5.34) uses the identifications

$$\begin{aligned} q &\longleftrightarrow (P_0^+)^{-1}\,, \\ \mathrm{Tr} &\longleftrightarrow \int_{\Sigma_\tau} |\mathrm{d}^2\sigma| \sqrt{w(\sigma)}\,, \\ [\,.\,,\,.\,] &\longleftrightarrow \{\,.\,,\,.\,\}\,. \end{aligned} \tag{4.2.26}$$

Note that the first correspondence implies that the M-theory longitudinal momentum is identified with N times the longitudinal membrane momentum. The spinors agree by the gauge fixing condition on the supermembrane side and the Weyl condition on the matrix theory side. Note that the fermionic kinetic term in the supermembrane $\overline{\theta}\Gamma^-\nabla_0\theta$ becomes $-\sqrt{2}\theta_2^T\nabla_0\theta_2$ and the interaction term turns into $-\sqrt{2}\theta_2^T\Gamma^a\{X^a,\theta_2\}$. The correspondence therefore is established by the field redefinition

$$\theta \longrightarrow \frac{i\theta}{\sqrt[4]{2}}\,. \tag{4.2.27}$$

Applying these substitutions one quickly finds equivalence of the Lagrangians (4.2.11) and (3.5.34), the Hamiltonians (4.2.11) and (3.5.42), the supercharges (3.5.45) and (4.2.18) in both theories and the constraints. However, the constraints that arise from the harmonic vector fields have no counterpart: we shall return to this issue later.

The natural question that arises is how the algebra of Hamiltonian vector fields relates to the unitary algebras. The super Yang-Mills theory has a finite-dimensional gauge algebra $\mathfrak{u}(N)$, while on the other side the gauge algebra of area-preserving diffeomorphisms is infinite-dimensional. This problem is already encountered when relating the Lagrangians: the supermembrane embedding coordinates are *functions* on the spacesheet, which under harmonic decomposition correspond to matrices of infinite size, opposed to the fields constituting the matrix model, which are of a finite size. We will explicitly investigate the correspondence of these algebras in two simple cases: the spherical and toroidal spacesheet [6, 53, 64]. On a spherical Σ every curve is contractible and the Poisson algebra of functions is a central extension of $\mathfrak{X}_\nu(\Sigma)$ by $\mathbb{R}$. Let $\sigma_1 = \phi \in [0, 2\pi)$ and $\sigma_2 = \theta \in [0, \pi)$ be local coordinates on the spacesheet representing the angle with the vertical axis and the angle with the X-axis in the horizontal plane[13]. The area metric for such a sphere with normalised area reads

$$w_{rs}(\phi,\theta) = \frac{1}{4\pi}(\sin^2\theta\delta_{r1}\delta_{s1} + \delta_{r2}\delta_{s2})\,. \tag{4.2.28}$$

A complete set of globally defined functions on Σ are the *spherical harmonics*: denote $X^i(\phi,\theta)$ the 3 cartesian embedding coordinates, $X(\phi,\theta) = \sin\theta\sin\phi\mathbf{i} + \sin\theta\cos\phi\mathbf{j} + \cos\theta\mathbf{k}$ and define $X^{\alpha\beta}(\phi,\theta) = \sum_{i=1}^3 X^i(\phi,\theta)(\sigma_2\sigma_i)^{\alpha\beta}$ where $\sigma^1,\sigma^2,\sigma^3$ are the (unitary and traceless) Pauli matrices. The X^i are symmetric and satisfy the reality condition $X_{\alpha\beta} \equiv (X^{\alpha\beta})^* = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}X^{\gamma\delta}$. Let $\alpha(2n)$ denote a set a $2n$ matrix indices $\{\alpha_1,\ldots,\alpha_n\}$. Such a set determines a spherical harmonic function,

$$Y^{\alpha(2n)}(\phi,\theta) = X^{(\alpha_1\alpha_2}(\phi,\theta)X^{\alpha_3\alpha_4}(\phi,\theta)\ldots X^{\alpha_{2n-1}\alpha_{2n})}(\phi,\theta)\,, \tag{4.2.29}$$

where the brackets denote symmetrisation with unit weight. Note that this basis is not normal with respect to the L^2 inner product (4.1.1). The eigenvalues of the Laplace-Beltrami operator

[13] The coordinate patch cannot contain the north pole, but this has no significance for the definition of a function basis

may be shown to be $-n(n+1)$, and we define ${D_k}^{mn}$ to be the tensor associated to pointwise multiplication of functions,

$$Y^{\alpha(2m)}Y^{\beta(2n)} = \sum_{k\in K_0(m,n)} {D_k}^{mn}\varepsilon^{\alpha\beta(m+n-k)}Y^{\alpha(m-n+k)\beta(n-m+k)}\,, \tag{4.2.30}$$

where $K_0(m,n) = \{m+n, m+n-2, \ldots, 2+|m-n|, |m-n|\}$, $\varepsilon^{\alpha\beta(2m)} = \varepsilon^{\alpha_1\beta_1}\ldots\varepsilon^{\alpha_{2m}\beta_{2m}} + (\alpha_1 \leftrightarrow \ldots \leftrightarrow \alpha_{2m})$ and $Y^{\alpha(2n)\beta(2m)} \equiv X^{(\alpha_1\alpha_2}\ldots X^{\alpha_{2n-1}\alpha_{2n}}X^{\beta_1\beta_2}\ldots X^{\beta_{2m-1}\beta_{2m})}$. Keeping in mind that each $X^{\alpha\beta}$ contributes factor $e^{\pm i\theta}$ and $e^{\pm i\phi}$ if it is diagonal, we see that the product is a sum of terms with highest wave number $2(m+n)$ and lowest one $2|m-n|$. The Clebsch-Gordon coefficients are uniquely determined by the initial value ${D_{m+n}}^{mn} = 1$ and the recursive relation

$$D^{mn}_{k-2} = \frac{(m+n+k+1)(k^2-(m-n)^2)}{(m+n-k+2)(4k^2-1)}D^{mn}_k\,. \tag{4.2.31}$$

Writing out the matrix $X^{\alpha\beta}(\phi,\theta)$, one easily verifies

$$\frac{\partial X^{\alpha\beta}}{\partial\phi}\frac{\partial X^{\gamma\delta}}{\partial\theta} - \frac{\partial X^{\alpha\beta}}{\partial\theta}\frac{\partial X^{\gamma\delta}}{\partial\phi} = -(\sin\theta)X^{\alpha\gamma}\varepsilon^{\beta\delta}\,. \tag{4.2.32}$$

Hence, using the chain rule we find

$$\{A,B\} = -8\pi X^{\alpha\beta}\varepsilon^{\gamma\delta}\frac{\partial A}{\partial X^{\alpha\gamma}}\frac{\partial B}{\partial X^{\beta\delta}}\,. \tag{4.2.33}$$

Applying this to the harmonic basis functions one finds after some algebra

$$\{Y^{\alpha(2n)}, Y^{\beta(2m)}\} = \sum_{k\in K_1(m,n)} {f_k}^{mn}\varepsilon^{\alpha(m+n-k)\beta(m+n-k)}Y^{\alpha(m-n+k)\beta(n-m+k)}\,, \tag{4.2.34}$$

where $K_1(m,n) = \{m+n-1, m+n-3, \ldots, |m-n|+1\}$. The structure constants, which are vanishing for *even* k, are related to the Clebsch-Gordon coefficients by

$${f_k}^{mn} = -8\pi\frac{mn(2k+1)}{m+n+k}{D_{k-1}}^{m-1n-1}\,. \tag{4.2.35}$$

Let us now relate to this algebra to the algebras $\mathfrak{su}(N)$. It is a fact that irreducible representations of $SO(3)$ can take all dimensions; the k-fold symmetrised 3-dimensional vector representation, $[1,\ldots,1]$ (j times) has dimension $3k$. Taking the irreducible harmonic subspace reduces this number by $k-1$, leaving $2j+1$ degrees of freedom. The spin of the representation is then j. Taking the tensor product with a fundamental 2-dimensional spinor representation yields a spin $j+\frac{1}{2}$ representation of dimension $2(2j+1)$. However, imposing the harmonic condition (2.4.2) fixes $2k$ components, leaving $2j+2$ independent parameters. These results can also immediately be obtained from the character of an irreducible (half-)integer spin j representation,

$$\chi_j(\exp(i\phi M_{12})) = \frac{\sin((j+\frac{1}{2})\phi)}{\sin(\frac{1}{2}\phi)}\,. \tag{4.2.36}$$

In fact, decomposing each irreducible spin-j representation into $2j+1$ eigenspaces of $J^3 = M_{12}$ is equivalent to decomposing the character ring with spin j into a basis of spherical harmonics $Y_{jm}(\phi,\theta)$, $m=-j,\ldots,j$. So let us consider an irreducible N-dimensional representation of $SO(3)$ and let $J^i \in \mathrm{End}(V^N)$, $i=1,2,3$ be its generators, satisfying

$$[J^i, J^j] = i\varepsilon^{ijk}J^k\,, \quad (J^i)^\dagger = J^i\,, J^2 = \frac{N^2-1}{4}\mathbf{1}\,. \tag{4.2.37}$$

The latter equation comes from the fact that the spin-j representation is an eigenspace of the Casimir invariant $J^2 = (J^1)^2 + (J^2)^2 + (J^3)^2 \in U(\mathfrak{so}(3))$ with eigenvalue $j(j+1)$. We may lift this to a representation of the double covering $SU(2)$ of the rotation group by setting $J^{\alpha\beta} = J^i(\sigma_2\sigma_i)^{\alpha\beta}$. Define the $N \times N$ matrices

$$T^{\alpha(2n)} = \left(\frac{4}{N^2-1}\right)^{(n-1)/2} J^{(\alpha_1\alpha_2}J^{\alpha_3\alpha_4}\dots J^{\alpha_{2n-1}\alpha_{2n})}\,. \tag{4.2.38}$$

These matrices satisfy the important property that $T^{\alpha(2n)} = 0$ whenever $n \geq N$. This can be seen as follows: decompose the spin j irreducible $SO(3)$ representation into eigenspaces of J^3: $J^2|j,m\rangle = j(j+1)|j,m\rangle$ and $J^3|j,m\rangle = m|j,m\rangle$ with $m \in [-j,j]\cap\mathbb{Z}$. Then we define the creation and annihilation operators $J^\pm = (J^1 \pm iJ^2)/\sqrt{2} \in \mathrm{End}(V^N_{\mathbb{C}})$. These may be shown to act on such an eigenstate as

$$J^\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle\,. \tag{4.2.39}$$

Hence the operators $(J^+)^{2j+1}$ and $(J^-)^{2j+1}$ in the universal enveloping algebra must be represented by zero on V^N. A quick calculation yields that in the $\mathfrak{su}(2)$ basis, $J^{11} = -iJ^+$ and $J^{22} = iJ^-$. Using the commutator $[J^+,J^-] = J^3$, one quickly verifies that sequences with more than $N-1$ entries yield zero and that all the $T^{\alpha(2n)}$ with $n < N$ are linearly independent. Furthermore we have the property $T^{\alpha(2n)})^\dagger = \varepsilon_{\alpha(2n)\beta(2n)}T^{\beta(2n)}$ and the generators are traceless, $\mathrm{Tr}(T^{\alpha(2n)}) = 0$ for $0 > n > N$. Since there are $\sum_{n=1}^{N-1}(2n+1) = N^2-1$ linearly independent $N\times N$ matrices $T^{\alpha(2n)}$, we have by the properties above found a complete basis of $\mathfrak{su}(N)$. Then we denote the 3-tensor associated to matrix multiplication by

$$T^{\alpha(2n)}T^{\beta(2m)} = \sum_{k\in K}\begin{Bmatrix} m & n \\ & k \end{Bmatrix}\varepsilon_{\alpha\beta(m+n-k)}T^{\alpha(m-n+k)\beta(n-m+k)}\,, \tag{4.2.40}$$

where $k = m+n, m+n-1,\dots,|m-n|$. Hence we should include the unit $T^0 = \sqrt{(N^2-1)/4}\mathbf{1}$ in the universal enveloping algebra to let the multiplication close. One easily derives that the coefficients $\{{}^m{}_k{}^n\}$ are symmetric in the indices m and n and satisfy the initial conditions

$$\begin{Bmatrix} m & & n \\ & m+n & \end{Bmatrix} = \sqrt{\frac{N^2-1}{4}}\,,\qquad \begin{Bmatrix} m & & n \\ & m+n-1 & \end{Bmatrix} = -imn\,, \tag{4.2.41}$$

which together with the recursive relation

$$\begin{aligned}&\frac{m+n-k+2}{m+n+k+1}\begin{Bmatrix} m & & n \\ & k-2 & \end{Bmatrix} - \left(\frac{N^2-k^2}{N^2-1}\right)\frac{k^2-(m-n)^2}{4k^2-1}\begin{Bmatrix} m & n \\ & k \end{Bmatrix}\\ &= \frac{i}{\sqrt{N^2-1}}\frac{k(k-1)-m(m+1)-n(n+1)}{m+n+k+1}\begin{Bmatrix} m & & n \\ & k-1 & \end{Bmatrix}\end{aligned} \tag{4.2.42}$$

uniquely determines all the coefficients. Observe that in the limit $N\to\infty$ the initial values and recursion relation for $\sqrt{4/N^2-1}\{{}^m{}_k{}^n\}$ become those of $D_k{}^{mn}$. We find that the in the large-N regime the coefficients of $\mathfrak{su}(N)$ contain both the structure constants as well as the Clebsch-Gordon coefficients in the leading order:

$$\begin{Bmatrix} m & n \\ & k \end{Bmatrix} = \begin{cases} \sqrt{\frac{N^2-1}{4}}\left(D_k{}^{mn} + \mathcal{O}(N^{-2})\right), & \text{for } m+n-k \text{ even} \\ \frac{i}{8\pi}\left(f_k{}^{mn} + \mathcal{O}(N^{-2})\right), & \text{for } m+n-k \text{ odd}\end{cases}\,. \tag{4.2.43}$$

To see that the latter are also the structure coefficients of the special unitary algebra we invoke reality arguments; using antisymmetry of the permutation symbol and symmetry of $T^{\alpha(m-n+k)\beta(n-m+k)}$ under a switch $\alpha(2m) \leftrightarrow \beta(2n)$ we see that the commutator under matrix multiplication gives twice the right hand side of (4.2.40). However, the structure constants of $\mathfrak{su}(N)$ should be strictly imaginary, as $[U,V]^\dagger = -[U,V]$. From the identifications above it is clear that only terms for which $m+n-k \in 2\mathbb{N}+1$ fulfill this condition. So the leading order in the asymptotic N-expansion of the $\mathfrak{su}(N)$ structure constants are proportional to those of the Poisson algebra on the unit sphere, or equivalently the algebra of Hamiltonian vector fields as harmonic vector fields are absent. However, we should not draw too strong conclusions from this observation, for it is clear that we have not constructed Lie algebra homomorphisms from $\mathfrak{X}_\nu(S^2)$ to $\mathfrak{su}(N)$. For such a construction we would need some kind Lie-bracket equivariant projections within the Poisson algebra with images isomorphic to the unitary algebras. The obstacle is the nature of the APD bracket, which always decomposes into higher frequency modes, making such a factorisation impossible. This is directly related to the nonlinearity of the equations of motion, in which low-frequency modes always excite higher states because of the appearance of these brackets. In conclusion, we have constructed a series of *vector space* isomorphisms, $\phi_N : \mathfrak{X}|_N(S^2) \longrightarrow \mathfrak{su}(N)$, where $\mathfrak{X}|_N(S^2)$ is the vector space spanned by the vector fields $\eta_{\alpha(2n)} = \mathrm{grad}_\nu(Y^{\alpha(2n)})$ on the unit sphere, and eventually in the limit $N \longrightarrow \infty$, the domain becomes a Lie algebra and the isomorphism becomes a Lie algebra isomorphism.

4.2.5 Toroidal Membranes from Matrices

The 'weak' approximation of a Lie algebra by its structure constants above has been given the name *quasilimit* in [65]. The ingredients for such an approximation are: an unbounded subset I of $\mathbb{N}$, a sequence of (real or complex) Lie algebras $(\mathfrak{g}_\alpha, [\,.\,,\,.\,]_\alpha)_{\alpha\in I}$ equipped with a positive-definite norm $||\,.\,||_\alpha$ (taking values in $\mathbb{R}$), a limiting Lie algebra $(\mathfrak{X}, , [\,.\,,\,.\,])$ and a collection of surjective maps $f_\alpha : \mathfrak{X} \longrightarrow \mathfrak{g}_\alpha$ such that for all $g, h \in \mathfrak{X}$,

$$\lim_{\alpha\to\infty} ||f_\alpha(g) - f_\alpha(h)||_\alpha = 0 \Rightarrow g = h\,, \tag{4.2.44}$$

$$\lim_{\alpha\to\infty} ||[f_\alpha(g), f_\alpha(h)]_\alpha - f_\alpha([g,h])||_\alpha = 0\,. \tag{4.2.45}$$

In the original article the definition was weaker in the sense that the sequence $\mathfrak{g}_\alpha$ does only have to be equipped with a bilinear metric d_α, not necessarily originating from a positive definite norm. It was also shown that for slight uniform deformations of this metric, say d'_α such that there exist positive real numbers r and s such that for all $\alpha \in I$ and $g, h \in \mathfrak{g}_\alpha$ we have $r d_\alpha(g,h) \le d'_\alpha(g,h) \le s d_\alpha(g,h)$, the conditions above remain valid if they are satisfied by d_α. From the mathematical point of view this approximation scheme is rather general: it provides a much weaker relation between the algebra structures of the sequence and the limiting Lie algebra than for instance the usual inductive limit; even stronger, it does not guarantee uniqueness of the limiting algebra. In particular, there has been found an infinite number of pairwise non-isomorphic quasi-limits of the special unitary algebra [66]. In the case of a toroidal spacesheet, there is however a well-defined mathematical framework that describes our approximation scheme [67]. Let T^2 be (globally) coordinated by $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$, representing the angles around the two circles. We may embed the torus in $\mathbb{R}^3$ by the coordinate functions $\mathbf{x}(\theta,\phi) = ((1+\cos\theta)\cos\phi, (1+\cos\theta)\cos\phi, \sin\theta)$, but this is by no means necessary. A complete basis of functions on T^2 is given by the Fourier modes $Y_{\boldsymbol{m}}(\theta,\phi) = e^{im_1\theta + im_2\phi}$, where $\boldsymbol{m} = (m_1, m_2)$, the index set being the lattice $\mathbb{Z}\times\mathbb{Z}$. The Riemannian metric in this coordinate system looks like $w = (4\pi^2)^{-1}(\mathrm{d}\theta\otimes\mathrm{d}\theta + \mathrm{d}\phi\otimes\mathrm{d}\phi)$ and the symplectic form is $\nu = (4\pi)^{-1}\mathrm{d}\theta\wedge\mathrm{d}\phi$. Hence this set of basis functions is normalised w.r.t. the L^2 inner product (4.1.1) and eigenfunctions of the Laplace-Beltrami operator with eigenvalues $\boldsymbol{m}\cdot\boldsymbol{m} = m_1^2 + m_2^2$. Hence the symplectic gradient acts as $\mathrm{grad}_\nu f = 4\pi((\partial f/\partial\phi)\partial/\partial\theta - (\partial f/\partial\theta)\partial/\partial\phi)$, and the Fourier basis gives rise to a basis of globally Hamiltonian vector fields [68]

$$L_{\boldsymbol{m}} = 4\pi i e^{im_1\theta+im_2\phi}\Big(m_2\frac{\partial}{\partial\theta} - m_1\frac{\partial}{\partial\phi}\Big)\,. \tag{4.2.46}$$

Since the 2-torus is a genus one manifold, there are two linearly independent harmonic vector fields, $P_1 = (2\pi)^{-1}\partial/\partial\theta$, $P_2 = (2\pi)^{-1}\partial/\partial\phi$. The APD algebra on the torus then looks like

$$\begin{aligned} [L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] &= (\boldsymbol{m}\times\boldsymbol{n})L_{\boldsymbol{m}+\boldsymbol{n}}\,,\\ [P_\lambda, L_{\boldsymbol{m}}] &= m_\lambda L_{\boldsymbol{m}}\,,\\ [P_1, P_2] &= 0\,, \end{aligned} \tag{4.2.47}$$

where $\boldsymbol{m} \times \boldsymbol{n} = m_1 n_2 - m_2 n_1$. Again we observe that the gradient vector fields form an ideal $\mathfrak{X}^H_\nu(T^2)$, and that their Lie algebra structure allows no restriction to low-laying modes. Note that the vanishing commutation relation of the harmonic vector fields is an artefact of the chosen manifold: for more complicated topologies the P_λ components will depend on the coordinates, and their mutual brackets may give non vanishing globally Hamiltonian vector fields. One easily computes the various quantities η, f, d and c associated to the L^2 inner product, the Lie bracket, pointwise function multiplication and the Riemannian inner product introduced in the previous section,

$$\begin{aligned}
\eta_{\boldsymbol{mn}} &= \delta_{\boldsymbol{m}+\boldsymbol{n}}\,, & d_{\boldsymbol{mnk}} &= \delta_{\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{k}}\,,\\
f_{\boldsymbol{mnk}} &= -4\pi^2(\boldsymbol{m}\times\boldsymbol{n})\delta_{\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{k}}\,, & f_{\lambda\boldsymbol{mn}} &= 2\pi i\delta m_\lambda \delta_{\boldsymbol{m}+\boldsymbol{n}}\,,\\
c_{\boldsymbol{mnk}} &= -\frac{\boldsymbol{m}\cdot\boldsymbol{n}}{\boldsymbol{m}\cdot\boldsymbol{m}}\delta_{\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{k}}\,, & c_{\lambda\boldsymbol{mn}} &= -4\pi i\varepsilon^{\lambda\lambda'}m_{\lambda'}\delta_{\boldsymbol{m}+\boldsymbol{n}}\,,
\end{aligned} \tag{4.2.48}$$

where we have denoted $\boldsymbol{m}\cdot\boldsymbol{n} = m_1 n_1 + m_2 n_2$ the standard Euclidean inner product on the index lattice. Tensor components with 2 or more λ-indices are zero. Observe that the appearance of numbers such as m_λ, using the coincidence that there are as many harmonic diffeomorphism generators as there are index components, makes such a formulation for higher genus surfaces impossible. The subalgebra of globally Hamiltonian vector fields can easily be seen to form a quasi-limit of $\mathfrak{su}(N)$ algebras by exploiting 't Hooft's so-called *twist matrices* [69]

$$\Omega_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & \omega & 0 & \dots & 0\\ 0 & 0 & \omega^2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & \omega^{N-1}\end{pmatrix}, \tag{4.2.49}$$

where $\omega = \exp(-4\pi i/N)$ (although any other N-th root of unity would suffice). These matrices can easily be verified to be unitary and obeying $\Omega_1\Omega_2 = \omega\Omega_2\Omega_1$. Let us now define the $N\times N$-matrices

$$T_{\boldsymbol{m}} = N\omega^{m_1 m_2/2}\Omega_1^{m_1}\Omega_2^{m_2}\,. \tag{4.2.50}$$

The double index $\boldsymbol{m}$ can be restricted to the finite lattice $[0,N]\times[0,N]\cap\mathbb{Z}^2$ since $T_{\boldsymbol{m}} = T_{\boldsymbol{n}}$ if and only if $\boldsymbol{m} = \boldsymbol{n} \bmod N$ (by this we mean $m_i = n_i \bmod N$). One quickly verifies that except for $T_{\boldsymbol{0}} = N\boldsymbol{1}$, all the $T_{\boldsymbol{m}}$ are traceless and $T^\dagger_{\boldsymbol{m}} = T_{-\boldsymbol{m}} = T_{\overline{\boldsymbol{m}}}$, where $\overline{\boldsymbol{m}} = (N-m_1, N-m_2)$. Hence the matrices $R_{\boldsymbol{m}} = T_{\boldsymbol{m}} - T_{\overline{\boldsymbol{m}}}$ and $S_{\boldsymbol{m}} = i(T_{\boldsymbol{m}} + T_{\overline{\boldsymbol{m}}})$ are antihermitean; since this linear transformation of the space spanned by the matrices $T_{\boldsymbol{m}}$ is isomorphic and the $T_{\boldsymbol{m}}$ are all linearly independent, they form a subspace of matrix representation of the Lie algebra $\mathfrak{su}(N)$. Since the dimension of this spanning is (excluding the unit matrix $T_{\boldsymbol{0}}$) N^2-1, they actually span the whole Lie algebra. Using the multiplication properties of the twist matrices, one finds the multiplication rule $T_{\boldsymbol{m}}T_{\boldsymbol{n}} = N\omega^{-(\boldsymbol{m}\times\boldsymbol{n})/2}T_{\boldsymbol{m}+\boldsymbol{n}}$ and hence

$$\begin{aligned}
\tilde{\eta}_{\boldsymbol{mn}} &\equiv N^{-3}\mathrm{Tr}(T_{\boldsymbol{m}}T_{\boldsymbol{n}}) = \delta_{\boldsymbol{n}+\boldsymbol{m}}\,,\\
\tilde{f}_{\boldsymbol{mnk}} &\equiv -iN^{-3}\mathrm{Tr}([T_{\boldsymbol{m}},T_{\boldsymbol{n}}]T_{\boldsymbol{k}}) = 2N\sin\Big(\frac{2\pi(\boldsymbol{m}\times\boldsymbol{n})}{N}\Big)\delta_{\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{k}}\,,\\
\tilde{d}_{\boldsymbol{mnk}} &\equiv N^{-4}\mathrm{Tr}(\{T_{\boldsymbol{m}},T_{\boldsymbol{n}}\}T_{\boldsymbol{k}}) = 2\cos\Big(\frac{2\pi(\boldsymbol{m}\times\boldsymbol{n})}{N}\Big)\delta_{\boldsymbol{m}+\boldsymbol{n}+\boldsymbol{k}}\,,
\end{aligned} \tag{4.2.51}$$

where $\delta_{\boldsymbol{m}} = \delta_{m_1 \bmod N,0}\delta_{m_2 \bmod N,0}$. Again the leading order in N^{-1} of these quantities is proportional to the corresponding untilded quantities in (4.2.48); in particular, we have $\tilde{f}_{\boldsymbol{mnk}} = \pi^{-1}f_{\boldsymbol{mnk}} + \mathcal{O}(N^{-2})$ and $\tilde{f}_{\boldsymbol{mnk}} = 2d_{\boldsymbol{mnk}} + \mathcal{O}(N^{-2})$. Several important remarks are to be made regarding this approximation. Using as a norm on $\mathfrak{su}(N)$ the trace of the product and a collection of linear surjective vector space homomorphisms $\phi_N : \mathfrak{X}^G_\nu(T^2) \longrightarrow \mathfrak{su}(N)$ defined by $\phi_N(Y_{\boldsymbol{m}}) = T_{\boldsymbol{m} \bmod N}$, one sees that this indeed defines a quasi-limit of the Poisson algebra on the torus. Since the unit matrix must be included at each finite N stage to close the algebra (it will approximate the zero modes in the Poisson algebra), it is actually the extension $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ which consistently approximates the Poisson algebra. Under the gradient the constant modes are projected out, so the traceless unitary matrix algebra defines a quasi-limit of the ideal of globally Hamiltonian vector fields. The construction of $\mathfrak{su}(N)$ by the twist matrices is particularly convenient to show the weakness of the quasi-limit definition earlier and the ambiguity in the definition of unitary matrices of infinite size. Taking for ω another primitive root of unity, say $\omega = \exp(4\pi i N/M)$ where N is odd and $M < N$ are relatively prime, one obtains generators $T_{\boldsymbol{m}}$ of $\mathfrak{su}(N)$ satisfying the algebra relation $[T_{\boldsymbol{m}}, T_{\boldsymbol{n}}] = 2(N/M)\sin(2\pi N(\boldsymbol{m}\times\boldsymbol{n})/M)T_{\boldsymbol{m}+\boldsymbol{n}}$. Now we take a limit $N, M \longrightarrow \infty$ such that $M/N \longrightarrow \lambda < 1$. We obtain the algebra $\mathfrak{g}_\lambda$, spanned by an infinite number of $T_{\boldsymbol{m}}$, $\boldsymbol{m} \in \mathbb{N}\times\mathbb{N}$, satisfying the algebra relation

$$[T_{\boldsymbol{m}}, T_{\boldsymbol{n}}] = \frac{2}{\lambda}\sin(2\pi\lambda(\boldsymbol{m}\times\boldsymbol{n}))T_{\boldsymbol{m}+\boldsymbol{n}}\,. \tag{4.2.52}$$

Although each $\mathfrak{g}_\lambda$ can be shown to be a quasi-limit of $(\mathfrak{su}(N)\oplus\mathfrak{u}(1))_{N\in 2\mathbb{N}+1}$, it was shown in [66] that if $\lambda \neq \lambda'$ are *irrational* (and smaller than 1/4), $\mathfrak{g}_\lambda$ and $\mathfrak{g}_{\lambda'}$ are non-isomorphic. If M/N approaches a rational number, $\mathfrak{g}_\lambda$ will contain an ideal which is isomorphic to the Poisson algebra on the torus.

Furthermore we notice that the harmonic vector fields, represented by the Lie algebra elements P_1 and P_2 seem to play no rôle in the approximation: it is the Poisson algebra which admits a quasi-limit by unitary matrices, not the full APD algebra. This phenomenon can be rigorously explained from a Lie algebra cohomology viewpoint, as we shall see below. Finally we mention that adopting these approximations in the physical theory will modify the Lie algebra of Noether currents by terms of the order N^{-2}. This follows from the fact that the Lie-algebra valued tensor identities (4.1.34) receive contributions for finite N. In particular, the bracket $\{M^{-a}, \mathscr{M}^2\}_D = 0$ on shell, which guarantees nonmanifest Lorentz invariance, relies upon these identities, and will therefore be violated in the finite-N approximation.

4.3 Regularisation as a Deformation

In this section we shall approach the regularisation procedure from an algebra-theoretic point of view. Although the quasi-limit characterisation is rigorously defined, its interpretation is cumbersome and it is not clear whether it captures all the information of the approximation in its definition. There exist algebraic theories covering modifications of Lie brackets which, as we shall see below, is the process that underlies the limits of the paragraphs above. Roughly speaking there are two mathematical approaches to the modification of Poisson algebras [70], both of which applicable to matrix regularisation: *geometric quantisation* and *deformation quantisation.* The former lifts the Poisson brackets to a complex line bundle on the spacesheet, while the latter has a more algebraic approach, deforming the algebra in an associative fashion.

4.3.1 Lie Algebra Cohomology

A consistent modification of a Lie bracket is called a *deformation.* Such a one-parameter deformation of a Lie algebra $\mathfrak{g}$ (possibly infinite-dimensional, but equipped with a topology) with Lie bracket $[\,.\,,\,.\,]$ is determined by a smooth mapping $\phi : \mathfrak{g}\times\mathfrak{g}\times\mathbb{R} \longrightarrow \mathfrak{g}$ such that $\phi(g_1, g_2, 0) = [g_1, g_2]$ and for all $t \in \mathbb{R}$ the map $g_1, g_2 \mapsto [g_1, g_2]_t = \phi(g_1, g_2, t)$ defines a Lie algebra structure on $\mathfrak{g}$. Two deformations h_1, h_2 are said to be *equivalent* if there exists a map $f : \mathfrak{g}\times\mathbb{R} \longrightarrow \mathfrak{g}$, smooth in the first argument and linear in the second, which is intertwining in the sense that $h_1(g_1, g_2, t) = h_2(f(g_1, t), f(g_2, t), t)$. Given a Lie algebra deformation h, we can take the derivative at the origin to the deformation parameter, giving a map

$$\eta : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g} : \eta(g_1, g_2) = \frac{\partial}{\partial t}h(g_1, g_2, t)|_{t=0}\,, \tag{4.3.1}$$

which is referred to as the associated *infinitesimal deformation*. By the antisymmetry and Jacobi identity of the deformed Lie bracket it is antisymmetric and satisfies

$$\begin{aligned}&[\eta(g_1,g_2),g_3]+[\eta(g_2,g_3),g_1]+[\eta(g_3,g_1),g_2]+\\&\eta([g_1,g_2],g_3)+\eta([g_2,g_3],g_1)+\eta([g_3,g_1],g_2)=0\,.\end{aligned} \tag{4.3.2}$$

This is the defining property of a 2-*cocycle* which play a central rôle in the *Chevalley-Eilenberg cohomology* theory of Lie algebras [71]. Given a Lie algebra $\mathfrak{g}$ and a right module V, a k-linear mapping $c:\bigwedge^k\mathfrak{g}\longrightarrow V$ is called an V-valued k-*cochain*. We denote by $C^k(\mathfrak{g};V)$ the vector space of these mappings and define $\partial: C^k(\mathfrak{g};V)\longrightarrow C^{k+1}(\mathfrak{g};V)$ by

$$\begin{aligned}(\partial c)(g_0,\ldots,g_1)=&\sum_{i=0}^{k}(-1)^i g_i\cdot c(g_0,\ldots,\hat{g}_i,\ldots,g_k)\\&+\sum_{i<j}^{k}(-1)^{i+j}c([g_i,g_j],g_0,\ldots,\hat{g}_i,\ldots,\hat{g}_j,\ldots,g_k)\,,\end{aligned} \tag{4.3.3}$$

where the dot denotes the action of the Lie group on V. This map defines a *coboundary*, i.e. it is linear and satisfies $\partial\circ\partial=0$. It gives rise to an associated cohomology complex with coefficients in V,

$$H^k(\mathfrak{g};V)=\frac{\ker(\partial: C^k(\mathfrak{g};V)\longrightarrow C^{k+1}(\mathfrak{g};V))}{\mathrm{Im}(\partial: C^{k-1}(\mathfrak{g};V)\longrightarrow C^{k}(\mathfrak{g};V))}\,. \tag{4.3.4}$$

A closed k-cochain w.r.t. ∂ is called a k-*cocycle* with coefficients in V. We can take $V=\mathfrak{g}$ and let the algebra action be the adjoint representation, yielding the so-called Chevalley-Eilenberg cohomology sequence $H^*(\mathfrak{g};\mathfrak{g})$. The zeroth cohomology space is quickly seen to be $H^0(\mathfrak{g};\mathfrak{g})=\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$, the complement of the maximal ideal of the Lie algebra. For the algebra of Hamiltonian vector fields (3.5.25) on the spacesheet this space is spanned by the harmonic vector fields. The first cohomology space consists of the *derivations* of the algebra, satisfying $c([g_1,g_2])=[c(g_1),g_2]+[g_1,c(g_2)]$. The zero vector represents the class of *inner derivations*, given by $g\mapsto[g_0,g]$ for some fixed g_0, the other classes consist of *outer derivations* which are not related to each other by an inner derivation. Observing that the harmonic vector fields act as a derivation on the globally Hamiltonian vector fields, we see that they provide outer derivations on the Poisson algebra (letting them act trivially on the central extension of constant functions). The low-dimensional cohomology spaces of the Poisson algebra and APD algebra therefore satisfy

$$H^1(P(\Sigma);P(\Sigma))\cong H^1_\Delta(M)\cong H^1_{dR}(\Sigma,\mathbb{R})\cong H^0(\mathfrak{X}_\nu(\Sigma);\mathfrak{X}_\nu(\Sigma))\,. \tag{4.3.5}$$

The first CE-cohomology vector space defines the set of classes of one-dimensional *right extensions* of $\mathfrak{g}$, i.e. the exact sequences

$$0\longrightarrow\mathbb{K}\longrightarrow\mathfrak{g}\oplus\mathbb{K}\longrightarrow\mathfrak{g}\longrightarrow 0\,, \tag{4.3.6}$$

where $\mathbb{K}$ is the field underlying the Lie algebra. The two maps in the middle are given by $g\mapsto(g,0)$ and $(g,\lambda)\mapsto\lambda$, and a cocycle $c\in C^1(\mathfrak{g};\mathfrak{g})$ gives a Lie algebra structure to $\mathfrak{g}\oplus\mathbb{K}$ by

$$[(g_1,\lambda_1),(g_2,\lambda_2)]=([g_1,g_2]+\lambda_2 c(g_1)-\lambda_1 c(g_2),0)\,. \tag{4.3.7}$$

Then the Jacobi identity of the bracket above asserts that c is indeed a cocycle and vice versa. Equivalence of right extensions is defined by intertwining Lie algebra isomorphisms between the extended Lie algebras, and these equivalence classes correspond to linear dependence in $H^1(\mathfrak{g};\mathfrak{g})$ of the cocycles. If c is cohomologous to zero, the extension is trivial. As an example consider the sine algebras (4.2.52), which have 2 independent cocycle $c_r\in C^1(\mathfrak{g};\mathfrak{g})$ $(r=1,2)$ defined by

$$c_r(T_{\boldsymbol{m}})=m_r T_{\boldsymbol{m}}\,, \tag{4.3.8}$$

which can be shown not cohomologous to zero. It gives rise to a class of central extensions, given by the Lie bracket

$$[T_{\boldsymbol{m}}, T_{\boldsymbol{n}}] = \frac{2}{\lambda} \sin(2\pi\lambda(\boldsymbol{m} \times \boldsymbol{n}))T_{\boldsymbol{m}+\boldsymbol{n}} + \boldsymbol{a} \cdot \boldsymbol{m} \delta_{\boldsymbol{m}+\boldsymbol{n}} \mathbf{1} \,, \tag{4.3.9}$$

where $\boldsymbol{a}$ is an arbitrary complex 2-vector. Note the relation to the harmonic vector fields, which define central extensions on $P_\nu(\Sigma)$ in a similar way. The key point is that the cocycles above vanish under a finite mode truncation. To see this, take $\lambda = N^{-1} \in \mathbb{N}^{-1}$ and restrict the indices $\boldsymbol{m}$ to the lattice

$$\Lambda = \{-N, -N+1, \ldots, N-1, N\} \times \{-N, -N+1, \ldots, N-1, N\} - \{(0,0)\} \,, \tag{4.3.10}$$

to obtain $\mathfrak{su}(N)$ in the basis generated by the twist matrices (see previous section). The mappings (4.3.8) then lose their property of being a derivation: take $\boldsymbol{m}$ and $\boldsymbol{n}$ such that $\boldsymbol{m}+\boldsymbol{n}$ lays outside the lattice Λ: $\boldsymbol{m}+\boldsymbol{n} \neq (\boldsymbol{m}+\boldsymbol{n})$ mod (N). Then

$$[c_r(T_{\boldsymbol{m}}), T_{\boldsymbol{n}}] + [T_{\boldsymbol{m}}, c_r(T_{\boldsymbol{n}})] = 2N(m_r + n_r) \sin\Big(\frac{2\pi(\boldsymbol{m}\times\boldsymbol{n})}{N}\Big) T_{(\boldsymbol{m}+\boldsymbol{n}) \bmod N} \,, \tag{4.3.11}$$

while on the other hand

$$c_r([T_{\boldsymbol{m}}, T_{\boldsymbol{N}}]) = 2N(m_r + n_r) \bmod N \sin\Big(\frac{2\pi(\boldsymbol{m}\times\boldsymbol{n})}{N}\Big) T_{(\boldsymbol{m}+\boldsymbol{n}) \bmod N} \,, \tag{4.3.12}$$

which by our assumption is not equal to (4.3.11). In fact, it turns out $\mathfrak{su}(N)$ has no first-order cocycles which are not cohomologous to zero:

Lemma (Whitehead) 4.1 *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra, V a finite-dimensional vector space and $\rho : \mathfrak{g} \longrightarrow End(V)$ a nontrivial irreducible representation. Then*

$$H^p(\mathfrak{g}, V) = 0 \,. \tag{4.3.13}$$

This shows that for the familiar Lie algebras only the cohomology space produced by the trivial representation are interesting (when the first summation in 4.3.3 vanishes). Obviously, the special unitary algebras are semi-simple. In particular, the $\mathfrak{su}(N)$ are simple Lie algebras and hence the adjoint representation acts irreducibly, so the lemma above is applicable and we deduce there are no nontrivial $\mathfrak{su}(N)$-valued one-cycles. It is therefore an artefact of infinite-dimensionality that the 2 cocycles c_r are nontrivial for the sine algebras. It is therefore very tricky to deduce properties of the cohomology of $\mathfrak{su}(\infty)$ from the finite-dimensional unitary algebras, as is done in [72]. For example, in the reference the well-known fact that under the trivial representation

$$H^p(\mathfrak{su}(N), \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } p \text{ is odd and } 1 < p \leq 2N-1 \\ 0 & \text{otherwise} \end{cases} \tag{4.3.14}$$

is used to show that the real-valued cohomology of $\mathfrak{su}(\infty)$ is $\mathbb{R}$ if p is odd and bigger then one and zero otherwise. This is however not true if one considers the sine algebras above as the infinite limit of the unitary algebras. There appear two nontrivial 2-cycles in the infinite-dimensional situation which are closely related to the cocycles above and are given by

$$c_r : \mathfrak{su}(N) \wedge \mathfrak{su}(N) \longrightarrow N : c_r(T_{\boldsymbol{m}}, T_{\boldsymbol{n}}) = m_r \mathrm{Tr}(T_{\boldsymbol{m}} T_{\boldsymbol{n}}) \,, \tag{4.3.15}$$

which is seen to be antisymmetric by using $\mathrm{Tr}(T_{\boldsymbol{m}} T_{\boldsymbol{n}}) \propto \delta_{\boldsymbol{m}+\boldsymbol{n}}$.

4.3.2 Deformation of the Algebra of Hamiltonian Vector Fields on the Torus

Let us now apply this formalism to the theory of deformations of a Lie algebra. We have seen that the differential of the deformation in $t = 0$ must be a 2-cocycle. Higher derivatives of the deformation provide more restrictions; suppose the deformation map h is analytic around $t = 0$. A Taylor expansion gives

$$h(g_1, g_2, t) = [g_1, g_2] + \sum_{n=1}^{\infty} t^n \eta_n(g_1, g_2) \,. \tag{4.3.16}$$

Imposing the Jacobi identity to all orders in t yields an infinite numbers of equations

$$\Delta_n(g_1, g_2, g_3) \equiv \sum_{m=0}^{n} \varepsilon^{ijk} \eta_m(g_i, \eta_{n-m}(g_j, g_k)) = 0 \,, \tag{4.3.17}$$

where $m_1, m_2 \geq 0$ and $i, j, k \in \{1, 2, 3\}$. These equations can be written as

$$\Delta_n = \Xi_n + \partial \eta_n = 0 \,, \tag{4.3.18}$$

where

$$\Xi_n(g_1, g_2, g_3) \equiv \sum_{m=1}^{n} \varepsilon^{ijk} \eta_m(g_i, \eta_{n-m}(g_j, g_k)) \,, \quad n \geq 1 \,. \tag{4.3.19}$$

This suggests a recursive construction of a deformation: given the cochains $\eta_1, \dots, \eta_{n-1}$ such that $\Delta_1, \dots, \Delta_{n-1} = 0$, one can compute Ξ_n using (4.3.19), and a tedious computation will show that it is a cocycle. Imposing (4.3.18) is only consistent iff Ξ_n is cohomologous to zero, and one finds η_n up to exact 2-cocycles by solving it. Apart from convergence questions, which we don't consider here, a deformation is always possible if the third cohomology class vanishes. Our main application of this result is the deformation of the Poisson algebra on the torus. This algebra can be shown to have a trivial third cohomology vector space (with coefficients in the algebra), but it has a nontrivial 2-cocycle given by [73]

$$c(T_{\boldsymbol{m}}, T_{\boldsymbol{n}}) = (\boldsymbol{m} \times \boldsymbol{n})^3 T_{\boldsymbol{m}+\boldsymbol{n}} \,. \tag{4.3.20}$$

This coincides with the second term (of order N^{-2}) in the approximation of the Poisson algebra on the torus by $\mathfrak{su}(N) \ltimes \mathfrak{u}(1)$ (cf. 4.2.51). One can recursively continue the deformation and obtain

$$[T_{\boldsymbol{m}}, T_{\boldsymbol{n}}]_t = \frac{2}{t} \sin(2\pi t (\boldsymbol{m} \times \boldsymbol{n})) T_{\boldsymbol{m}+\boldsymbol{n}} \,. \tag{4.3.21}$$

Rescaling the generators by a factor 4π and letting t tend to zero exactly reproduces the Poisson algebra on the torus in Fourier basis. Now suppose t is a rational number smaller then 1: $t = M/N$ with M and N relatively prime. Then the sine structure constants will have zeros: $T_{\boldsymbol{m}}$ with $\boldsymbol{m} \in N(\mathbb{Z} \oplus \mathbb{Z})$ are all central elements of the algebra. Hence these can be consistently by modded out; defining the equivalence relation

$$T_{(m,n)} \sim T_{(m+N,n)} \sim T_{(m,n+N)} \tag{4.3.22}$$

makes the resulting quotient algebra $P(T^2)/\sim$ finite-dimensional. In particular, only the N^2 generators $T_{\boldsymbol{m}}$ with $0 \leq m_1, m_2 \leq N$ generate the different equivalence classes. The resulting Lie algebra is isomorphic to $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$. This puts the quasilimit approximation in a whole new perspective: the sequence of vector space homomorphisms tending to a Lie algebra isomorphism has been replaced by an algebra deformation with parameter t and a series of Lie algebra homomorphisms for rational values of the deformation parameter. This procedure can be directly pushed forward to the subalgebra of globally Hamiltonian vector fields, yielding a deformation of the algebra and a sequence of Lie algebra homomorphisms to $\mathfrak{su}(N)$ (the constant modes, which correspond to the semidirect $\mathfrak{u}(1)$-factor, are in the kernel of the gradient). This situation is illustrated in fig. 6. Note that in this figure only homomorphisms corresponding to the rational values $t = 1/N$ are drawn. In reality, there are homomorphisms for all rational values < 1 and the image of the morphism does not depend on the size of this parameter, but on the prime nominator. Let

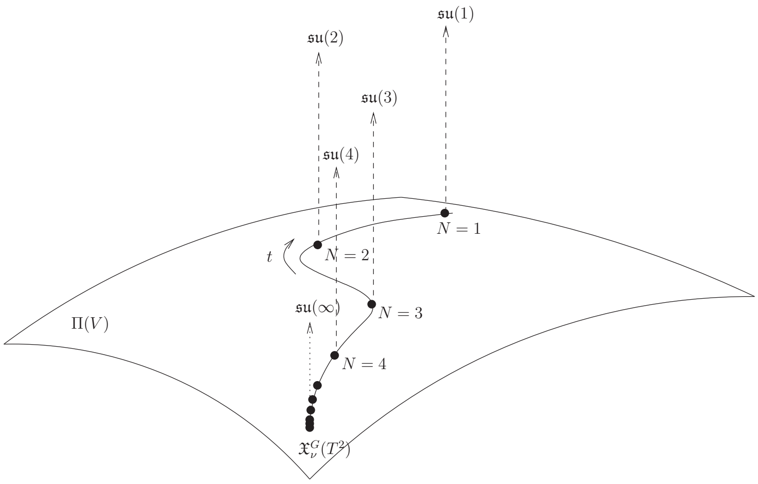


Figure 6: Deformation of the algebra of exact Hamiltonian vector fields on the 2-torus. Only homomorphisms for values $1/N$ of the deformation parameters are drawn. The total space $\Pi(V)$ represents he space of all Lie algebra structures on the infinite-dimensional space V.

us investigate how the two harmonic vector fields on the torus are included in this framework. Let $P_{M/N}(T^2)$ denote the deformed Poisson algebra (4.3.21) at $t = M/N$ with $M < N \in \mathbb{N}$ relatively prime and denote $C_N(T^2)$ the infinite-dimensional subspace spanned by Fourier modes with wave numbers on the sublattice $N(\mathbb{Z} \oplus \mathbb{Z})$. The matrix truncation is then given by the sequence of Lie algebra homomorphisms

$$P_{M/N}(T^2) \xrightarrow{\pi} \frac{P_{M/N}(T^2)}{C_N(T^2)} \xrightarrow{\simeq} \mathfrak{su}(N) \oplus \mathfrak{u}(1)\,, \tag{4.3.23}$$

where π is the canonical projection given by taking the equivalence class (which is by construction preserved under the deformed bracket for rational values of t). The second map is an isomorphism and induces an isomorphism between the cohomology spaces. The first map however is projection, and can not be used to push forward the cocycles, as these elements are in the dual of Lie algebra. The problem of finding two cocycles in $C^1(\mathfrak{su}(N); \mathfrak{su}(N))$ which are pulled back by the mapping above to the harmonic vector fields P_r has no solution, simply because, as argumented in the previous paragraph, the finite-dimensional unitary algebras have no nontrivial $\mathfrak{su}(N)$-valued cocycles. In other words, applying the deformation above to the total algebra of Hamiltonian vector fields on T^2 (where the brackets involving harmonic vector fields are not deformed) yields a sequence of algebras isomorphic to the sine algebras extended by their first cohomology space. This extension is an obstruction for a consistent projection onto the unitary algebra [74], as the modes $T_{N\boldsymbol{m}}$ are no longer central when one includes the cocycles: $[c_r, T_{N\boldsymbol{m}}] = N\boldsymbol{m}T_{N\boldsymbol{m}} \neq 0$. Observe that since the c_r act on each mode differently, any truncation to a finite-dimensional algebra by identifying blocks of generators is inconsistent. This phenomenon was already noticed in [6], where it was phrased in terms of aperiodicity of the structure constants corresponding to the harmonic vector fields. The conclusion that the harmonic vector fields cannot be regularised of course implicitly assumes the brackets involving these Lie algebra elements are not deformed. This is because the second cohomology space (with coefficients in the algebra) of the Poisson algebra on T^2, extended by the harmonic vector fields is one-dimensional, generated by a trivial extension of the cocycle (4.3.20) by $c(P_r, T_{\boldsymbol{m}}) = c(P_1, P_2) = 0$.

4.3.3 Regularisation and the Moyal Star

The deformation of the Poisson algebra on the torus can be induced by a deformation of the universal associative enveloping algebra $U(P_\nu(\Sigma))$. If we denote with $*_t$ the deformed product with parameter t, we write

$$f *_t g = \sum_{n=0}^{\infty} t^n C_n(f,g)\,, \tag{4.3.24}$$

where C_n are bilinear forms on the enveloping algebra. These are severely restricted when one requires the preservation of associativity: writing out $(f *_t g) *_t h - f *_t (g *_t h)$ yields a power expansion in t with coefficients

$$D_k(f,g,h) = \sum_{m=0}^{k} \sum_{n=0}^{k-m} \big(C_m(C_n(f,g),h) - C_m(f, C_n(g,h))\big)\,. \tag{4.3.25}$$

So these must all vanish for an associative deformation. Writing $E_k(f,g,h)$ as the expression above without the boundary terms $(m,n) = (0,k)$ or $(k,0)$, one finds $D_k = E_k - \partial^H C_k$, where ∂^H is the *Hochschild coboundary* on the graded vector space of cochains. For an associative algebra $\mathcal{A}$ such a cochain is simply defined as a k-linear mapping from $\mathcal{A}^k$ to $\mathcal{A}$, and the coboundary acts as

$$\begin{aligned}(\partial^H C)(a_0,\ldots,a_k) \;&=\; a_0 \cdot C(a_1,\ldots,a_k) + \sum_{i=0}^{k-1} (-1)^{i+1} C(a_0,\ldots,a_i\cdot a_{i+1},\ldots,a_k) \\ &\quad + (-1)^{k+1} C(a_0,\ldots,a_{k-1})\cdot a_k\,. \end{aligned} \tag{4.3.26}$$

Composing ∂^H with itself is zero by associativity. As usual, this coboundary turns the graded vector space of cochains into an exact sequence and as before, we call the cochains in the kernels of ∂^H *cocycles* and the associated cohomology complex the *Hochschild cohomology*, $H^k_H(\mathcal{A})$. Completely analogously to the deformation theory of the Lie bracket, one can use $D_k = 0, 0 \le k \le n \Rightarrow \partial^H E_{n+1} = 0$, and hence nontriviality of $H^3_H(U(P(\Sigma)))$ forms an obstruction to an associative deformation. For the space of functions on a manifold, we may look at the subcomplex of cochains given by differential operators which act trivially on constants (which is a reasonable assumption, since in general one wants to keep the product of real numbers). If we denote this complex by $H^k_{H,D}(C^\infty)$, it was shown by J. Vey that these spaces are $H^k_{H,D}(C^\infty) \simeq \Gamma(\bigwedge^k T^*M)$. This shows that in our 2-dimensional case the obstruction space $H^3_{H,D}(C^\infty)$ vanishes. These statements have recently been generalised: every finite dimensional Poisson manifold admits a deformation of its Poisson algebra, and a canonical construction is given in [75]. The simplest example of an associative product constructed from $C^2_{H,D}(C^\infty)$ is the *Moyal star product* with constant coefficients:

$$f *_t g = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{R(n),S(n)} \alpha^{r_1 s_1} \ldots \alpha^{r_n s_n} (\partial_{r_1}\ldots\partial_{r_n} f)(\partial_{s_1}\ldots\partial_{s_n} g)\,, \tag{4.3.27}$$

where $I(n) = (i_1,\ldots,i_n)$ and $J(n) = (j_1,\ldots,j_n)$ are n-tuples of indices, in our case each i_k, j_l running over $\{1,2\}$. Note that this product is fully specified by the constant 2-tensor α, which by associativity is required to be antisymmetric: $\alpha \in \bigwedge^2 \mathbb{R}$. In our case of a toroidal 2-manifold, it turns out we can choose $\alpha^{rs} = 2\pi\epsilon^{rs}$, and we conveniently shorten the notation of the deformed product to

$$f *_t g = f \exp(2\pi t \epsilon^{rs} \overleftarrow{\partial}_r \overrightarrow{\partial}_s) g\,. \tag{4.3.28}$$

A quick calculation of the deformed product of 2 Fourier modes shows that the star product and the Lie algebra deformation (4.3.21) are related by

$$[f,g]_t = \frac{1}{it}(f *_t g - g *_t f)\,. \tag{4.3.29}$$

The factor $1/it$ is obvious, since by construction to zeroth order in t the star product is commutative. Observe the analogy with the quantisation of a free particle on a line: the phase space of this system has 2 coordinates p, q which under the classical Poisson bracket are conjugate to each other, $\{p,q\}_P = 1$. Quantisation from the algebraic point of view is a deformation of the algebra of *scalar observables*, whose universal associative enveloping algebra is the polynomial ring $\mathbb{C}[p,q]$ (there is no gauge symmetry). The deformation of this algebra is given by the star product $f *_{i\hbar} g = f \exp(\frac{1}{2} i\hbar \epsilon^{rs} \overleftarrow{\partial}_r \overrightarrow{\partial}_s) g$, where $\partial_1 = \partial/\partial q$ and $\partial_2 = \partial/\partial p$ and the deformation parameter is taken imaginary (the functions therefore complex-valued). By simple substitution one finds the well-known quantum relation between position and momentum, $[q,p]_{i\hbar} \equiv q *_{i\hbar} p - p *_{i\hbar} q = i\hbar$. Up to first order in Planck's constant one finds Dirac's quantisation principle, $[q,p]_{i\hbar} = i\hbar\{p,q\}_P$, which was shown inconsistent to higher orders by Weyl. Comparing to the matrix regularisation of $P(T^2)$, where we have the star commutator $[\sigma^1,\sigma^2]_* = 2\pi t$, we associate (σ^1,σ^2) with the phase space variables (q,p) and t with $i\hbar/2\pi$. However, one should keep in mind that the classical variables p and q are not bounded and the full equivalence would be reached when considering a toroidal phase space.

4.3.4 General Spacesheet Geometries

One can imagine that the deformation quantisation may as well be applied to the algebra generating area-preserving diffeomorphisms on the sphere. The Fourier modes $e^{im_1\theta + im_2\phi}$ are smooth functions on the unit sphere, which are all linearly independent (not orthogonal or normalised, nor eigenfunctions of the Laplace-Beltrami operator however). By linearity of the gradient, these properties are also fulfilled by the Hamiltonian vector fields they induce. Then one may construct a deformation of the Poisson algebra such that the modes on the N-multiple lattice become central and divide the algebra by the infinite-dimensional span of these modes. This procedure can be transformed to a basis of spherical harmonics, in which case we mod out by an infinite-dimensional linear subspace of the Poisson algebra.

It is however not clear how to implement this approximation scheme into membranes modeled on base manifolds of arbitrary topology. For such spacesheets the Fourier modes $e^{m_r\sigma^r}$ are in general not smooth; for higher genus Riemann surfaces a global coordinate system is generally absent (as for the sphere). The approximation of the Poisson algebra shall therefore require a different approach. It is Berezin's quantisation of a Poisson algebras on Riemann surfaces [76] which gives the correct solution to this deformation problem. We shall discuss this from n algebraic topological point of view, exploiting the machinery of automorphic forms, as well as from the geometric perspective, using the theory of Hermitian line bundles [65]), equipped with Kähler metrics.

The Berezin quantisation scheme was first applied to the Poisson algebra on the (open) complex unit disc [77], and based on this method, a discrete sequence of Poisson algebra deformations on any closed higher genus surface may be constructed which is a quasilimit of the ordinary Poisson algebra [78, 65, 79]. Finally, this discrete sequence may be embedded in a continuous deformation of the Poisson algebra on a particular covering of the manifold [80]. The equivalence of Berezin's deformation with the approach to the regularisation of the torus algebra of previous paragraphs was exhibited in [65], and we shall give a short outline of these results in the upcoming paragraph. We end the section with a short discussion of a method introduced by Bars [81], which claims to be a straightforward generalisation of the torus algebra deformation.

For the treatment of the general case we note that our constructions are applicable to all smooth compact surfaces homeomorphic to respectively the 2-sphere or the 2-torus. For smooth 2-manifolds, being homeomorphic is equivalent to being diffeomorphic and the Poisson algebra of a surface N diffeomorphic to M may be pulled back to the Poisson algebra of M, yielding an algebra isomorphism. It is therefore natural to work in topology classes, since topological properties are invariant under homeomorphisms. In conclusion, we have treated the matrix regularisation for all smooth compact 2-manifolds of genus 0 and 1. We shall use the tools of the uniformisation theory of Riemann surfaces to simplify the regularisation procedure as a problem of group-invariant forms on the complex unit disc. This powerful machinery uses the fact that every smooth closed orientable 2-manifold is a *complex* manifold. Formally, any even-dimensional orientable manifold admits an *almost complex structure*, which is a globally defined map $J : TM \longrightarrow TM$ such that $J^2 = -\mathrm{Id}_{TM}$ on the fibres. For 2-manifolds, J is locally given by a rotation of tangent space by 90 degrees. For this to be able to be consistently patched together, the manifold must be orientable. The morphism J locally induces a decomposition of the tangent spaces into real and imaginary eigenspaces, and it induces a bigrading on the complexified exterior algebras:

$$\Omega^r(M)_{\mathbb{C}} \simeq \bigoplus_{0\leq p+q=r} \Omega^{(p,q)}(M)\,, \tag{4.3.30}$$

where $\Omega^{(p,q)}(M)$ consists of the wedge product of p forms in the real eigenspace and q forms in the imaginary eigenspace of J (acting on the cotangent bundle). Such a structure is called *complex* if the real-imaginary decomposition is integrable, i.e. there exist local coordinate patches $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, such that

$$J\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu}\,, \quad J\frac{\partial}{\partial y^\mu} = -\frac{\partial}{\partial x^\mu}\,. \tag{4.3.31}$$

However, in general the coordinate transition mappings will not preserve this property. If they do, we can view M^{2n} as a manifold based on $\mathbb{C}$, equipped with coordinates $z^\mu = x^\mu + iy^\mu$ inducing local bases on the complexified tangent bundle $TM_{\mathbb{C}}$ such that

$$J\frac{\partial}{\partial z^\mu} = i\frac{\partial}{\partial z^\mu}\,, \quad J\frac{\partial}{\partial \overline{z}^\mu} = -i\frac{\partial}{\partial \overline{z}^\mu}\,, \tag{4.3.32}$$

and such that the coordinate transition functions are holomorphic mappings between open domains in $\mathbb{C}^n$. If $n = 1$ and these domains are homeomorphic to the open unit disc in the complex plane, the manifold is called a *Riemann surface*. Recall that the supermembrane spacesheet is both symplectic and Riemannian. Let us denote with w^* the duality map $TM^* \longrightarrow TM$ induced by the metric w. An almost-complex structure is defined by

$$J(X) = w^*(\iota_X \nu)\,, \tag{4.3.33}$$

which fulfills the *compatibility conditions*

$$w(X, Y) = \nu(X, J(Y))\,. \tag{4.3.34}$$

It can be shown that J is integrable to a complex structure if Σ is closed, orientable and smooth. Such complex manifolds containing compatible symplectic and Riemannian structures are called *Káhler*. Obviously, all Riemann surfaces are Kähler manifolds. The tangent bundle to a Riemann surface is an example of what we call a *complex line bundle* L, a 2-dimensional vector bundle with additional structure J making its fibres isomorphic to $\mathbb{C}$. We call such a bundle *Hermitian* if its fibres are equipped with a Hermitian scalar product, given by a section g of $L^* \vee L^*$ such that

$$g(J(X), J(Y)) = g(X, Y)\,. \tag{4.3.35}$$

In the local complex coordinates (4.3.32), this tensor is of the form $f(z, \overline{z})\mathrm{d}z \otimes \mathrm{d}\overline{z}$, where f is real-valued. Of particular importance will be the space of *holomorphic sections* of a complex line bundle: globally defined forms in $\Omega^{(p,0)}(M, L)$ with components holomorphic functions. On a compact Kähler manifold these spaces of sections are finite-dimensional, and isomorphic to the spaces of harmonic differential forms:

$$\Omega^{(p,0)}_{\mathrm{hol}}(\Sigma, L) \simeq H^p_\Delta(\Sigma, \mathbb{R})\,. \tag{4.3.36}$$

In particular, the (real) dimension of holomorphic one-forms on a Riemann surface is $2g$.

A *conformal mapping* between 2 topological inner product spaces is a bijective map which preserves local angles between intersecting curves. The typical example is the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \cup \{\infty\}$ equipped with the metric $g = (z - \overline{z})^{-2}\mathrm{d}z \otimes \mathrm{d}\overline{z}$, whose group of conformal mappings to itself consists of the fractional transformations

$$z \mapsto \frac{az+b}{cz+d}\,, \quad A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}(2)\,, \quad \det(A) = 1\,. \tag{4.3.37}$$

Naively one might think this conformal group is $SL(2,\mathbb{R})$. However, multiplying the matrix A by a real constant such that the determinant is left unchanged yields the same transformation, so we may identify each matrix with its negative: the conformal group is the *projective* special linear group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathbf{1}\}$. Another example of particular importance to us is the unit disc, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, equipped with the Bergman metric

$$g = 4\frac{\mathrm{d}z \otimes \mathrm{d}\overline{z}}{(1-|z|^2)^2}\,. \tag{4.3.38}$$

whose group of conformal mappings to itself consists of

$$z \mapsto \frac{az+\overline{c}}{cz+\overline{a}}\,, \quad |a|^2 - |c|^2 = 1\,. \tag{4.3.39}$$

Multiplying the complex numbers a and c by a real parameter yields the same transformation; we may therefore assume $|c| < 1$ without loss of generality. Then let us denote $\theta_c = -\log(|c|)$ and $b = (\sqrt{\tanh\theta_c})a$. Then the matrix

$$A = \begin{pmatrix} b & c \\ -\overline{c} & \overline{b} \end{pmatrix} \tag{4.3.40}$$

satisfies $A^\dagger A = \mathbf{1} = aa^\dagger$ and $\det(A) = 1$, i.e. is an element of $SU(2)$. Under our assumptions there is a one-to-one correspondence between the Caley-Klein parameters b, c and the complex numbers a, c. However composing this isomorphism with the mapping $(b, c) \mapsto (-b, -c)$ yields the same conformal transformation on the unit disk. Analogously to the upper half plane, we define this group as the *projective special unitary group* $PSU(2) = SU(2)/\{\pm\mathbf{1}\}$. By the isomorphisms $SU(2) \simeq SO(3) \simeq SL(2,\mathbb{R})$ we see that the conformal group of the unit disk is isomorphic to that of the upper half plane; in fact the half-plane and the unit disc are conformally equivalent: there is a holomorphic bijective map between them (for example, $z \mapsto (z-i)/(z+i)$). Generalising to Riemann surfaces, a mapping $f : M \longrightarrow N$ is holomorphic if for every coordinate charts $\phi_U : M \longrightarrow U \subseteq \mathbb{C}$ and $\psi_V : N \longrightarrow V \subseteq \mathbb{C}$ the complex function $\psi_V \circ f \circ \phi_U^{-1} : U \longrightarrow V$ is holomorphic, and it is conformal if it is bijective. Every Riemann surface defines an orientable 2-manifold, and a conformal mapping corresponds to real analytic mapping with Jacobi matrix everywhere a scalar times a rotation matrix. In other words, the pull-back of the Riemann metric by a conformal mapping is a multiplication by a scalar function. If M and N admit such a mapping, they are said to be *conformally equivalent*. The most important result in the theory of Riemann surfaces is the following,

Uniformisation Theorem 4.2 *If a Riemann surface is homeomorphic to a sphere then it is conformally equivalent to the Riemann sphere. If not, it is either conformally equivalent to $\mathbb{C}/\Gamma$ or $\mathbb{D}/\Gamma$, where Γ is a discrete subgroup of isometries acting freely on resp. $\mathbb{C}$ or $\mathbb{D}$.*

By the statements above, the unit disc may be replaced by the upper-half complex plane. The isometries are w.r.t. the Riemannian metric $\mathrm{d}z \otimes \mathrm{d}\overline{z}$ on $\mathbb{C}$ and the metrics mentioned above for the upper half plane and the unit disc, and the subgroup Γ is called the *Fuchsian* group. For example, in the toroidal case it is generated by 2 translations along linearly independent directions in $\mathbb{C}$. The Riemann metrics the covering spaces are equipped with can be shown to yields constant Gauss curvature: the uniformisation theorem may therefore be restated as: every smooth closed orientable 2-manifolds admits a (up to a constant unique) metric with Gauss curvature -1, 0 or 1. Riemann surfaces conformally equivalent to the Riemann sphere may have constant positive curvature and are said to be *elliptic*, in the other cases they are called *parabolic* (complex plane) and *hyperbolic* (unit disc). As real compact orientable 2-manifolds, the elliptic surfaces are homeomorphic to the sphere, the parabolic ones homeomorphic to the 2-torus. So the interesting Riemann surfaces are the hyperbolic ones, since they represent higher genus surfaces.

4.3.5 Hyperbolic Riemann Surfaces

A continuous function on an arbitrary genus g surface Σ has a unique lift to a real-valued Γ-invariant, or *automorphic* function on $\mathbb{D}$, and this will be the way to deal with the problem of globally undefined basis functions on these surfaces. So the strategy shall be to repeat the method above in a completely Γ-invariant way. For compact, smooth finite genus g surfaces the group Γ is finitely generated by $A_1, B_1, \ldots, A_g, B_g$ satisfying the relations $A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} = I$. A holomorphic scalar function on Σ corresponds to a holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that $f(A(z)) = f(z)$ for all $A \in \Gamma$. A holomorphic section of the cotangent bundle has different transformation behaviour however. Let $L \longrightarrow \Sigma = \mathbb{D}/\Gamma$ be a line bundle over the Riemann surface with fibre projection map π and denote with $\rho : \mathbb{D} \longrightarrow \Sigma$ the canonical projection of the covering space onto the surface. This gives rise to a pullback bundle $\rho^* L$ on $\mathbb{D}$, which should be thought of as copies of L on each inverse image of the fundamental polygon under Γ. Since $\mathbb{D}$ is an open neighbourhood of the complex plane, a complex line bundle on it is topologically trivial, i.e. there is a global isomorphism $\Phi : \rho^* L \longrightarrow \mathbb{D} \times \mathbb{C}$. For $z \in \mathbb{D}$, let Φ_z denote Φ restricted to the fibre above z. Then for $A \in \Gamma$,

$$\sigma_A(z) = \Phi_{A(z)} \circ \Phi_z \tag{4.3.41}$$

is an isomorphism, and hence given by multiplication by a nonzero complex number, also denoted with $\sigma_A(z)$. These complex numbers satisfy

$$\sigma_B(A(z))\sigma_A(z) = \sigma_{AB}(z) = \sigma_A(B(z))\sigma_B(z)\,, \qquad A, B \in \Gamma\,, \quad z \in \mathbb{D}\,, \tag{4.3.42}$$

and are called a system of *multipliers*. They define the line bundle, in a way that transition functions define a real vector bundle. For L of complex dimension one, $\sigma_A(z)$ can be written as $j_A(z)^2 v(A)$ where $j_A(z)$ is the square-root of the Jacobian of $A(z)$:

$$j_A(z) = (cz + \overline{a})^{-1}\,, \tag{4.3.43}$$

If A is given by the projective unitary transformation (4.3.39). However, other tensors will transform with higher powers of this Jacobian, and this leads to the general concept of a multiplier of weight $r > 0$, which we define as a map $v : \Gamma \longrightarrow \mathbb{C}$ such that $|v| = 1$ and

$$\sigma_A(z) = (j_A(z))^r v(A) \tag{4.3.44}$$

obeys the condition (4.3.42). It is not clear how to interpret these functions on $\mathbb{D}$ in terms of transition functions of line bundles. When r is an even integer $2m$, it can be seen to define the trivialisation of the m-th tensor power $L^{\otimes m}$ of the line bundle L. When r is odd or rational, the multipliers can be associated to bundles over coverings of Σ. There is only a discrete sequence of values of r for which a v exists such that (4.3.44) can fulfill (4.3.42):

Corollary 4.3 *For $\Sigma = \mathbb{D}/\Gamma$ a compact hyperbolic Riemann surface of genus $g \geq 2$, a multiplier λ of weight r can only exist if $r = n/(g-1)$ for some $n = 1, 2 \ldots$.*

This is a result of the famous *Riemann-Roch theorem.* If λ_1, λ_2 are multipliers of the same weight, their ratio $\chi = \lambda_1/\lambda_2 : \Gamma \longrightarrow S^1$ satisfies $\chi(AB) = \chi(a)\chi(B)$; such a mapping is called a *character* of the group. Writing the multipliers as exponential functions, one quickly proofs that the space Λ_r of multipliers of weight r, if it is nonempty, is isomorphic to the space of characters of Γ. This is a $2g$-dimensional torus T^{2g} (the *Jacobian* associated to Γ). An automorphic form of weight $r > 0$ and multiplier v is a holomorphic function $\phi : \mathbb{D} \longrightarrow \mathbb{C}$ such that

$$\phi(A(z)) = v(A) j_A(z)^{-r} \phi(z)\,, \tag{4.3.45}$$

for all $A \in \Gamma$ and $z \in \mathbb{D}$. If the Riemannian manifold is $\mathbb{H}/PSL(2,\mathbb{Z})$, the functions above are called *modular forms* and play an important rôle in number theory and string theory. We denote the complex vector space of such forms by $\mathscr{H}^r(\Gamma, v)$. Again, the Riemann-Roch theorem implies that this space is finite-dimensional. In particular, their dimension does not depend on the choice of v,

$$\dim(\mathscr{H}^r(\Gamma, v)) = (g-1)(2r-1)\,. \tag{4.3.46}$$

So for $r = n/(g-1)$ we can choose a v of weight r and this gives rise to a space of automorphic forms of dimension

$$N = 2n - g + 1 \tag{4.3.47}$$

Let U be a *fundamental domain* of the Riemann surface, the smallest domain in $\mathbb{D}$ which, under the action of Γ generates the entire surface. The *Petersson* inner product on the space of automorphic forms of weight r associated to λ is given by

$$(\phi, \psi) = \int_U \overline{\phi(z)} \psi(z) \omega_r(z)\,. \tag{4.3.48}$$

with

$$\omega_r(z) = i\frac{r-1}{\pi}(1-|z|^2)^{r-2} \mathrm{d}z \wedge \mathrm{d}\bar{z}\,, \tag{4.3.49}$$

which is an automorphic form of 'negative' weight:

$$\omega(A(z)) = j_A(z)^{2r} \omega(z)\,. \tag{4.3.50}$$

Hence the integral (4.3.48) is independent of the choice of the fundamental domain U. It turns the finite dimensional vector space $\mathscr{H}^r(\Gamma, v)$ into a Hilbert space. Given a bounded holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C}$, there is a canonical method to construct a weight r automorphic form using the *Poincaré theta series*,

$$\theta_v^r(\phi)(z) = \sum_{A\in\Gamma} \frac{j_A(z)^r}{v(A)} \phi(A(z))\,. \tag{4.3.51}$$

It has been shown by Poincaré that this sum converges almost uniformly. Denote $K^r : \mathbb{D}\times\mathbb{D} \longrightarrow \mathbb{C} : K^r(z,w) = (1-\overline{w}z)^{-r}$ for $r > 0$ and denote the theta-series

$$K_v^r(z,w) = \theta_v^r(K^r(\,.\,,w))(z) = \sum_{A\in\Gamma} \frac{j_A(z)^r}{v(A)} K^r(A(z), w)\,. \tag{4.3.52}$$

This is the *Bergman kernel* associated to the Hilbert space $\mathscr{H}^r(\Gamma, v)$. In general, Bergman kernels are used to project integrable complex functions on a domain in $\mathbb{C}$ to *holomorphic* functions. The key properties of the kernel $K_v^r(z,w)$ are that it is holomorphic in the z-variable, anti-holomorphic in the w and

$$\overline{K_v^r(z,w)} = K_v^r(w,z)\,. \tag{4.3.53}$$

In our case, it should preserve the automorphic properties of argument, which is established by taking the theta series of a kernel. If we denote $\mathscr{L}^r(\Gamma, v) = \{f : \mathbb{D} \longrightarrow \mathbb{C} \mid (f,f) < \infty,\ f(A(z)) = v(A) j_A(z)^{-r} f(z) \text{ for all } A \in \Gamma,\ z \in \mathbb{D}\}$ then we define the projector

$$P_v^r : \mathscr{L}^r(\Gamma, v) \longrightarrow \mathscr{H}^r(\Gamma, v) : P_v^r \phi(z) = \int_U K_v^r(z,w)\phi(w)\omega_r(w)\,. \tag{4.3.54}$$

Now we would like to associate to each smooth function on the compact hyperbolic manifold $\Sigma \simeq \mathbb{D}/\Gamma$ a linear unitary operator on a Hilbert space. As already mentioned, such a function corresponds to a bounded, Γ-invariant real-valued function on the unit disc. To each element of the Banach space $\mathscr{C}_\Gamma = \{f : \mathbb{D} \longrightarrow \mathbb{R} \mid f(A(z)) = f(z) \text{ for all } A \in \Gamma,\, z \in \mathbb{D}\}$ we associate the linear Toeplitz operator $T^r_v f$ determined by

$$[T^r_v f(\phi)](z) = P^r_v(f\phi)(z) = \int_U K^r_v(z,w) f(w)\phi(w)\omega_r(w)\,. \tag{4.3.55}$$

One then sees that $T^r_v f$ acts reducibly on $\mathscr{L}^r(\Gamma, v)$ since the image is holomorphic: $T^r_v f \in \mathrm{End}(\mathscr{H}^r(\Gamma, v))$. In fact, it can be shown that the space $\mathscr{T}^r(\Gamma, v)$ generated by all the Toeplitz operators spans the whole endomorphism group on the Hilbert space of automorphic forms. Furthermore, the representation is unitary, $(\phi, (T^r_v \overline{f})\psi) = ((T^r_v f)\phi, \psi)$ so that for f real-valued $(T^r_v f)^\dagger = T^r_v f$. Multiplying by i yields the defining property of $\mathfrak{u}(N)$-matrices. By subtracting the trace we can map functions onto $SU(N)$-generators:

$$T^r_v : C^\infty(\Sigma, \mathbb{R}) \longrightarrow \mathfrak{su}((g-1)(2r-1))\,. \tag{4.3.56}$$

Important is the approximation of the Poisson algebra, reflected by the estimates

$$\begin{aligned} \|f\|_\infty \leq \|T^r_v f\| &\leq \|f\|_\infty + \mathcal{O}(r^{-1})\,, \\ \|[T^r_v f, T^r_v g] - ir^{-1} T^r_v\{f,g\}\| &\leq C r^{-3/2} \|f\|_4 \|g\|_4\,. \end{aligned} \tag{4.3.57}$$

Here $\|\,.\,\|_\infty$ denotes the supremum norm, C is a constant (generically depending on the functions), and $\|f\|_n = \sum_{k+\ell\leq n} \|\partial^k_z \partial^\ell_{\overline{z}} f(z)\|_\infty$, which is bounded on the space of smooth functions on the surface. The second equation implies that the algebra $\mathscr{T}^r(\Gamma, v)$ equipped with the bracket induced by composition of endomorphisms represents a deformation of the Poisson algebra with deformation parameter $t = r^{-1}$:

$$\{f,g\}_t(z,\overline{z}) = t(1 - z\overline{z})^2 (\partial_z f(z,\overline{z}) \partial_{\overline{z}} g(z,\overline{z}) - \partial_z g(z,\overline{z}) \partial_{\overline{z}} f(z,\overline{z})) + \mathcal{O}(t^2)\,. \tag{4.3.58}$$

4.3.6 Geometric Approach

The by S. Chern generalised Gauss-Bonnet theorem states that for a even-dimensional orientable vector bundle on a closed even-dimensional manifold, equipped with a metric and a connection ∇ compatible with this connection, then

$$\frac{1}{2\pi} \oint_S F_\nabla \in \mathbb{Z} \tag{4.3.59}$$

for arbitrary smooth, closed 2-dimensional submanifolds S of M. This statement follows from the more general fact that for a complex Hermitian vector bundle E of complex rank n on an even-dimensional closed smooth manifold the *Chern classes* $c_i(E)$ defined by

$$\det\left(\frac{itF_\nabla}{2\pi} + \mathbf{1}_n\right) = \sum_{k=1}^n c_k(E) t^k \tag{4.3.60}$$

are integer-cohomology valued: $c_k \in H^{2k}(M, \mathbb{Z})$. The integral (4.3.59) may thus be recognised as the integral of the first Chern class on the vector bundle (viewed as a complex Hermitian bundle). Conversely, A. Weil showed that an even-dimensional closed orientable manifold M containing a 2-form θ such that $(2\pi)^{-1}\theta \in H^2(M, \mathbb{Z})$, may be endowed with a Hermitian line bundle L, and a connection ∇, compatible with the fibre metric with curvature F_∇ on L such that

$$F_\nabla = -i\theta\,. \tag{4.3.61}$$

This only holds for line bundles: such manifolds are completely classified by the first (and simultaneously top) Chern class. For the supermembrane, there is only one closed 2-dimensional submanifold of Σ, namely Σ itself, and the condition $\nu \in H^2(\Sigma, \mathbb{Z})$ is trivially met by a suitable normalisation of $\sqrt{w}$. By the uniformisation theorem Σ is conformally equivalent to $\mathbb{D}/\Gamma$, equipped with the Poincaré metric (4.3.38). This gives rise to a Levi-Civita connection ∇ on its tangent bundle, a Hermitian line bundle, whose curvature is given by the symplectic form

$$\omega = \frac{2}{(1-z\bar{z})^2}\mathrm{d}z \wedge \mathrm{d}\bar{z} = -i\nu\,. \tag{4.3.62}$$

where ν is the volume form corresponding to g. As the Gauss curvature of the Poincaré metric constant $+1$, the first Chern number of the line bundle is determined by the Gauss-Bonnet theorem,

$$\frac{i}{2\pi}\int_{\mathbb{D}/\Gamma}\omega = \frac{1}{2\pi}\int_{\Sigma} K\nu = (2g-2)\,, \tag{4.3.63}$$

where g is the genus of Σ. The Hilbert space is taken to be $\Gamma_{\text{hol}}(\Sigma, L)$, the space of sections s of the line bundle satisfying the *polarisation condition*

$$\nabla_{\bar{\partial}} s = 0\,. \tag{4.3.64}$$

This space can be shown finite-dimensional, and in particular $\dim(\Gamma_{\text{hol}}(\Sigma, L)) = 2g$. Now we consider the m-fold tensor product $L^m = L \otimes \ldots \otimes L$. We equip its fibres with the tensor product $g^{(m)} = g \otimes \ldots \otimes g$ and define the connection

$$\nabla^{(m)}_X (s_1 \otimes \ldots \otimes s_n) = \sum_{k=1}^{m} s_1 \otimes \ldots \otimes \nabla_X s_k \otimes \ldots \otimes s_m\,. \tag{4.3.65}$$

A quick calculation shows that the curvature of this connection reduces to

$$F_{\nabla^{(m)}}(X,Y) = mF_{\nabla} = m\omega(X,Y)\,. \tag{4.3.66}$$

Integrating both sides and using the integral Chern class condition, $[iF_{\nabla^{(m)}}/2\pi] \in H^2(\Sigma, \mathbb{Z})$ yields

$$2m(g-1) \in \mathbb{Z} \tag{4.3.67}$$

This is essentially the condition on r (cf. corollary 4.3), the weight of the automorphic forms which formed the Hilbert spaces on which the Toeplitz operators act. It is because automorphic forms define global holomorphic sections of the tensor product line bundles; under an $A \in \Gamma$, such a global holomorphic section $f(z)(\mathrm{d}z)^m$ satisfies

$$f(z)(\mathrm{d}z)^m = v(A)A^*(f(z)(\mathrm{d}z)^m) = v(A)(j_A(z))^{2m} f(A(z))(\mathrm{d}z)^m\,, \tag{4.3.68}$$

for some multiplier v of Γ. Hence f is an automorphic function of weight $2m$. We denote with $\Gamma_{\text{hol}}(\Sigma, L^m)$ the subspace of holomorphic sections of L^m which satisfy the polarisation condition $\nabla^{(m)}_{\bar{\partial}} s = 0$. With the Riemann-Roch theorem one can deduce

$$\dim(\Gamma_{\text{hol}}(\Sigma, L^m)) = (4m-1)(g-1) \equiv N\,, \tag{4.3.69}$$

which agrees with (4.3.46). This space turns into a finite-dimensional Hilbert space if we equip it with the bilinear pairing

$$\langle s_1 | s_2 \rangle = \int_{\Sigma} g^{(m)}(s_1, s_2)\omega\,. \tag{4.3.70}$$

Note however that the geometric construction allows only regularisation on Hilbert spaces of automorphic forms with integer weight, while the algebraic theory of previous section takes forms into account of fractional weights $n/(g-1)$.

The rest of the story is similar to the procedure before: one associates to each smooth function on $\mathbb{D}/\Gamma$ a Toeplitz operator on $\Gamma_{\text{hol}}(\Sigma, L^m)$. This is done by starting with the representation

$$P : C^{\infty}(\Sigma, \mathbb{C}) \longrightarrow \mathrm{End}(\Gamma(\Sigma, L)) : f \mapsto P_f = -\nabla_{\text{grad}_{\omega} f} + if\,, \tag{4.3.71}$$

where the second term in P_f denotes multiplication with if. This is just a homomorphism between infinite-dimensional Lie algebras:

$$[P_f, P_g] = P_{\{f,g\}}\,, \tag{4.3.72}$$

The regularisation proceeds by projecting these operators on the (finite-dimensional) space of holomorphic sections of L fulfilling the polarisation condition (4.3.64). Obviously, there exists a canonical projection $\rho : \Gamma(\Sigma, L) \longrightarrow \Gamma_{\rm hol}(\Sigma, L)$, and we define

$$Q_f = \rho \circ P_f\,. \tag{4.3.73}$$

These operators no longer fulfill (4.3.72). Given a basis $s_1, \ldots, s_N$ of $\Gamma_{\rm hol}(\Sigma, L)$ orthonormal w.r.t. the inner product (4.3.70), these operators were shown in [65] to be able to be expressed as

$$Q_f = \sum_{i,j=1}^{N} |s_i\rangle\langle s_i|P_f|s_j\rangle\langle s_j| = i \sum_{i,j=1}^{N} |s_i\rangle\langle s_i|f - \tfrac{1}{2}\Delta f|s_j\rangle\langle s_j|\,, \tag{4.3.74}$$

where Δ is the Laplacian w.r.t. the Kähler structure on Σ, obeying the property

$$\langle s_1|\Delta f|s_2\rangle = -2i\langle s_1|\nabla_{{\rm grad}_\omega f}|s_2\rangle\,. \tag{4.3.75}$$

From (4.3.74) we read off that if f is real-valued, Q_f is an antihermitian operator on a finite-dimensional vector space. Now consider the same procedure on the m-th tensor power of the complex line bundle. The covariant derivative $\nabla^{(m)}$ was shown above to have curvature $im\omega$. Hence, the bracket of the Toeplitz operators $P_f = -\nabla^{(m)}_{{\rm grad} f} + if$ will not be the Toeplitz operator of the spacesheet Poisson bracket, but $1/m$ times this operator, and therefore we rescale the Toeplitz operators as

$$P_f^{(m)} = -\nabla^{(m)}_{{\rm grad}_\omega f} + imf\,, \tag{4.3.76}$$

such that $[P_f^{(m)}, P_g^{(m)}] = P_{\{f,g\}}^{(m)}$. The operators on the space of holomorphic sections satisfying the polarisation condition $\nabla^{(m)}_{\bar\partial} s = 0$ are constructed as before:

$$Q_f^{(m)} = \rho^{(m)} \circ P_f^{(m)}\,, \tag{4.3.77}$$

where $\rho^{(m)}$ denotes the orthogonal projection of the space of smooth sections of L^m onto $\Gamma_{\rm hol}(\Sigma, L^m)$. Now we define the operator norm on $\mathfrak{gl}(N, \mathbb{C})$ (which the $Q_f^{(m)}$ belong to for N given by (4.3.69):

$$\|A\|_m = \frac{1}{m} \sup_{s\neq 0} \sqrt{\frac{\langle s|A|s\rangle}{\langle s|s\rangle}}\,. \tag{4.3.78}$$

We now have established a quasilimit:

Theorem 4.4 *With the definitions above, we have*

$$\lim_{m\to\infty} \|P_f^{(m)}\| = \|f\|_{\rm sup}\,, \qquad \lim_{m\to\infty} \|[P_f^{(m)}, P_g^{(m)}] - P_{\{f,g\}}^{(m)}\| = 0\,. \tag{4.3.79}$$

The proof of this theorem (generalised to arbitrary quantisable even-dimensional Kähler manifolds) may be found in [79]. In the supermembrane theory, we replaced the spacesheet integral by the trace of the products of the matrices. This is justified by the theorem above, as it leads to the estimate

$$\frac{1}{N} {\rm Tr}(P_{f_1}^{(m)} \ldots P_{f_p}^{(m)}) = \frac{1}{{\rm vol}(\Sigma)} \int_\Sigma f_1 \ldots f_p\, \omega + \mathcal{O}(m^{-1})\,. \tag{4.3.80}$$

which is also derived in [79]. At this point it is not clear whether the regularisation procedure above is the one proposed for the Poisson algebra on the torus of previous paragraphs. The parabolic surface T^2 is written as the complex plane modulo the lattice $\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$, where λ_1

and λ_2 are nonzero complex numbers which are not on the same ray. For simplicity we assume $\lambda_2 = i\tau\lambda_1$ for $\tau \in \mathbb{R}_0$ (in the literature [65] this is called 'principal polarisation'). A system of multipliers is generated by

$$\sigma_{\lambda_1}(z) = z\,, \quad \sigma_{\lambda_2}(z) = e^{\pi\tau - 2\pi i z} \tag{4.3.81}$$

and the multiplication rule (4.3.42). Note that a single-valued function should fulfill $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$, a section of L should fulfill $s(z+\lambda) = \sigma_\lambda(z)s(z)$, and a fibre metric g should transform as $g(z+\lambda) = |\sigma_\lambda(z)|^{-2}g(z)$ to have a globally well-defined pairing of sections. The space $\Gamma_{\mathrm{hol}}(T^2, L)$ is one-dimensional, spanned by the theta series

$$\Theta(z) = \sum_{k\in\mathbb{Z}} \exp(\pi i k^2 \tau + 2\pi i k z)\mathrm{d}z\,. \tag{4.3.82}$$

As a fibre metric we choose

$$g = \exp(\frac{\pi}{2\tau}(z-\overline{z})^2)\mathrm{d}z \otimes \mathrm{d}\overline{z}\,. \tag{4.3.83}$$

which gives rise to the curvature and Laplacian

$$\omega = -\frac{\pi}{\tau}\mathrm{d}z \wedge \mathrm{d}\overline{z}\,, \quad \Delta = \frac{2}{\pi\tau}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\,. \tag{4.3.84}$$

Looking at tensor powers of the line bundle, we construct a compete linearly-independent basis of $\Gamma_{\mathrm{hol}}(T^2, L^m)$ by the theta functions

$$\Theta_a(z) = \frac{(2m\tau)^{1/4}}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \exp\Big(\pi i \tau m\big(k + \frac{a}{m}\big)^2 + 2\pi i m\big(k + \frac{r}{m}\big)z\Big)\,, \tag{4.3.85}$$

where $a = 1, \ldots, m$. A straightforward calculation shows these sections are orthonormal w.r.t. the Riemannian structure defined above: $\langle\Theta_a, \Theta_b\rangle = \delta_{ab}$. Now denote $z = x + iy$ and consider the Fourier modes

$$F_{(r,s)} = \exp\Big(2\pi i\big(rx + \frac{s}{\tau}y\big)\Big)\,. \tag{4.3.86}$$

Let us denote $X_{(r,s)} = \mathrm{grad}_{-i\omega}F_{r,s}$ and define the (rescaled) Toeplitz operators

$$P^{(m)}_{(r,s)} = i\Big(1 + \frac{\pi\tau}{m}\big(r^2 + \frac{s^2}{\tau^2}\big)\Big)\exp\Big(-\frac{\pi\tau}{2m}\big(r^2 + \frac{s^2}{\tau}\big)\Big)\rho^{(m)} \circ \Big(-\nabla^{(m)}_{X_{(r,s)}} + imF_{(r,s)}\Big)\,. \tag{4.3.87}$$

Then it is shown in [65] that these operators constitute the unitary algebra on the m-dimensional space of holomorphic sections of $L^{(m)}$:

$$\langle\Theta_a|P^{(m)}_{(r,s)}|\Theta_b\rangle = m\omega^{(m-r)s/2}(\Omega_1^{m-r}\Omega_2^s)_{ab}\,, \qquad \omega = e^{-2\pi i/m} \tag{4.3.88}$$

where Ω_1 and Ω_2 are the 't Hooft clock and shift matrices, given in (4.2.49). Consequently these fulfill the $\mathfrak{su}(m) \oplus \mathfrak{u}(1)$ algebra relation

$$[P^{(m)}_{\boldsymbol{r}}, P^{(m)}_{\boldsymbol{s}}] = \frac{m}{\pi}\sin\Big(\frac{\pi(\boldsymbol{r}\times\boldsymbol{s})}{m}\Big)P^{(m)}_{\boldsymbol{r}+\boldsymbol{s}}\,, \tag{4.3.89}$$

which shows that the Berezin deformation of Poisson algebras is the generalisation of the regularisation methods of the torus algebras considered in the previous paragraphs.

We already mentioned that we have not given a geometrical meaning of automorphic forms of fractional weight. Moreover, in the deformation procedures of the previous paragraphs the parameter t (which is here identified as r) could take arbitrary values in $\mathbb{R}$, though for irrational values the deformation remained infinite-dimensional. The question how to generate automorphic forms of arbitrary weight was solved in [80]. In this paper, the authors construct a noncompact covering of the Riemann surface which supports automorphic forms of arbitrary weights, but for which the values $r = n/(g-1)$ yield Hilbert spaces that are isomorphic to the spaces on the Riemann surface. The covering is constructed by taking an infinitely generated subgroup Γ_0 of Γ and considering the covering of Σ which has this group as its first fundamental group. If Γ is generated by $A_1, B_1, \ldots, A_g, B_g$ obeying $A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} = 1$, the subgroup Γ_0 is taken to be generated by

$$A_g\,, \quad B_g^{-n} A_i B_g^n\,, \quad B_g^{-n} B_i B_g^n\,, \qquad n \in \mathbb{N}\,, \quad i = 1, \ldots, g-1\,. \tag{4.3.90}$$

The covering $U(\Sigma)$ of the surface can then be shown to be noncompact, and should be thought of as an infinite cylinder with an infinite number of handle bodies attached to, each body containing $g-1$ handles. The existence of multipliers is essentially a cohomologous question, and it is obstructed by elements in the second cohomology space of the Fuchsian group. A multiplier is a 1-cochain, a mapping $\Gamma \longrightarrow \mathbb{C}$, and the requirement (4.3.42) can be restated as the coboundary of this mapping being a 2-cocycle cohomologous to zero in $H^2(\Gamma, \mathbb{Z})$. However, the Eilenberg-MacLane theorem states that

$$H^*(\Gamma, \mathbb{Z}) \simeq H^*(M, \mathbb{Z}) \tag{4.3.91}$$

Hence, for the covering $U(\Sigma)$ we have by Poincaré duality $H^2(U(\Sigma), \mathbb{Z}) \simeq H_c^0(U(\Sigma), \mathbb{Z}) = 0$, since constant functions have no compact support on $U(\Sigma)$. This is the reason why on $U(\Sigma)$ multipliers on all weights exist. Properties of the resulting deformed Poisson algebra (such as its expected infinite-dimensionality) remain to be investigated for these values of r.

Finally, we mention a method developed in [81], based on a discrete set of translations on the Riemann surface, acting on a space of multi-valued functions. The author finds explicitly a g-fold tensor products of representations of $\mathfrak{su}(N)$ on a space of Jacobi theta functions by cutting each of the complex holomorphic cycles in N pieces, $i = 1, \ldots, g$, and applying the translations along these pieces to the theta functions. There are however a number questions left concerning this procedure. First, Bars finds a representation of $\otimes^g \mathfrak{su}(N)$, but a morphism to the algebra of Hamiltonian vector fields is missing. An idea would be to construct the small translations from the flow of the gradient vector fields, because this identification preserves the Lie bracket. However, it is then not clear how to truncate these flows to a lattice of translations along the homology cycles. Secondly, the dimension of the unitary algebras depends differently on the genus of the surface than the regularisation described above. The latter procedure yields matrices of a size which depends linearly on g, namely $\mathfrak{su}((g-1)(4m-1))$, while the former approach yields $\mathfrak{su}(N^g)$. It seems rather that the author has found a regularisation of the algebra of Hamiltonian vector fields on the $2g$-dimensional torus T^{2g} (compare e.g. with the treatment of tori in [65]). It seems therefore not convenient to pursue this path.

4.4 Compactification

4.4.1 Wrapping Membranes around Target-Space Tori

In this section we assume the bosonic sector of the target space to have one or more compactified directions. Geometrically, these directions are assumed to be flat, so we replace the transverse and longitudinal target manifold $\mathbb{R}^{10}$ by $\mathbb{R}/2\pi R_1\mathbb{Z} \times \ldots \times \mathbb{R}/2\pi R_k\mathbb{Z} \times \mathbb{R}^{10-k}$ as it would be unphysical to replace the (lightcone) time direction by some compact loop. A contractible loop on the spacesheet of the membrane is homotopy equivalent to a point. Assuming continuity of the bosonic variable X^a, the composition with the homotopy on the spacesheet gives a homotopy on the bosonic image, which is therefore contractible. So noncontractible loops on the bosonic image only exist for membranes modeled on higher genus spacesheets. If the bosonic target space is compactified, such a loop may wind around the wrapped dimensions. For C_λ, $\lambda = 1, \ldots 2g$ a basis of the homology of the surface and X^i a compactified coordinate on the flat unit circle, we obtain its winding numbers by

$$\oint_{C_\lambda} dX^i = 2\pi n_{i,\lambda}\,. \tag{4.4.1}$$

This immediately shows the modifications: we are no longer working with real-valued differential forms, but with $U(1)$-valued forms, which do not automatically give zero when closed and integrated along a closed loop. The closed form above, which is an element of $d\Omega^0(\Sigma, \mathbb{R}/2\pi\mathbb{Z})$ may actually be expanded in a basis of $H^1(\Sigma, \mathbb{Z}) \oplus d(\Omega^0(\Sigma, \mathbb{R}))$; if $\varphi_{\lambda r} \mathrm{d}\sigma^r$ is a basis of cocycles orthogonal w.r.t. the homology basis C_λ, such a decomposition looks like

$$dX^a = 2\pi n^{a\lambda} \varphi_\lambda + X^{aM} dY_M \,, \tag{4.4.2}$$

where $\{Y_M\}$ is a complete basis of $C^\infty(\Sigma, \mathbb{R})$. In the dual vector field language, the symplectic gradient of a circle-valued object can be decomposed into integer multiples of harmonic vector fields and the gradient of a real-valued function:

$$\mathrm{grad}_\nu(X^a) = 2\pi n^{a\lambda} \phi_\lambda + \mathrm{grad}_\nu f \,. \tag{4.4.3}$$

Consequently the sequence (3.5.22) is no longer exact if one replaces the $C^\infty(\Sigma, \mathbb{R})$ by the Poisson algebra of circle-valued smooth functions. It should be modified to

$$0 \longrightarrow H^0(\Sigma) \longrightarrow C^\infty(\Sigma_\tau, \mathbb{R}/2\pi\mathbb{Z}) \stackrel{\mathrm{grad}_\nu}{\longrightarrow} \mathfrak{X}_\nu(\Sigma_\tau) \stackrel{\gamma}{\longrightarrow} \frac{H^1(\Sigma_\tau, \mathbb{R})}{H^1(\Sigma_\tau, 2\pi\mathbb{Z})} \longrightarrow 0 \,. \tag{4.4.4}$$

Now the map γ is given by $\gamma(\xi) = [[\iota_\xi \nu]]$ where the inner brackets are the equivalence class resulting from modding out exact forms and the outer denote the equivalence class under $\alpha \sim \alpha + 2\pi n^\lambda \varphi_\lambda$, taking the $2g$-dimensional flat unit torus in the cohomology vector space. This space is equipped with the zero bracket to make it an exact sequence of Lie algebras.

If 2 coordinates X^a, X^b are nontrivially wound around the periodic directions in the target space, the exterior product of their differentials is not exact,

$$\int_\Sigma \mathrm{d}^2\sigma \sqrt{w} \{X^a, X^b\} = \int_\Sigma dX^a \wedge dX^b = 2\pi n \in 2\pi\mathbb{Z} \setminus \{0\} \,. \tag{4.4.5}$$

Configurations with $n \neq 0$ are called *irreducible*. By a theorem of A. Weil, there exists a principal $U(1)$ bundle and a connection on it such that $dX^a \wedge dX^b$ is its curvature. If X^a and X^b are the only compactified coordinates, this makes the doubly-wrapped membrane equivalent to a gauge theory on the space sheet coupled to the remaining scalars and fermions [82]. Formulated on a Riemann surface $\Sigma = \mathbb{D}/\Gamma$, a closed form dZ, where Z is a complex function from the surface to the unit circle with winding numbers

$$\frac{1}{2\pi} \oint_{A_i} dZ = n_i \,, \qquad \frac{1}{2\pi} \oint_{B_i} dZ = m^i \tag{4.4.6}$$

can be expanded in the basis $\varphi_1, \ldots, \varphi_g$ of $\Omega^1_H(\Sigma, \mathbb{C})$ plus an exact real-valued form,

$$dZ = \lambda^i \varphi_i + \overline{\lambda}^i \overline{\varphi}_i + df \,, \tag{4.4.7}$$

where

$$\lambda^i = \frac{i\pi}{\det(\mathrm{Im}(\Omega))} (n_i - \sum_j m_j \Omega_{ji}) \,, \tag{4.4.8}$$

where Ω is the $g \times g$ complex *periodicity matrix* of the Riemann surface, which can be expressed as the integrals over the B-cycles of a system of holomorphic differentials that are normalised over the A-cycles:

$$\Omega_{ij} = \int_{B_i} \varphi_j \,, \quad \int_{A_i} \varphi_j = \delta_{ij} \,. \tag{4.4.9}$$

The $2g$ real harmonic forms φ_λ are given in terms of these holomorphic basis vectors by $\varphi_i + \overline{\varphi}_i$ and $(\Omega^{-1})^{ij} \varphi_i + \overline{(\Omega^{-1})^{ij}} \overline{\varphi}_i$. We construct a winding function $f : \Sigma \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ with winding numbers

n_λ by requiring that for each generator A_λ of the discrete group Γ we have $f(A_\lambda(z)) = f(z)+2\pi n_\lambda$. The Fuchsian group of the 2-torus is generated by 2 translations $\sigma^1 = \mathrm{Re}(z) \mapsto \sigma^1 + 2\pi$ and $\sigma^2 = \mathrm{Re}(z) \mapsto \sigma^2 + 2\pi$, so a function a winding function on the torus can be expanded as

$$f(\sigma^1, \sigma^2) = n_1\sigma^1 + n_2\sigma^2 + \sum_{k,\ell\in\mathbb{Z}} c_k e^{2\pi i(k\sigma^1+\ell\sigma^2)} \,. \tag{4.4.10}$$

Suppose now we have performed the gauge-fixing from section (3.4), with the requirement that X^+ takes its values in $\mathbb{R}$. The first nontrivial property to check when a number of embedding coordinates is compactified is the independence of the mass of the CM modes. This is quickly verified to hold since the integral of $dX^a \wedge df$, where f is an ordinary real-valued function, is zero. The first term in $d\gamma^a$ (cf. (3.4.38)) is of this form because

$$\begin{aligned} d(\{X^a, X^b\}dX_b) &= d*(dX^a \wedge dX^b) \wedge dX_b \\ &= -*d^*\Big[(df^a + 2\pi n^{a\lambda}\varphi_\lambda) \wedge (df^b + 2\pi n^{b\lambda'}\varphi_{\lambda'})\Big] \wedge dX_b \\ &= d*(df^a \wedge df^b) \wedge dX_b \,. \end{aligned} \tag{4.4.11}$$

The last line is obtained using harmonic properties of the nonexact APD generators: $d^*\varphi_\lambda = 0$. More important is the construction of constraints from equation (3.5.1). If there is a nontrivial winding in the X^- coordinate, the right hand side of this equation is closed and may be expanded into exact and harmonic forms with the X^- winding numbers n_λ, yielding constraints

$$\psi \equiv d(\partial_0 X_a) \wedge dX^a + d\bar{\theta}\Gamma^- \wedge d\theta = 0 \,, \tag{4.4.12}$$

$$\psi_\lambda \equiv \int_{\Sigma_\tau} \varphi_\lambda \wedge *(\partial_0 X_a dX^a + \bar{\theta}\Gamma^- d\theta) = 2\pi n_\lambda \,. \tag{4.4.13}$$

Note that by consistency, the winding number is not a dynamical variable. Clearly (cf. equation (4.4.3)), to describe a regularisation of nontrivially wound bosonic coordinates one needs a representation of the harmonic vector fields. Where in the uncompactified case we were able to describe the theory in the gauge $A_\lambda = 0$, symmetries generated by the constraints corresponding to these auxiliary fields not taken into account, now the cocycles don't act as outer derivations on these fields and therefore should find themselves as adjoint transformations in the matrix regularised action. We have introduced new physical variables, the winding numbers, which can only be measured by integrating the bracket with (normalised) harmonic vector fields. Since physical observables such as the mass depend on these numbers, the regularised theory should contain the winding data as well, and consequently we should find operators in matrix theory corresponding to the cocycles.

4.4.2 APD Gauge Theory and Supersymmetry Algebra in Compactified Target Space

If we denote $\tilde{X}^a$ to be the real valued function satisfying $d\tilde{X}^a = dX^a - 2\pi n^{a\lambda}\varphi_\lambda$ (uniquely defined up to a constant), an APD generating vector field $\xi = \mathrm{grad}_\nu f_\xi + \chi^\lambda\phi_\lambda$ acts on this field by

$$\mathfrak{L}_\xi X^a = \{f_\xi, \tilde{X}^a\} + \phi^r_\lambda(2\pi n^{a\lambda}\partial_r f_\xi - \chi^\lambda\partial_r\tilde{X}^a) - 2\pi\sqrt{w}\chi^\lambda n^{a\lambda'}\varepsilon_{rs}\phi^r_\lambda\phi^s_{\lambda'} \,. \tag{4.4.14}$$

Gauge theory of area-preserving diffeomorphisms is constructed to exhibit these kind of symmetries. Assuming the winding number is a constant, the covariant derivative receives two contributions

$$\nabla_0 X^a = \nabla_0\tilde{X}^a - 2\pi n^{a\lambda}\phi^r_\lambda\partial_r\omega + 2\pi\sqrt{w}A^\lambda n^{a\lambda'}\varepsilon_{rs}\phi^r_\lambda\phi^s_{\lambda'} \,. \tag{4.4.15}$$

This has its impact on the momentum eigenvalues: the second term vanishes when multiplied by $\sqrt{w}$ and integrated, but the third term does not. Hence the total momentum corresponding to this coordinate becomes

$$P^a_0 = \tilde{P}^a_0 + 2\pi P^+_0 A^\lambda n^a_\lambda \,, \tag{4.4.16}$$

where

$$\tilde{P}_0^a = P_0^+ \int_\Sigma \mathrm{d}^2\sigma \sqrt{w} \partial_0 \tilde{X}^a \tag{4.4.17}$$

is the non-winding momentum part. The modification $\psi_\lambda = 2\pi n_\lambda$ can be implemented in an area-preserving diffeomorphism gauge theory using a simple trick: one just adds a total derivative term proportional to $\sqrt{w}\nabla_0 X^-$ to the Lagrangian density,

$$\begin{aligned} L = \mathrm{d}\tau \wedge |\mathrm{d}^2\sigma| P_0^+ \sqrt{w} \Big[& \tfrac{1}{2}\nabla_0 X^a \nabla_0 X_a + \bar{\theta}\Gamma_- \nabla_0 \theta - \tfrac{1}{4}(P_0^+)^{-2}\{X^a, X^b\}\{X_a, X_b\} \\ & + (P_0^+)^{-1}\bar{\theta}\Gamma^- \Gamma_a \{X^a, \theta\} + \nabla_0 X^- \Big] . \end{aligned} \tag{4.4.18}$$

Using the expansion of the covariant derivative in compact directions (4.4.15) we see that adding this term corresponds to adding 2 ordinary total derivatives (a nonphysical gauge change) and a term $2\pi P_0^+ w A^\lambda n^{\lambda'} \varepsilon_{rs} \phi_\lambda^r \phi_{\lambda'}^s$. This adds a term $2\pi n^\lambda P_0^+$ to the right hand side of the equation of motion of the gauge field A^λ. Furthermore adding such a total derivative preserves the manifest APD-invariance of the Lagrangian and its supersymmetry. An essential feature of compactification of supersymmetric field theories is the appearance of central charges in the supersymmetry algebra. Under nontrivial winding the boundary integrals in (3.5.49) may take nonzero values. In particular, a short analysis shows that $d\beta_a$ and $d\beta_{abcd}$ are exact as real-valued forms, but $d\beta_{ab}$ has a nontrivial decomposition into harmonics if the transverse coordinates wind around compact dimensions,

$$\begin{aligned} \int_\Sigma d\beta_{ab} = \int_\Sigma dX^a \wedge dX^b = 4\pi^2 n_a^\lambda n_{\lambda b} \equiv Z_{ab} \, , \\ \int_\Sigma dX^a \wedge \psi = 4\pi^2 n_a^\lambda n_{\lambda -} \equiv Z_a \, . \end{aligned} \tag{4.4.19}$$

These are integer multiples of $(2\pi)^2$. They become the nonzero central charges in the supersymmetry algebra:

$$\begin{aligned} \{Q_\alpha^-, Q_\beta^-\}_D &= -2(\Gamma^-)_{\alpha\beta} P_0^+ \, , \\ \{Q_\alpha^+, Q_\beta^-\}_D &= -(\Gamma_a \Gamma^+ \Gamma^-)_{\alpha\beta} P_0^a - (\Gamma^{ab}\Gamma^+\Gamma^-)_{\alpha\beta} Z_{ab} \, , \\ \{Q_\alpha^+, Q_\beta^+\}_D &= 2(\Gamma^+)_{\alpha\beta} \mathscr{H}_\tau - 2(\Gamma^a \Gamma^+)_{\alpha\beta} Z_a \, . \end{aligned} \tag{4.4.20}$$

It is well-known that central charges 'lift' the mass spectrum of a supersymmetric field theory. This can be seen by defining the zero-mode supercharges $Q_\alpha^{+(0)} = 2P_0^a \Gamma_a \theta_0$, which generate supersymmetry transformations on the CM coordinates. These variables obey the algebra [58]

$$\{Q_\alpha^{+(0)}, Q_\beta^{+(0)}\} = (\Gamma^+)_{\alpha\beta}(\tilde{P}_0{}^a \tilde{P}_{0a} + \tfrac{1}{2} Z_{ab} Z^{ab}) + 2(\Gamma^+\Gamma^a)_{\alpha\beta} \tilde{P}_0{}^b Z_{ab} \, , \tag{4.4.21}$$

and they commute with the remaining part of the supercharges, $Q_\alpha^{+(1)} = Q_\alpha^+ - Q_\alpha^{+(0)}$. The mass operator for the winding supermembrane is defined as

$$\mathscr{M}_\tau^2 = 2\mathscr{H}_\tau - \tilde{P}_0{}^a \tilde{P}_{0a} - \tfrac{1}{2} Z_{ab} Z^{ab} \, . \tag{4.4.22}$$

Subtracting the brackets of the CM-mode supercharges from the expression (4.4.20) yields the commutator

$$\{Q_\alpha^{+(1)}, Q_\beta^{+(1)}\}_D = (\Gamma^+)_{\alpha\beta} \mathscr{M}_\tau^2 - 2(\Gamma^+\Gamma^a)_{\alpha\beta}(Z_a + \tilde{P}_0{}^b Z_{ab}) \, . \tag{4.4.23}$$

Since the supercharges are Majorana spinors, the left hand side is equal to $\{Q_\alpha^{+(1)}, (Q_\beta^{+(1)})^c\}_D$, which becomes a positive definite operator under quantisation. Hence we find for half of the spinor indices that for a simultaneous mass-central charge eigenstate $|\phi\rangle$

$$\langle\phi|\mathscr{M}^2|\phi\rangle \geq (\Gamma^a)_{\alpha\beta}\langle\phi|Z_a + \tilde{P}_0{}^b Z_{ab}|\phi\rangle \, . \tag{4.4.24}$$

So membrane states with two or more coordinates irreducibly[14] winded cannot be massless. However, if only one target space direction is compactified, both Z_a and Z_{ab} are the zero operators and the spectrum is not lifted. Note that this does not mean the spectrum of (irreducibly) wrapped supermembranes becomes discrete, because also such configurations can grow string-like spikes and there is no reason to believe that in this setting supersymmetry and zero-point quantum fluctuations remove this property. In connection with matrix string theory this poses no problem: viewing the membrane as limit of Dirichlet particles may even give a physical interpretation of a continuous spectrum.

4.4.3 Compactifying the Matrix Model

How does compactification work in matrix theory? A matrix variable X is wrapped by imposing gauge equivalence under a discrete set of translations along the unit matrix [83, 84],

$$U^{-1}XU = X + 2\pi R\mathbf{1}\,. \tag{4.4.25}$$

Here U is a unitary matrix and R is a constant bigger than zero, the compactification radius. Taking the trace of both sides immediately leads to the conclusion that (excluding the trivial representation) a finite-dimensional matrix representation cannot satisfy this winding criterium. Moreover, elements proportional to the unit matrix are not included in the Lie algebras of $U(N)$. The condition (4.4.25) will be fulfilled by a set of linear operators on an infinite-dimensional vector space, called the *loop algebra* of $\mathfrak{u}(N)$ [85],

$$\mathfrak{u}(N)_{\text{loop}} = C^\infty(S^1, \mathfrak{u}(N))\,, \tag{4.4.26}$$

consisting of smooth mappings of the unit circle S^1 in the complex plane into the unitary algebra. The elements of the loop algebra may be written as the tensor product of a $\mathbb{C}$-valued function on S^1 and a matrix in $\mathfrak{u}(N)$. For $Y_1, Y_2 \in \mathfrak{u}(N)_{\text{loop}}$, we define the loop algebra element $[Y_1, Y_2]$ by

$$[Y_1, Y_2](\rho) = [Y_1(\rho), Y_2(\rho)]\,, \tag{4.4.27}$$

where $\rho \in [0, 2\pi)$ is the (real) angle parameterising S^1. This turns $\mathfrak{u}_{\text{loop}}(N)$ into an infinite-dimensional Lie algebra. The element U is considered to be a linear operator on the loop algebra too and will be required to act unitarily on this space with respect to the inner product

$$\langle Y_1, Y_2\rangle = \oint d\rho \operatorname{Tr}(Y_1(\rho)Y_2^*(\rho))\,. \tag{4.4.28}$$

These definitions now allow a solution for the condition (4.4.25): take $V = \mathfrak{u}_{\text{loop}}(N)$, and define $U, X \in \operatorname{End}(V)$ by

$$\begin{aligned} X(Y)(\rho) &= 2\pi i R\frac{\partial Y(\rho)}{\partial\rho} + [A, Y](\rho)\,, \\ U(Y)(\rho) &= e^{-i\rho}Y(\rho)\,, \end{aligned} \tag{4.4.29}$$

where A is an arbitrary element in $\mathfrak{u}_{\text{loop}}(N)$. Note that U is an element of the loop group $C^\infty(S^1, U(N))$, acting on the loop algebra by left translation. More important is the appearance of the operator X, whose first term is an outer derivation on the loop algebra: $\partial_\rho \in H^1(\mathfrak{u}(N)_{\text{loop}}; \mathfrak{u}(N)_{\text{loop}})$ represents a nonzero cohomology class. This fundamentally changes the rôle of the compactified coordinate from a Lie algebra element to a derivation. However, we can extend the loop algebra to include this derivation, obtaining the so-called *affine* algebra

$$\tilde{\mathfrak{u}}(N) = \mathfrak{u}(N)_{\text{loop}} \oplus \mathbb{C}\frac{\partial}{\partial\rho}\,. \tag{4.4.30}$$

[14]Irreducibly winded means there is no linear transformation on the bosonic coordinates which makes all (but possibly one) winding numbers vanish. We shall discuss this requirement in more detail below.

A more conventional way of looking at it is to consider a principal $U(N)$-bundle over the unit circle: the unwrapped coordinates are sections of the associated adjoint bundle, while the winding coordinate X defines a covariant derivative on the adjoint bundle with connection form (gauge field) A. The generalisation of this procedure can be straightforwardly extended to compactification on higher-dimensional compact spaces [84], adding more and more loop variables (although only tori are considered because of calculational difficulties).

The simplest compactification of the matrix model is wrapping a single coordinate, say X^9, around a circle of radius R_9. With a coordinate substitution $\tilde{\rho} = (2\pi R_9 q)^{-1}\rho \in \mathbb{R}/q^{-1}\mathbb{Z}$, the brackets with the compactified coordinate become

$$\begin{aligned} q[X^i, X_9] &\mapsto i\partial_{\tilde{\rho}} X^i - q[A_1, X^i] \equiv \nabla_1 X^i\,, \quad i = 1, \dots 8\,, \\ \nabla_0 X^9 &\mapsto \partial_0 A_1 - i\partial_{\tilde{\rho}} A_0 - q[A_0, A_1] \equiv F_{01}\,, \end{aligned} \tag{4.4.31}$$

and similarly the bracket with matrix-valued spinors gets modified. Of course all field variables become dependent of time and the circle parameter. The trace in the D-particle Lagrangian gets replaced by the inner product (4.4.28) and one obtains the so-called *matrix string* Lagrangian

$$\begin{aligned} L = -\pi R_9 \oint \mathrm{d}\rho \mathrm{Tr}\Big[&- (\nabla_0 X^i)^2 + (\nabla_1 X^i)^2 - (F_{01})^2 + \theta^T(-\nabla_0 + \Gamma^9 \nabla_1)\theta \\ &+ \frac{q^2}{2}[X^i, X^j]^2 - \theta^T \Gamma_i [X^i, \theta]\Big]\,. \end{aligned} \tag{4.4.32}$$

Wherever a square in the formula above, a (Euclidean) summation over repeated Latin indices is understood. The Lagrangian above describes the low-energy effective behaviour of an ensemble of N Dirichlet 1-branes (strings), since it is just super-Yang-Mills theory reduced to the worldsheet of a string. Indeed, the Lagrangian above exhibits supersymmetry and the global $U(N)$ symmetry of the original model is extended to a full $U(N)$ gauge symmetry: a map $\Psi : C^\infty(S^1, \mathfrak{u}(N)) \longrightarrow \Gamma T\mathcal{F}$ which acts on the fields as

$$\iota_{\Phi(Y)}\delta X^a = [Y, X^a]\,, \quad \iota_{\Phi(Y)}\delta A_0 = \nabla_0 Y\,, \quad \iota_{\Phi(Y)}\delta A_1 = \nabla_1 Y\,. \tag{4.4.33}$$

The classical vacuum, consisting of the Cartan subalgebra of $\mathfrak{u}_{\mathrm{loop}}(N)$ of diagonal ρ-dependent matrices, has a residual symmetry group $U(1)_{\mathrm{loop}} \times \ldots \times U(1)_{\mathrm{loop}}$ (N times).

The Lagrangian above was derived in [86] by exploiting *T-duality* rather then imposing the periodicity condition (4.4.25). The T-dual of a system of D-strings is an infinite number of copies of the D-particle Lagrangian (4.2.11) on the real line and subsequently modding out the lattice $\mathbb{Z}$ of the resulting gauge theory, where the winding direction is imposed to fulfill an additional periodicity constraint as in (4.2.11). The infinite number of copies of D0-branes is described by an infinite number of $\mathfrak{u}(N)$ matrices X^a_n, θ^α_n where $a = 1, \ldots, 9$, $\alpha = 1, \ldots, 16$ and $n \in \mathbb{Z}$; these degrees of freedom may equally be captured by infinite blocks of matrices $X^a_{nm}, \theta^\alpha_{nm}$ satisfying $(X^a_{nm})^\dagger = X^a_{mn}$ and $(\theta^\alpha_{nm})^\dagger = \theta^\alpha_{mn}$ and fulfilling the boundary conditions

$$X^a_{nm} = X^a_{(n-1)(m-1)}\,, \quad \theta^\alpha_{nm} = \theta^\alpha_{(n-1)(m-1)}\,. \tag{4.4.34}$$

Consequently, all information is carried by the matrices $X^a_n \equiv X^a_{n0}$ and $\theta^\alpha_n \equiv \theta^\alpha_{0n}$. The Lagrangian of such a system of copies of D-particle ensembles is *not* simply taken the noninteracting sum of the Lagrangians of the subsystems: one takes a Lagrangian in which the copies interact with each other as if they were D-particles themselves; the bosonic part reads

$$\begin{aligned} L = \frac{1}{2q}\mathrm{Tr}\Big[&(\nabla_0 X^a)_m{}^n (\nabla_0 X_a)_n{}^m - \frac{q^2}{4}\big((X^a)_m{}^q (X^b)_q{}^n \\ &- (X^b)_m{}^q (X^a)_q{}^n\big)\big((X_a)_n{}^r (X_b)_r{}^m - (X_b)_n{}^r (X_a)_r{}^m\big)\Big]\,, \end{aligned} \tag{4.4.35}$$

where summation over repeated indices is understood. Equivalently, one can obtain this model by dividing the variables in the $U(\infty)$-matrix model into blocks of size $N \times N$ and imposing elements on the same diagonal to be equal. The winding of, say, the ninth direction is described by modifying the conditions (4.4.34) for this matrix variable into

$$X^9_{nm} = \begin{cases} 2\pi R_9 \mathbf{1}_N + X^9_{(n-1)(n-1)} & \text{if } n = m\,, \\ X^9_{(n-1)(m-1)} & \text{if } n \neq m\,. \end{cases} \tag{4.4.36}$$

Substituting these conditions into the Lagrangian above yields exactly the Fourier transform of the matrix string Lagrangian. Both approaches to the compactification of a matrix model are rather ad-hoc and a natural formalism seems not to exist.

A solution of the compactification condition (4.4.25) was given by a set of operators acting on the infinite-dimensional space of sections of a topologically trivial $U(N)$-bundle over the unit circle. Such a solution however is not unique. Since the unit circle is not simply connected, we might consider topologically nontrivial bundles, so-called *twisted* bundles, whose sections are single-valued on the circle *up to a (global) unitary transformation*,

$$X^i(\rho + 2\pi) = V X^i(\rho) V^\dagger\,. \tag{4.4.37}$$

Since the eigenvalues on both sides of this equation are the same, V acts on diagonalised matrices by a permutation of their eigenvalues (it is an element in the Weil group of $U(N)$). Hence the twisted sectors can be distinguished into equivalence classes corresponding to the conjugacy classes of the permutation group S_N [87]. Where the vacuum of the untwisted theory generated an ensemble of N closed strings of equal length, the twisted sectors are able to generate the full Type II spectrum because arbitrary string lengths can be achieved, depending on the conjugacy class. A particular sector is the full cyclic permutation of the N eigenvalues; for

$$X^a(\rho) = \begin{pmatrix} x^a_1(\rho) & & & 0 \\ & x^a_2(\rho) & & \\ & & \ddots & \\ 0 & & & x^a_N(\rho) \end{pmatrix}\,, \tag{4.4.38}$$

this sector imposes the twisted boundary condition

$$x^a_n(\rho + 2\pi R) = x^a_{n+1}(\rho)\,, \quad x^a_N(\rho + 2\pi R) = x^a_1(\rho)\,. \tag{4.4.39}$$

This kind of configuration is called a *long matrix string*. In the limit $N \longrightarrow \infty$ the eigenvalue functions may be 'glued' together to form a smooth function from $\mathbb{R}/2\pi NR$ to $U(1)$. Since every permutation may be decomposed into small cycles, every sector's Hilbert space is actually the tensor product of smaller long string Hilbert spaces, where the length of these individually wrapped strings is 2π times their winding number, the length of the cycle.

4.4.4 Hamiltonian Vector Fields from Affine $\mathfrak{su}(N)$

Wrapping a matrix model automatically introduces a continuous parameter in the symmetry group, and as a consequence a new outer derivation on the (loop) algebra given by differentiation w.r.t. this parameter. Observe that this is exactly what is needed to include harmonic vector fields in the regularisation procedure of the supermembrane. The strategy to find a matrix representation in the affine $\mathfrak{su}(N)$ will be to deform the ideal $\mathfrak{X}^G_\nu(\Sigma)$ and consistently project it into $\mathfrak{su}(N)_{\text{loop}}$ and find a representation of the harmonic vector fields as (independent) linear combinations of the outer derivations on the loop algebra such that the commutators among themselves are represented as well. Let us take the Poisson algebra on the torus to illustrate this. As we have seen, for rational values of the deformation parameter t, this algebra with rescaled brackets $[\,.\,,\,.\,]_t \to t[\,.\,,\,.\,]_t$ could

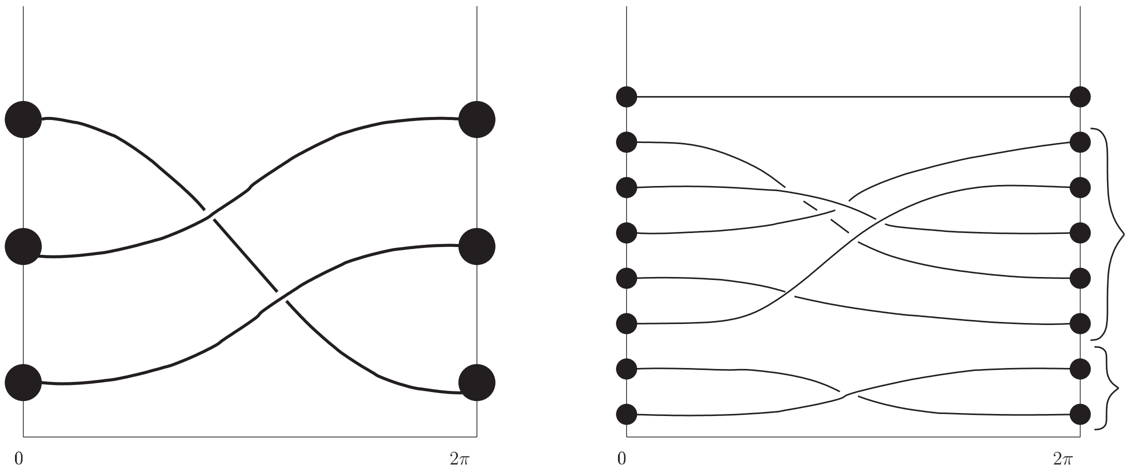


Figure 7: On the left: a graphical interpretation of the twisted sector of the $N = 3$ matrix string model representing a long matrix string. On the right: a configuration in a twisted sector of the $N = 8$ matrix string representing three strings of lengths 2π (top), 10π (middle) and 4π (bottom).

be consistently projected onto $\mathfrak{su}(N)$, where N is the denominator of t. For $t = 1/N$ the rescaled deformation of the torus algebra in Fourier basis is

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}]_* = 2\sin\Big(\frac{2\pi}{N}\boldsymbol{m}\times\boldsymbol{n}\Big)L_{\boldsymbol{m}+\boldsymbol{n}}\,, \quad [\phi_r, L_{\boldsymbol{m}}]_* = \frac{m_r}{N}L_{\boldsymbol{m}}\,, \quad [\phi_1, \phi_2]_* = 0\,, \tag{4.4.40}$$

where $r = 1, 2$ and $L_{\boldsymbol{m}}$ denotes the symplectic gradient of a Fourier mode with wave vector $\boldsymbol{m} \in \mathbb{Z}^2/N(\mathbb{Z}\oplus\mathbb{Z})$. Note that the last commutator differs in [88]. This comes from the fact that the authors approximate the algebra of circle-valued *functions*, which is a central extension of the algebra of Hamiltonian vector fields above (the zero in the third commutator representing the zero vector field). It is however more convenient to choose the former because the there is no notion of differential geometry in matrix models. Let $\tilde{\sigma}^r$ denote the circle-valued functions winding one time around the cocycles C^r, which should satisfy $\mathrm{grad}_\nu\tilde{\sigma}^r = \phi_r = \partial_r$. The definition of the Poisson bracket of such functions arises naturally from the Lie bracket of vector fields by keeping the symplectic gradient a Lie algebra homomorphism. This implies $\mathrm{grad}_\nu(\{\tilde{\sigma}^1, \tilde{\sigma}^2\}) = (2\pi)^{-2}[\partial_1, \partial_2] = 0$. The bracket is therefore a constant. This constant can be determined using the normalisation of the cocycles φ_λ,

$$\int_\Sigma \mathrm{d}^2\sigma\sqrt{w}\{\tilde{\sigma}^1, \tilde{\sigma}^2\} = \int_\Sigma d\tilde{\sigma}^1\wedge\mathrm{d}\tilde{\sigma}^2 = \int_\Sigma \varphi_1\wedge\varphi_2 = 1\,, \tag{4.4.41}$$

which determines this constant to be 1, provided $\sqrt{w}$ is normalised. Rather than representing (4.4.40) we shall seek a matrix representation of the star commutator algebra

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}]_* = 2\sin\Big(\frac{2\pi}{N}\boldsymbol{m}\times\boldsymbol{n}\Big)L_{\boldsymbol{m}+\boldsymbol{n}}\,, \quad [\tilde{\sigma}^r, L_{\boldsymbol{m}}]_* = \frac{m_r}{N}L_{\boldsymbol{m}}\,, \quad [\tilde{\sigma}^1, \tilde{\sigma}^2]_* = \frac{1}{N}\,. \tag{4.4.42}$$

Let us now seek a mapping into affine $\mathfrak{su}(N)$ based on the 2-torus. By this we mean the vector space $C^\infty(T^2, \mathfrak{su}(N))\otimes\oplus\mathbb{C}\partial_1\oplus\mathbb{C}\partial_2$, equipped with the brackets

$$\begin{aligned}
[g^{\boldsymbol{m}}(\rho^1,\rho^2)T_{\boldsymbol{m}}, h^{\boldsymbol{n}}(\rho^1,\rho^2)T_{\boldsymbol{n}}] &= f_{\boldsymbol{mn}}{}^{\boldsymbol{k}}g^{\boldsymbol{m}}(\rho^1,\rho^2)h^{\boldsymbol{n}}(\rho^1,\rho^2)T_{\boldsymbol{k}}\,,\\
[\partial_r, f^{\boldsymbol{m}}(\rho^1,\rho^2)T_{\boldsymbol{m}}] &= \partial_r f^{\boldsymbol{m}}(\rho^1,\rho^2)T_{\boldsymbol{m}}\,,\\
[\partial_1,\partial_2] &= 0\,,
\end{aligned} \tag{4.4.43}$$

where $f_{\boldsymbol{mn}}{}^{\boldsymbol{k}}$ are the structure constants of $\mathfrak{su}(N)$ and ρ^1, ρ^2 are the real angles parameterising the torus. Analogously to the regularisation procedures mentioned before, we can only approximate the algebra (4.4.42) with affine $\mathfrak{su}(N)\oplus\mathfrak{u}(1)$ since we have to include the zero modes in the Poisson algebra above. From previous sections we know that the 't Hooft clock and shift matrices Ω_1 and Ω_2 (cf. (4.2.49)) provide a representation of the central extended unitary algebra with sine-algebra structure constants. One possibility of a Lie algebra homomorphism into affine $\mathfrak{su}(N)\oplus\mathfrak{u}(1)$ with two derivations is

$$\begin{aligned} L_{\boldsymbol{m}} &\longrightarrow e^{im_1\rho_1/R_1}e^{im_2\rho_2/R_2}\otimes\omega^{-s_1s_2/2}\Omega_1^{s_1}\Omega_2^{s_2}\,,\\ \tilde{\sigma}^1 &\longrightarrow -\frac{iR_1}{N}\partial_1-\frac{i}{2R_2}\rho_2\otimes\mathbf{1}_N\equiv D_1\,,\\ \tilde{\sigma}^2 &\longrightarrow -\frac{iR_2}{N}\partial_2+\frac{i}{2R_1}\rho_1\otimes\mathbf{1}_N\equiv D_2\,, \end{aligned}\tag{4.4.44}$$

where we have denoted $s_r=m_r \bmod N$ and ω is some N-th root of unity. The parameters R_1 and R_2 denote the two radii of the torus T^2 on which the unitary loop algebra is constructed. At this stage, they can be completely absorbed in a rescaling of the torus parameters, but (as was the case for the matrix string) they will become prefactors when we regularise the supermembrane Lagrangian with the identifications above. The representation of the derivations differs from those in [88]. Writing $D_r=-(iR_r/N)\partial_r+iV_{rs}\rho^s$ (summation over s), the bracket $[D_1,D_2]=N^{-1}$ puts no restriction on the diagonal elements of V, while the off-diagonal are restricted to lie on the real line defined by $R_1x+R_2y=1$. In the Fourier basis of $C^\infty(T^2,\mathbb{R})$, it is quickly seen that the derivations ∂_r induce a double grading on the algebra, and each eigenspace is isomorphic to $\mathfrak{su}(N)$. Were the identifications above only show that the regularised algebra of Hamiltonian vector fields can be embedded in affine unitary algebras, it is shown in [74] that the former algebra with only one harmonic vector field included coincides with $C^\infty(S^1,\mathfrak{su}(N))\oplus\mathbb{C}\partial_1$. As was noted by Cederwall [67], the difficulty here is identifying the standard derivations of the affine special unitary algebra, whose eigenspaces are isomorphic to ordinary $\mathfrak{su}(N)$, with the harmonic vector fields in the APD algebra. Because this correspondence is not trivial, the root system of the APD algebra is 'slanted' w.r.t. the root system of affine $\mathfrak{su}(N)$.

4.4.5 $1+0$ Super Yang-Mills Theory from the Toroidal Supermembrane

With the results from the previous paragraph, it becomes a matter of substituting the generators of the gauge algebra to find the matrix regularised theory for winding membranes (based on toroidal spacesheets). We have gained even more: we are now able to include the harmonic vector fields for the non-compactified supermembrane as well. The APD gauge field $A=\mathrm{grad}_\nu\omega+A_1(\tau)\phi_1+A_2(\tau)\phi_2$, becomes in the regularised theory

$$A\longrightarrow A_0(\rho_1,\rho_2,\tau)+A_1(\tau)D_1+A_2(\tau)D_2\,,\tag{4.4.45}$$

where the affine algebra elements D_r are given in (4.4.44). This corresponds to a compactification of the gauge field A_0, the remaining component of the Yang-Mills gauge field upon dimensional reduction. The expression above of the gauge field follows from the periodicity conditions

$$U_rA(\rho^1,\rho^2,\tau)U_r^{-1}=A(\rho^1,\rho^2,\tau)+\frac{R_r}{N}A_r(\tau)\mathbf{1}\,,\quad r=1,2\,,\tag{4.4.46}$$

where the action of the unitary loop group elements U_r is multiplication by the unitary matrix $e^{i\rho^r}\mathbf{1}_N$. We want to relate the resulting theory to a non-winding membrane, so we impose

$$U_rX^aU_r^{-1}=X^a\,,\quad U_r\theta^\alpha U_r^{-1}=\theta^\alpha\,.\tag{4.4.47}$$

Hence the embedding coordinates will take values in the $\mathfrak{su}(N)_{\mathrm{loop}}$-subalgebra of the affine algebra. The matrix theory Lagrangian then becomes

$$L=\frac{1}{2q}\oint\mathrm{d}\rho_1\mathrm{d}\rho_2\mathrm{Tr}\Big[\nabla_AX^a\nabla_AX_a-\frac{q^2}{2}[X^a,X^b][X_a,X_b]+\theta^T\nabla_A\theta+q\theta^T\Gamma^a[X_a,\theta]\Big]\,,\tag{4.4.48}$$

where the covariant derivative acts as

$$\nabla_A X^a = \partial_0 X^a - \frac{iq}{N}\Big(A_1 R_1 \frac{\partial}{\partial\rho^1} + A_2 R_2 \frac{\partial}{\partial\rho^2}\Big) X^a - q[A_0, X^a]\,. \tag{4.4.49}$$

This can be simplified by a rescaling of the torus coordinates, $\tilde\rho_r = -(N/qR_r)\rho_r$, at the cost of a prefactor $(q/N)^2 R_1 R_2$ in front of the Lagrangian. The Lagrangian exhibits the usual supersymmetries of the original matrix model (4.2.13), with the transformation of the new gauge fields A_1, A_2 trivial. The corresponding supercharges are integrated over the loop group torus,

$$\begin{aligned} Q^+ &= -q^{-1}\oint \mathrm{d}\rho^1 \mathrm{d}\rho^2 \mathrm{Tr}\big[(\nabla_A X_a \Gamma^a + \tfrac{1}{2}[X_a, X_b]\Gamma^{ab})\theta\big]\,,\\ Q^- &= -q^{-1}\oint \mathrm{d}\rho^1 \mathrm{d}\rho^2 \mathrm{Tr}\,\theta\,. \end{aligned} \tag{4.4.50}$$

More important is the extended gauge symmetry of the model (4.4.48), which corresponds to the extended gauge symmetry of the supermembrane action on nontrivial spacesheet topologies. The gauge theory is constructed such that there is a map $\Phi : \Gamma(C^\infty(T^2, \mathfrak{su}(N)) \oplus \mathbb{C}\partial_1 \oplus \mathbb{C}\partial_2, M^1) \longrightarrow \Gamma T\mathcal{F}$ which satisfies $\mathfrak{L}_{\Phi(Y)} = d\alpha$. Writing for such a section of the gauge algebra bundle $Y = Y_0(\rho^1,\rho^2) + Y_1 D_1 + Y_2 D_2$, it acts on the matrix coordinates by

$$\iota_{\Phi(Y)}\delta X^a = [X^a, Y] = [X^a, Y_0] + \frac{i}{N}(R_1 Y_1 \partial_1 + R_2 Y_2 \partial_2) X^a\,, \tag{4.4.51}$$

and $\iota_{\Phi(Y)}\delta\theta^\alpha = [\theta^\alpha, Y]$. It acts on the gauge field A as $\iota_{\Phi(Y)}\delta A = \nabla_A Y$, or equivalently

$$\begin{aligned} \iota_{\Phi(Y)}\delta A_0 &= \partial_0 Y_0 - q[A_0, Y_0] + \frac{iqR_1}{N}(A_1\partial_1 Y_0 - Y_1 \partial_1 A_0)\\ &\qquad + \frac{iqR_2}{N}(A_2\partial_2 Y_0 - Y_2\partial_2 A_0) - \frac{q}{N}(A_1 Y_2 - A_2 Y_1)\mathbf{1}_N\,,\\ \iota_{\Phi(Y)}\delta A_1 &= \partial_0 Y_1\,, \quad \iota_{\Phi(Y)}\delta A_2 = \partial_0 Y_2\,. \end{aligned} \tag{4.4.52}$$

Notice the contribution to the zero mode of the gauge field due to winding of the section of the gauge bundle Y. In the supermembrane theory this contribution is absent, which is again a manifestation of the fact that the correspondence between the matrix theory and membrane theory does not take these constants into account. As usual, the gauge fields are not dynamical and should be taken on-shell when the transition to the Hamiltonian formalism is made. This corresponds to a set of constraints on the other fields, which are nothing but the Euler-Lagrange equations of A. This will amount to the Gauss-law constraint of the uncompactified matrix model, extended by 2 global constraints coming from the extra vector potentials A_1 and A_2. The structure of the symmetry group tells us these fields can only depend on time, so their equations of motion should be integrated over the whole loop group torus,

$$\begin{aligned} \psi &= [\nabla_A X^a, X_a] + [\theta^T, \theta] = 0\,,\\ \psi_r &= \oint \mathrm{d}\rho^1 \mathrm{d}\rho^2 \mathrm{Tr}\big[\nabla_A X^a \partial_r X_a + \theta^T \partial_r \theta\big] = 0\,, \quad r = 1, 2\,. \end{aligned} \tag{4.4.53}$$

where we have rescaled the spinor coordinates by a factor $\sqrt{2}$. The second and third constraints are the matrix-regularised versions of the Noether generators of the area-preserving diffeomorphisms along the two cocycles on the toroidal spacesheet. Note that a membrane theory with compactified longitudinal coordinate X^- is not generated , since this coordinate has no analog in the matrix models (coming from a dimensional reduction of *ten*-dimensional SYM). The constraints above ensure unique time evolution of a quantum mechanical state, governed by the Hamiltonian

$$H = q\oint \mathrm{d}\rho^1 \oint \mathrm{d}\rho^2 \mathrm{Tr}\Big[\frac{1}{2}P^a P_a + \frac{1}{4}[X_a, X_b][X^a, X^b] - \frac{1}{2q}\theta^T\Gamma^a[X_a, \theta]\Big]\,, \tag{4.4.54}$$

where $P^a = q^{-1}\nabla_A X^a$. In conclusion, the harmonic vector fields change the regularisation of the supermembrane drastically, as for each independent spacesheet cocycle a new parameter should be introduced. The question arises whether these vector fields, which are essentially artifacts of the chosen space to model the membrane upon, can have such an impact. The answer lies in the way the original matrix model is 'embedded' in the theory above: upon 2 gauge choices one recovers it! One can simply use the extended gauge symmetry to fix

$$A_1 = A_2 = 0\,. \tag{4.4.55}$$

This fixes the gauge parameters Y_1 and Y_2 in (4.4.52) and reduces the symmetry under the full affine algebra to a symmetry under the loop algebra,

$$\iota_{\Phi(Y)}\delta X^a = [Y_0, X^a]\,, \quad \iota_{\Phi(Y)}\delta A_0 = \partial_0 Y_0 - q[A_0, Y_0]\,. \tag{4.4.56}$$

Implementing this gauge fixing, the Lagrangian remains unchanged but the covariant time derivative reduces to the familiar expression $\nabla_A X^a = \partial_0 X^a - q[A_0, X^a]$. Hence the gauge-fixed Lagrangian posses no longer derivatives in the spatial torus directions, and one can fix the system to any point on the torus to describe the dynamics. More precisely, the extended gauge symmetry (4.4.52) consists of local (torus-position and time dependent) unitary transformations and time dependent translations along the torus. The latter can be used to fix the gauge fields. The gauge-fixed theory possesses a residual symmetry under global (time-independent) translations. However, the absence of terms containing spatial derivatives which do *not* couple to the gauge fields, cause the gauge-fixed Lagrangian to possess an extra local translation symmetry along the torus, $\Pi : \Gamma T(T^2) \longrightarrow \Gamma T\mathcal{F}$ such that for a vector field $\xi(\rho_1, \rho_2)$ on T^2,

$$\iota_{\Pi(\xi)}\delta X^i(\tau,\rho) = \xi^r(\rho)\partial_r X^i(\tau,\rho)\,, \quad \iota_{\Pi(\xi)}\delta A_0(\tau,\rho) = \xi^r(\rho)\partial_r A_0(\tau,\rho)\,, \tag{4.4.57}$$

and similarly it acts on the spinor matrices. A local translation symmetry as above allows a set of gauge-fixing conditions

$$\partial_r X^a = \partial_r \theta^\alpha = \partial_r A_0 = 0\,. \tag{4.4.58}$$

And consequently the gauge-fixed Lagrangian becomes the torus volume times the space-independent integrand, the matrix model Lagrangian. The space-dependent translation invariance is expected to originate from some space-dependent translation symmetry of the Lagrangian (4.4.48). Because all space derivatives are coupled to gauge fields, one would proceed by letting the variations $\delta_{\Pi(\xi)} A_r$ to consist of the correct counter terms to cancel the space derivative variations. However, such variations would be in contradiction with the constraint that these fields are space-independent. Hence we restrict the translational vector field ξ such that no terms have to be canceled at all. This is equivalent to letting Y_1 and Y_2 in the gauge transformations (4.4.52) to be space-dependent too: this will only modify the transformation laws of the fields A_r. The restriction on the (times-dependent) vector field $Y^r\partial_r$ on the torus comes from the preservation of the equations $\partial_1 A_r = \partial_2 A_r = 0$, yielding

$$(\partial_0 + A_1\tilde{\partial}_1 + A_2\tilde{\partial}_2)W_{rs} = 0\,, \quad W_{rs} = \tilde{\partial}_r Y_s\,, \tag{4.4.59}$$

where we have used the rescaled coordinates $\tilde{\rho}_r$. Imposing the gauge-fixing conditions (4.4.55), one quickly sees that this symmetry breaks up in time-independent translations with arbitrary space-dependence, as was claimed above.

4.4.6 $1+1$ Super Yang-Mills Theory from the Toroidal Supermembrane

The embedding of the Lie algebras of Hamiltonian vector field in affine unitary algebras on tori allows a straightforward regularisation method for winding supermembranes. The regularised models will always yield a dimensionally reduced super Yang-Mills theory upon fixing the gauge fields. Let us start with the single-wrapped toroidal membrane; we shall restrict ourselves to configurations where the membrane coordinate X^9 winds w_1 times around the first homology loop. The regularised theory will be the torus matrix model of the previous paragraph, with the exception of the ninth coordinate, which is imposed to be of the form

$$X^9 \longrightarrow w_1 D_1 + B\,, \quad B = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} X^9_{\boldsymbol{m}} e^{im_1\rho_1/R_1} e^{im_2\rho_2/R_2} T_{\boldsymbol{m} \bmod (N)}\,. \tag{4.4.60}$$

The APD gauge theory action is regularised to

$$S = \frac{1}{2q}\int_{M^1} \mathrm{d}\tau \oint \mathrm{d}\rho^1 \oint \mathrm{d}\rho^2 \mathrm{Tr}\Big[(F_{AB})^2 + \nabla_A X^i \nabla_A X_i - \nabla_B X^i \nabla_B X_i - \frac{q^2}{2}[X^i, X^j][X_i, X_j] \\ + \theta^T(\nabla_A - \Gamma^9\nabla_B)\theta + q\theta^T\Gamma^i[X_i,\theta]\Big]\,, \tag{4.4.61}$$

where the covariant derivative ∇_A is given by (4.4.49) and we have replaced

$$\nabla_B X_i = q[X_i, X^9] = -\frac{iR_1 w_1}{N}\frac{\partial}{\partial\rho^1}X_i + [B, X^i]$$

$$F_{AB} = \nabla_A X^9 = \partial_0 B - \frac{iq}{N}\Big(A_1 R_1\frac{\partial}{\partial\rho^1} + A_2 R_2 \frac{\partial}{\partial\rho^2}\Big)B - q[A_0, B] - \frac{iw_1 q R_1}{N}\partial_1 A_0 + \frac{qA_2 w_1}{N}\mathbf{1}_N\,.$$

The winding data is contained in the flux integral

$$\oint \mathrm{d}\rho^1 \oint \mathrm{d}\rho^2 \mathrm{Tr}[F_{AB}] = 4\pi^2 R_1 R_2 q A_2 w_1\,, \tag{4.4.62}$$

which gives a nonzero contribution to the total momenta. Except for this contribution, there are no other central charges which arise from compactifying a single matrix coordinate. In particular, the mass spectrum is not raised by these winding configurations. The Lagrangian above exhibits gauge invariance (4.4.52), where the B-field transforms as

$$\iota_{\Phi(Y)}\delta B = -\nabla_B Y_0 - \frac{iR_1}{N}Y_1\partial_1 B - \frac{w_1}{N}Y_1\,. \tag{4.4.63}$$

Examining the local translational symmetries along the torus, one observes that the term $(\nabla_B X)^2$-term breaks this symmetry down to local translations depending only on the ρ^2-coordinate, satisfying

$$(\partial_0 + A^2\tilde{\partial}_2)\tilde{\partial}_2 Y_r = 0\,. \tag{4.4.64}$$

Performing the gauge-fixing $A_1 = A_2 = 0$, the gauge symmetry breaks up in ρ^2-dependent translations and (τ, ρ^1, ρ^2)-dependent unitary transformations. The local translational invariance now only allows the gauge fixing

$$\partial_2 X^i = \partial_2\theta^\alpha = \partial_2 A_0 = \partial_2 B = 0\,, \tag{4.4.65}$$

which further reduces the gauge group to (τ, ρ_1)-dependent unitary transformations. The coordinate ρ^2 can be integrated out of the Lagrangian, giving a factor $2\pi R_2$ and a single integral; what remains is (after some rescalings) exactly the matrix string Lagrangian (4.4.35).

4.4.7 $1+2$ Super Yang-Mills Theory from the Toroidal Supermembrane

As an introduction to the general case, consider the doubly-wrapped supermembrane. First suppose that the ninth coordinate wraps around both homology cycles of the torus with winding numbers w_1 and w_2. The matrix regularisation then becomes $X^9 \longrightarrow w_1 D_1 + w_2 D_2 + B$. This system is equivalent to the winding configuration above upon a linear transformation on the torus

$$\begin{pmatrix} \tilde{\rho}^1 \\ \tilde{\rho}^2 \end{pmatrix} = \begin{pmatrix} w_1^{-1} & -\frac{R_2 w_2}{R_1 w_1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \rho^1 \\ \rho^2 \end{pmatrix}\,, \tag{4.4.66}$$

provided $w_1 \neq 0$. If so, one simply interchanges the torus coordinates and will end up with matrix string theory on a circle of radius R_2. Note that with such a linear transformation comes a Jacobian in front of the action and a redefinition of the gauge fields A_1 and A_2. On the other hand, suppose both the eighth and ninth coordinate wrap around the first homology cycle: $X^8 \longrightarrow w_1^8 D_1 + C$ and $X^9 \longrightarrow w_1^9 D_1 + B$. Now let O denote the element in the transverse galilean group (which is a symmetry of the action) which acts by a rotation in the $X^8 - X^9$-plane over an angle θ which is determined by

$$\begin{pmatrix} \sqrt{(w_1^8)^2 + (w_1^9)^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} w_1^8 \\ w_1^9 \end{pmatrix} . \tag{4.4.67}$$

The resulting theory is equivalent to the matrix string, with winding number $\sqrt{(w_1^8)^2 + (w_1^9)^2}$ (this factor can then be scaled away). Finally, let us consider the general case

$$\begin{pmatrix} X^8 \\ X^9 \end{pmatrix} = W \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \begin{pmatrix} B \\ C \end{pmatrix} , \tag{4.4.68}$$

where W is a 2×2-matrix with integer entries. We can act independently from the left on W with orthogonal transformations embedded in the super-Galilean group and from the right with linear transformations on the loop algebra torus. This allows us to restrict ourselves to two classes of winding configurations: if $\det(W) = 0$ (as was the case for the two examples above), we can diagonalise W to the form $\mathrm{diag}(\lambda, 0)$, where λ may be assumed real and bigger than zero (as we shall see below): the resulting regularised theory becomes the (untwisted) matrix string. If W is invertible, we can diagonalise it to $\mathrm{diag}(\lambda_1, \lambda_2)$ where $\lambda_1 \geq \lambda_2 > 0$ and the regularised theory changes fundamentally; we call such configurations *irreducible*. The matrix model Lagrangian becomes

$$\begin{aligned} L = \frac{1}{2q} \oint \mathrm{d}\rho^1 \oint \mathrm{d}\rho^2 \mathrm{Tr}\Big[& (F_{AB})^2 + (F_{AC})^2 - (F_{BC})^2 + (\nabla_A X^i)^2 \\ & - (\nabla_B X^i)^2 - (\nabla_C X^i)^2 - \frac{q^2}{2}([X^i, X^j])^2 \\ & + \theta^T (\nabla_A - \Gamma^8 \nabla_B - \Gamma^9 \nabla_C)\theta + q\theta^T \Gamma^i [X_i, \theta]\Big] , \end{aligned} \tag{4.4.69}$$

where now i runs from 1 to 7. The covariant derivatives are given by

$$\nabla_B X^i = \frac{iqR_1\lambda_1}{N}\partial_1 + q[B, X^i] , \quad \nabla_C X^i = \frac{iqR_2\lambda_2}{N}\partial_2 + q[C, X^i] , \tag{4.4.70}$$

and the curvatures are

$$\begin{aligned} F_{AB} &= \partial_0 B - \frac{iq}{N}(A_1 R_1 \partial_1 + A_2 R_2 \partial_2)B - q[A_0, B] + \frac{iqR_1\lambda_1}{N}\partial_1 A_0 + \frac{q\lambda_1}{N} A_2 \mathbf{1}_N , \\ F_{AC} &= \partial_0 C - \frac{iq}{N}(A_1 R_1 \partial_1 + A_2 R_2 \partial_2)C - q[A_0, C] + \frac{iqR_2\lambda_2}{N}\partial_2 A_0 - \frac{q\lambda_2}{N} A_1 \mathbf{1}_N , \\ F_{BC} &= -\frac{iq}{N}(\lambda_1 R_1 \partial_1 C - \lambda_2 R_2 \partial_2 B) + q[B, C] + \frac{q\lambda_1\lambda_2}{N}\mathbf{1}_N . \end{aligned} \tag{4.4.71}$$

The transition to the Hamiltonian formalism is made by substituting P^8 by $q^{-1}F_{AB}$ and P^9 by $q^{-1}F_{AC}$. The Gauss constraint receives a Bianchi-identity contribution,

$$\psi = q[P^i, X_i] + [\theta^T, \theta] - \nabla_B F_{AB} - \nabla_C F_{AC} , \tag{4.4.72}$$

and the global constraints ψ_r receive contributions which are proportional to the components of the Yang-Mills field,

$$\psi_r = 2\pi^2 i \Big(\sum_{s=1,2} \varepsilon_{rs} R_s \lambda_s^2 \Big) A_r + \oint \mathrm{d}\rho^1 \oint \mathrm{d}\rho^2 \mathrm{Tr}\big[F_{AB}\partial_r B + F_{AC}\partial_r C + qP^i \partial_r X_i + \theta^T \partial_r \theta\big] . \tag{4.4.73}$$

The Hamiltonian takes the form

$$H_\tau = \frac{1}{2q}\oint \mathrm{d}\rho^1 \oint \mathrm{d}\rho^2 \mathrm{Tr}\Big[q^2(P^i)^2 + (\nabla_B X^i)^2 + (\nabla_C X^i)^2 + \frac{q^2}{2}([X^i, X^j])^2 - q\theta^T\Gamma^i[X_i, \theta] + \theta^T(\Gamma^8\nabla_B + \Gamma^9\nabla_C)\theta + (F_{BC})^2\Big]\,. \tag{4.4.74}$$

The model above is nothing but super Yang-Mills theory dimensionally reduced to a torus; to see this first break time-dependent translational invariance to fix the gauge

$$A_1 = A_2 = 0\,. \tag{4.4.75}$$

The residual translation invariance is global (independent of the coordinates (ρ^1, ρ^2)) because of the covariant spatial derivatives in the Lagrangian. Subsequently rescale the torus coordinates $\rho^r \mapsto (N/qR_r\lambda_r)\rho^r$ and construct the vector field A_μ, $\mu = 0, 1, 2$ from the gauge fields by setting $A_1 = B$ and $A_2 = C$. After these substitutions one obtains SYM reduced to the loop algebra torus, with an additional constant term in the spacelike curvature F_{BC}, resulting in a constant magnetic flux,

$$\oint_{T^2} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \mathrm{Tr}[F_{\mu\nu}] = 4\pi^2 q R_1 R_2 \lambda_1\lambda_2\,. \tag{4.4.76}$$

This nonzero constant is a direct consequence of the irreducibility of the winding configuration. It is nothing but the central charge in the supersymmetry algebra, in accordance with the membrane's supersymmetry algebra bracket

$$\{Q^+_\alpha, Q^-_\beta\}_D = -(\Gamma^a\Gamma^+\Gamma^-) - 4\pi^2 R_1 R_2(\Gamma^{89}\Gamma^+\Gamma^-)\lambda_1\lambda_2\,, \tag{4.4.77}$$

where R_1 and R_2 are the compactification radii in the eighth and ninth direction of the target space. Another essential feature of irreducible winding is the appearance of an energy gap in the spectrum. The flat valleys of the classical potential of the transverse bosonic variables correspond to directions in the Cartan subalgebra of the gauge algebra. For the non-winding membrane this it was spanned by a set of Fourier modes depending on a single linear combination of the two spacesheet coordinates and for the D-particle Lagrangian the set of diagonal matrices. In the regularised theory above, the potential is modified with spatial covariant derivatives, which act nontrivially on different modes in the Cartan subalgebra, causing the flat valleys to disappear (at the classical level). Take for example the set $L_{(m,0)}$, $m \in \mathbb{Z}$ as a maximal commutative torus in the Poisson algebra on the membrane spacesheet. These correspond the affine $\mathfrak{su}(N)$ elements $e^{im\rho^1/R_1} \otimes T_{(m \bmod N, 0)}$. Although the Lie bracket in the loop algebra vanishes as it is proportional to the sine of the outer product of the indices, the space-like covariant derivative $(\nabla_B X^i)^2$ in the Hamiltonian (4.4.74) contributes terms

$$\frac{q^3}{2N}\lambda_1^2 \sum_{p,q\in\mathbb{Z}, n\in E_N} (X^i)^{pN+n}(X_i)^{pN-n}(pN+n)(pN-n)e^{i(p+q)N\rho^1/R_1}\,, \tag{4.4.78}$$

where $E_N = \{0, 1, \ldots, N-1\}$ and $(X^i)^n$ are the Fourier components of the i-th matrix variable w.r.t. the first coordinate. The potential is therefore seen to depend explicitly on these components, and obviously no other choice of Cartan subalgebra removes this dependence. Using this argument it is shown in [89] that all the modes get confined, i.e. the growth of zero-energy density spikes is suppressed and the spectrum of the supermembrane becomes discrete. In [90] this result was obtained by geometrical methods.

The results above allow a straightforward generalisation to membranes based on spacesheets of arbitrary genera, with any number of embedding coordinates wrapping around the homology cycles. Suppose g is the genus of Σ and let W denote the integer-valued $(9 \times 2g)$-matrix capturing the winding information,

$$\oint_{C_\lambda} dX^a = 2\pi R^a W^{a\lambda}\,, \tag{4.4.79}$$

where no summation over a is understood. Because each (co-)homology cycle represents an outer derivation on the Poisson algebra, this system is expected to allow a regularisation to $C^\infty(T^{2g}, \mathfrak{su}(N)) \oplus \mathbb{C}\partial_1 \ldots \oplus \mathbb{C}\partial_{2g}$, where the coordinates take the form

$$X^a \longrightarrow W^{a\lambda} D_\lambda + B^a \,. \tag{4.4.80}$$

where the D_λ are constructed from the canonical derivations ∂_λ such that the harmonic part of the Lie algebra of Hamiltonian vector fields is generated:

$$[D_\lambda, T_M] = f_{\lambda M}{}^N T_N \,, \qquad [D_\lambda, D_{\lambda'}] = P_{\lambda\lambda'} = f_{\lambda\lambda'}{}^M T_M \,, \tag{4.4.81}$$

where T_M are the generators of the unitary algebra spanned by the Toeplitz operators on the Riemann surface and $P_{\lambda\lambda}$ is the matrix determined by the Toeplitz operator corresponding to the hamiltonian vector field $\Phi_{\lambda\lambda'} = [\phi_\lambda, \phi_{\lambda'}]$. To simplify the theory we have orthogonal transformations acting on X^a, and therefore acting from the left on W and invertible transformations on the $2g$-dimensional torus, acting from the right on the winding matrix. The following well-known theorem [91] provides this simplification,

Theorem (Singular Value Decomposition) 4.5 *Let A be a real $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix of the form*

$$X = \begin{pmatrix} D & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix} \quad \textit{with} \quad D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_r \end{pmatrix} , \tag{4.4.82}$$

where all λ_i are real and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ and there exist an $m \times m$ orthogonal matrix U and an $n \times n$ orthogonal matrix V such that

$$A = UXV^T \,. \tag{4.4.83}$$

So if $r = \mathrm{rank}(W)$ we can apply a Lorentz transformation V on the embedding coordinates and a rotation in the covering space of the $2g$-dimensional torus to obtain a regularisation

$$\begin{aligned} X^a &\longrightarrow \lambda_a D_a + B_a \,, && 1 \leq a \leq r \leq 9 \,, \\ X^a &\longrightarrow X^a \,, && r < a \leq 9 \,, \\ A &\longrightarrow A^\lambda D_\lambda + A_0 \,. \end{aligned} \tag{4.4.84}$$

since all the λ_i are nonzero, we can scale the loop algebra torus and put them equal to one. Note that these simplifications can already be done at the level of the supermembrane by performing a Lorentz transformation on the embedding coordinates and change of basis vectors in the harmonic sector of the Lie algebra of Hamiltonian vector fields. Furthermore, the affine generators D_λ may be replaced by the usual partial derivatives because the C_λ fields may be absorbed in the gauge fields by redefining

$$B_a \mapsto B_a + C_a \,, \quad A_0 \mapsto A_0 + A^\lambda C_\lambda \,. \tag{4.4.85}$$

The matrix model Lagrangian becomes

$$\begin{aligned} L = \Big(\frac{N}{q}\Big)^{2g} (2qR_1 R_2 \ldots R_{2g})^{-1} \oint_{T^{2g}} \mathrm{d}^{2g}\rho \mathrm{Tr}\Big[&- \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - (\nabla_\mu X^i)(\nabla^\mu X_i) - \frac{q^2}{2}([X^i, X^j])^2 \\ &- \theta^T \Gamma^\mu \nabla_\mu \theta + q\theta^T \Gamma^i [X_i, \theta]\Big] \,, \end{aligned} \tag{4.4.86}$$

where $\mu = 0, 1, \ldots r$ and $i = r+1, r+2, \ldots, 9$ and T^{2g} is the torus with all radii equal to $2\pi N/q$. The summation over μ is with respect to the metric in r-dimensional Minkowski space with signature $(- + + \ldots +)$. The covariant derivatives are given by

$$\nabla_0 X^i = \partial_0 X^i - iA^\lambda \partial_\lambda X^i - q[A_0, X^i] \,, \quad \nabla_a X^i = i\partial_a X^i + q[B_a, X^i] \,, \tag{4.4.87}$$

where a runs from 1 to r. The curvatures are

$$F_{0a} = \partial_0 B_a + i\partial_a A_0 + q[A_0, B_a] + iA^\lambda \partial_\lambda B_a\,, \quad F_{ab} = i\partial_{[a} B_{b]} + q[B_a, B_b]\,. \tag{4.4.88}$$

Now one uses the $SU(N)$ gauge symmetry on the torus to fix $A^\lambda = 0$. The resulting theory possesses local translation invariance in the directions $\rho^r, \rho^{r+1}, \ldots, \rho^{2g}$. This allows us to take all the fields independent of these coordinates and integrate them out. We obtain

$$L \propto \oint_{iT^r} \mathrm{d}^r \rho \mathrm{Tr}\Big[-\frac{1}{2}(F_{\mu\nu})^2 - (\nabla_\mu X^i)^2 - \frac{q^2}{2}([X^i, X^j])^2 - \theta^T \Gamma^\mu \nabla_\mu \theta + q\theta^T \Gamma^i [X_i, \theta]\Big]\,, \tag{4.4.89}$$

where iT^r denotes the imaginary r-dimensional torus with radii $2\pi i N/q$ and the connections and curvatures are the usual Yang-Mills expressions,

$$\nabla_\mu X^i = \partial_\mu + q[B_\mu, X^i]\,, \quad F_{\mu\nu} = \partial_{[\mu} B_{\nu]} + q[B_\mu, B_\nu]\,. \tag{4.4.90}$$

So after some gauge fixing the theory of a rank-r wrapped supermembrane reduces to super-Yang-Mills theory reduced to an r-dimensional imaginary torus. Note that we did not need the explicit function basis and Poisson algebra deformation on the Riemannn surface to establish this result, due to the following three results: (i) all the Hamiltonian vector fields represent independent derivations not cohomologous to zero or each other, and extending the loop group $C^\infty(T^k, \mathfrak{su}(N))$ has k-dimensional second cohomology space, (ii) the general expression of the regularised Hamiltonian vector field is by previous statement one of the $2g$ outer derivations on $C^\infty(T^{2g}, \mathfrak{su}(N))$ plus some inner derivation, but the latter are irrelevant since they can always be absorbed into the gauge fields, (iii) the Hamiltonian vector field only give rise to propagating degrees of freedom along the torus if embedding coordinates wind around their respective homology cycles, and consequently the gauge-fixed action is SYM reduced to a space which is determined by the rank of the winding, as is explained above.

4.4.8 From Membranes to the Twisted Sectors

Finally we discuss how the regularised membrane is embedded in the twisted sectors of the gauge theories above. Let us again start with the toroidal case $\Sigma = T^2$, governed by the generalised torus star-commutator algebra

$$[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}]_* = 2\sin\Big(\frac{2\pi M}{N} \boldsymbol{m} \times \boldsymbol{n}\Big) L_{\boldsymbol{m}+\boldsymbol{n}}\,, \quad [\tilde{\sigma}^r, L_{\boldsymbol{m}}]_* = \frac{m_r}{N} L_{\boldsymbol{m}}\,, \quad [\tilde{\sigma}^1, \tilde{\sigma}^2]_* = \frac{1}{N}\,. \tag{4.4.91}$$

There exists an embedding of this algebra into the twisted sector of affine $\mathfrak{su}(N)$. By this we mean the space of functions $f : \mathbb{R}^2 \longrightarrow \mathfrak{su}(N)$ which obey twisted boundary conditions:

$$f(\rho^1 + 2\pi R_1, \rho^2) = \mathrm{Ad}_U(f(\rho^1, \rho^2))\,, \quad f(\rho^1, \rho^2 + 2\pi R_2) = \mathrm{Ad}_V(f(\rho^1, \rho^2))\,, \tag{4.4.92}$$

for some $U, V \in SU(N)$, which may depend on f. We have already seen that these mappings, together with the derivatives ∂_r, $r = 1, 2$ meet the compactification conditions of the matrix model. Obviously there is a subalgebra which, together with the derivatives, is isomorphic to the deformed membrane algebra, namely the untwisted mappings (which have $U = V = \mathbf{1}$). With the new sectors there appear more subalgebras which fulfill this property:

$$\begin{aligned} L_{\boldsymbol{m}} &\longrightarrow \exp\Big(\frac{iM}{N}\big(m_1 \frac{\rho^1}{R_1} - m_2 \frac{\rho^2}{R_2}\big)\Big) \otimes \omega^{s_1 s_2/2} \Omega_1^{s_1} \Omega_2^{s_2}\,, \quad s_r = m_r \bmod N\,, \\ \tilde{\sigma}^r &\longrightarrow D_r = -\frac{iR_r}{M} + iV_{rs}\rho^s \otimes \mathbf{1}_N\,, \end{aligned} \tag{4.4.93}$$

where M and N are required to be co-prime, $\omega = \exp(2\pi i M/N)$ and $\Omega_2 = \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{N-1})$. The matrix V_{rs} may be taken off-diagonal with entries laying on the real line

$$R_1 V_{12} + R_2 V_{21} = \frac{M}{N}\,. \tag{4.4.94}$$

The periodicity matrices U and V mentioned above are just 't Hooft's twist matrices:

$$
\begin{aligned}
L_{\boldsymbol{m}}(\rho^1 + 2\pi R_1, \rho^2) &= \Omega_1 L_{\boldsymbol{m}}(\rho^1, \rho^2)\Omega_1^\dagger\,, \\
L_{\boldsymbol{m}}(\rho^1, \rho^2 + 2\pi R_2) &= \Omega_2 L_{\boldsymbol{m}}(\rho^1, \rho^2)\Omega_2^\dagger\,.
\end{aligned}
\tag{4.4.95}
$$

The twisted configurations above may be viewed as the 'N/M-th root' of untwisted ones, as $L_{\boldsymbol{m}}(\rho^1 + 2\pi N R_1/M, \rho^2) = L_{\boldsymbol{m}}(\rho^1, \rho^2 + 2\pi N R_2/M) = L_{\boldsymbol{m}}(\rho^1, \rho^2)$. Recall the conclusion of the deformation theory of the Poisson algebra on the torus: for each rational value of the deformation parameter there was a Lie algebra homomorphism to $\mathfrak{su}(N) \oplus \mathfrak{u}(1)$, where N was the denominator of this rational value. The mappings above generalise this result to the full algebra of Hamiltonian vector fields: deforming the subalgebra of gradient vector fields yields a Lie algebra homomorphism for each rational value of the deformation parameter to twisted affine $\mathfrak{su}(N)$. The number N, the denominator of the parameter, is also the maximal number of twists a mode can make, but this is a purely algebraic result (it is the order of the Weil group, the permutation group of the N eigenvalues).

One can now use the identifications above to build matrix models from membrane theory on the torus. Without winding, the twisting is irrelevant as all the (ρ^1, ρ^2)-dependence drops out. If one (or more) embedding coordinates wind around one of the homology cycles[15] the winding matrix rank is one, and one obtains matrix string theory, including its twisted sectors generating multiple D-strings with various lengths. Finally if there is irreducible double wrapping, one obtains super-Yang-Mills reduced to a torus with twisted configurations, which is interpreted as an ensemble of toroidal D-membranes of various areas. The generalisation to supermembranes based on spacesheets of higher genus again lacks an explicit expression of the regularised modes. We expect however a g-fold tensor product of the twisted matrix representations above, which allows a mapping to all the twisted sectors of the Dp-brane theories for $p \leq 10$. It seems that the maximal twisting number may depend on the direction on the dual torus, which indicates a deformation of the Poisson algebra with g independent parameters $t_1, \ldots, t_g$, which reduces to twisted affine unitary algebras if all the parameters are rational.

[15] or vice versa, one embedding coordinate winds around both cycles

5 Conclusion and Outlook

For the lightcone gauge-fixed Green-Schwartz supermembrane [53] based on a toroidal spacesheet, the regularisation procedure is explicitly calculable and discussed in the thesis; there is a consistent mathematical procedure, based on a deformation of the Poisson algebra, together with a series of projections onto $\mathfrak{su}(N)$, regularising the theory to the corresponding matrix models. Because only the Poisson algebra is approximated, which is a central extension the ideal in the Lie algebra of Hamiltonian vector fields spanned by the gradient vector fields, the remaining generators have no place in this procedure. These additional vector fields, originating from the nontrivial topology of the spacesheet, are called harmonic vectors and they represent the outer derivations on the Poisson algebra. Here appears the underlying reason why they cannot be regularised within a finite N matrix model: the unitary algebras possess no outer derivations.

Super-Yang-Mills theory, dimensionally reduced from ten-dimensional Minkowski space to the time line is a regularisation of the membrane in a particular gauge, such that the freedom generated by the harmonic vector fields is fixed. This is a well-defined mapping of field theories as long as the target space directions are noncompact. If there are circular directions in the embedding space, coordinates may wind around them several times as they run along a noncontractible loop on the spacesheet. The 'embedding' coordinates of the membrane transform under the adjoint representation of their symplectic gradient; for winding mappings into compactified target spaces, the gradient is nontrivially decomposed into a gradient of an ordinary, real-valued function plus the winding numbers times the respective harmonic vector fields. This poses a fundamental problem for regularising wrapped supermembranes: variables take values in the Lie algebra which cannot be approximated.

The solution comes from wrapping the matrix model; this is done upon postulating certain periodicity conditions on the matrix variables [83], which turn out to be solvable only if one introduces extra continuous variables, replacing $\mathfrak{su}(N)$ by the loop algebra $C^\infty(T^2, \mathfrak{su}(N))$. The wrapped matrix variables turn out to be represented by the independent outer derivations on this algebra, the derivatives along the loop algebra torus T^2. As such, they exactly fulfill the requirements a regularised Hamiltonian vector field should (so there is no deformation of the brackets involving these vector fields). Another, perhaps more instructive way to compactify the matrix model to a higher dimension is by taking the $N \longrightarrow \infty$ limit of the $SU(N)$ matrix model and dividing the matrices into blocks of size $n \times n$. As such, one arrives at an interacting system of ensembles of D-particles, which is the T-dual of a system of type IIA matrix strings. Imposing certain periodicity conditions on the blocks yields the Fourier transform of the matrix string model [86].

The appearance of derivatives along the compactification torus allows us to regularise the supermembrane in noncompact target space without fixing the gauge freedom represented by the Hamiltonian vector fields. This is established by requiring only the gauge fields representing this symmetry to be compactified. The result is, as expected, equal to super Yang-Mills dimensionally reduced to a point, upon a gauge fixing. This is because the derivatives along the torus are all multiplied by gauge fields in the Lagrangian; subsequently one can fix the gauge and put these gauge fields zero, such that the action contains no longer propagating degrees of freedom along the torus, making it equivalent to a dimensional reduction to a single point on it. Secondly, the identification of the Hamiltonian vector fields makes it possible to regularise the theory constrained by arbitrary winding configurations. Here the rank of the winding matrix determines the effective dimension of the reduced super-Yang-Mills theory, as well as the possible appearance of a mass gap in and the discreteness of the spectrum of the supermembrane [89, 90]. It should be noted that the mapping from the gradient vector fields into the loop algebra is by no means surjective: only Fourier modes of which the wave numbers and the powers of the 't Hooft twist matrices differ by multiples of N are generated. In this sense matrix model states may be used to approximate membrane states, but not vice versa.

For membranes based on spacesheets of higher genus, an explicit deformation of the algebra of Hamiltonian vector fields is not known. There is however an axiomatical approach, using the Bergman kernel to associate to each function a unitary operators on a Hilbert space of automorphic forms [78]. This can be viewed as a discrete sequence of deformations of the Poisson algebras, where the parameter takes a countable number of values corresponding to the weights of these automorphic forms. This may be embedded in a deformation with a continuous parameter by considering the spaces of automorphic forms on a particular covering of the spacesheet, which by non-compactness allows automorphic forms of arbitrary real weights [80]. If the deformation parameter takes integer values m, the deformation corresponds to the operator algebra of covariant derivatives w.r.t. the Hamiltonian vector fields, acting on the holomorphic sections of the m-fold tensor product of a Hermitian line bundle on the Riemann surface [65, 79]. Applied to the Poisson algebra on the torus, it was shown that this regularisation method reduces to the noncommutative deformation of the torus [67]. Without knowing the explicit regularisation, we can already state that the genus-g spacesheet with harmonic vector fields should be regularised to a matrix theory on a $2g$-dimensional torus. Again, the rank of the winding matrix determines possible reduction of this theory. This rank is at most 9 (as there are 9 transverse bosonic membrane coordinates), so a membrane based on a spacesheet of genus greater than 9 with irreducible winding regularises to ordinary 9-dimensional super-Yang-Mills theory.

It remains to be investigated what happens in the arbitrary-genus situation. The difficulty here is the absence of a Fourier analysis on arbitrary Riemann surfaces. The deformation on the torus can be induced by replacing the pointwise product of functions by the Moyal star product [67]. This leads to the consideration of gauge theories on noncommutative Riemann surfaces [92], which for certain rational values of the noncommutativity parameter reduce to (twisted) matrix models. Another outstanding problem is the application of the regularisation procedure to membranes in curved superspaces [47]. Although under certain conditions on the background supergravity fields the system can still be described by a gauge theory of area-preserving diffeomorphisms [47, 93], many problems appear, such as the explicit dependence of the Lagrangian on the longitudinal coordinate X^-, and it is not known how matrix regularisation can be reconciled with general target space covariance. These problems are closely related to the ones encountered when one tries to compactify the matrix model to curved spaces [94, 95], as opposed to flat tori.

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