Digital Parametric Equalizer Design With Prescribed Nyquist-Frequency Gain[†]

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Abstract

A new type of second-order digital parametric equalizer is proposed whose frequency response matches closely that of its analog counterpart throughout the Nyquist interval and does not suffer from the prewarping effect of the bilinear transformation near the Nyquist frequency. Closed-form design equations and direct-form and lattice realizations are derived.

1. Introduction

Conventional bilinear-transformation-based methods of designing second-order digital parametric equalizers [1–11] result in frequency responses that fall off faster than the corresponding analog equalizers near the Nyquist frequency due to the prewarping nature of the bilinear transformation. This effect becomes particularly noticeable when the peak frequencies and widths are relatively high. Figure 1 illustrates this effect.

In this paper, we introduce an additional degree of freedom into the design, namely, the gain at the Nyquist frequency, and derive a new class of digital parametric equalizers that closely match their analog counterparts over the entire Nyquist interval and do not suffer from the prewarping effect of the bilinear transformation.

The design specifications are the quantities $\{f_s, f_0, \Delta f, G_0, G_1, G, G_B\}$, namely, the sampling rate f_s, the boost/cut peak frequency f_0, the bandwidth Δf, the reference gain G_0 at DC, the gain G_1 at the Nyquist frequency $f_s/2$, the boost/cut peak gain G at f_0, and the bandwidth gain G_B (that is, the level at which the bandwidth Δf is measured.)

All previous methods of designing second-order equalizers assume $G_1 = G_0$ (usually set equal to unity.) In these methods, the bilinear transformation is used to transform an analog equalizer with equivalent specifications into the digital one. As remarked by Bristow-Johnson [9], all of these designs are essentially equivalent to each other, up to a different definition of the bandwidth Δf and bandwidth gain G_B. For the equivalent analog equalizer, the quantity $G_0 = G_1$ represents the gain at DC and at infinity, with the latter being mapped onto the Nyquist frequency $f_s/2$ by the bilinear transformation.

In the method proposed here, we allow G_1 to be different from G_0. In particular, we set G_1 equal to the gain an analog equalizer would have at $f_s/2$ if it were not bilinearly transformed. This condition on G_1, together with the requirements that the gain at DC be G_0, that there be a peak maximum (or minimum) at f_0, that the peak gain be G, and that the bandwidth be Δf at level G_B, provide five constraints that fix uniquely the five coefficients of the second-order digital filter.

The resulting digital filter matches the corresponding analog filter as much as possible, given that there are only five parameters to adjust. The matching is exact at $f = 0, f_0, f_s/2$, and the two filters have the same bandwidth Δf. These design goals are illustrated in Fig. 2.

[†]Presented at the 101st AES Convention, Los Angeles, November 1996, and published in JAES, vol.45, p.444, June 1997.

Thus, such a digital equalizer can be used to better emulate the sound quality achieved by an analog equalizer. This is the main motivation of this paper. Moreover, setting $G_0 = 0$, we also obtain more realistic modeling of resonant filters of prescribed peaks and widths for use in music and speech synthesis applications.

In the following sections, we summarize the conventional analog and digital equalizer designs, present the new design and some simulations, and discuss direct and lattice form realizations, and the issue of bandwidth. We also give a small MATLAB function for the new design.

2. Conventional Analog and Digital Equalizers

Here, we review briefly the design of analog and digital equalizers, following the discussion of Ref. [11]. A second-order analog equalizer with gain G_0 at DC and at infinity has transfer function:

$$H(s) = \frac{G_0 s^2 + Bs + G_0\Omega_0^2}{s^2 + As + \Omega_0^2} \tag{1}$$

and magnitude response:

$$|H(\Omega)|^2 = \frac{G_0^2(\Omega^2 - \Omega_0^2)^2 + B^2\Omega^2}{(\Omega^2 - \Omega_0^2)^2 + A^2\Omega^2} \tag{2}$$

where $\Omega = 2\pi f$ is the physical frequency in rads/sec and $\Omega_0 = 2\pi f_0$ the peak frequency. The filter coefficients A and B are fixed by the two requirements that the gain be G at Ω_0 and that the bandwidth be measured at level G_B. These requirements can be stated as follows:

$$|H(\Omega_0)|^2 = G^2, \qquad |H(\Omega)|^2 = G_B^2 \tag{3}$$

where the solutions of the second equation are the right and left bandedge frequencies, say Ω_2 and Ω_1. They satisfy the geometric-mean property:

$$\Omega_1\Omega_2 = \Omega_0^2 \tag{4}$$

Defining the bandwidth $\Delta\Omega = 2\pi\Delta f$ as the difference of the bandedge frequencies, $\Delta\Omega = \Omega_2 - \Omega_1$, the two conditions in Eq. (3) determine the filter coefficients as follows:

$$A = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\Delta\Omega\,, \qquad B = GA \tag{5}$$

The equalizer's gain at a desired Nyquist frequency $f_s/2$ can be obtained by evaluating Eq. (2) at $\Omega_s = 2\pi(f_s/2) = \pi f_s$, giving:

$$G_1^2 = \frac{G_0^2(\Omega_s^2 - \Omega_0^2)^2 + B^2\Omega_s^2}{(\Omega_s^2 - \Omega_0^2)^2 + A^2\Omega_s^2} \tag{6}$$

A digital equalizer can be designed by applying the bilinear transformation to an equivalent analog filter of the form of Eq. (1). The bilinear transformation is defined here as:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}\,, \qquad \Omega = \tan\left(\frac{\omega}{2}\right), \qquad \omega = \frac{2\pi f}{f_s} \tag{7}$$

where Ω is now the prewarped version of the physical frequency ω. The physical peak and bandwidth frequencies are in units of radians/sample:

$$\omega_0 = \frac{2\pi f_0}{f_s}\,, \qquad \Delta\omega = \frac{2\pi\Delta f}{f_s} \tag{8}$$

The prewarped versions of the peak and bandedge frequencies are $\Omega_0 = \tan(\omega_0/2)$, $\Omega_1 = \tan(\omega_1/2)$, and $\Omega_2 = \tan(\omega_2/2)$. They satisfy the prewarped geometric-mean property:

$$\tan\left(\frac{\omega_1}{2}\right)\tan\left(\frac{\omega_2}{2}\right) = \tan^2\left(\frac{\omega_0}{2}\right) \tag{9}$$

and the following relationship between the physical bandwidth $\Delta\omega = \omega_2 - \omega_1$ and its prewarped version $\Delta\Omega = \Omega_2 - \Omega_1$:

$$\Delta\Omega = (1 + \Omega_0^2)\tan\left(\frac{\Delta\omega}{2}\right) \tag{10}$$

Replacing s by its bilinear transformation in Eq. (1), gives after some algebraic simplifications the digital transfer function:

$$H(z) = \frac{\left(\dfrac{G_0 + G\beta}{1+\beta}\right) - 2\left(\dfrac{G_0\cos\omega_0}{1+\beta}\right)z^{-1} + \left(\dfrac{G_0 - G\beta}{1+\beta}\right)z^{-2}}{1 - 2\left(\dfrac{\cos\omega_0}{1+\beta}\right)z^{-1} + \left(\dfrac{1-\beta}{1+\beta}\right)z^{-2}} \tag{11}$$

where the parameter β is given by

$$\beta = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\tan\left(\frac{\Delta\omega}{2}\right) \tag{12}$$

By design, the gain of this digital filter at the Nyquist frequency is equal to G_0, whereas that of a physical analog filter is G_1, as given by Eq. (6). This can be seen directly from Eq. (11) by setting $z = -1$, or from the equivalent analog filter by taking the limit of Eq. (1) as $s \to \infty$. Figure 1 compares the conventional analog and digital equalizer designs.

3. Digital Equalizer with Prescribed Nyquist-Frequency Gain

Because the bilinear transformation maps $z = -1$ onto $s = \infty$, in order to design a digital filter with prescribed Nyquist-frequency gain G_1, we may start by designing an equivalent analog filter whose gain at $s = \infty$ is G_1. The transfer function of such filter is the modified form of Eq. (1):

$$H(s) = \frac{G_1 s^2 + Bs + G_0 W^2}{s^2 + As + W^2} \tag{13}$$

It has gain G_1 at $s = \infty$, and G_0 at $s = 0$. Its magnitude response is:

$$|H(\Omega)|^2 = \frac{(G_1\Omega^2 - G_0 W^2)^2 + B^2\Omega^2}{(\Omega^2 - W^2)^2 + A^2\Omega^2} \tag{14}$$

The parameter W is no longer equal to the peak frequency Ω_0, but is related to it. The filter coefficients A, B, W^2 can be determined by requiring the three conditions that $|H(\Omega)|^2$ have a maximum (or minimum) at Ω_0, that the peak gain be G, and that the bandedge frequencies be measured at level G_B:

$$\frac{\partial}{\partial\Omega^2}|H(\Omega_0)|^2 = 0\,, \qquad |H(\Omega_0)|^2 = G^2\,, \qquad |H(\Omega)|^2 = G_B^2 \tag{15}$$

The solutions of the third equation are the left and right bandedge frequencies Ω_1, Ω_2, which define the analog bandwidth as the difference $\Delta\Omega = \Omega_2 - \Omega_1$. Solving Eqs. (15) (see Appendix A for details), gives the filter coefficients:

$$W^2 = \sqrt{\frac{G^2 - G_1^2}{G^2 - G_0^2}}\,{\Omega_0}^2\,, \qquad A = \sqrt{\frac{C + D}{|G^2 - G_B^2|}}\,, \qquad B = \sqrt{\frac{G^2 C + G_B^2 D}{|G^2 - G_B^2|}} \tag{16}$$

where C, D are given in terms of the center frequency Ω_0, bandwidth $\Delta\Omega$, and gains as follows:

$$\begin{aligned} C &= (\Delta\Omega)^2\,|G_B^2 - G_1^2| - 2W^2\left(|G_B^2 - G_0 G_1| - \sqrt{(G_B^2 - G_0^2)(G_B^2 - G_1^2)}\right) \\ D &= 2W^2\left(|G^2 - G_0 G_1| - \sqrt{(G^2 - G_0^2)(G^2 - G_1^2)}\right) \end{aligned} \tag{17}$$

Moreover, the bandedge frequencies satisfy the modified geometric-mean property:

$$\Omega_1\Omega_2 = \sqrt{\frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}}\,W^2 = \sqrt{\frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}}\sqrt{\frac{G^2 - G_1^2}{G^2 - G_0^2}}\,{\Omega_0}^2 \tag{18}$$

Equations (16) and (17) implement the design of a second-order analog filter of given peak frequency and width, Ω_0, $\Delta\Omega$, and prescribed DC, high-frequency, peak, and bandwidth gains, G_0, G_1, G, G_B. Note that the absolute values in (16) and (17) are needed only when designing a cut, as opposed to a boost.

The desired digital filter can be designed now by the bilinear transformation applied to the above analog filter. To complete the design, the given physical frequency parameters ω_0, $\Delta\omega$ of Eq. (8) must be mapped onto those of the equivalent analog filter. This can be done via the transformations (see Appendix A):

$$\Omega_0 = \tan\left(\frac{\omega_0}{2}\right), \qquad \Delta\Omega = \left(1 + \sqrt{\frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}}\sqrt{\frac{G^2 - G_1^2}{G^2 - G_0^2}}\,{\Omega_0}^2\right)\tan\left(\frac{\Delta\omega}{2}\right) \tag{19}$$

Applying the bilinear transformation (7) to Eq. (13), gives rise to the digital filter transfer function:

$$H(z) = \frac{\left(\dfrac{G_1 + G_0 W^2 + B}{1 + W^2 + A}\right) - 2\left(\dfrac{G_1 - G_0 W^2}{1 + W^2 + A}\right)z^{-1} + \left(\dfrac{G_1 + G_0 W^2 - B}{1 + W^2 + A}\right)z^{-2}}{1 - 2\left(\dfrac{1 - W^2}{1 + W^2 + A}\right)z^{-1} + \left(\dfrac{1 + W^2 - A}{1 + W^2 + A}\right)z^{-2}} \tag{20}$$

To summarize, given the set of digital filter specifications $\{\omega_0, \Delta\omega, G_0, G_1, G, G_B\}$, use Eqs. (19) to calculate the prewarped analog frequencies. Then, use Eqs. (16) and (17) to calculate the parameters $\{A, B, W^2\}$, from which the digital filter coefficients of Eq. (20) are determined.

In the special case $G_1 = G_0$, we recover the results of Section 2. Indeed, we have, $W = \Omega_0$, $D = 0$, $C = (\Delta\Omega)^2|G_B^2 - G_0^2|$, and A, B reduce to Eq. (5). Similarly, Eq. (20) reduces to Eq. (11).

So far, the Nyquist-frequency gain G_1 has been chosen arbitrarily. However, for the digital filter to match the corresponding (physical) analog filter as much as possible, the gain G_1 must match the analog filter's gain at $f_s/2$, as given by Eq. (6). Using Eq. (5), we can rewrite Eq. (6) in terms of the normalized digital frequencies ω_0, $\Delta\omega$ of Eq. (8) as follows:

$$G_1^2 = \frac{G_0^2(\omega_0^2 - \pi^2)^2 + G^2\pi^2(\Delta\omega)^2(G_B^2 - G_0^2)/(G^2 - G_B^2)}{(\omega_0^2 - \pi^2)^2 + \pi^2(\Delta\omega)^2(G_B^2 - G_0^2)/(G^2 - G_B^2)} \tag{21}$$

Figure 2 compares the new digital equalizer with the conventional analog and digital designs. The overall design method contained in Eqs. (16–21) is implemented by the MATLAB function `peq.m` of Appendix B.

For cascadable parametric equalizers, the DC reference gain must be set equal to unity, $G_0 = 1$. For resonator filters, it must be set to zero, $G_0 = 0$, and the peak gain set to unity, $G = 1$. The

Nyquist-frequency gain is still calculated by Eq. (21) (with $G_0 = 0$), and represents the gain an analog resonator has at $f_s/2$. Thus, such a digital resonator filter can better emulate physical resonances. Digital notch filters emulating analog ones can also be designed by setting $G_0 = 1$, $G = 0$.

4. Realizations

The digital filter of Eq. (20) was given in terms of the direct-form numerator and denominator coefficients, and can be written in the compact form:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{22}$$

It can be realized in any of the standard direct-form realizations, such as direct-form I, II, or transposed forms. The direct-form II should perhaps be avoided since it requires special care to prevent internal overflows.

The filter can also be realized in its lattice/ladder form [12,13], which has good numerical properties. For the conventional design, the Regalia-Mitra realization [5,7,8,10] uses an allpass filter realized in its lattice form and allows the independent control of the three parameters of center frequency ω_0, bandwidth $\Delta\omega$, and peak gain G.

In this section, we discuss the lattice realization of Eq. (20) and find that it leads to a generalization of the Regalia-Mitra form. The lattice realization is built out of the lattice recursions of the denominator polynomial $A(z)$, that is, iterating up to order two:

$$\begin{aligned}
A_0(z) &= 1 \\
A_1(z) &= A_0(z) + k_1 z^{-1} A_0^R(z) = 1 + k_1 z^{-1} \\
A_2(z) &= A_1(z) + k_2 z^{-1} A_1^R(z) = 1 + k_1(1 + k_2) z^{-1} + k_2 z^{-2}
\end{aligned} \tag{23}$$

where the reversed polynomials are:

$$\begin{aligned}
A_0^R(z) &= 1 \\
A_1^R(z) &= k_1 + z^{-1} \\
A_2^R(z) &= k_2 + k_1(1 + k_2) z^{-1} + z^{-2}
\end{aligned} \tag{24}$$

Identifying $A_2(z)$ with the direct-form denominator $A(z)$ of Eq. (22), gives for the coefficients $k_2 = a_2$ and $k_1(1 + k_2) = a_1$, which may be solved for the reflection coefficients:

$$k_1 = \frac{a_1}{1 + a_2}, \qquad k_2 = a_2 \tag{25}$$

Using the coefficients of Eq. (20), we find:

$$k_1 = -\frac{1 - W^2}{1 + W^2}, \qquad k_2 = \frac{1 + W^2 - A}{1 + W^2 + A} \tag{26}$$

We can express now the numerator polynomial $B(z)$ of Eq. (20) in terms of the reflection coefficients k_1 and k_2 in the following form:

$$B(z) = \frac{1}{2}\overline{G}_0(1 + k_2)(1 + 2k_1 z^{-1} + z^{-2}) + \frac{1}{2}\overline{G}(1 - k_2)(1 - z^{-2}) + \Delta G(k_1 + 2z^{-1} + k_1 z^{-2}) \tag{27}$$

where we defined the quantities:

$$\overline{G}_0 = \frac{1}{2}(G_0 + G_1)\,, \qquad \overline{G} = \frac{B}{A} = \sqrt{\frac{G^2 C + G_B^2 D}{C + D}}\,, \qquad \Delta G = -\frac{1}{4}(G_1 - G_0)(1 + k_2) \tag{28}$$

The three polynomial terms of $B(z)$ can be written in terms of the lattice polynomials $A_1(z)$, $A_2(z)$ and their reverse, as follows:

$$\begin{aligned}
A_2(z) + A_2^R(z) &= (1 + k_2)(1 + 2k_1 z^{-1} + z^{-2}) \\
A_2(z) - A_2^R(z) &= (1 - k_2)(1 - z^{-2}) \\
A_1^R(z) + z^{-1}A_1(z) &= k_1 + 2z^{-1} + k_1 z^{-2}
\end{aligned} \tag{29}$$

Thus, Eq. (27) can be written in the form:

$$\begin{aligned}
B(z) &= \frac{1}{2}\overline{G}_0\big(A_2(z) + A_2^R(z)\big) + \frac{1}{2}\overline{G}\big(A_2(z) - A_2^R(z)\big) + \Delta G\big(A_1^R(z) + z^{-1}A_1(z)\big) \\
&= \frac{1}{2}(\overline{G}_0 + \overline{G})A_2(z) + \frac{1}{2}(\overline{G}_0 - \overline{G})A_2^R(z) + \Delta G\big(A_1^R(z) + z^{-1}A_1(z)\big)
\end{aligned}$$

The transfer function of Eq. (20) will be then:

$$H(z) = \frac{B(z)}{A(z)} = \frac{1}{2}(\overline{G}_0 + \overline{G}) + \frac{1}{2}(\overline{G}_0 - \overline{G})\frac{A_2^R(z)}{A_2(z)} + \Delta G\,\frac{A_1^R(z) + z^{-1}A_1(z)}{A_2(z)} \tag{30}$$

A block diagram realization of Eq. (30) is shown in Fig. 3, where the allpass transfer function $A_2^R(z)/A_2(z)$ has been realized in its lattice form. As a consequence of the lattice structure, it can be verified easily that the terms $A_2^R(z)/A_2(z)$, $A_1^R(z)/A_2(z)$, and $A_1(z)/A_2(z)$ are the transfer functions from the input x to the signals y_2, y_1, and x_1, respectively. It might appear strange that we introduced a third delay into the realization of this second-order filter. However, this was done for convenience in order to make use of the successive outputs of the lattice sections.

In the limiting case when $G_1 = G_0$, we have $\Delta G = 0$, and Eq. (30) and Fig. 3 reduce to the Regalia-Mitra realization for the conventional design of Eq. (11). Indeed, we have in this limit:

$$\overline{G}_0 = G_0\,, \qquad \overline{G} = G\,, \qquad k_1 = -\cos\omega_0\,, \qquad k_2 = \frac{1-\beta}{1+\beta} \tag{31}$$

where β is given by Eq. (12). In the general case, the realization coefficients k_1, k_2, $\overline{G}_0$, $\overline{G}$, and ΔG do not quite provide independent control of the equalizer's parameters. Thus, the value of this lattice realization lies mostly in its numerical properties.

Strictly speaking, the conventional Regalia-Mitra realization with parameters given by Eq. (31) is not completely decoupled either, because β depends on both G and $\Delta\omega$, unless one defines G_B^2 as the weighted arithmetic mean of Eq. (36), as discussed in the next section.

An alternative realization—which is the standard lattice/ladder realization [12,13]—can be obtained by expressing $B(z)$ as a linear combination of the three reverse filters $A_0^R(z)$, $A_1^R(z)$, and $A_2^R(z)$ in the form:

$$B(z) = c_0 A_0^R(z) + c_1 A_1^R(z) + c_2 A_2^R(z) \tag{32}$$

where the expansion coefficients can be obtained from the direct-form coefficients $\{b_0, b_1, b_2\}$ of $B(z)$ via the backward substitution:

$$\begin{aligned}
c_2 &= b_2 \\
c_1 &= b_1 - a_1 c_2 \\
c_0 &= b_0 - k_1 c_1 - k_2 c_2
\end{aligned} \tag{33}$$

Then, the equalizer transfer function becomes:

$$H(z) = \frac{B(z)}{A(z)} = c_0\,\frac{A_0^R(z)}{A_2(z)} + c_1\,\frac{A_1^R(z)}{A_2(z)} + c_2\,\frac{A_2^R(z)}{A_2(z)} \tag{34}$$

The transfer functions of the three terms are obtained at the lattice section output signals y_0, y_1, and y_2 of Fig. 3, which can then be linearly combined with the c-coefficients. This structure also has good numerical properties and it is straightforward to modify the MATLAB function `peq.m` to compute the coefficients $\{k_1, k_2, c_0, c_1, c_2\}$.

We note finally that the transfer function (20) is a minimum phase transfer function, so that both $H(z)$ and its inverse $1/H(z)$ are stable and causal. This follows [12,13] from the fact that the denominator reflection coefficients have magnitudes less than one, $|k_1| \le 1$, $|k_2| \le 1$, and so do the numerator reflection coefficients, which are

$$k_{b1} = \frac{b_1/b_0}{1 + b_2/b_0} = -\frac{G_1 - G_0 W^2}{G_1 + G_0 W^2}\,, \qquad k_{b2} = \frac{b_2}{b_0} = \frac{G_1 + G_0 W^2 - B}{G_1 + G_0 W^2 + B}$$

5. Bandwidth

As discussed by Bristow-Johnson [9], there is considerable variation in the literature in the definition of bandwidth $\Delta\omega$ and bandwidth gain G_B. For example, one can define $\Delta\omega$ to be the difference of the bandedge frequencies in linear frequency units, or define it in octaves in log units.

As seen in Fig. 1, for a conventional digital equalizer, the gain G_B must always be defined to lie somewhere between the reference and the peak gains, that is,

$$\begin{array}{ll} G_0 < G_B < G & \text{(boost)} \\ G_0 > G_B > G & \text{(cut)} \end{array} \tag{35}$$

For a boost, one may define G_B to be 3-dB below the peak gain, $G_B^2 = G^2/2$, or take it to be 3-dB above the reference, $G_B^2 = 2G_0^2$, or, define it as the arithmetic mean of the peak and reference gains, $G_B^2 = (G_0^2 + G^2)/2$, or as the geometric mean, $G_B^2 = G_0 G$, which is the arithmetic mean of the gains in dB scales. The 3-dB definitions are possible only if the boost gain G is itself greater than 3 dB, that is, $G^2 > 2G_0^2$.

For a cut, one may take G_B to be 3-dB above the cut gain, $G_B^2 = 2G^2$, or 3-dB below the reference $G_B^2 = G_0^2/2$, or use the arithmetic/geometric means. Again, the first two definitions are possible only if the cut gain is at least 3-dB below the reference, $G^2 < G_0^2/2$.

In the special cases of a resonator ($G_0 = 0$, $G = 1$) and a notch filter ($G_0 = 1$, $G = 0$), the 3-dB definitions of G_B are always possible, and in fact, they coincide with the arithmetic mean.

The arithmetic/geometric mean definitions are always possible for any value of the boost/cut gain G. They can be replaced by the more general weighted arithmetic or geometric means:

$$\begin{aligned} G_B^2 &= \alpha G_0^2 + (1-\alpha)G^2 \\ G_B &= G_0^{\alpha} G^{1-\alpha} \end{aligned} \tag{36}$$

where $0 < \alpha < 1$. The conventional means have $\alpha = 1/2$. The weighted geometric mean is equivalent to a weighted arithmetic mean of the dB gains.

The weighted geometric mean is attractive because a boost and a cut by equal and opposite gains in dB cancel exactly [9] (their transfer functions are inverses of each other.) The arithmetic mean is attractive because it makes the conventional Regalia-Mitra realization truly independently controllable by the three EQ parameters $\{\omega_0, \Delta\omega, G\}$.

Indeed, if G_B^2 is given by the weighted arithmetic mean of Eq. (36), then the square root factor in the definition of β in Eq. (12) becomes independent of G:

$$\sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}} = \sqrt{\frac{1-\alpha}{\alpha}}$$

and it is equal to unity when $\alpha = 1/2$.

For an analog equalizer, as well as for the new digital design, it is evident from Fig. 2 that the peak gain G must be greater than the Nyquist-frequency gain G_1. Thus, the minimum requirement for the choice of G_B is that it lie in the intervals:

$$\begin{aligned} &G_0 < G_1 < G_B < G \qquad \text{(boost)} \\ &G_0 > G_1 > G_B > G \qquad \text{(cut)} \end{aligned} \tag{37}$$

It follows from these inequalities that the arguments of all the square roots in Eqs. (16–21) are always positive, for either a boost or a cut.

The result that a boost and a cut of equal and opposite dB gains cancel each other can be generalized to the new design as follows. Given a set of design gains $\{G_0, G_1, G_B, G\}$ for a boost, we can get a design set for a cut by the transformation:

$$\{G_0, G_1, G_B, G\} \rightarrow \{G_0^{-1}, G_1^{-1}, G_B^{-1}, G^{-1}\} \tag{38}$$

Indeed, if the gains $\{G_0, G_1, G_B, G\}$ satisfy the boost inequalities in Eq. (37), then the inverted gains $\{G_0^{-1}, G_1^{-1}, G_B^{-1}, G^{-1}\}$ will satisfy the cut inequalities.

For a unity reference gain ($G_0 = 1$), the transformation $G \rightarrow G^{-1}$ implies that the boost and cut will have equal and opposite peak gains in dB. The transformation $G_B \rightarrow G_B^{-1}$ means that the bandwidth gain for the cut must be measured at a dB level which is the negative of the bandwidth level of the boost.

Under the boost-to-cut transformation (38), and for fixed values of center frequency ω_0 and bandwidth $\Delta\omega$, it follows that the cut transfer function will be exactly the inverse of the boost:

$$H_{\text{cut}}(z) = \frac{1}{H_{\text{boost}}(z)} \tag{39}$$

To see this, we note that under the transformation $\{G_0, G_B, G\} \rightarrow \{G_0^{-1}, G_B^{-1}, G^{-1}\}$, the Nyquist-frequency gain of Eq. (21) transforms according to $G_1 \rightarrow G_1^{-1}$. Moreover, the prewarped bandwidth $\Delta\Omega$ of Eq. (19) remains invariant. It follows from Eqs. (16)–(17) that the analog filter coefficients will transform as:

$$W^2 \rightarrow G_0 G_1^{-1} W^2, \qquad A \rightarrow G_1^{-1} B, \qquad B \rightarrow G_1^{-1} A$$

These and Eq. (38) imply that the equivalent analog transfer function of Eq. (13) will map into its inverse:

$$\frac{G_1 s^2 + Bs + G_0 W^2}{s^2 + As + W^2} \rightarrow \frac{G_1^{-1} s^2 + (G_1^{-1} A)s + G_0^{-1}(G_0 G_1^{-1} W^2)}{s^2 + (G_1^{-1} B)s + (G_0 G_1^{-1} W^2)} = \frac{s^2 + As + W^2}{G_1 s^2 + Bs + G_0 W^2}$$

or, $H_{\text{cut}}(s) = 1/H_{\text{boost}}(s)$. Then, the bilinear transformation implies Eq. (39).

Although the boost and cut levels G_B, G_B^{-1} are equal and opposite in dB, they do not have to be measured at the arithmetic-mean dB level, $G_B = \sqrt{G_0 G} = \sqrt{G}$. (This may not even be possible if G is so small that $\sqrt{G} < G_1 < G$.) A better choice might be $G_B = \sqrt{G_1 G}$.

Finally, we discuss the modifications to the design when the bandwidth is to be specified in octaves. If the bandedge frequencies are related by $\omega_2 = 2^{\Delta y}\omega_1$, so that the bandwidth is Δy octaves, then for the design of a physical analog equalizer, the bandedge frequencies will lie symmetrically with respect to the center frequency ω_0 in a log-frequency scale and their difference $\Delta\omega = \omega_2 - \omega_1$ can be expressed in terms of the octave width Δy as follows [9] (in units of rads/sample):

$$\Delta\omega = 2\omega_0 \sinh\left(\frac{\ln 2}{2}\,\Delta\gamma\right) \tag{40}$$

For the conventional digital equalizer design, Bristow-Johnson [9] suggests the following approximation for the prewarped difference $\Delta\Omega$, which effectively amounts to linearizing the bilinear transformation mapping $\ln\Omega = \ln\tan(\omega/2)$ about the center frequency $\Omega_0 = \tan(\omega_0/2)$:

$$\Delta\Omega = 2\Omega_0 \sinh\left(\frac{\omega_0}{\sin\omega_0}\,\frac{\ln 2}{2}\,\Delta\gamma\right) \tag{41}$$

Using Eqs. (10) and (12), the design parameter β can be expressed as

$$\beta = \sqrt{\frac{G_B^2 - G_0^2}{G^2 - G_B^2}}\,\sinh\left(\frac{\omega_0}{\sin\omega_0}\,\frac{\ln 2}{2}\,\Delta\gamma\right)\sin\omega_0 \tag{42}$$

The approximation (41) works well also for the new digital equalizer. The only difference is that now the quantity $\Delta\Omega$ of Eq. (17) must be calculated from Eq. (41) instead of Eq. (19), and Eq. (40) must be used in Eq. (21) to calculate G_1.

An alternative approach, which leads to an exact solution, is to specify the center frequency ω_0 and one (but not both) of the bandedge frequencies, say ω_2, and give it in linear or octave scales. For example, $\omega_2 = 2^{\gamma_2}\omega_0$ lies γ_2 octaves above ω_0. Then, solve for the other bandedge frequency using the prewarped geometric-mean rule (18) or (43). Then, calculate $\Delta\omega = \omega_2 - \omega_1$, and proceed with Eqs. (16)–(21) to complete the design. Eq. (18) can be written in terms of the physical frequencies as follows:

$$\tan\left(\frac{\omega_1}{2}\right)\tan\left(\frac{\omega_2}{2}\right) = \sqrt{\frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}}\sqrt{\frac{G^2 - G_1^2}{G^2 - G_0^2}}\,\tan^2\left(\frac{\omega_0}{2}\right) \tag{43}$$

6. Design Examples

Next, we present some design examples. Fig. 4 shows an equalizer with a 12 dB boost (and a 12 dB cut), peak frequency of $\omega_0 = 0.5\pi$ rads/sample, bandwidth of $\Delta\omega = 0.2\pi$ rads/sample, and reference DC gain of 0-dB. The bandwidth is measured at 3-dB below the peak, that is, at a level of 9 dB. Thus, the design gains will be:

$$G_0 = 10^{0/20} = 1\,,\quad G = 10^{12/20} = 3.9811\,,\quad G_B = 10^{9/20} = 2.8184$$

Fig. 4 shows the magnitude responses in dB, that is, $20\log_{10}|H(\omega)|$, of the new design of Eq. (20), the old design of Eq. (11), and the analog design of Eq. (1). The frequency axis extends over one complete Nyquist interval, $0 \le \omega \le 2\pi$. The Nyquist-frequency gain is calculated to be $G_1 = 1.369 = 2.725$ dB. For the cut case, all gains are the inverses of the above, (or, the negatives in dB), and the corresponding transfer functions become the inverses, according to Eq. (39).

Fig. 5 shows two more examples with the same specifications as the above, except the peak frequency is now $\omega_0 = 0.3\pi$ for the left figure and $\omega_0 = 0.7\pi$ for the right one. The corresponding Nyquist-frequency gains are 2.053 dB and 4.420 dB, respectively.

Fig. 6 illustrates the dependence of the Nyquist-frequency gain G_1 given by Eq. (21) on the design parameters $\{\omega_0, \Delta\omega, G\}$. The left figure shows G_1 in dB as a function of the bandwidth $\Delta\omega$, varying over the range $0.01\pi \le \Delta\omega \le 0.5\pi$. The peak frequency was fixed at $\omega_0 = 0.5\pi$ and the following three values of the peak gain were chosen: $G = 12, 9, 6$ dB. The bandwidth was always measured at 3-dB below each peak.

The right figure shows the dependence of G_1 on the peak frequency ω_0, being varied over the interval $0.01\pi \le \omega_0 \le 0.95\pi$. The same three peak gains were used. The bandwidth was fixed at $\Delta\omega = 0.1\pi$ and measured at 3-dB below each peak.

Figs. 7–9 illustrate the nature of the approximations of Eq. (41). The frequency axis is $\log_2(\omega/\pi)$, and is measured in octaves below the Nyquist frequency. In all cases, the peak gains are 12 dB and the bandwidths are measured at 9 dB.

The designs of Fig. 7 have center frequency $\omega_0 = 2^{-1}\pi$ and octave widths $\Delta\gamma = 1$ and $\Delta\gamma = 0.5$. Fig. 8 has $\omega_0 = 2^{-0.5}\pi$ and octave widths $\Delta\gamma = 0.5$ and $\Delta\gamma = 0.25$. Fig. 9 has $\omega_0 = 2^{-0.25}\pi$, $\Delta\gamma = 0.25$ on the left, and $\omega_0 = 2^{-0.125}\pi$, $\Delta\gamma = 0.125$ on the right.

In all cases, the analog design has symmetric bandwidth about the center frequency and the new digital design attempts to follow the analog one as much as possible.

7. Discussion

The design method of this paper results in the most general type of second-order digital parametric equalizer, because the five filter coefficients are fixed uniquely by five different design constraints.

The method encompasses the conventional design as a special case. For low center frequencies and widths, the new method will be almost identical to the conventional method, because the Nyquist-frequency gain G_1 is almost equal to G_0. The differences of the two methods are felt only for high frequencies and widths. Fig. 6 gives an idea of how high is "high."

The method allows various ways of defining the bandwidth in linear- and log-frequency scales and of defining the bandwidth gain G_B. Given the wide variety of possibilities in choosing G_B, it is perhaps best to leave G_B as a free parameter to be chosen by the user, as long as it satisfies Eq. (37).

8. Acknowledgements

This paper was motivated by a question raised by R. Silfvast on `comp.dsp`. I would like to thank Mark Kahrs for discussions.

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Appendix A

Here, we present some of the derivations of the design equations (16)–(21). The derivative of Eq. (14) can be written in the form:

$$\frac{\partial}{\partial \Omega^2}|H(\Omega)|^2 = \frac{2G_1(G_1\Omega^2 - G_0W^2) + B^2 - |H(\Omega)|^2\big(2(\Omega^2 - W^2) + A^2\big)}{(\Omega^2 - W^2)^2 + A^2\Omega^2}$$

The first two conditions of Eq. (15) applied at $\Omega = \Omega_0$ give:

$$\begin{aligned} 2G_1(G_1{\Omega_0}^2 - G_0W^2) + B^2 - G^2\big(2({\Omega_0}^2 - W^2) + A^2\big) &= 0 \\ \frac{(G_1{\Omega_0}^2 - G_0W^2)^2 + B^2{\Omega_0}^2}{({\Omega_0}^2 - W^2)^2 + A^2{\Omega_0}^2} &= G^2 \end{aligned}$$

Solving these equations for the quantities W^2 and $B^2 - G^2A^2$, gives Eq. (16) for W^2 and

$$B^2 - G^2A^2 = -2W^2\left((G^2 - G_0G_1) - \sqrt{(G^2 - G_0^2)(G^2 - G_1^2)}\right) \equiv -D \tag{A.1}$$

Next, consider the bandedge condition:

$$\frac{(G_1\Omega^2 - G_0W^2)^2 + B^2\Omega^2}{(\Omega^2 - W^2)^2 + A^2\Omega^2} = G_B^2$$

It can be written as the quartic equation:

$$\Omega^4 - \left[\frac{G_B^2 - G_0G_1}{G_B^2 - G_1^2}\,2W^2 + \frac{B^2 - G_B^2A^2}{G_B^2 - G_1^2}\right]\Omega^2 + \frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}W^4 = 0$$

It follows that the two bandedge frequencies will satisfy:

$$\begin{aligned} \Omega_1^2 + \Omega_2^2 &= \frac{G_B^2 - G_0G_1}{G_B^2 - G_1^2}\,2W^2 + \frac{B^2 - G_B^2A^2}{G_B^2 - G_1^2} \\ \Omega_1\Omega_2 &= \sqrt{\frac{G_B^2 - G_0^2}{G_B^2 - G_1^2}}\,W^2 \end{aligned} \tag{A.2}$$

Using Eq. (A.2), we find for the difference $\Delta\Omega = \Omega_2 - \Omega_1$:

$$\Delta\Omega^2 = \Omega_2^2 + \Omega_1^2 - 2\Omega_2\Omega_1 = \frac{B^2 - G_B^2 A^2}{G_B^2 - G_1^2} + \frac{G_B^2 - G_0 G_1 - \sqrt{(G_B^2 - G_0^2)\,(G_B^2 - G_1^2)}}{G_B^2 - G_1^2}\, 2W^2$$

which can be rewritten as:

$$B^2 - G_B^2 A^2 = (\Delta\Omega)^2 (G_B^2 - G_1^2) - 2W^2 \left((G_B^2 - G_0 G_1) - \sqrt{(G_B^2 - G_0^2)\,(G_B^2 - G_1^2)} \right) \equiv C \tag{A.3}$$

Solving (A.1) and (A.3) for A and B, gives Eq. (16). Finally, we derive the prewarped bandwidth in Eq. (19). Using a trigonometric identity and the bilinear transformation, we have for the physical bandwidth difference $\Delta\omega = \omega_2 - \omega_1$:

$$\tan\left(\frac{\Delta\omega}{2}\right) = \tan\left(\frac{\omega_2 - \omega_1}{2}\right) = \frac{\tan\left(\frac{\omega_2}{2}\right) - \tan\left(\frac{\omega_1}{2}\right)}{1 + \tan\left(\frac{\omega_2}{2}\right)\tan\left(\frac{\omega_1}{2}\right)} = \frac{\Omega_2 - \Omega_1}{1 + \Omega_2\Omega_1} = \frac{\Delta\Omega}{1 + \Omega_2\Omega_1}$$

which leads to Eq. (19).

Appendix B

The following MATLAB function, `peq.m`, implements the design equations (16–21):

```
% peq.m - Parametric EQ with matching gain at Nyquist frequency
%
% Usage:  [b, a, G1] = peq(G0, G, GB, w0, Dw)
%
% G0 = reference gain at DC
% G  = boost/cut gain
% GB = bandwidth gain
%
% w0 = center frequency in rads/sample
% Dw = bandwidth in rads/sample
%
% b  = [b0, b1, b2] = numerator coefficients
% a  = [1,  a1, a2] = denominator coefficients
% G1 = Nyquist-frequency gain
%
% Available from: www.ece.rutgers.edu/~orfanidi/intro2sp/mdir/peq.m

function [b, a, G1] = peq(G0, G, GB, w0, Dw)

F   = abs(G^2  - GB^2);
G00 = abs(G^2  - G0^2);
F00 = abs(GB^2 - G0^2);

num = G0^2 * (w0^2 - pi^2)^2 + G^2 * F00 * pi^2 * Dw^2 / F;
den = (w0^2 - pi^2)^2 + F00 * pi^2 * Dw^2 / F;

G1 = sqrt(num/den);

G01 = abs(G^2  - G0*G1);
```

```
G11 = abs(G^2  - G1^2);
F01 = abs(GB^2 - G0*G1);
F11 = abs(GB^2 - G1^2);

W2 = sqrt(G11 / G00) * tan(w0/2)^2;
DW = (1 + sqrt(F00 / F11) * W2) * tan(Dw/2);

C = F11 * DW^2 - 2 * W2 * (F01 - sqrt(F00 * F11));
D = 2 * W2 * (G01 - sqrt(G00 * G11));

A = sqrt((C + D) / F);
B = sqrt((G^2 * C + GB^2 * D) / F);

b = [(G1 + G0*W2 + B), -2*(G1 - G0*W2), (G1 - B + G0*W2)] / (1 + W2 + A);
a = [1, [-2*(1 - W2), (1 + W2 - A)] / (1 + W2 + A)];
```

Its inputs are the gains G_0, G, G_B in absolute units, and the digital frequencies ω_0, $\Delta\omega$ in units of rads/sample. Its outputs are the Nyquist-frequency gain G_1 given by Eq. (21), and the numerator and denominator coefficient vectors $\mathbf{b} = [b_0, b_1, b_2]$, $\mathbf{a} = [1, a_1, a_2]$, defining the transfer function of Eq. (20) or (22).

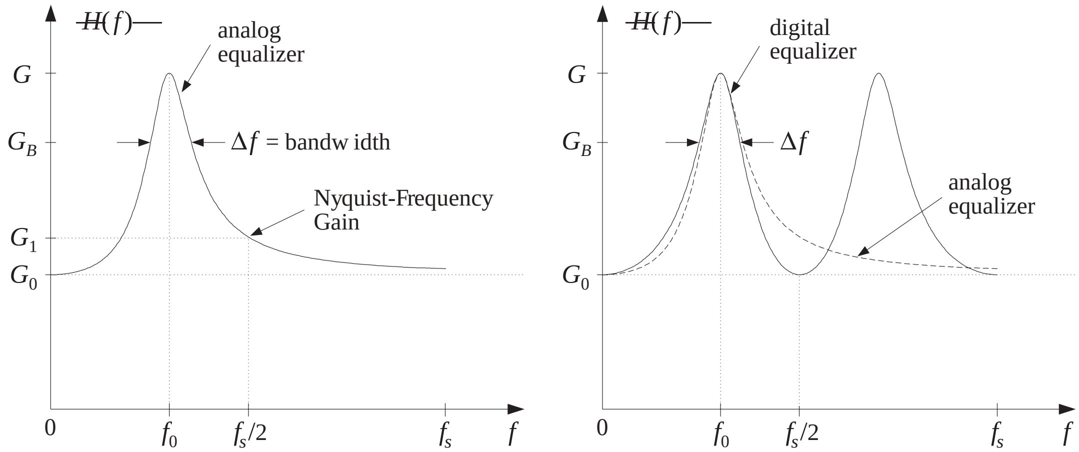

Fig. 1 Conventional analog and digital equalizers. Digital design has $G_1 = G_0$ at $f_s/2$.

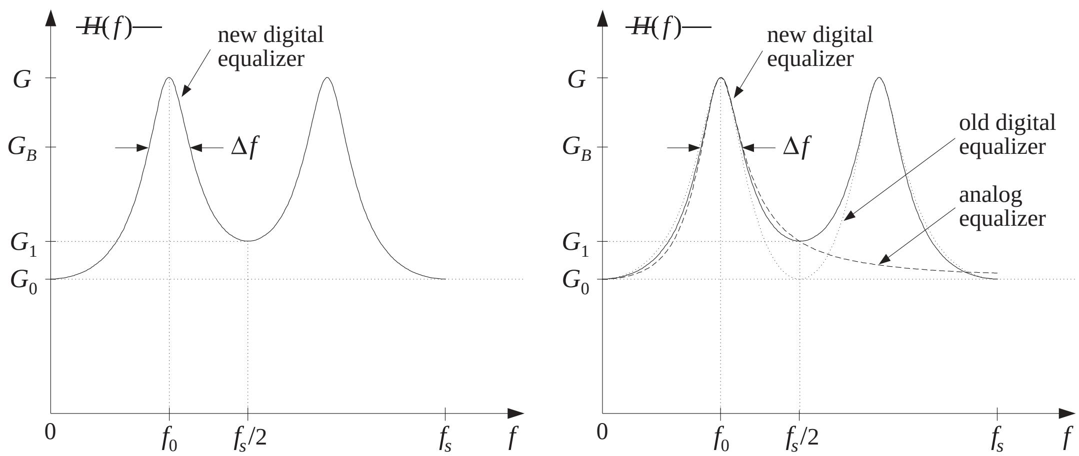

Fig. 2 New digital equalizer matches the Nyquist-frequency gain of the corresponding analog equalizer.

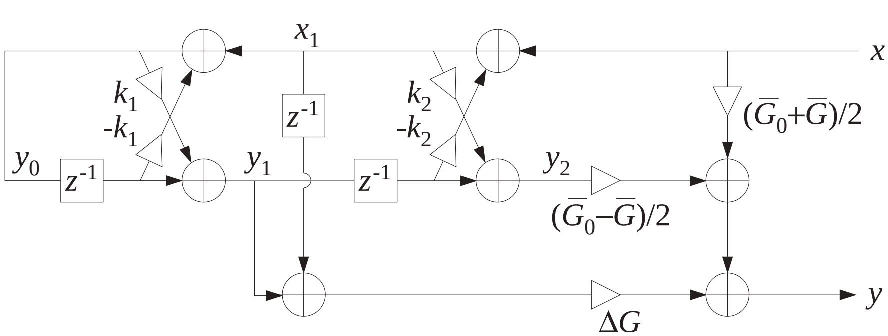

Fig. 3 Lattice realization of Eq. (30).

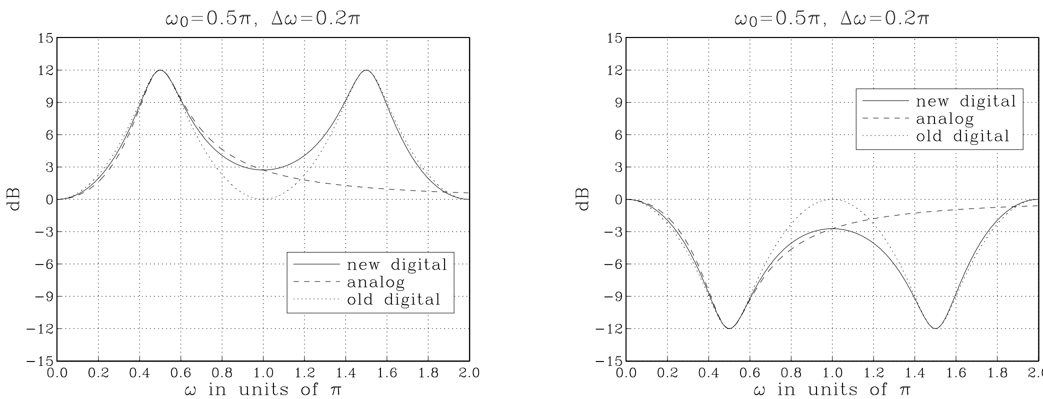

Fig. 4 12-dB boost and cut at $\omega_0 = 0.5\pi$. Bandwidth $\Delta\omega = 0.2\pi$ is measured at ± 9 dB.

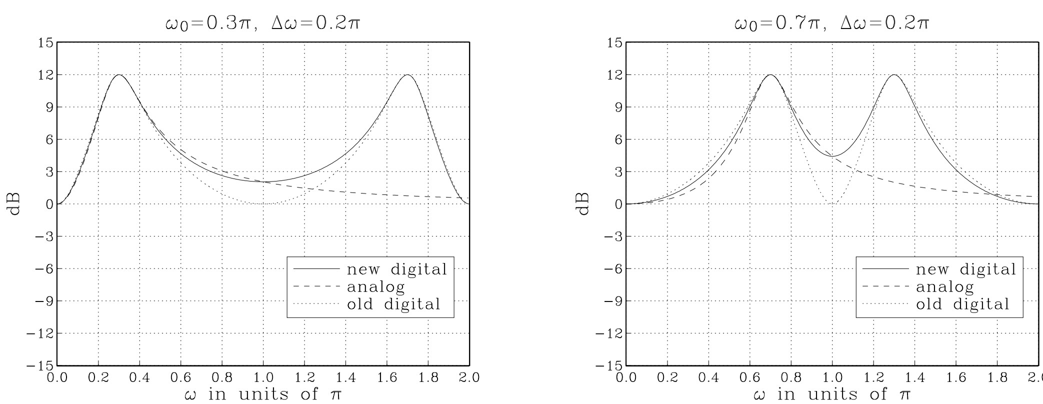

Fig. 5 12-dB boost at $\omega_0 = 0.3\pi$ and $\omega_0 = 0.7\pi$. Bandwidth $\Delta\omega = 0.2\pi$ is measured at 9 dB.

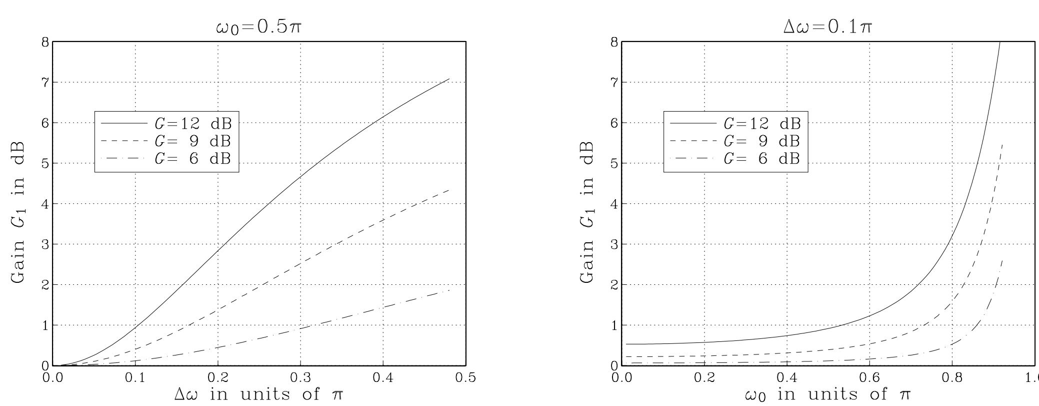

Fig. 6 Nyquist-Frequency gain G_1 as a function of ω_0 and $\Delta\omega$.

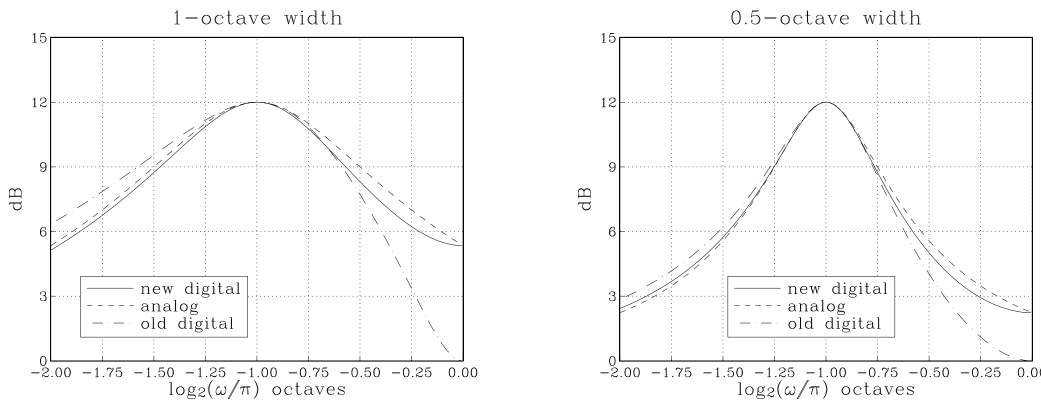

Fig. 7 Center frequency $\omega_0 = 2^{-1}\pi$. Octave widths $\Delta\gamma = 1$ and $\Delta\gamma = 0.5$.

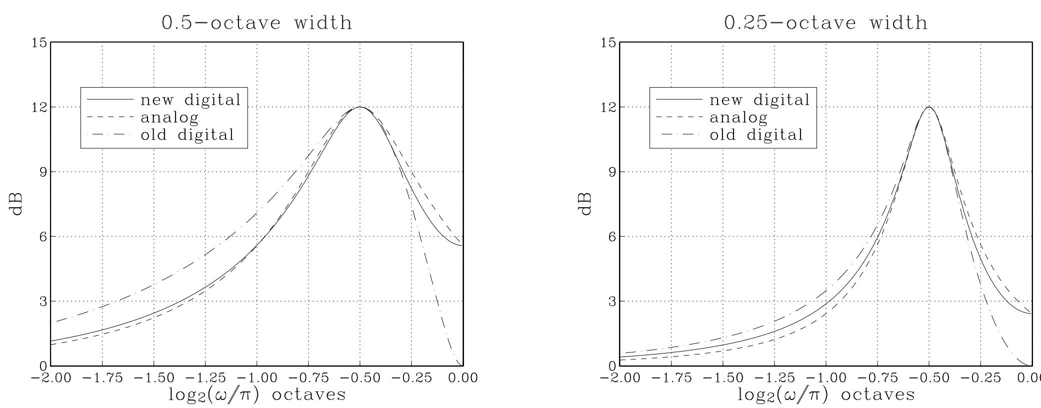

Fig. 8 Center frequency $\omega_0 = 2^{-0.5}\pi$. Octave widths $\Delta\gamma = 0.5$ and $\Delta\gamma = 0.25$.

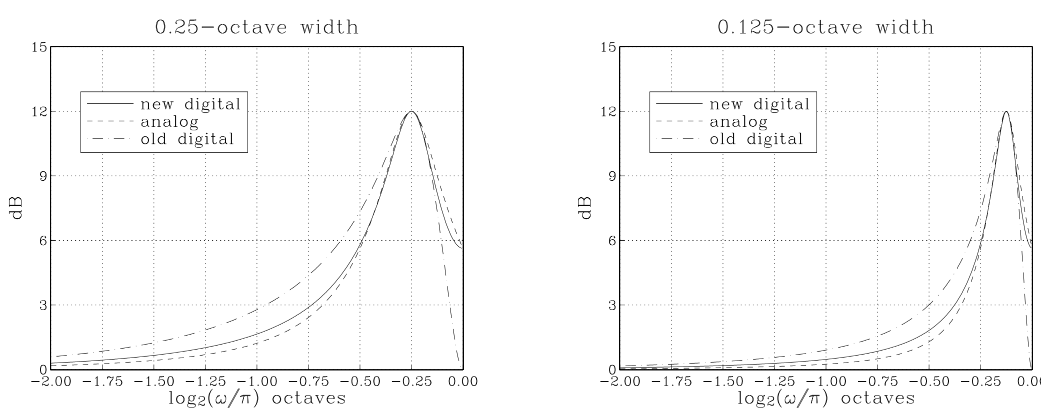

Fig. 9 Left: $\omega_0 = 2^{-0.25}\pi$, $\Delta\gamma = 0.25$. Right: $\omega_0 = 2^{-0.125}\pi$, $\Delta\gamma = 0.125$.