# Symmetry, Defects, and Gauging of Topological Phases of Matter

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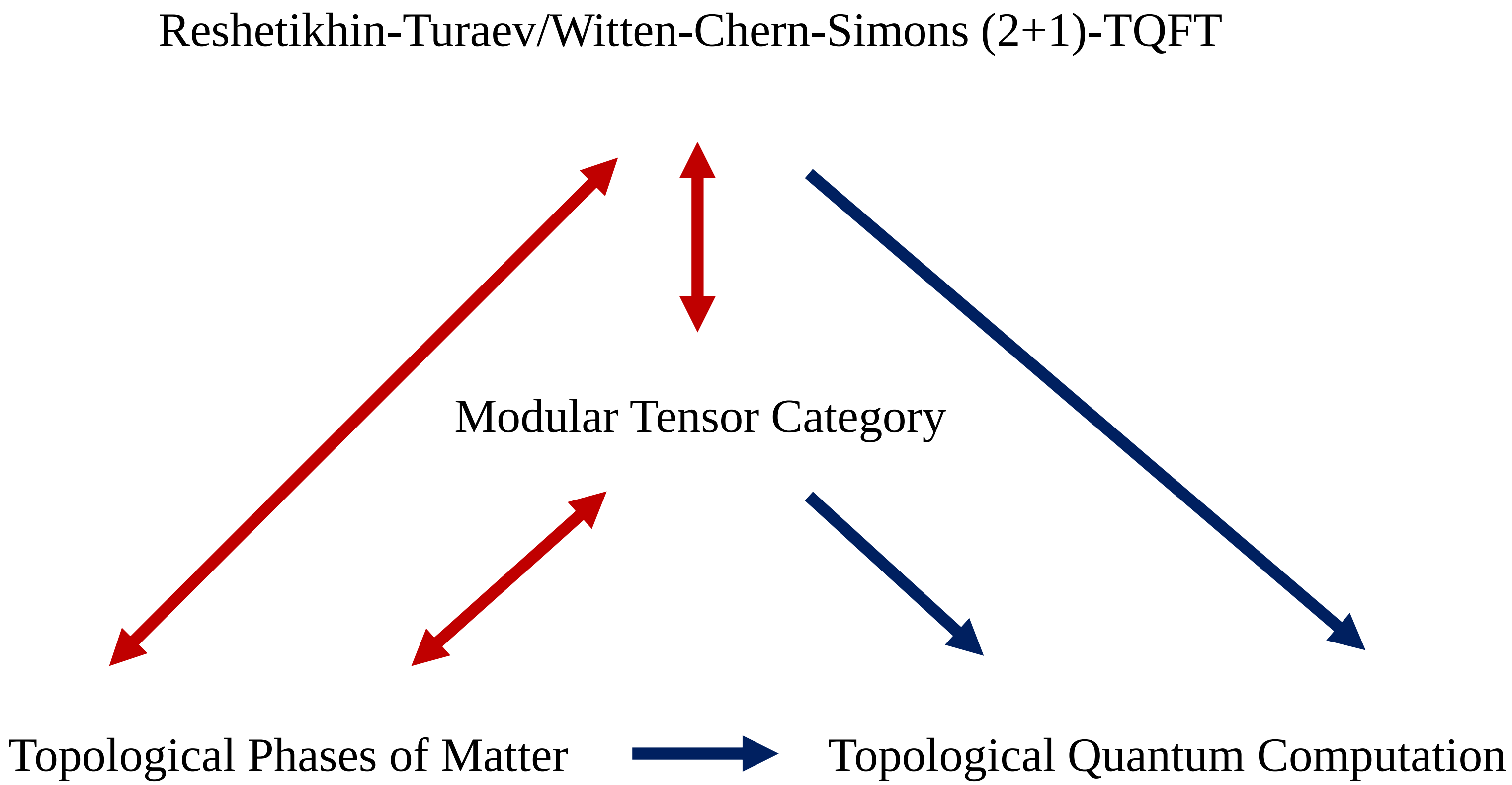
# Ad: Low Dimensional Higher Categories and Applications

- Math:

  Classification of (2+1)- and (3+1)-TQFTs, not fully extended---

  Invariants of low dimensional manifolds (especially smooth 4D)
- Physics:

  Classification of 2D and 3D symmetry enriched topological order (SET) and symmetry protected topological order (SPT)




**Topological phases of matter are TQFTs in Nature and hardware for hypothetical topological quantum computers.**

# Symmetry and 2D Topological Phases of Matter

We develop a general framework to classify 2D topological order in topological phases of matter with symmetry by using G-crossed braided tensor category.

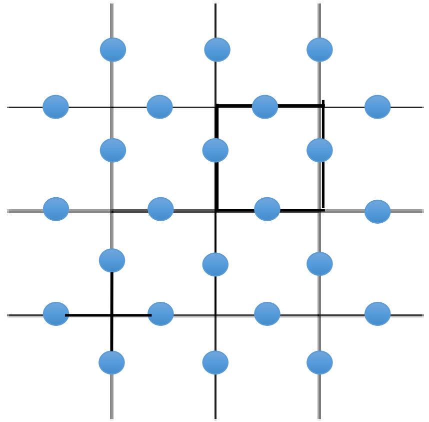
Given a 2D topological order $\mathcal{C}$ and a global symmetry G of $\mathcal{C}$, **three intertwined themes** on the interplay of symmetry group G and intrinsic topological order $\mathcal{C}$

- **Symmetry Fractionalization**---topological quasi-particles carry fractional quantum numbers of the underlying constituents
- **Defects**---extrinsic point-like defects. Many are non-abelian objects
- **Gauging**---deconfine defects by promoting the global symmetry G to a local G gauge theory

# Examples of Topological Phases with Symmetry

## $Z_2$ Toric Code (Kitaev):

$$H_{Z_2} = \sum_s A_s + \sum_p B_p$$



$$B_p = \prod_{i \in \mathrm{boundary}(p)} \sigma_i^z \qquad A_s = \prod_{i \in \mathrm{star}(s)} \sigma_i^x$$

Topological order: $\mathcal{C} = D(Z_2) = \{1, e, m, \psi\}$

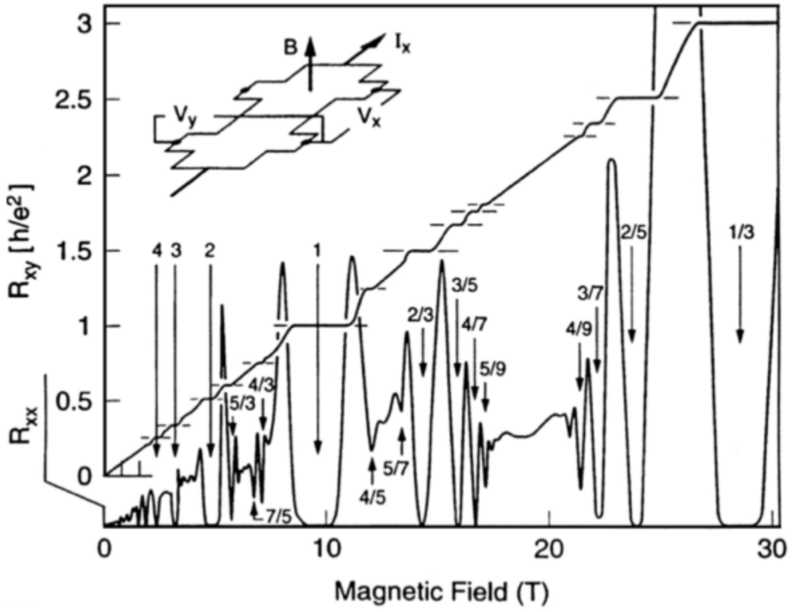
Electric-magnetic duality: e ↔ m

# Examples of 2D Topological Phases with Symmetry

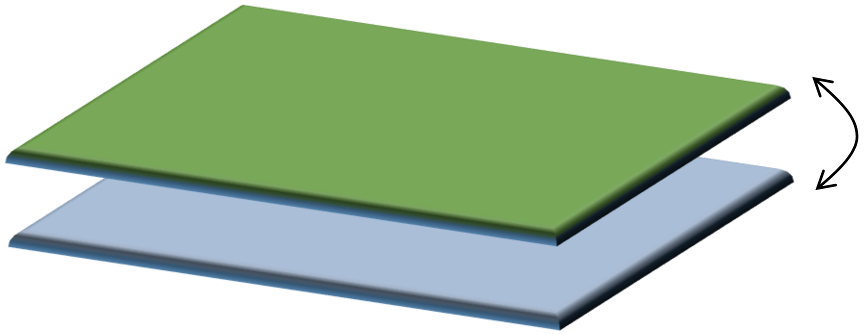
1/*m*-Laughlin state $\Psi(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_B^2}$

**Topological order is encoded by $U(1)_m$~~x{1,e}~~**

**Topological particle-hole symmetry: a↔-a**



# $Z_2$-Layer Exchange Symmetry: Bilayer FQH States

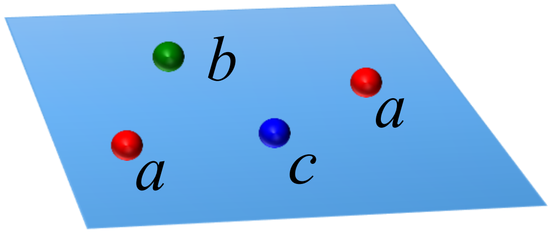


E.g. Halperin (mml) state

$$\Psi(\{z_i\},\{w_i\}) = \prod_{i<j}(z_i - z_j)^m (w_i - w_j)^m \prod_{i,j}(z_i - w_j)^l$$

# Topological Phases of Matter

Finite-energy topological quasiparticle excitations=anyons




Anyons a, b, c

Anyons are of the same type if they differ only by local operators

Anyons in 2+1 dimensions described mathematically by a Unitary Modular Tensor Category $\mathcal{C}$ = Anyon Model

# Symmetry of Quantum Systems ($\mathcal{L}$, H)

Microscopic Symmetry G:

$$R_g \colon \mathcal{L} \longrightarrow \mathcal{L} \qquad\qquad R_g H = H R_g$$

Should preserve locality of $\mathcal{L}$

Symmetry is on-site if: $R_g = \prod_{i=1}^{N} R_g^{(i)}$

# Assumptions and Work In Progress

**1) The global symmetry G is a finite group**

**2) Bulk 2D topological order in boson/spin systems=UMTC=anyon model**

**3)** ***Global symmetry G can be realized as on-site unitary symmetries of the microscopic Hamiltonian, at least at low energies***

**Partial results in our paper:**

- **Continuous symmetries such as U(1) charge conservation and SO(3) spin rotation (2/3)**
- **Fermion systems (1 or 0)**
- **Time-reversal (1/3)**
- **Spatial (1/6+1/6)**
- **Fermion parity (0)**

# Classification of 2D SETs Topologically

- Given a 2D topological order=UMTC=anyon model $\mathcal{C}$, and a finite group G, then G-SETs=G-crossed braided extensions of $\mathcal{C}$
- **SETs are in 1-1 correspondence with set [BG, B$\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$] of homotopy classes of maps between classifying spaces BG and B$\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$,** where B$\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$ is the classifying space of the categorical 2-group $\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$ with $\pi_1 = \mathrm{Aut}(\mathcal{C})$, $\pi_2 = \mathcal{A}$, $\pi_3 = \mathbb{C}\backslash\{0\}$, and $\pi_i = 0$ for i>3. (**ENO 2010**)
- Note that [BG, B$\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$] = $\pi_0(X^Y)$, where X= B$\underline{\underline{\mathrm{Pic}(\mathcal{C})}}$, Y=BG. **Do higher homotopy groups $\pi_i(X^Y)$, i>0, of the mapping space $X^Y$ have physical meaning and significance?**

# Classification of G-crossed Extensions of a UMTC $\mathcal{C}$ Algebraically

Etingof, Nikshych, Ostrik (2010)

$\mathcal{C}_G^\times$ classified by $([\rho],[t],[\alpha])$

$$[\rho] : G \to \mathrm{Aut}(\mathcal{C})$$

If a primary obstruction in $H^3_\rho(G,\mathcal{A})$ vanishes, then choose

$$[t] \in H^2_\rho(G,\mathcal{A})$$

If a secondary obstruction in $H^4(G,U(1))$ vanishes, then choose $[\alpha] \in H^3(G,U(1))$

# Fine print: Symmetry, Defects, and Gauging

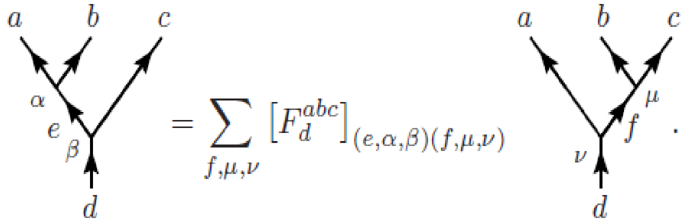
1. Skeletonizing G-Crossed Braided Tensor Category to Obtain Numerical Version of G-Crossed Braided Tensor Category

2. Applying G-Crossed Braided Tensor Categories to Physics:

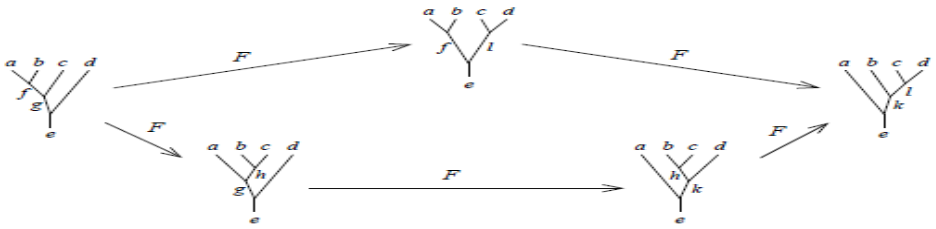
   General Classification and Characterization of Symmetry-Enriched 2D Topological Order

# 2D Topological Order = UMTC = Anyon Model $\mathcal{C}$

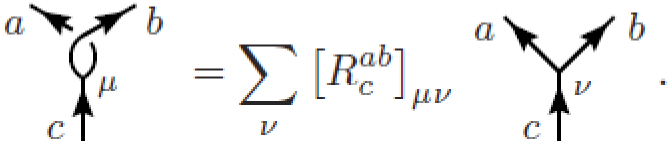
**A modular tensor category = a non-degenerate braided spherical fusion category:**
**a collection of numbers {L, $N_{ab}^{c}$, $F_{d;nm}^{abc}$, $R_{c}^{ab}$} that satisfy some polynomial constraint equations.**



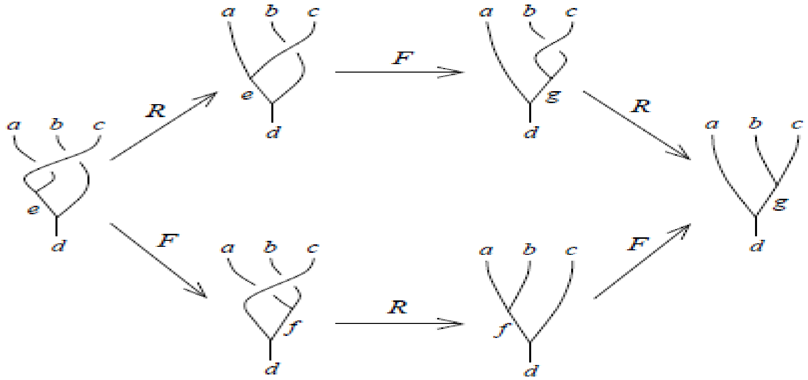

6j symbols for recoupling




Pentagons for 6j symbols




R-symbol for braiding



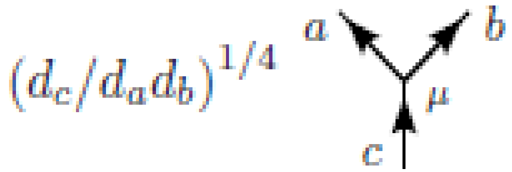

Hexagons for R-symbols

# Examples

- Pointed: $\mathcal{C}(A, q)$, $A$ finite abelian group, $q$ non-deg. quadratic form on $A$.
- $\mathrm{Rep}(D^{\omega}G)$, $\omega$ a 3-cocycle on $G$ a finite group.
- Quantum groups/Kac-Moody algebras: subquotients of $\mathrm{Rep}(U_q\mathfrak{g})$ at $q = e^{\pi i/l}$ or level $k$ integrable $\hat{\mathfrak{g}}$-modules, e.g.
  - $\mathrm{SU}(N)_k = \mathcal{C}(\mathfrak{sl}_N, N+k)$,
  - $\mathrm{SO}(N)_k$,
  - $\mathrm{Sp}(N)_k$,
  - for $\gcd(N,k) = 1$, $\mathrm{PSU(N)_k} \subset \mathrm{SU}(N)_k$ "even half"
- Drinfeld center: $\mathcal{Z}(\mathcal{D})$ for spherical fusion category $\mathcal{D}$.
- Rank-finiteness (see E. Rowell's poster).

# Topological and Global Symmetry

**The** **categorical symmetry group** **<u>Aut($\mathcal{C}$)</u> of an anyon model $\mathcal{C}$ consists of all permutations of anyon types and transformations of fusion states {|a,b,c,$\mu$>} that preserve all defining data up to gauge freedom. In math jargon, all braided tensor auto-equivalences of $\mathcal{C}$.**




**Given an anyon model $\mathcal{C}$, its <u>Aut($\mathcal{C}$)</u> is classified by a triple**

$$(\Pi_1, \Pi_2, \kappa),$$

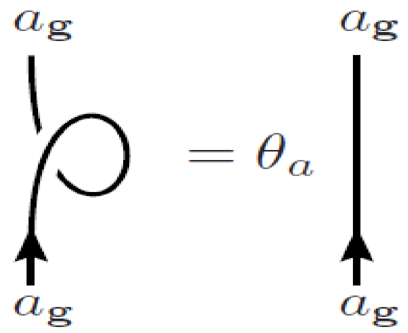
**where $\Pi_1$ is the** **classes of** **braided tensor auto-equivalences of $\mathcal{C}$ , $\Pi_2$=$\mathcal{A}$ the abelian anyons of $\mathcal{C}$ , and $\kappa \in H^3(\Pi_1,\Pi_2)$ a cohomology class.**

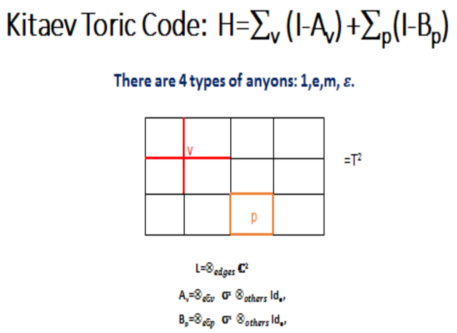
**$\Pi_1$=Aut($\mathcal{C}$) will be called the** **topological symmetry group** **of $\mathcal{C}$.**

**Given a group G, a** **global G-symmetry** **of $\mathcal{C}$ is $\rho$: G→Aut($\mathcal{C}$) --- a group homomorphism.**

# Symmetries of Abelian Anyon Models

- An **abelian anyon model** is given by **a pair $\mathcal{C}$=(A,q),**
  where A is a commutative finite group and
  q(x) is the topological twist of anyon type x ∈A, q: A→U(1).
- The topological symmetry group $\Pi_1$=Aut($\mathcal{C}$)
  is the group O(A,q)={s∈ Aut(A): q(s(x))=q(x) for all x∈A}
  and $\Pi_2$=A
- $U(1)_3$: A=$Z_3$, q(x)={1,$e^{\frac{2\pi i}{3}}$, $e^{\frac{2\pi i}{3}}$}, $\Pi_1$=$Z_2$, $\Pi_2$=$Z_3$.
- Toric code and 3fermion: both A=$Z_2 \oplus Z_2$ ={1,,e,m,$\psi$} and
  q(x)={1,1,1,-1} or q(x)={1,-1,-1,-1}, so $\Pi_1$= $Z_2$ or $S_3$.



# Origin of Symmetry Fractionalization: Topological Symmetry Is Categorical

Given a global symmetry (G, $\rho$) realized as symmetries $R_g$ of a Hamiltonian with a local Hilbert space L(Y;l), then L(Y;l)=$\bigoplus L_{\lambda_i}$ according to energy levels $\lambda_i$. The ground state manifold $L_{\lambda_0}$ further decomposes as V(Y;t)$\otimes L_{\lambda_0}^{loc}$(Y;l), where V(Y;t) is the topological part and $L_{\lambda_0}^{loc}$(Y;l) the local part. On-site symmetries $R_g$ act on $L_{\lambda_0}$= V(Y;t)$\otimes L_{\lambda_0}^{loc}$(Y;l) split as $\rho_g \otimes \prod_l R_g^l$.

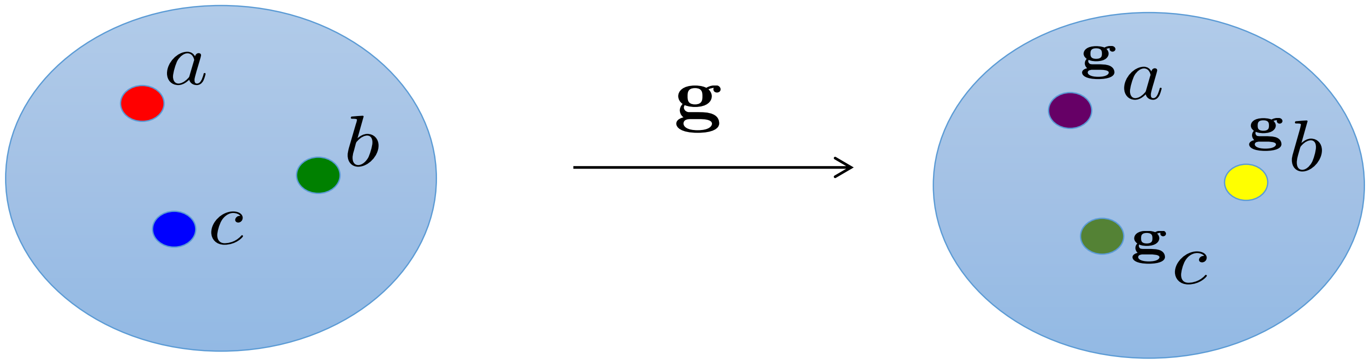
**Anyon states in V(Y;t) are universality classes up to local actions, so global symmetry actions are not exact.** Hence, projective local actions on $L_{\lambda_0}^{loc}$(y;l) are allowed to compensate for the overall phases from the global actions. Since projective representations of G are classified by $H^2$(G, U(1)), can symmetry factionalizations be classified by $H^2$(G, U(1))?

**The separation of global symmetry into topological and local parts requires subtle consistency:**

1. **A potential obstruction;**
2. **The coefficient for $H^2$ is not U(1), but $\Pi_2$={abelian anyons}.**

# Global Symmetry G

$\rho : G \longrightarrow Aut(\mathcal{C})$




$${}^{\mathbf{g}}a \equiv \rho_{\mathbf{g}}(a)$$

Natural Isomorphism

$$\rho_{\mathbf{gh}} = \kappa_{\mathbf{g},\mathbf{h}} \rho_{\mathbf{g}} \rho_{\mathbf{h}}$$

$$\rho_{\mathbf{g}}(|a,b;c\rangle) = U_{\mathbf{g}}({}^{\mathbf{g}}a, {}^{\mathbf{g}}b; {}^{\mathbf{g}}c)|{}^{\mathbf{g}}a, {}^{\mathbf{g}}b; {}^{\mathbf{g}}c\rangle$$

$\rho$ leads to an obstruction $o_3(\rho) \in H^3_\rho$ (G,$\mathcal{A}$)

$\mathcal{A} \subseteq \mathcal{C}$

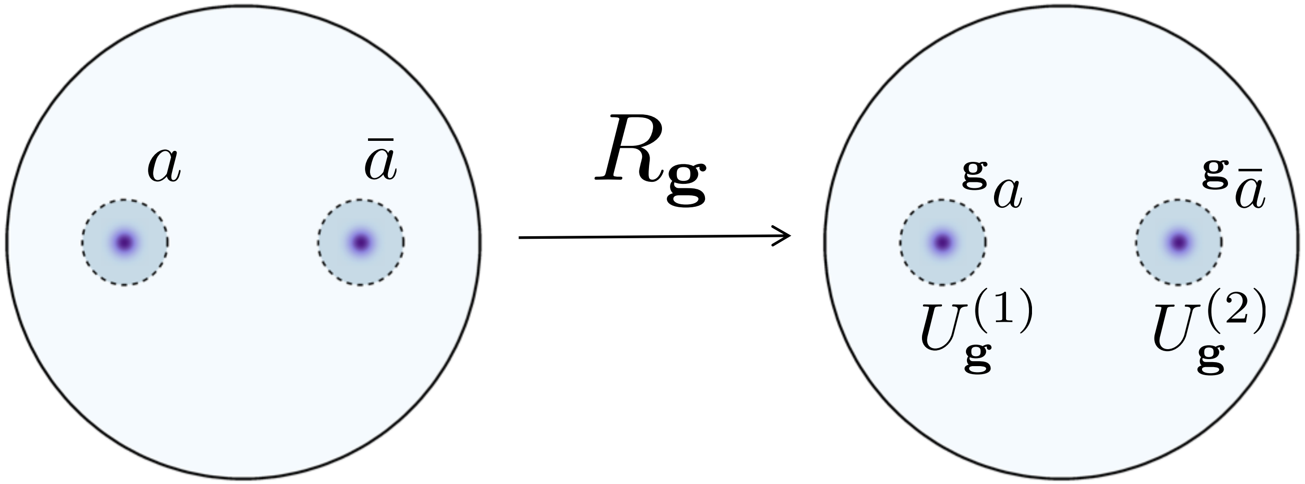
Abelian anyons

# Symmetry Localization

Ground state is symmetric: $R_{\mathbf{g}}|\Psi_0\rangle = |\Psi_0\rangle$

Consider state with two anyons:

$$R_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle = U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}\,\rho_{\mathbf{g}}|\Psi_{a,\bar{a};0}\rangle$$
$$= U_{\mathbf{g}}^{(1)}U_{\mathbf{g}}^{(2)}U_{\mathbf{g}}({}^{\mathbf{g}}a, {}^{\mathbf{g}}\bar{a};0)|\Psi_{{}^{\mathbf{g}}a,{}^{\mathbf{g}}\bar{a};0}\rangle$$



# Symmetry Fractionalization

Anyons can form a projective representation

$$U_{\mathbf{g}}^{(j)} U_{\mathbf{h}}^{(j)} \neq U_{\mathbf{gh}}^{(j)}$$

Even if $R_{\mathbf{g}} R_{\mathbf{h}} = R_{\mathbf{gh}}$

General Result: Symmetry Fractionalization

1. Requires $o_3(\rho) = 0$ ( $H^3_\rho(G, \mathcal{A})$ obstruction must vanish)
2. Classified by $H^2_\rho(G, \mathcal{A})$

$\mathcal{A} \subseteq \mathcal{C}$

Abelian anyons

# Symmetry Fractionalization Mathematically

The obstruction $o_3(\rho) = \rho^*(\kappa) \in H^3(G, \Pi_2)$ :

**the pull back of the class κ in $(\Pi_1, \Pi_2, \kappa)$ to $H^3(G, \Pi_2)$ by the global symmetry $\rho$ : G→$\Pi_1$.**

If $o_3(\rho) = 0$, then possible symmetry fractionalizations form a torsor over $H^2(G, \Pi_2)$.

A set X is a torsor over a group G if X has a transitive free action of G.

# Vanishing of Symmetry Fractionalization Obstruction

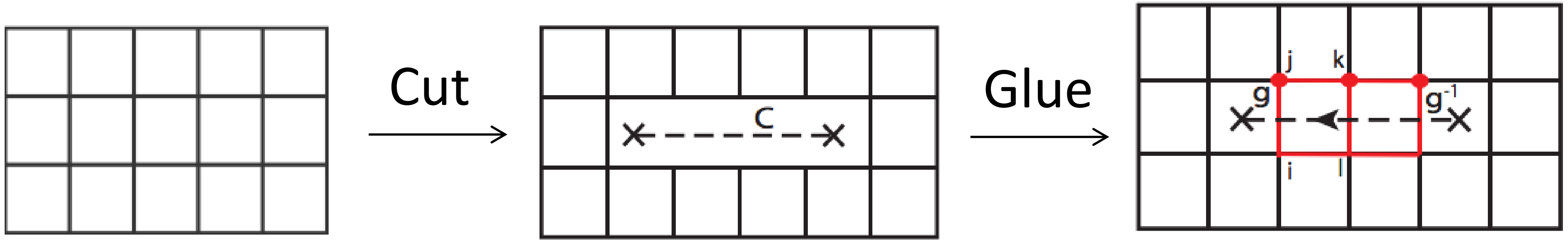
**Theorem:**

The obstruction to symmetry fractionalization vanishes if either

1) the global symmetry $\rho$ does not permute anyon types or
2) the anyon model is abelian with all 6j symbols trivial, i.e. the associativity 3-cocyle $\omega$ is trivial.

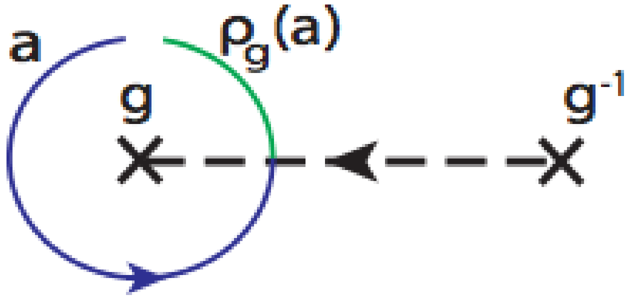
**It follows that the obstructions to symmetry fractionalizations for toric code and 3fermion all vanish.**

# Symmetry Defects




$$H_0 = \sum_i h_i + \sum_{\langle ij \rangle} h_{ij}$$

$$H_{\mathbf{g},\mathbf{g}^{-1}} = H_0 + \sum_{\substack{\langle ij \rangle : \\ i \in C_l ; j \in C_r}} [R_{\mathbf{g}}^{(j)} h_{ij} R_{\mathbf{g}}^{(j)-1} - h_{ij}]$$

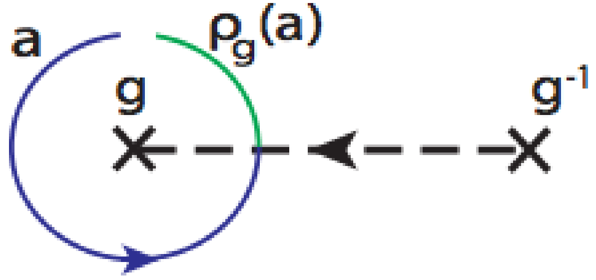



**Given a topological phase with symmetry G, extrinsic point-like defects can be introduced by modifying the original Hamiltonian**

# Defects Confined

Defects are NOT finite-energy quasiparticle excitations/anyons

Cannot be described by original UMTC

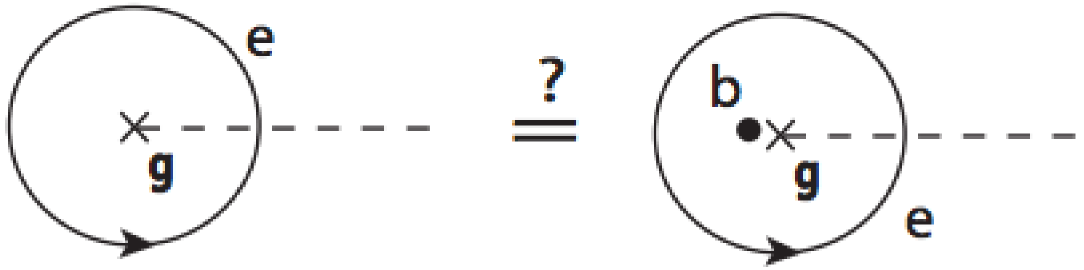



Mathematics: G-Crossed Braided Tensor Category

We would like to have methods to systematically compute all properties of defects (fusion rule, braiding ,etc)

# G-Graded Fusion

Topologically distinct types of g-defects




$$\mathcal{C}_G = \bigoplus_{\mathbf{g} \in G} \mathcal{C}_{\mathbf{g}}$$

$\mathcal{C}_{\mathbf{g}}$ contains collection of g-defects. Module category

$$a_{\mathbf{g}} \times b_{\mathbf{h}} = \sum_{c \in \mathcal{C}_{\mathbf{gh}}} N_{ab}^{c} c$$

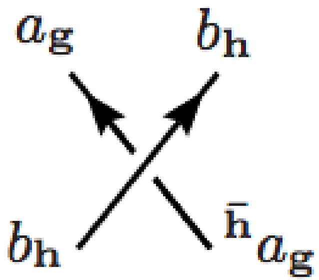
$$\boxed{\mathcal{D}_{\mathbf{g}} = \mathcal{D}_{\mathbf{0}}}$$

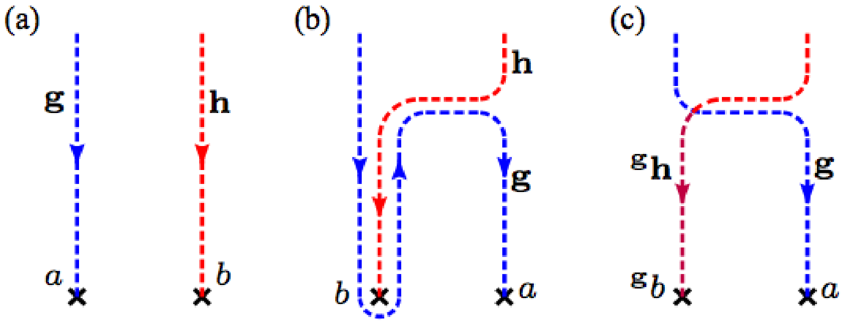
# Obstructions to Defectification

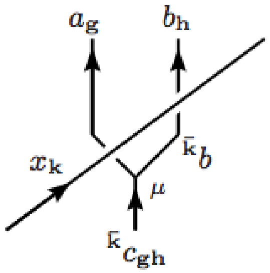
- Obstruction $o_3(\rho)$ to symmetry fractionalization is also the obstruction to a consistent fusion rule for $\mathcal{C}_g$. If $o_3(\rho)=0$ , then consistent fusion rules are in 1-1 correspondence with symmetry fractionalization classes ($\rho$,t).
- Pentagons lead to a secondary obstruction $o_4(\rho,t) \in H^4$(G, U(1)) to consistently defectify.
- If $o_4(\rho,t)=0$, possible defectifications form a torsor over $H^3$(G,U(1)).
- If both obstructions=0, a defect theory is determined by (G, $\rho$, t, $\alpha$), where $\alpha \in H^3$(G,U(1)).

# G-Crossed Braiding

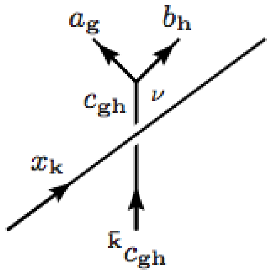
$$R^{a_{\mathbf{g}} b_{\mathbf{h}}} =$$



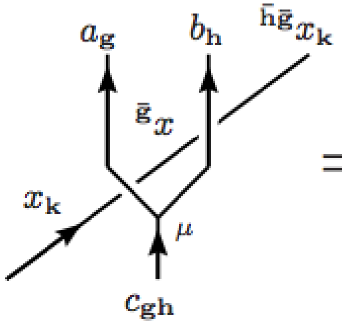

$$= \sum_{\nu} \left[U_{\mathbf{k}}(a,b;c)\right]_{\mu\nu}$$



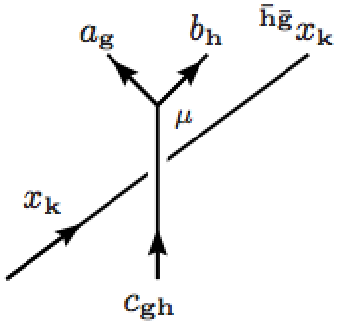

$$\left[U_{\mathbf{0}}(a,b;c)\right]_{\mu\nu} = \delta_{\mu\nu}$$

$$U_{\mathbf{k}}(0,0;0) = U_{\mathbf{k}}(a,0;a)$$

$$= U_{\mathbf{k}}(0,b;b) = 1$$




$$= \eta_x(\mathbf{g},\mathbf{h})$$

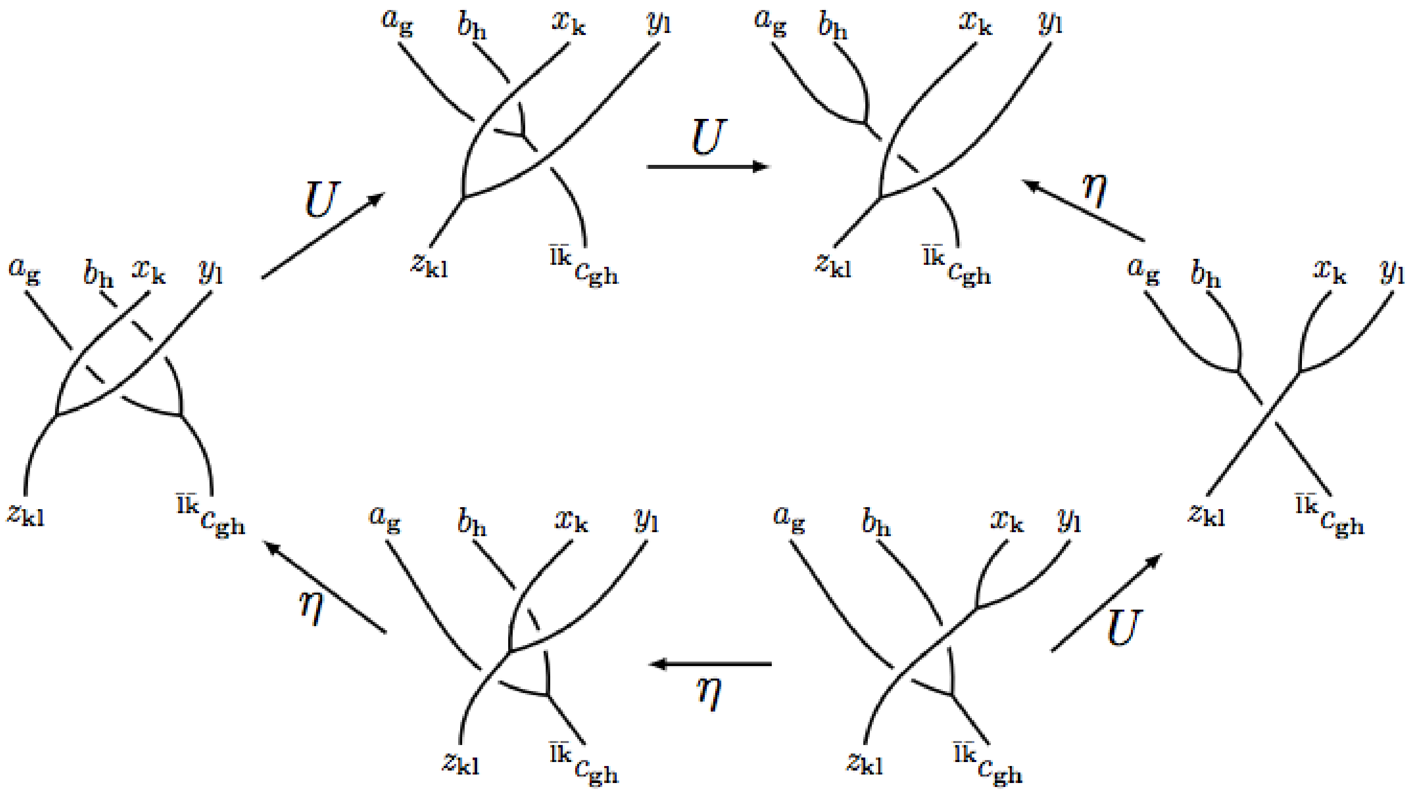



$$\eta_0(\mathbf{g},\mathbf{h}) = 1$$

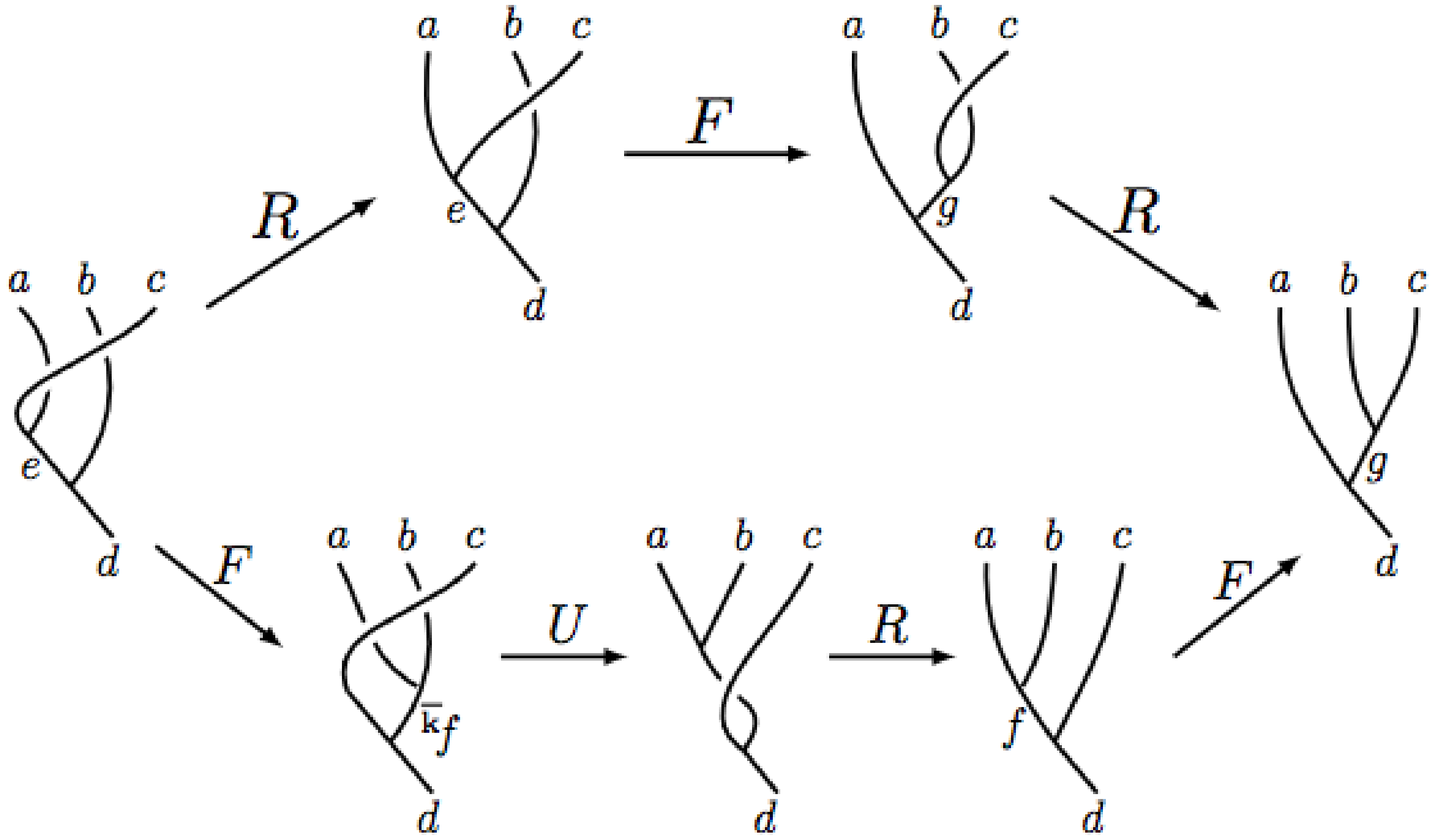
$$\eta_x(\mathbf{0},\mathbf{0}) = \eta_x(\mathbf{g},\mathbf{0})$$

$$= \eta_x(\mathbf{0},\mathbf{h}) = 1$$

# Sliding Consistency



# G-Crossed Heptagon




G-Crossed version of hexagon equation

# G-Crossed Data: Skeletonization

G-Crossed UBTC $\mathcal{C}_G^{\times}$ characterized by data

$$\{L, N_{ab}^{c}, F_{d}^{abc}, R_{c}^{ab}, \eta_a(\boldsymbol{g}, \boldsymbol{h}), U_{\boldsymbol{k}}(a, b; c)\}$$

Subject to consistency equations

Inequivalent solutions $\longleftrightarrow$ Distinct SET phases

**Gauge-Invariant quantities = Topological invariants of SET**

# Gauge Transformations

(1) Vertex basis gauge transformations (Old type)

$$\widetilde{|a,b;c,\mu\rangle} = \sum_{\mu'} \left[\Gamma_c^{ab}\right]_{\mu\mu'} |a,b;c,\mu'\rangle$$

$$\left[\widetilde{R}_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}}\right]_{\mu\nu} = \sum_{\mu',\nu'} \left[\Gamma_c^{b\,{}^{\bar{\mathbf{h}}}a}\right]_{\mu\mu'} \left[R_{c_{\mathbf{gh}}}^{a_{\mathbf{g}} b_{\mathbf{h}}}\right]_{\mu'\nu'} \left[\left(\Gamma_c^{ab}\right)^{-1}\right]_{\nu'\nu}$$

$$\left[\widetilde{U}_{\mathbf{k}}(a,b;c)\right]_{\mu\nu} = \sum_{\mu',\nu'} \left[\Gamma_{{}^{\bar{\mathbf{k}}}c}^{{}^{\bar{\mathbf{k}}}a\,{}^{\bar{\mathbf{k}}}b}\right]_{\mu\mu'} \left[U_{\mathbf{k}}(a,b;c)\right]_{\mu'\nu'} \left[\left(\Gamma_c^{ab}\right)^{-1}\right]_{\nu'\nu}$$

$$\left[\widetilde{F}_d^{abc}\right]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sum_{\alpha',\beta',\mu',\nu'} \left[\Gamma_e^{ab}\right]_{\alpha\alpha'} \left[\Gamma_d^{ec}\right]_{\beta\beta'} \left[F_d^{abc}\right]_{(e,\alpha',\beta')(f,\mu',\nu')} \left[\left(\Gamma_f^{bc}\right)^{-1}\right]_{\mu'\mu} \left[\left(\Gamma_d^{af}\right)^{-1}\right]_{\nu'\nu}$$

# Gauge Transformations

(2) Symmetry Action Gauge Transformations (New Type)

Associated with natural isomorphism $\check{\rho}_{\mathbf{g}} = \Upsilon_{\mathbf{g}} \rho_{\mathbf{g}}$

$$\left[\check{U}_{\mathbf{k}}\left(a,b;c\right)\right]_{\mu\nu} = \frac{\gamma_a(\mathbf{k})\gamma_b(\mathbf{k})}{\gamma_c(\mathbf{k})}\left[U_{\mathbf{k}}\left(a,b;c\right)\right]_{\mu\nu}$$

$$\left[\check{R}^{a_{\mathbf{g}} b_{\mathbf{h}}}_{c_{\mathbf{gh}}}\right]_{\mu\nu} = \gamma_a(\mathbf{h})\left[R^{a_{\mathbf{g}} b_{\mathbf{h}}}_{c_{\mathbf{gh}}}\right]_{\mu\nu}$$

$$\check{\eta}_x(\mathbf{g},\mathbf{h}) = \frac{\gamma_x(\mathbf{gh})}{\gamma_{\bar{\mathbf{g}}x}(\mathbf{h})\gamma_x(\mathbf{g})}\eta_x(\mathbf{g},\mathbf{h}) \qquad \gamma_0(\mathbf{h}) = \gamma_a(\mathbf{0}) = 1$$

# Invariants of Modular Tensor Category

MTC $\mathcal{C}$ ⇄ RT (2+1)-TQFT $(V, Z)$

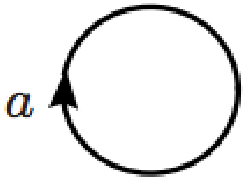
- Pairing $\langle Y^2, \mathcal{C}\rangle = V(Y^2; \mathcal{C}) \in \mathrm{Rep}(\mathcal{M}(Y^2))$ for a surface $Y^2$, $\mathcal{M}(Y^2)$ = mapping class group
- Pairing $Z_{X,L,\mathcal{C}} = \langle (X^3, L_C), \mathcal{C}\rangle \in \mathbb{C}$ for colored framed oriented links $L_c$ in 3-mfd $X^3$

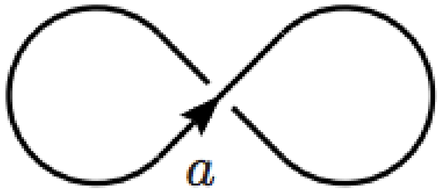
  fix $\mathcal{C}$, $Z_{X,L,\mathcal{C}}$ invariant of $(X^3, L_c)$

  *fix $(X^3, L_c)$, $Z_{X,L,\mathcal{C}}$ invariant of $\mathcal{C}$*

  *fix $Y^2$, $V(Y^2; \mathcal{C})$ invariant of $\mathcal{C}$*

# Quantum Dimensions, Twists, and S-matrix: Unknot and Hopf Link

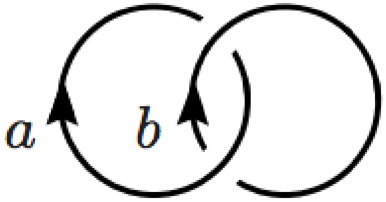
Quantum Dimension $d_a =$ 


Twist $\theta_a = \frac{1}{d_a}$ 


$$= \sum_{c,\mu} \frac{d_c}{d_a} [R_c^{aa}]_{\mu\mu}$$

Total Quantum Dimension $\mathcal{D} = \sqrt{\sum_{a \in \mathcal{C}} d_a^2}$

$$S_{ab} = \mathcal{D}^{-1} \sum_c N_{\bar{a}b}^c \frac{\theta_c}{\theta_a \theta_b} d_c = \frac{1}{\mathcal{D}}$$

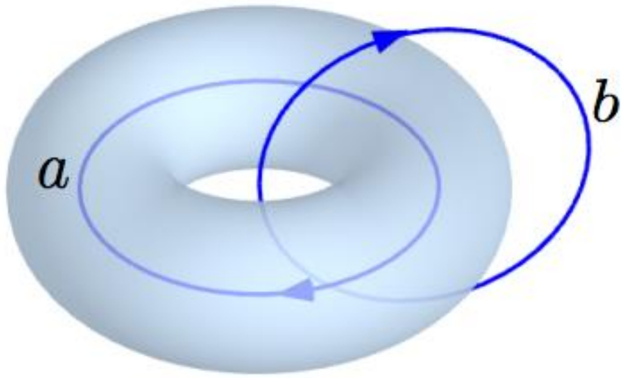


# Verlinde Algebra and Modularity: Rep. of SL(2,$\mathbb{Z}$)=MCG of $T^2$

For a Unitary Modular Tensor Category,

$$(ST)^3 = e^{i\pi c/4} C \qquad S^2 = C \qquad C^2 = \mathbb{I}$$

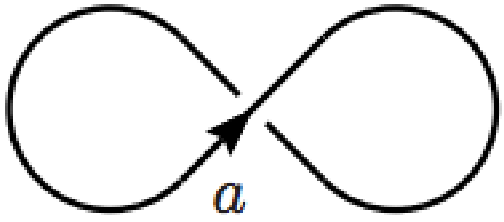
$$T_{ab} = \theta_a \delta_{ab}$$



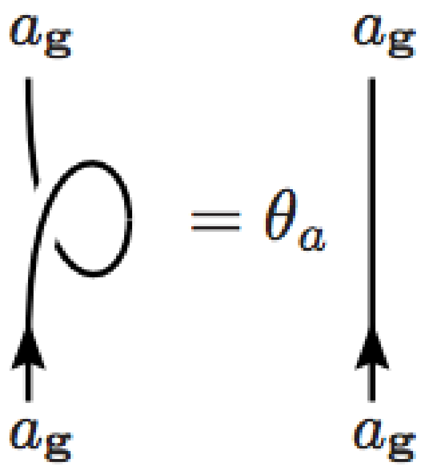

Dimension of ground state Hilbert space on torus = $|\mathcal{C}|$

$$|a\rangle_l = \sum_{b\in\mathcal{C}} S_{ab} |b\rangle_m$$

# Topological Twists

$$\theta_a = \frac{1}{d_a}$$ 


$$= \sum_{c,\mu} \frac{d_c}{d_a} [R_c^{aa}]_{\mu\mu}$$

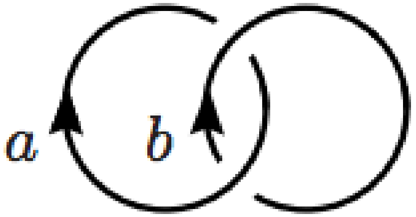



Type (2) Gauge transformations: $\check{\theta}_{a_{\mathbf{g}}} = \gamma_{a_{\mathbf{g}}}(\mathbf{g})\theta_{a_{\mathbf{g}}}$

Twist of defects is not gauge-invariant, as expected

# Topological S-Matrix

$$S_{a_{\mathbf{g}}b_{\mathbf{h}}} = \frac{1}{\mathcal{D}_0}$$




$$= \frac{1}{\mathcal{D}_0} \sum_{c,\mu} d_c \frac{\theta_c}{\theta_{\bar{a}}\theta_b} \frac{\left[U_{\bar{\mathbf{g}}\mathbf{h}}(\bar{a}, b; c)\right]_{\mu\mu}}{\eta_{\bar{a}}(\bar{\mathbf{g}}, \mathbf{h})\eta_b(\mathbf{h}, \bar{\mathbf{g}})}$$

Type II Gauge transformations: $\check{S}_{a_{\mathbf{g}}b_{\mathbf{h}}} = \gamma_{\bar{a}}(\mathbf{h})\gamma_b(\bar{\mathbf{g}})S_{a_{\mathbf{g}}b_{\mathbf{h}}}$

G-Crossed Verlinde Formula:

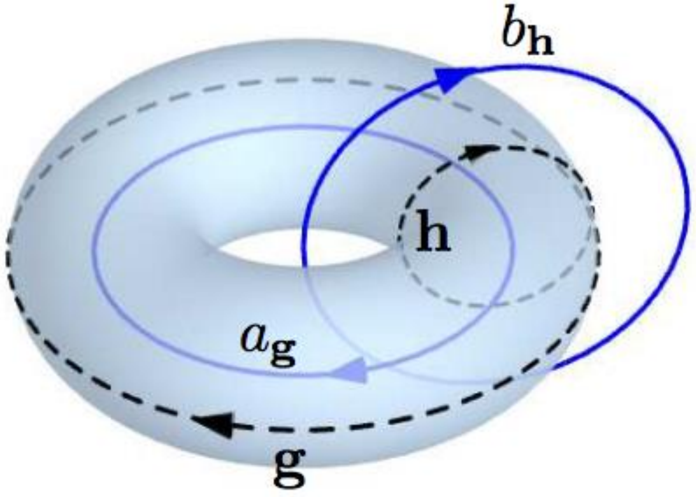
$$N_{a_{\mathbf{g}}b_{\mathbf{h}}}^{c_{\mathbf{gh}}} = \sum_{x_0 \in \mathcal{C}_0^{\mathbf{g},\mathbf{h}}} \frac{S_{a_{\mathbf{g}}x_0} S_{b_{\mathbf{h}}x_0} S^{*}_{c_{\mathbf{gh}}x_0}}{S_{0x_0}} \eta_x(\bar{\mathbf{h}}, \bar{\mathbf{g}})$$

# Extended Verlinde Algebra

$$\mathcal{V}^{\text{ext}} = \bigoplus_{(\mathbf{g},\mathbf{h}),\mathbf{gh}=\mathbf{hg}} \mathcal{V}_{(\mathbf{g},\mathbf{h})}$$

$$\mathcal{S}^{(\mathbf{g},\mathbf{h})} \;:\; \mathcal{V}_{(\mathbf{g},\mathbf{h})} \to \mathcal{V}_{(\mathbf{h},\bar{\mathbf{g}})}$$

$$\mathcal{T}^{(\mathbf{g},\mathbf{h})} \;:\; \mathcal{V}_{(\mathbf{g},\mathbf{h})} \to \mathcal{V}_{(\mathbf{g},\mathbf{gh})}$$




$$\dim \mathcal{V}_{(\mathbf{g},\mathbf{h})} = |\mathcal{C}_{\mathbf{g}}^{\mathbf{h}}| \qquad \mathcal{C}_{\mathbf{g}}^{\mathbf{h}} = \{\, a \in \mathcal{C}_{\mathbf{g}} \mid {}^{\mathbf{h}}a = a \,\}$$

$$|\mathcal{C}_{\mathbf{h}}^{\mathbf{g}}| = |\mathcal{C}_{\mathbf{g}}^{\mathbf{h}}| \qquad \boxed{|\mathcal{C}_{\mathbf{g}}| = |\mathcal{C}_{\mathbf{0}}^{\mathbf{g}}|}$$

# G-Crossed Modularity

For G-Crossed UBTC, define modular matrices:

$$\mathcal{S}^{(\mathbf{g},\mathbf{h})}_{a_{\mathbf{g}}b_{\mathbf{h}}} = \frac{S_{a_{\mathbf{g}}b_{\mathbf{h}}}}{U_{\mathbf{h}}(a,\bar{a};0)} \qquad \mathcal{T}^{(\mathbf{g},\mathbf{h})}_{a_{\mathbf{g}}b_{\mathbf{g}}} = \eta_a(\mathbf{g},\mathbf{h})\theta_{a_{\mathbf{g}}}\delta_{a_{\mathbf{g}}b_{\mathbf{g}}}$$

$$C^{(\mathbf{g},\mathbf{h})}_{a_{\mathbf{g}}b_{\bar{\mathbf{g}}}} = \frac{1}{U_{\mathbf{h}}(\bar{b},b;0)\eta_b(\mathbf{h},\bar{\mathbf{h}})}\delta_{a_{\mathbf{g}}\overline{b_{\bar{\mathbf{g}}}}}$$

$$(\mathcal{S}\mathcal{T})^3 = e^{i\pi c/4}C \quad \mathcal{S} = \mathcal{S}^{\dagger}C \quad C^2 = \mathbb{1}$$

Unitarity of S → Representation of SL(2, $\mathbb{Z}$): Homotopy TQFT

# Gauging Global Symmetry G

**Given a topological order $\boldsymbol{\mathcal{C}}$, then gauging (G, $\boldsymbol{\rho}$, $\mathrm{t}$, $\alpha$) of $\boldsymbol{\mathcal{C}}$ is:**

**Step I:**

Defectify $\boldsymbol{\mathcal{C}}$, $\boldsymbol{\mathcal{C}}_G^{\times}=\bigoplus_g \boldsymbol{\mathcal{C}}_g$, where $\boldsymbol{\mathcal{C}}_e=\boldsymbol{\mathcal{C}}$.

**Step II:**

Orbifold $\boldsymbol{\mathcal{C}}_G^{\times}$, a new topological order $\boldsymbol{\mathcal{C}}/\mathrm{G}=(\boldsymbol{\mathcal{C}}_G^{\times})^G$.

**Gauging deconfines defects and leads to a topological phase transition from $\boldsymbol{\mathcal{C}}$ to $\boldsymbol{\mathcal{C}}/\mathrm{G}$.**

# Gauged Theory

Objects in $\mathcal{C}/G$

$$[a] = \{{}^{\mathbf{g}}a, \forall \mathbf{g} \in G\}$$

$$G_a = \{\, \mathbf{g} \in G \,|\, {}^{\mathbf{g}}a = a\,\}$$

$\pi_a$ = irreducible projective representation of $G_a$

$$\pi_a(\mathbf{g})\pi_a(\mathbf{h}) = \eta_a(\mathbf{g}, \mathbf{h})\pi_a(\mathbf{gh}) \quad \mathbf{g}, \mathbf{h} \in G_a$$

$$([a], \pi_a) \in \mathcal{C}/G$$

Flux-Charge composite

# General Results

- The anyon model $\mathcal{C}/\mathrm{G}$ =$(\mathcal{C}_G^{\times})^G$ contains a sub-category Rep(G).
- ${D^2}_{\mathcal{C}/G}$=$D_{\mathcal{C}}^2\ |G|^2$. Same central charge.
- Gauging done sequentially if N⊂G normal: first N and then G/N.
- If $\mathcal{C}$ is a quantum double, then $\mathcal{C}/\mathrm{G}$ a double.
- $\mathcal{C}$ and $\mathcal{C}/\mathrm{G}$ same up to doubles.

- **Inverse process of gauging:**

  When Rep(G) in $\mathcal{C}/\mathrm{G}$ =$(\mathcal{C}_G^{\times})^G$ condensed, $\mathcal{C}$ recovered.

# Particle-Hole Symmetry of Bosonic $\boldsymbol{Z_3}$

**Consider p-h symm. of $\boldsymbol{Z_3}$---No symm. fractionalization as $\boldsymbol{H^2(Z_2, Z_3)}$=0.**

**Defectification:**

Only one twist defect g in $C_1$: g⨂g=1+a+$\bar{a}$. This theory is NOT braided---Tambara-Yamagami theory for $Z_3$. But it has a G-crossed braiding. There are two ways to have an defect as $H^3(Z_2,\mathrm{U}(1))=Z_2$.

**Gauging:**

Taking the equivariant quotient results either $SU(2)_4$ or its cousin Jones-Kauffman theory at r=6---two metaplectic theories corresponding to the two classes in $H^3(Z_2,\mathrm{U}(1))=Z_2$ as above.

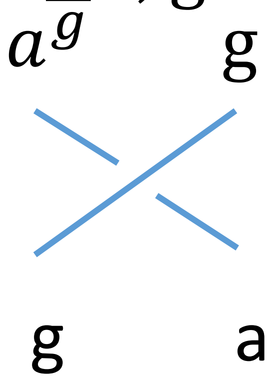
# Braided G-crossed $\boldsymbol{Z_3}$-Tambara-Yamagami

The 6j symbols for the $Z_3$-Tambara-Yamagami theory is (unlisted admissible 6j symbols and R-symbols=1):

$\boldsymbol{F_g^{agb}}$=$\boldsymbol{F_b^{gag}}$=$\boldsymbol{\chi}$**(a,b),** $\boldsymbol{F_{g,ab}^{ggg}}$=$\boldsymbol{\frac{\kappa}{\sqrt{3}}\chi^{-1}}$**(a,b),**

where $\chi$(a,b) is a symmetric bi-character of $Z_3$ and $\kappa$=$\pm$1, g=defect and a,b $\in Z_3$.

It is known that this theory is NOT braided.




But it is G-crossed braided:

$\boldsymbol{R_g^{ga}}$=$\boldsymbol{R_g^{ag}}$=$\boldsymbol{\omega^{2a^2}}$ **and** $\boldsymbol{R_a^{gg}}$=$\boldsymbol{(-i\kappa)^{1/2}\omega^{a^2}}$**, a=0,1,2.**

# Modular G-crossed Category

- The extended Verlinde algebra has 4 sectors: $V_{0,0}$, $V_{0,1}$, $V_{1,0}$, $V_{1,1}$, and $\tilde{s}$-, $\tilde{t}$-matrices form a rep. of SL(2,$\mathbb{Z}$). Below the **s,t** are those of the $Z_3$ theory.
- The extended $\tilde{s}$-matrix $\tilde{s}=\begin{pmatrix} \boldsymbol{s} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\kappa \end{pmatrix}$
- The extended $\tilde{t}$ matrix $\tilde{t}=\begin{pmatrix} \boldsymbol{t} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (-i\kappa)^{1/2} \\ 0 & 0 & (-i\kappa)^{1/2} & 0 \end{pmatrix}$

# Gauging As Construction of New UMTCs

- 3-fermion theory (toric code sister): $SO(8)_1$ with G=$S_3$
- S-,T-matrices:

$$\begin{bmatrix}
1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\
1 & 1 & 2 & 3 & 3 & 4 & 4 & 4 & -3\sqrt{2} & -3\sqrt{2} & -3\sqrt{2} & -3\sqrt{2} \\
2 & 2 & 4 & 6 & 6 & -4 & -4 & -4 & 0 & 0 & 0 & 0 \\
3 & 3 & 6 & -3 & -3 & 0 & 0 & 0 & -3\sqrt{2} & -3\sqrt{2} & 3\sqrt{2} & 3\sqrt{2} \\
3 & 3 & 6 & -3 & -3 & 0 & 0 & 0 & 3\sqrt{2} & 3\sqrt{2} & -3\sqrt{2} & -3\sqrt{2} \\
4 & 4 & -4 & 0 & 0 & b & c & a & 0 & 0 & 0 & 0 \\
4 & 4 & -4 & 0 & 0 & c & a & b & 0 & 0 & 0 & 0 \\
4 & 4 & -4 & 0 & 0 & a & b & c & 0 & 0 & 0 & 0 \\
3\sqrt{2} & -3\sqrt{2} & 0 & -3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 6 & -6 \\
3\sqrt{2} & -3\sqrt{2} & 0 & -3\sqrt{2} & 3\sqrt{2} & 0 & 0 & 0 & 0 & 0 & -6 & 6 \\
3\sqrt{2} & -3\sqrt{2} & 0 & 3\sqrt{2} & -3\sqrt{2} & 0 & 0 & 0 & 6 & -6 & 0 & 0 \\
3\sqrt{2} & -3\sqrt{2} & 0 & 3\sqrt{2} & -3\sqrt{2} & 0 & 0 & 0 & -6 & 6 & 0 & 0
\end{bmatrix}$$

$$a = -8\cos\frac{2\pi}{9}, b = -8\sin\frac{\pi}{9}, c = 8\cos\frac{\pi}{9}.$$

| Label | $d$ | $\theta$ |
|---|---|---|
| $(I,+)$ | 1 | 1 |
| $(I,-)$ | 1 | 1 |
| $\{a,\overline{a}\}$ | 2 | 1 |
| $(Y,+)$ | 3 | $-1$ |
| $(Y,-)$ | 3 | $-1$ |
| $\{w,\overline{w}\}$ | 4 | $\alpha^{-1/3}$ |
| $\{wa,\overline{wa}\}$ | 4 | $\omega\alpha^{-1/3}$ |
| $\{w\overline{a},\overline{w}a\}$ | 4 | $\omega^2\alpha^{-1/3}$ |
| $(\sigma_+,+)$ | $3\sqrt{2}$ | $e^{\frac{i\pi\nu}{8}}$ |
| $(\sigma_-,+)$ | $3\sqrt{2}$ | $-e^{-\frac{i\pi\nu}{8}}$ |
| $(\sigma_+,-)$ | $3\sqrt{2}$ | $-e^{\frac{i\pi\nu}{8}}$ |
| $(\sigma_-,-)$ | $3\sqrt{2}$ | $e^{-\frac{i\pi\nu}{8}}$ |

$\nu = 1, \omega = e^{2\pi i/3}, \alpha = e^{4\pi i/3}$

# Summary

**We skeletonize an existing mathematical theory and formulate it into a physical theory with full computational power for symmetry, defects, and gauging of 2D topological phases.**

**It provides a general framework to classify symmetry enriched 2D topological phases of matter.**

$$\mathcal{C} \underset{\text{Confinement}}{\overset{\text{Defectification}}{\rightleftarrows}} \mathcal{C}_G^{\times} \underset{\text{Condensation}}{\overset{\text{Gauging}}{\rightleftarrows}} \mathcal{C}/G$$