9 Ancient Greek Mathematics

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This chapter is addressed to those who wish to read the primary sources of Greco-Roman mathematics, either in the original languages or in modern translations. Hence, it focuses on the kinds of mathematics that was disseminated in treatises written by scholars who were members of a relatively small literary elite. This theoretical style of mathematics was not the only kind of mathematics practised in Greco-Roman antiquity, and, indeed, the total number authors of philosophical mathematics must have been dwarfed by the number of individuals who used practical mathematics in their daily work, and who passed on such mathematical skills to their sons, disciples, and apprentices.[1] Nevertheless, the literary works produced by this self-selected group of individuals have elicited the admiration and study of mathematical scholars throughout the centuries, and have justly been regarded as one of the most important products of ancient scholarship.

The modern reader who encounters these works in their original presentation may, however, have the uncanny feeling of experiencing something that is at once both reassuringly familiar and yet strangely alien. Much of the mathematics that we learn in school derives from Greco-Roman origins, but many of the actual interests and methods of ancient mathematicians are no longer part of our approach to mathematics. In order to read the ancient sources, however, we must try to recreate their interests and to follow through with their methods. This chapter is meant to be an introduction to this process.

It begins with a discussion of the evidence itself, with emphasis on how far removed this often is from the mathematical activity we are trying to understand. It then situates the production of literary mathematical texts in a broader context of mathematical activities, including oral presentation and material practices. Finally, special topics of mathematical practice are addressed: the overall role of structure, the production of various types of argument, the function of constructions and constructivist thinking, and the execution of operations and algorithms.

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SOURCES FOR THEORETICAL MATHEMATICS

Our evidence for ancient Greek mathematical activity comes, almost exclusively, from texts that were passed down through the medieval period in contexts that were often not devoted to mathematical activity and by individuals who generally did not themselves produce original mathematics. Although this is true for all of the ancient Greek theoretical sciences, the situation is perhaps more pronounced in the case of mathematics.

The manuscript sources are divided by modern scholars into direct and indirect traditions. The direct tradition consists of manuscripts of source texts, in Greek, while the indirect traditions are made up of commentaries and summaries in Greek along with translations and their commentaries, largely in Arabic and Latin (Lorch 2001). For understanding Greek mathematics, the most important indirect traditions are the Arabic translations that derive from the eighth- and ninth-century translation activity in Baghdad, and the twelfth- and thirteenth-century Latin translations, from either Greek or Arabic. From this description, it might seem that the direct tradition could be treated as the principal source, so that the indirect traditions could be neglected except in cases where the direct tradition was deficient.

The difficulty with this assumption, however, is that even in the direct tradition the mathematical texts were subjected to numerous revisions over the centuries, the details of which are now mostly lost to us. In the case of religious and literary texts, the actual words of the original author were considered sacrosanct and the ancient and medieval editors conceived of their role as the preservation of these words themselves. In the case of the exact sciences, however, the texts were often edited by scholars who were themselves expert in the fields that the texts transmitted. These scholars often took the scope of their role to include a correction of the words of the text based on their own understanding of the ideas that the words conveyed. Hence, the Greek mathematical texts must be understood as canonical in the sense that the canon was somewhat flexible and subject to repeated reinterpretation. Both the selection of texts that we are now able to read, and the specific words in which we read them, are the result of this repeated reworking and re-examination of the canon. For these reasons, in order to determine how Greek mathematics was actually practised, we are often in the position of having to reconstruct a lost context of mathematical activity on the basis of both the direct and indirect traditions. In order to get a sense for some of the vagaries of this transmission, we will consider a few illustrative cases in some detail.[2]

The work of Archimedes (ca. 280s–212) will furnish our first example (Heiberg and Stanatis 1972). We know of this corpus through a number of early modern copies of a lost Byzantine manuscript, a thirteenth-century Latin translation by William of Moerbeke (ca. 1215–86) made on the basis of this and another lost Byzantine manuscript, and a third Byzantine manuscript that was made into the famous palimpsest in the twelfth century (Clagett 1976; Netz and Noel 2007; Netz et al. 2011). Neither the Arabic nor the pre-Moerbeke Latin tradition is crucial in our assessment of Archimedes' writings. Although the Arabic tradition is important for some of the minor works,[3] it appears that *On the Sphere and Cylinder* was the only one of Archimedes' substantial treatises that was translated into Arabic. Hence, our knowledge of Archimedes is based on three, presumably independent, Greek manuscripts that were probably produced as part of the Byzantine revival of the ninth century and one or two other Greek manuscripts that were in Baghdad around this same time. In fact, compared with other major Greek mathematical sources, such as the works of Apollonius (late third century BCE) or Pappus (early fourth century CE), this is a fairly rich basis. One thing that we notice immediately, however, is that a number of treatises – including those on which Eutocius (early sixth century CE) wrote commentaries – are written in Koine, whereas other treatises are partially written in Archimedes' native Doric.[4] Since Eutocius himself, and others in his milieu, edited the works they studied, we may presume that these changes in dialect were introduced by such editorial work. We cannot now know what other changes were introduced in this process. We cannot be certain that the texts were not already edited before the late ancient period and we also do not know what changes were introduced around the ninth century in Constantinople when the three Byzantine manuscripts for which we now have any direct evidence were produced. Nevertheless, it is clear that late ancient and medieval editors felt that they were fully justified in making fairly extensive changes without comment.

The next example that we will look at is that of Apollonius (Heiberg 1891–3; Toomer 1990; Decorps-Foulquier and Federspiel 2008–10; Rashed 2008–10; Rashed and Bellosta 2010). We know the Greek version of Apollonius' *Conics* through a single Byzantine manuscript, of which all other extant manuscripts are copies (Decorps-Foulquier 1999). What we find in this manuscript, however, is not an original work by Apollonius, but an edition of the first four books of the original eight made by Eutocius, over six centuries later, as part of his project to expound classical works of advanced Hellenistic geometry. A second important Greek source for Apollonius' activity comes from another single Byzantine manuscript containing Pappus' *Collection* – a loose grouping of writings on various mathematical topics. From this text we learn about aspects of Apollonius' work for which we would not otherwise have any evidence, such as his interest in systems of large numbers, or his adherence to Euclid's organisation of geometry into those fields that can be handled with elementary constructions (straight lines and circles), with conic sections (parabolas, hyperbolas, and ellipses), and those that require more involved curves (spirals, quadratrixes, cissoids, and so forth).

For our understanding of Apollonius' mathematics, however, the Arabic tradition is as important a source as the Greek. In the ninth century, a group of scholars around the Banū Mūsā acquired a copy of the *Conics* in a version which had not been modified by Eutocius, but from which the eighth book had already gone missing. Through the mathematical work of al-Ḥasan ibn Mūsā (mid-ninth century CE), the chance discovery of a copy of the Eutocius version in Damascus, and the philological and mathematical expertise of Thābit ibn Qurra (ca. 830s–901) and others, an Arabic version of the seven remaining books was eventually completed (Toomer 1990, 621–9; Rashed 2008–10, 1.1.501–7). When we compare this version with the Greek, there are a number of differences, but it is not clear which one is closest to whatever Apollonius wrote (Rashed 2008–10, 1.1.12–25, 44–5). Indeed, we no longer posses the *Conics* that Apollonius wrote. We have the descendent of an edition made by Eutocius, in Greek, and another of that made by the scholars in the circle of the Banū Mūsā, in Arabic. The Arabic tradition has also preserved *On Cutting off a Ratio*, a text in what Pappus calls the 'field of analysis', otherwise only known from a discussion in Pappus' *Collection* 7. Hence, in order to try to evaluate Apollonius' mathematics, it is necessary to read a variety of texts, none of which he actually wrote, and some of which are not even translations or summaries of his work.

As these two examples serve to show, the significance of the manuscript tradition for interpreting the received text has to be evaluated independently in each case. Nevertheless, it is clear that the texts with which we have been working have been modified over the centuries. This is even more pronounced in the case of the texts that were more often read, such as the *Elements* or the treatises of the so-called *Little Astronomy*.[5] The Greek text of the *Elements* is preserved in two main versions: that in most of the manuscripts is called 'the edition of Theon' although there is some disagreement among the principal sources, while another, non-Theonine version, is extant in one manuscript. At the end of the ninth century, there were apparently a number of Arabic versions – two translations by al-Ḥajjāj (late eighth/early ninth century) and a translation by Isḥaq ibn Ḥunayn (830–910) that was revised by Thābit ibn Qurra – of which only Thābit's correction remains, but not without substantial incorporation of the older versions (Lo Bello 2003, xiii–xxix; De Young 2005, 176–7). All of the various Arabic texts, however, are different, in places, from the Greek, and it is not clear that the Greek versions have not also undergone some changes since the Baghdad translations were made (Knorr 1996; Rommevaux, Djebbar, and Vitrac 2001). This means that the Arabic versions should also be used to assess the original source, but this is made difficult by the numerous variants in the Arabic tradition and the fact that only parts of the text have been published (Engroff 1980; De Young 1981; Brentjes 1994).

A similarly complicated assortment of variants can be found in the sources for the group of texts known as the *Little Astronomy*, in the late ancient period, or the *Middle Books*, during the medieval period. By the late ancient period, these texts were grouped together by teachers like Pappus and described as the texts to be mastered between Euclid's *Elements* and Ptolemy's *Almagest*. Hence, these treatises, like the *Elements*, were often studied, and thus often edited. For example, there are two substantially different Greek versions of Euclid's *Optics* and *Phenomena* (Jones 1994; Knorr 1994), while there are at least three early Arabic versions of the *Spherics* by Theodosius (early second/midfirst century BCE) (Kunitzsch and Lorch 2010) and at least two of *On the Sizes and Distances of the Sun and the Moon* by Aristarchus (early third century BCE) (Berggren and Sidoli 2007). Once again, there are differences between the Greek and Arabic traditions, such as extra propositions in the Arabic *On the Sizes and Distances* or in the Greek *Spherics*. Moreover, it is often difficult to decide, in any objective way, which variant should be ascribed to the older source.

Because the texts of Greek mathematics have been subjected to repeated editorial work, we must regard them as historically contingent objects – in some ways created by the process of transmission itself. The texts, as we find them, are the products of a literary culture, produced by literary practices and made for literary consumption. Nevertheless, the mathematics that they contain was originally produced in a context of activity, now mostly lost to us, of which the production and consumption of literary texts formed only a part.

MATHEMATICAL PRACTICES

Although almost none of the surviving documents tell us how Greek mathematicians actually taught and produced mathematics, we can make some conjectures about this based on what the sources do say, and the types of mathematics that are preserved. In the following, we will examine three primary nexuses of mathematical activity: oral practices, material practices, and literary practices. In our sources, we can perceive a gradual transition from a more oral tradition, based around public arguments made about diagrams and instruments, to a more literary tradition that involved reading and writing texts containing elaborate arguments, tables, and special symbols that would have been difficult for anyone to follow without engaging the written works as material objects.

Of the formative, primarily oral, period of Greek mathematics, we know very little. It is now generally accepted that the Greeks produced little or no deductive mathematics before the mid-fifth century BCE, when Hippocrates was active. It was also around this time that Greek mathematicians began writing down their results (Netz 2004, 243–86). Nevertheless, it is clear that the practice of mathematics at this time was still highly oral. John Philoponus (mid-sixth century CE) tells us that Hippocrates learned mathematics during his time in Athens, by associating with philosophers, while he was waiting for the resolution of his lawsuit against certain pirates who had plundered his cargo (*in Phys.* A2185a16). From Plato's writings, we have the images of Socrates teaching Meno's slave boy mathematics by discussing diagrams in a public square, and Theaetetus (early fourth century BCE) and Socrates working through a question pertaining to commensurability, which was presumably meant to be reminiscent of the way Theaetetus had studied mathematics under Theodorus (late fifth century BCE; *Meno* 82a–85b, *Tht.* 147d–148b). When we reflect on the fact that deductive mathematics arose during the period of the sophists, when Greek, and particularly Athenian, culture put a premium on the ability to persuade others of one's position in public forums, it is clear that mathematical practice also originally involved the oral presentation of arguments in public spaces.

Moreover, throughout the ancient period, the most common institutional location for mathematical activities was in schools that were predominantly devoted to teaching philosophy and rhetoric. Since there were no schools of higher mathematics, we must assume that the bulk of the higher education of mathematicians, like other intellectuals, took place in schools of philosophy, where they studied the skills of winning others to their position through oral disputation and rational argument. We still find considerable evidence for such oral practices in the elementary texts, such as Euclid's *Elements* or Theodosius' *Spherics*. The format of the propositions and the repetitive language lends itself to oral presentation and memorisation,[6] and the fact that earlier propositions are often referenced by repeating the enunciation indicates that the listener was expected to learn the propositions by memorising the enunciations.[7]

As well as drawing diagrams and making arguments about them, Greek mathematicians engaged in a range of material practices involving specialised instruments, of which we now have only indirect evidence. It has long been recognised that the constructive methods of Euclid's *Elements* are a sort of abstraction of procedures that can actually be carried out with a straightedge and collapsing compass. More recently, it has been recognised that the constructions of Theodosius' *Spherics* are also meant to be applicable to actual globes (Sidoli and Saito 2009). The construction of mechanical globes was brought to a high level by the most mechanically orientated of the ancient mathematicians, Archimedes. We are told that the consul Marcus Claudius Marcellus brought back to Rome two devices built by Archimedes for modelling the heavens (Cic. *Rep.* 1.14), and, according to Pappus, Archimedes wrote a book on *Sphere-Making* (*Collection* 8.3). Hence, as with oral practices, we find that the material practices have left their mark in the preserved texts.

In a number of places, mathematical authors explicitly describe the sorts of instruments that they used in the course of their research. Eutocius attributes to Plato, somewhat dubiously, a sort of mechanical sliding square, which could be used to find two mean proportionals between two given lines (Heiberg 1891–3, 3.56–8; Knorr 1989, 78–80). Diocles (early second century BCE), in *On Burning Mirrors*, describes how we can use a flexible ruler, made of horn, to draw an accurate parabola (Prop. 4). Nicomedes (mid-third century BCE) is said to have built a mechanical device for inserting a line of a given length between two given objects, known as a *neusis* construction (Heiberg 1891–3, 3.98–106). These and many other passages make it clear that Greek mathematicians were engaged in a range of material practices that involved the accurate reproduction of the objects that they studied.

Interest in the mathematisable properties of instruments is also evidenced from texts like Pappus' *Collection* 8, which shows how to carry out geometric constructions with a straightedge and a compass set at a fixed opening (Jackson 1980). Moreover, a number of fields of applied, or mixed, mathematics were based around the set of operations that could be carried out with specific instruments. Ancient gnomonics, the study of sundials, was based on constructions that could be carried out with a set-square and a normal compass (Vitr. *De arch.* 9.7; Ptol. *Analemma* 11–14). The methods developed for projecting the objects in the surface of a sphere on to a plane were closely related to the practices involved in drawing star maps in the plane (Ptol. *Planisphere* 14–20). Finally, the analemma methods of spherical astronomy involved the use of analogue calculations that were carried out by performing physical manipulations on a prepared plate, and in some cases a hemisphere (Ptol. *Planisphere* 9–13; Sidoli 2005).

Whereas these activities were mostly employed in research and teaching, there were also material practices that involved the production and use of literary texts. One important area of this activity involved the production of literary diagrams. Whereas we have descriptive evidence that Greek mathematicians were concerned with the visual accuracy of their drawings, the figures that we find preserved in our manuscript sources are so far from such accuracy that it seems there must have been special principles operative in the production of these literary diagrams. In the manuscript sources, we find, for example, a square representing any rectangle, a regular pentagon representing any polygon, circular arcs representing conic sections, straight lines representing curved lines, curved lines representing straight lines, and so forth.

The two most consistent features of these diagrams are the use of an unnecessarily regular object to represent a general class of objects, and a basic disregard for visual accuracy in favour of the representation of key mathematical relationships (Saito and Sidoli 2012). It seems that ancient authors developed a type of diagram that would be easy to copy and which could be used as a schematic, in conjunction with the text, to produce a more accurate diagram when the need arose. The literary diagram was an object of communication that served to mediate between the readers and the mathematical objects under investigation. As Greek mathematics became literary, the diagram secured a central place in the production of mathematical texts so that we find diagrams even in places like *Elements* 7–9, on number theory, where they often do not convey essential information and appear to be merely a literary trope.[8]

All of our preserved texts, however, come from the fully literary period, and hence we see little change in the use of diagrams in our sources. This can be contrasted with the use of tables. Whereas it would be possible to follow the mathematical details in Euclid's *Elements* in an oral presentation, in order to verify even a simple calculation in Ptolemy's *Almagest* one needs to have access to a copy of the chord table. While the proto-trigonometry of Aristarchus' *On the Sizes and Distances of the Sun and Moon* can be followed in detail with just a working knowledge of geometry and some arithmetical calculation, the chord-table trigonometry developed by Hipparchus (mid-second century BCE) and others in the late Hellenistic period was a literary practice involving the consultation and manipulation of written sources.

The literary practices of Greek mathematicians naturally extended to the production of the texts themselves. Greek mathematicians were members of a small group of individuals in Greco-Roman society who produced works of high literature and they took pains to secure this social role. It has been argued that there are parallels between both the language and the structure of Greek mathematical works and other types of literary production (Netz 1999b, ch. 4; 2009). It is also clear that Greek mathematicians, like other ancient intellectuals, engaged in various editorial and pedagogical projects to revise their works and to make them more accessible to students and less specialised readers (Cameron 1990; Mansfeld 1998).

This was true not only for the structure and overall presentation of their works, but also for the language itself. Although individual authors had their own personal style, Greek mathematicians developed a distinctive mathematical style that can be recognised in all their theoretical texts (Mugler 1959; Aujac 1984; Federspiel 1995; Acerbi 2011b). This style becomes especially conspicuous when we see it mishandled by a non-mathematician, such as in an argument by Theon of Smyrna (early second century CE) that if the parameters of Hipparchus' solar model are given, the position of the sun is determined (Hiller 1878, 157–8).

It is not clear to what extent the homogeneity of mathematical style was due to the attention of the original authors or to the care of their later editors; nevertheless, already by the middle of the Hellenistic period the production of mathematics had become a fully literary activity, closely involved with the careful study of written works. This must have formed yet another barrier to entry into the small group of individuals who produced original mathematics. While a fair number of people may have studied the mathematical sciences by attending lectures and sessions at various schools, only a small number of these could have advanced to the study of written mathematical texts, either by being wealthy enough to buy their own books or by being bright enough to be considered worthy of studying their teacher's books.

STRUCTURES

A striking feature of Greek mathematical texts is their organisation. Like other literary texts, Greek mathematical works were divided into books. The books often varied in length, which depended on the mathematical content they developed. In the case of the more elementary texts, such as the *Elements* and the treatises of the *Little Astronomy*, which would have often been used in teaching, these books began immediately with mathematical content. More advanced works, however, often began with an introduction, for example in the form of an epistle to a colleague or student, in the Hellenistic period, or, in the imperial period, more commonly a short address to a student or patron. These epistles provide introductory material meant to be useful for understanding the goal of the theory developed in the text and the tools used to develop it (Mansfeld 1998). The mathematics itself is then divided up into clear sections: an introduction, which often includes definitions or axioms, followed by various units of text. In the medieval manuscripts of most ancient mathematical texts these units are numbered as propositions; however, even in unnumbered texts, such as Ptolemy's *Planisphere*, the sections are clearly distinguished. Propositions, as a type of textual unit, are then grouped together into theories, which are only differentiated on the basis of mathematical content. For example, *Elements* 1 begins with a theory of triangles and their congruency, followed by a theory of parallelism and a theory of area, which are then used to prove the so-called Pythagorean theorem, *Elements* 1.47. The interweaving of, sometimes, obscure individual units to form an overall theory produced an element of narrative that Greek mathematicians had great skill in exploiting (Netz 2009, 66–114).

The most common types of unit are the two types of propositions that the ancients called *theorems* and *problems* – in which, starting with some set of initial objects, a *theorem* shows that some property is true of these objects, while a *problem* shows how something can be done, and then demonstrates that what has been done is satisfactory.[9] These are what we find making up the majority of theoretical treatises. There are, however, other types of units, such as *analysed propositions* (analysis/synthesis pairs), *metrical analyses*, *computations*, *tables*, *algorithms*, and *descriptions*. Not all of these types of texts are found in all works and some of them, such as tables, are rarely found outside the exact sciences. For example, a description is a discussion of a mathematical figure or model that explains the properties of the objects but contains little or no argument. While these are common in the exact sciences, they are rare in pure mathematics; an exception is Archimedes' *Sphere and Cylinder* 1.23.[10]

It was recognised already by Proclus (fifth century CE) that a Euclidean proposition is carefully structured (Friedlein 1873, 203; Netz 1999a; Acerbi 2011b, 1–117). He names the following parts: *enunciation* (*protasis*), *exposition* (*ekthesis*), *specification* (*diorismos*), *construction* (*kataskeuē*), *demonstration* (*apodeixis*), and *conclusion* (*sumperasma*).[11] This exact division, however, is limited to the theorems and problems of *Elements* 1. In more involved problems, such as *Elements* 3.1 – find the centre of a circle – or *Spherics* 2.15 – draw a great circle through a point tangent to a lesser circle – there are two further parts: first, there is a specification of the problem followed by a construction that solves the problem, then there is a specification of the demonstration followed by a second construction for the sake of the demonstration. Moreover, other elements can also be recognised. It has been noted that in many cases the beginning of the demonstration makes explicit reference to statements that are made possible by the exposition or the construction, in a section that modern scholars have called the *anaphor* (Federspiel 1995, 1999). Like the other parts of a proposition, the usage of the anaphora is not rigid and in some case, as *Elements* 3.1, it blends seamlessly into the demonstration.

The flexibility of these divisions must be emphasised. Outside texts that became pedagogical, such as the *Elements* or Theodosius' *Spherics*, these parts did not seem to exercise much constraint and we find Archimedes, Apollonius, and Ptolemy mixing them up or omitting them all together. Nevertheless, the realisation that Greek mathematical units are structured has led to useful insights and a number of scholars have put forward structures for various units. It has long been accepted that there are four parts in a standard problematic analysed proposition: *transformation*, *resolution*, *construction*, and *demonstration* (Hankle 1874, 137–50; Berggren and Van Brummelen 2000). More recently, the *diorism* has been added to these four, and it has been noted that in the case of theoretic analysed propositions the division is somewhat different: *construction*, *deduction*, *verification*, *inverse deduction* (Saito and Sidoli 2010; Sidoli and Saito 2012). In the case of Diophantus' problems, we can again recognise various parts although they are not always clearly distinguished: *enunciation*, *instantiation*, *treatment*, *solution*, and *test*.[12] All of these structures, however, seem to have functioned more as guidelines than as dictates for acceptable practice, and although their existence may owe much to later editorial efforts, they are still useful to us in reading and understanding Greek mathematical texts.

As well as the structures found in the arguments themselves, as discussed in the previous section, we also find structure at the level of individual verbal expressions. Greek mathematicians, of course, developed a system of technical idioms to handle their discipline.[13] As well as a nomenclature, which became fairly standardised over time, they developed formulaic expressions that allowed them to condense their texts and help their readers keep mindful of the mathematical objects themselves. Since Greek mathematics was still essentially rhetorical, the use of operations and constructions was facilitated by a highly abbreviated diction that relied on various features of the Greek language in order to function. The fact that ancient Greek is a gendered, inflected language with a definite article allowed Greek authors of all genres to condense their terminology through various types of ellipsis that would not be possible in languages like English, Latin, or Arabic. In mathematics, particularly, these expressions needed to be highly regular in order to still be intelligible. This process led to the use of formulaic expressions involving particularities of the language such as the use of prepositions and number-and-gender agreement between definite articles and nouns (Netz 1999b, 127–67). In this way, *hē hupo tōn* $AB\Gamma$ (the [feminine] between the $AB\Gamma$s) means *angle* $AB\Gamma$, whereas *to hupo tōn* $AB\Gamma$ (the [neuter] between the $AB\Gamma$s) means *rectangle* $AB \times B\Gamma$. These processes allowed Greek mathematicians to make involved statements without unnecessary verbiage so as to focus attention on the objects and their relations. These expressions, however, cannot be literally translated into English. For example, Archimedes uses a sentence that would be literally translated as 'the of the on the $A\Theta$ to the on the ΘB having gained the of the AB to the ΘB is the on the $A\Theta$ to the between the $\Gamma\Theta B$' to convey $(A\Theta^2 : \Theta B^2) \times (A\Theta : \Theta B) = A\Theta^2 : (\Gamma\Theta \times \Theta B)$ (Heiberg and Stanatis 1972, 3.220). It should be clear, however, that these expressions could be used to express patterns of abstract thought, even in the absence of symbolism.

ARGUMENTS

The core element of theoretical mathematics was the argument. As is well known, Greek mathematical texts often begin with definitions that state properties of various objects. In some cases, these properties can then be used to make further claims. If the objects involved in the assertion of a proposition are insufficient to actually carry through the proof, new objects are introduced through constructions – which are discussed below.

The argument itself begins with references back to the objects that are introduced and then named in the beginning, or brought in by the construction (Acerbi 2011b, 73–5). It then proceeds by making claims about instantiated objects, usually in the form of letter-names. It has often been claimed that a Greek proof is actually about a specific, instantiated object, but this neglects the manner in which these letter-names are introduced. When an object is introduced into the domain of discourse this is done with an expression such as 'let there be an object, *AB*' – so the object is *any* one of the type of objects under consideration and its letter-name is simply a way of referring to it (Federspiel 1995, 1999; Acerbi 2011b, 39–57). Hence, the letter-name is a sign, referring not to a specific object but to any member of the class of objects that the proposition concerns. This generality is maintained throughout the argument and becomes clear whenever the text refers back to what has gone before – for example, references to the construction, or a previous proposition are almost always general claims (Acerbi 2011b, 26–32).

The necessity of the argument is produced by a deductive chain of assertions that some relation, or property, holds for some object (Netz 1999b, ch. 5). Each step of this chain can be justified on the basis of either a definition or some previously established result. Often these previously established results are found earlier in the same work, but they may also be part of a broader set of theorems assumed to be known to the reader, known as the *toolbox* by modern scholars, and made up largely of propositions of the *Elements* (Saito 1997, 1998; Netz 1999b, 216–35). The standard form of synthetic argument is a chain of assertions that two objects have some relation, or that an object has a certain property, but analyses also contain deductive chains asserting that certain objects are *given*, and in various ways.

The simplest argument type is direct. After the assumed objects have been stated and any necessary constructions performed, some claims can then be asserted about these objects. In simple cases, these claims can lead directly, through a chain of implications and operations, to the conclusion. More often, however, other starting points must be introduced: either through a new appeal to the construction, introducing a new construction, appealing to a previous theorem in the same work, or in the toolbox. References to previous theorems, and to the theorems of the toolbox, are often invoked by a brief summary of the enunciation, or simply by a generalized claim of the mathematical fact.

A common form of argument is the indirect argument, in which a claim contrary to what the mathematician wishes to prove is assumed, followed by constructions and arguments leading to a contraction with the hypotheses, with established results, or with some feature of the mathematical objects in question that is taken to be inherently obvious. Although indirect arguments are very common in our sources, not all Greek mathematicians found them satisfactory: Menelaus (turn of second century CE) tells us in the introduction to his *Spherics* that he will avoid them (Krause 1936, 118; Sidoli and Kusuba 2014, 167; Rashed and Papadopoulos 2017: 697).

Another common argument type is the proof-by-cases. In some of the proofs-by-cases found in the geometrical books of the *Elements*, it is not clear whether the cases were part of the original composition or if they were added by later editors. In the number theory, however, it is certain that proof-by-cases was an integral part of the original approach. Arguments by cases were also combined with indirect argument, as in the double-indirect argument, often used by Archimedes, and sometimes called the method of exhaustion. Using this argument structure, one shows that two objects are equal by showing that one is neither greater nor less than the other. Archimedes uses this form of argument in *Measurement of the Circle* 3 to show that a circle is equal to a right triangle that has one leg equal to the circle's radius and the other leg equal to the circle's circumference.

Occasionally, we find an argument in two parts, or which extends beyond a single unit of text. The most common case of this is found in simple converses. In order to prove that *A* holds if and only if *B*, Greek mathematicians would often first show that *A* implies *B* and then that *B* implies *A*, in two separate propositions, for example in *Elements* 1.5 and 6, or 1.17 and 18. Another example of this sort of extended argument was a two-stage method of showing that four magnitudes are proportional, $a : b :: c : d$, which involved first showing that the proportion holds when *a* and *b* are assumed to be commensurable, and then, in a second argument using this and a double-indirect argument, showing that the proportion also holds when *a* and *b* are incommensurable. This structure was used in Archimedes' *Equilibrium of Planes* 1.6 and 7, Theodosius' *Spherics* 3.9 and 10, and Pappus' *Collection* 5.12 and 6.7–9 (Knorr 1978; Mendell 2007).

An interesting type of argument that extends beyond a single unit is the analysis/synthesis pair, which consists of an assumed construction, a deductive argument concerning *givens*, a treatment of the limits to the possibility of solution and of the total number of solutions, a construction, and a deductive argument concerning the relations and properties of geometric objects. From a purely deductive perspective, the analysis is unnecessary, so that the reasons for providing it must have been expository. A *problematic analysis*, can provide a motivation for the initial construction steps of the synthesis and point the way towards an articulation of the limits to possibility of solution, while a *theoretic analysis* can provide insight into how the main relation of the theorem was obtained (Saito and Sidoli 2010; Sidoli and Saito 2012).

The argument was the locus of deduction and hence can be regarded as the core of theoretical mathematical activity. Although mathematicians clearly put considerable care into making arguments, they left few discussions about what constitutes a valid argument. Hence, in order to understand their philosophy of mathematics, we must read the mathematical texts themselves, paying attention to the overall structures which the individual arguments compose.

CONSTRUCTIONS

One of the most distinctive methodological features of Greek mathematics is the use of constructions. Construction, or construction-based thinking, is found not only in geometry but also in number theory, pre-modern algebra, the exact sciences, and in general investigations of what, and how, mathematical objects are given.

Constructive techniques are conspicuous in elementary geometry. Recently, much attention has been paid to the role of diagrams in Greek mathematical thought (Manders 2008; Netz 1999b, 12–67; Le Meur 2012); however, it is only through the mediating process of construction that the diagram has any deductive force (Avigad, Dean, and Mumma 2009; Sidoli and Saito 2009). Constructions are used in very nearly every proposition in order to introduce new objects whose defined properties are then used as starting points in chains of deductive inference.

Constructions played a different role in *problems* than in *theorems* and in the demonstration section of a *problem*. In the *Elements*, construction postulates are introduced to justify the construction procedures that are used in *problems*, but not necessarily those used in *theorems*. More elaborate construction procedures are set out and then justified in *problems*. The construction of a problem uses postulates, or previously established problems, to show that there is an effective algorithm for producing the sought object. The construction of a theorem, or of the demonstration of a problem, however, can call on a wider range of constructive assumptions (*Elements* 1.4, 6, 8), or even involve impossible, or counterfactual, constructions (*Elements* 1.6, 7 and 3.1). In other geometrical texts, a variety of constructive processes are used that are never explicitly postulated, such as setting a line of a given length between two given objects (a *neusis* construction), or passing a plane through a solid object – for examples, see Archimedes *Spiral Lines* 5–9, or Apollonius *Conics* 1.4–14.

In geometric texts, different verbs were used to denote various types of constructions, depending on what the geometer intended to do. Lines could be *produced* (*agō*) between two given points, circles *drawn* (*graphō*) with a given centre and passing through a given point, solid objects *cut* (*temnō*) by a passing plane, diameters in spheres *set out* (*ektithēmi*) on plane survaces, parallelograms *erected* (*sunistēmi*) on given lines, and so forth. In the enunciation of *problems*, these operations were stated in the infinitive, whereas in the construction of either type of proposition they were generally stated in the perfect imperative passive. Despite the fact that the construction is explicitly stated as complete, it is clear that it represents the most active part of mathematical practice. Moreover, the construction is often the most creative part of a mathematical argument, since it introduces new objects into the domain of discourse, which are entirely at the mathematician's discretion.

Although construction is generally associated with geometry, constructivist thinking permeated other branches of Greek mathematics as well. All of the problems in Euclid's number theory, *Elements* 7–9, show us how to *find* (*euriskō*) numbers, which are hence assumed to exist from the beginning (Mueller 1981, 60). The problems in Euclid's number theory, however, are the active components and provide the algorithms that are used in the rest of the theory. In geometric texts, as well, objects that are assumed to exist are sometimes *found* by construction – such as the centre of a circle, in Euclid's *Elements* 3.1, or of a sphere, in Theodosius' *Spherics* 1.2 – so that these problems are conceptually related to the algorithms in the number theory. Furthermore, in works of pre-modern algebra, such as the *Arithmetics* of Diophantus (ca. third century CE), a number of different constructive operations are invoked such as *find* (*euriskō*), *separate* (*diaireō*), *add* (*prostithēmi*), and *make* (*poieō*). Although these expressions assume the existence of the numbers involved, because rational numbers are expected, limits to the possibility of solution must sometimes be invoked. One of Diophantus' problems, such as *Arithmetics* 2.8 – '*Separate* a proposed square into two squares' – can later function as an algorithm used in further problems, in much the same way as a problem in the geometric texts (Tannery 1893, 90). One difference is that for Diophantus the constructive procedures are themselves joined together, making a series of conditions for the solution to the problem, such as *Arithmetics* 1.7 – '*Take away* two given numbers from the same number and *make* the remainders have a given ratio to one another' – whereas in the geometric texts only one construction is stipulated and all the conditions are expressed as modifications of the objects, such as Theodosius' *Spherics* 2.15 – 'Given a lesser circle in a sphere and some point on the surface of the sphere that is between it and the circle equal and parallel to it, *draw* through the point a great circle tangent to the given circle'

(Tannery 1893, 24; Czinczenheim 2000, 102). This difference occurs both at the grammatical level and also in terms of the procedures of solution. In geometric problems, there is only one verb, in the infinitive, and a single geometric object is constructed satisfying all of the conditions, which are stated as modifications of the nouns. In Diophantus, the conditions are stated as a series of verbs and they are satisfied individually as the problem proceeds.[14]

The notion of construction was so fundamental for Greek mathematicians that they developed a sort of conceptual framework to handle constructive processes as a theory of *givens*, formalised in Euclid's *Data* (Taisbak 2003). Given objects exist in a definite and often unique way and their properties are known and manageable (Acerbi 2011a). Given objects, or properties, are those that are found at the start of the discourse, that are constructed at the discretion of the mathematician, or that can be inferred to be given based on these. The *Data* shows us how to make inferences about given objects or properties. The late-Platonic commentator Marinus of Neapolis (late fifth century BCE) reports a number of definitions of the concept of *given*, which he attributes to various mathematicians (Menge 1896, 234–6). After discussing various ways that we can understand the notion of *given*, Marinus settles on the concepts of *known* (*gnorimon*) and *provided* (*porimon*), claiming that what is provided is that which we are able to *make* or to *construct*, for example drawing a circle or finding three expressible lines that are only commensurable in square (Menge 1896, 250, 240). This agrees with Euclid's definition of given in the *Data*. Def. 1 reads, '*Given* is said of regions, lines and angles of which we are able to provide an equal' (Menge 1896, 2). The notion of *provision* was an attempt to formalise the productive processes through which the mathematician gained mastery of the subject. Its formalisation was meant to facilitate the types of inferences that mathematicians made in geometrical analysis – starting with the *analytical assumption* that a certain configuration exists containing the sought object, one then started with objects in this configuration that were either already or assumed to be known, or could be readily constructed, and then proceeded, through a chain of inferences, to show that the sought object was also *given*.

In later readings of the *Data*, the notion of *given*, and hence of *provision*, was expanded to include computations and other sorts of deductive inferences. Later authors, such as Heron and Ptolemy, constructed *chains of givens*, where each step can be referred to a purely geometric theorem of the *Data*, but which are in fact used to justify computational procedures, involving arithmetical operations and tabular functions.[15] We can call these chain of inferences *metrical analyses*, since they show how to construct a sought number.[16]

In these ways, construction fulfilled a number of important roles for Greek mathematicians. On a practical level, construction formalised and abstracted various active procedures that were necessary in actually doing mathematics. On a more theoretical level, it allowed mathematicians to introduce new objects whose properties could then be used to prove theorems or solve problems. On a fundamental level, it provided instantiations of objects with known properties to be used in mathematical discourse.

OPERATIONS AND PROCEDURES

Although, in a general sense, we can regard constructions as operations, in this section we focus on those operations that can be performed on a statement, expression, or number. While there is relatively little operational mathematics in elementary geometrical treatises, such as the early books of the *Elements* or Theodosius' *Spherics*, as soon as we begin to read higher geometry, number theory, pre-modern algebra, or the exact sciences, we encounter long passages of deductive reasoning in the form of chains of mathematical operations.

From both a theoretical and practical perspective, Greek mathematicians privileged operations on ratios and proportions over arithmetic operations. A theoretical justification for many of the common ratio manipulations that were in use was provided by *Elements* 5, which is thought to have been formulated by Eudoxus. Almost all of the theorems of the second half of this book deal with manipulations that can be carried out on proportions. For example, the operation of *separation* (*deilōn*), justified in *Elements* 5.17, entails inferring from a proportion of the form $a : b :: c : d$ one of the form $a - b : b :: c - d : d$, where $a > b$. The operation of *combination* (*sunthentos*), justified in *Elements* 5.18, is the converse. Although in the *Elements* these operations are only justified for proportions, Greek mathematicians also applied them to ratio equalities and, occasionally, equations or inequalities.[17]

This gives the impression that Greek mathematicians sharply distinguished between proportions and equations, and there is some truth to this. Equations were taken to be statements about different things that were *equal in quantity*, whereas proportions were claims that two ratios were *the same*. Nevertheless, despite this distinction, Greek mathematicians were aware that equations and proportions could be interchanged, and occasionally subjected equations to ratio manipulations, or proportions to arithmetic operations.[18] Of course, all of the ratio manipulations can be rewritten as arithmetic operations, but Greek mathematicians apparently had no interest in doing so. In fact, even in places where one might expect to find only arithmetic operations, such as in the calculation of the size of a length or an angle by Aristarchus or Archimedes, we still encounter the use of ratio manipulations.

Greek mathematicians, of course, also performed arithmetic operations; however, they did not spend much effort attempting to formalise or justify these operations. More difficult operations, such as taking roots, are not explicitly discussed in much detail in our sources before the late ancient period.[19] Arithmetic operations were performed on individual terms, whole proportions and ratio inequalities, and equations.

The three ancient and medieval algebraic operations were probably regarded as special cases of such arithmetic operations, applicable under certain specified conditions. In the introduction to his *Arithmetics*, Diophantus describes the two primary pre-modern algebraic operations that can be performed on an equation to solve for an unknown number. He says that if 'a kind (*eidē*)[20] becomes equal to the same kind but not of the same quantity, it is necessary to take away the similar from the similar on each of the sides, in order that that kind should be equal to kind' (Tannery 1893, 14). In other words, given an equation in which numbers, some number of unknowns, or higher terms are found on both sides of the equation, to subtract the common term from both sides, so as to bring it to the other side – as we would say. The second operation is 'to add a kind lacking from either of the sides, in order that an extant kind will come to be for each of the sides' (Tannery 1893, 14). That is – as we would say – to make all our terms positive. These operations may be repeated as necessary until only one of a number, some number of unknowns, or higher terms are found on each side of the equation. The third operation is not stated until it is needed at the beginning of the Arabic Book 4, which follows the Greek Book 3. The text says that, if after the other two operations have been performed we have a statement equating unknowns of higher degree, then, 'we divide the whole by a unit of the lesser in degree of the two sides, until there results for us one kind equal to a number' (Sesiano

1982, 88). In other words, we reduce the equation to the lowest degree possible. There is no attempt in the text to formally relate these operations to operations of arithmetic or to develop further operations to be carried out on equations on analogy with the other arithmetic operations. Hence, they appear to have been treated separately as an operation for eliminating lacking (negative) terms, an operation for grouping like terms on one side of the equation, and an operation for reducing certain equations of higher degree.

Series of operations were also arranged in algorithms. In *Elements* 7 there are a number of problems that involve algorithms, for example *Elements* 7.2 – find the greatest common measure of two numbers – or 7.34 – find the least common multiple of two numbers. The only actual operations involved in these problems, however, are arithmetical and they are not postulated, but simply assumed as obvious. As with all problems, following a presentation of the algorithm, there is a proof that the algorithm accomplishes its goal.

In other authors, such as Diophantus and Ptolemy, we have less formal approaches to algorithms and computational procedures that involve a series of arithmetic operations and are generally unjustified (Acerbi 2012, 183–9). These algorithms proceed by a chain of instructions, in the second person imperative, and may involve the use of parameters of calculation and entries into tables, as well as arithmetic operations. Parameters of calculation are often distinguished from the data for any particular procedure with the word 'always' (*aei*). The results of table-entries and calculations can be set aside and then brought back in at some later stage of the procedure.

There are also algorithms in which each successive operation is carried out directly on the result of the proceeding operation, which are often presented in a context of justification (Acerbi 2012, 190–9). For example, Heron, in *Measurements* 1.8, gives a general algorithm for finding the area of a triangle given its sides, which is followed not by a proof, but by a computed example (Taisbak 2014). This justificatory section ends with a short *metrical analysis* before the example, which for Heron functions as the synthetic construction of a particular number given some assumed values, and which he calls the 'synthesis'. This was Heron's general approach to *metrical analysis*: first an 'analysis', justified by steps of the *Elements* and the *Data* that if certain values are assumed as given, the sought value can be shown to also be given, followed by a 'synthesis', in which the sought value is computed from some values assumed as given.

An interesting type of *metrical analysis* is found in trigonometric texts, such as Ptolemy's *Analemma* or *Almagest*. Here we find general statements of an algorithmic procedure using the *givens* terminology, where each step can be justified by a theorem of the *Data*, but actually refers to arithmetical operations, ratio manipulations, entries into a chord table, and so forth. Ptolemy apparently thought of these arguments as justifying a computational procedure, by referring each of its steps to a theorem of the *Data*. On the other hand, he does not seem to have regarded arguments by givens and computations as an analysis/synthesis pair, since he always only gives one or the other, and he sometimes later refers to a calculation as a proof.

CONCLUSION

It is now generally recognised by historians of ancient and medieval mathematics that ancient Greek mathematics is not *our* mathematics (Høyrup 1996). Nevertheless, Greek mathematics was one of the great productions of ancient scholarship – particularly in its desire to produce arguments that established both generality and necessity, in its endeavour to formalise mathematical knowledge through structure and regularity, in its goal of producing problem-solving techniques through constructive processes under the mathematician's control. It is for these reasons that Greek mathematical texts were read and reread over many centuries by creative mathematicians such as Ibn al-Haytham (ca. 965–ca. 1040), Abū Naṣr Manṣūr ibn ʿIrāq (ca. 960–1036), Jordanus of Nemore (thirteenth century), Francesco Maurolyco (1494–1575), Pierre Fermat (1601–65), and Isaac Newton (1642–1726).

I hope that this chapter will have highlighted some of the characteristics of Greek mathematics that make it distinctive, so that we can study this material as a style of mathematics different from our own, but nonetheless, as belonging to mathematics. In this way, we can more effectively compare Greek mathematics with other ancient ways of doing mathematics, and with the medieval approaches to mathematics that built on, and broke away from, ancient Greek works.

NOTES

This chapter was written in 2015.

1 Asper 2009 provides a discussion of the differences between the theoretical and practical traditions of mathematics.

2 For the Greek manuscript tradition, see Acerbi 2010, 269–375; Vitrac n.d.; for the Arabic tradition, see Sezgin 1974–9.

3 For example, Knorr 1989, 375–816, has argued for the importance of the medieval tradition of *Measurement of the Circle*.

4 The question of Archimedes' dialect is made difficult by the fact that much of the Doric in the received text was produced by the editor, J. L. Heiberg, in response to the fact that the manuscripts contain a strange mixture of Koine and Doric; Heiberg and Stamatis 1972, II: x–xviii; Netz 2012).

5 Vitrac 2012 gives a discussion of the editorial production of the *Elements*.

6 Netz 1999b, 127–67, discusses the formulaic nature of Greek mathematical prose. Although he focuses on the cognitive roles of formulae, it is also clear that they would have facilitated memorisation and oral presentation.

7 This is also supported by the format of three of the papyri containing material from the *Elements*: *P. Oxy.* I 29 (*Elements* 2.4 and 5), *P. Oxy*. 5299 (*Elements* 1.4, 8–11, 14–25), and *P. Berol.* 17469 (*Elements* 1.8–10); see Sidoli 2015, 392–3.

8 It should be noted, however, that there are some cases where the diagrams help us understand both the proposition and the argument (Bajri, Hannah and Montelle 2015, 559–68).

9 Definitions of *theorem* and *problem* are given by Pappus and Proclus (Hultsch 1876, 30–2; Friedlein 1873, 200–1).

10 This unit is unnumbered in the manuscripts.

11 The conclusion is almost certainly a late addition in the Greek tradition of the *Elements* (Acerbi 2011b, 38–9).

12 The names given to these parts by A. Bernard and J. Christianidis are different, but the parts appear to be the same (Christianidis 2007; Bernard and Christianidis 2012). See also Christianidis and Oaks 2013, 130–4.

13 Heath 1896, clvii–clxx; 1912, clv–clxxxvi, provides useful introductions to Greek mathematical terminology, and there have been a number of studies of the language of Greek mathematics (Sidoli 2014, 29).

14 For *Arithmetics* 1.7, the two conditions are rather simple, but they are still handled sequentially. For more involved problems the conditions are solved individually. For example, in *Arithmetics* 3.1, after satisfying two of the conditions, Diophantus says 'two of the conditions (*epigmata*) are now solved (*lelumena*)' (Tannery 1893, 138).

15 The expression 'catena dei dati' is due to Acerbi 2007, 455.

16 This type of argument is called an 'analysis' by Heron throughout his *Measurements*, and by Pappus in his commentary on Ptolemy's *Almagest* 5 (Rome 1931–43, 35).

17 See, for example, Aristarchus' *On the Sizes and Distances of the Sun and Moon* 4, or Archimedes' *Sand Reckoner* (Heath 1913, 367; Heiberg and Stamatis 1972, 2.216–58).

18 See, for examples, Aristarchus' *On the Sizes and Distances of the Sun and Moon* 4, Apollonius' *Conics* 1.15, or Ptolemy's *Almagest* 1.10 (Heath 1913, 367; Heiberg 1891–3, 1.63; Heiberg 1898–1903, 1.45–6).

19 The extraction of square roots is described by a scholium to *Elements* 10, and Theon in his *Commentary to the Almagest*. Heron gives an example of taking a cube root, but does not give his method in detail (Heath 1921/81, I: 60–2).

20 In Diophantus' terminology, a *kind* is mathematically related to what we would call a term of a polynomial, although he does not appear to have conceived of a polynomial as a series of terms combined by operations.