DEFORMATIONS OF HOMOTOPY ALGEBRAS

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ABSTRACT. Let k be a field of characteristic zero, $\mathcal{O}$ be a dg operad over k and let A be an $\mathcal{O}$-algebra. In this note we suggest a definition of a formal deformation functor of A

$$\mathrm{Def}_A : \mathtt{dgart}^{\leq 0}(k) \to \Delta^0\mathtt{Ens}$$

from the category of artinian local dg algebras to the category of simplicial sets. This functor generalizes the classical deformation functor for an algebra over a linear operad. In the case $\mathcal{O}$ and A are non-positively graded, we prove that Def_A is governed by the tangent Lie algebra T_A which can be calculated as the Lie algebra of derivations of a cofibrant resolution of A.

An example shows that the result does not necessarily hold without the non-positivity condition.

1. Introduction

1.1. It is well-known that formal deformations of an associative algebra A over a field k of characteristic zero are governed by a differential graded Lie algebra $\mathfrak{g}$ which coincides, up to a shift, with the cohomological Hochschild complex of A. Another way to calculate the dg Lie algebra $\mathfrak{g}$ can be described as follows.

Let P be an associative free dg algebra-resolution of A. The collection $\mathrm{Der}(P, P)$ of graded algebra derivations of P form a dg Lie algebra. It is known that for a very specific choice of P this construction gives the cohomological Hochschild complex. On the other hand, according to [H2], the dg Lie algebra $\mathrm{Der}(P, P)$ does not depend, up to quasi-isomorphism, on the choice of the resolution P. Therefore, for any choice of P the dg Lie algebra $\mathrm{Der}(P, P)$ governs the deformations of A.

The above considerations suggest that there should exist a homotopy invariant formal deformation theory for dg algebras. In fact, suppose that each dg algebra defines a deformation functor so that quasi-isomorphic algebras define equivalent functors. Let $P \to A$ be a free algebra-resolution of A. Deformations of A and of P are the same since A and P are quasi-isomorphic ; P is free as a graded associative algebra, so one has nothing to deform in P, except for the differential. Since the deformations of the differential are described by the Maurer-Cartan elements in $\mathrm{Der}(P, P)$, this dg Lie algebra governs the deformations of A.

The aim of this paper is to construct such deformation theory in a more general context of dg operad algebras.

1.2. **Classical deformations.** "Classical" formal deformation theory over a field of characteristic zero deals with deformation functors which can be described as follows.

Let k be a field of characteristic zero, $\mathtt{art}(k)$ be the category of artinian local k-algebras with residue field k.

Let $\mathcal{C}$ be a category cofibred over $\mathtt{art}(k)$. Equivalently, this means that a 2-functor

$$\mathcal{C} : \mathtt{art}(k) \to \mathtt{Cat}$$

is given, that is a collection of categories $\mathcal{C}(R)$, $R \in \mathtt{art}(k)$, of functors $f^* : \mathcal{C}(R) \to \mathcal{C}(S)$ for each morphism $f : R \to S$ in $\mathtt{art}(k)$ and of isomorphisms $f^*g^* \xrightarrow{\sim} (fg)^*$ satisfying the cocycle condition. For instance, for the deformations of associative algebras, $\mathcal{C}(R)$ is the category of associative R-algebras flat over R.

Finally, let an object $A \in \mathcal{C}(k)$ be given. Then the deformation functor

$$\mathrm{Def}^{cl}_A : \mathtt{art}(k) \to \mathtt{Grp} \tag{1}$$

assigns to each $R \in \mathtt{art}(k)$ the groupoid whose objects are isomorphisms $\alpha : \pi^*(B) \to A$ where $\pi : R \to k$ is the natural map, and morphisms are isomorphisms $B \to B'$ compatible with α and α'.

1.3. **“Higher” deformations.** Let $\mathcal{O}$ be a dg operad. It is clear that in order to define a homotopy invariant deformation theory of $\mathcal{O}$-algebras, one has to take into account that the category of $\mathcal{O}$-algebras is endowed with extra structures. These are weak equivalences, homotopies between maps, higher homotopies between the homotopies etc. Therefore, it seems inevitable that one has to assign to an $\mathcal{O}$-algebra A and to an artinian local ring R some “higher” version of a groupoid.

1.3.1. Metaphorically speaking, the picture should be the following.

For each artinian local algebra R an ∞-category of $R \otimes \mathcal{O}$-algebras should be defined; denote it $\mathtt{Alg}^\infty(\mathcal{O}, R)$. The collection of $\mathtt{Alg}^\infty(\mathcal{O}, R)$ should form an ∞-category cofibred over the category of artinian local algebras.

Let now $A \in \mathtt{Alg}^\infty(\mathcal{O}, k)$. Then the deformation functor

$$\mathrm{Def}^{\infty}_A : \mathtt{art}(k) \to \mathtt{Grp}^\infty$$

should be a (∞-) functor to ∞-groupoids; its objects are ∞-isomorphisms $\alpha : \pi_*(B) \to A$ and morphisms — ∞-isomorphisms $B \to B'$ commuting with α and α'.

1.3.2. We do not know well what an ∞-category is and how to assign an ∞-category to the category of operad algebras. Therefore, we are looking for an appropriate substitute of this notion.

According to [H2], the category $\mathtt{Alg}(\mathcal{O}, R)$ of $R \otimes \mathcal{O}$-algebras admits a simplicial closed model category structure.

As a substitute to the ∞-category $\mathtt{Alg}^\infty(\mathcal{O}, R)$, we suggest considering the simplicial category of cofibrant $R \otimes \mathcal{O}$-algebras.

As a substitute for ∞-groupoids we use the category $\mathtt{Kan}$ of Kan simpicial sets. And for some reason (explained in 1.5) we use a more general class of non-positively graded dg artinian local algebras as bases for formal deformations.

In this way we define in 2.3.2 a deformation functor

$$\mathrm{Def}_A : \mathtt{dgart}^{\le 0}(k) \to \mathtt{Kan} \tag{2}$$

(defined uniquely up to homotopy).

1.4. **Dg Lie algebras.**

1.4.1. A functor

$$D : \mathtt{art}(k) \to \mathtt{Grp}$$

is said to be governed by a differential graded Lie algebra $\mathfrak{g}$ (in this definition one has to admit $\mathfrak{g}^i = 0$ for $i < 0$) if there is a functorial equivalence

$$D \xrightarrow{\sim} \mathrm{Del}_{\mathfrak{g}} : \mathtt{art}(k) \to \mathtt{Grp}$$

where Deligne groupoid $\mathrm{Del}_{\mathfrak{g}}(R)$ is defined as follows.

Let $\mathfrak{m}$ be the maximal ideal of R. The tensor product $\mathfrak{m} \otimes \mathfrak{g}$ is a nilpotent dg Lie algebra. Let $\mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g})$ denote the collection of elements $z \in (\mathfrak{m} \otimes \mathfrak{g})^1$ satisfying the Maurer-Cartan equation

$$dz + \frac{1}{2}[z, z] = 0.$$

The nilpotent Lie algebra $(\mathfrak{m} \otimes \mathfrak{g})^0$ acts on $\mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g})$ by vector fields:

$$\rho(y)(z) = dy + [z, y], \text{ where } y \in (\mathfrak{m} \otimes \mathfrak{g})^0,\ z \in \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}).$$

This defines an action of the nilpotent group $G = \exp((\mathfrak{m} \otimes \mathfrak{g})^0)$ on $\mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g})$. Then the groupoid $\mathrm{Del}_{\mathfrak{g}}(R)$ is defined by the formulas

$$\mathrm{Ob}\,\mathrm{Del}_{\mathfrak{g}}(R) = \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}); \tag{3}$$

$$\mathrm{Hom}(z, z') = \{g \in G | z' = g(z)\} \tag{4}$$

1.4.2. One could expect that in order to govern more general deformations as (2), one needs a new device. Fortunately, this is not so. In [H1] (see also [H3], Sect. 8) a *nerve* functor

$$\Sigma_{\mathfrak{g}} : \mathtt{dgart}^{\le 0}(k) \to \mathtt{Kan}$$

is defined for any dg Lie algebra $\mathfrak{g}$. Quasi-isomorphic dg Lie algebras give rise to equivalent nerve functors.

If a nilpotent dg Lie algebra $\mathfrak{m} \otimes \mathfrak{g}$ is non-negatively graded, the nerve $\Sigma_{\mathfrak{g}}(R)$ is homotopically equivalent to the groupoid $\mathrm{Del}_{\mathfrak{g}}(R)$ — see [H1].

In what follows we will say that a functor

$$D : \mathtt{dgart}^{\le 0}(k) \to \Delta^0\mathtt{Ens}$$

is governed by a dg Lie algebra $\mathfrak{g}$ if there is a natural weak equivalence

$$D(R) \xrightarrow{\sim} \Sigma_{\mathfrak{g}}(R).$$

1.4.3. In [H2], Sect. 7, we constructed for each algebra A over a dg operad $\mathcal{O}$ a tangent dg Lie algebra $T_A \in \mathtt{dglie}(k)$. It is defined as the dg Lie algebra $\mathrm{Der}(\widetilde{A},\widetilde{A})$ of derivations of a cofibrant resolution $\widetilde{A}$ of A.

The main result of this paper says that T_A governs the formal deformations of A defined by the functor (2), provided $\mathcal{O}$ and A belong to $C^{\leq 0}(k)$.

It turns out that the condition on the grading is important — see 4.3.

1.5. **Dg bases for formal deformations.** There are several (interrelated) reasons to consider differential graded artinian algebras as bases of formal deformations.

The nerve of a dg Lie algebra $\mathfrak{g}$ is represented by the coalgebra $\mathcal{C}(\mathfrak{g})$ which is the standard complex for $\mathfrak{g}$ with trivial coefficients. This means that the formal moduli space is a sort of dg stack and so it should naturally have R-points where R is a local dg artinian algebra. This is connected to another reason: Deligne groupoid

$$\mathrm{Del}_{\mathfrak{g}} : \mathtt{art}(k) \rightarrow \mathtt{Grp}$$

depends on a very small part of $\mathfrak{g}$ — only the segment

$$\mathfrak{g}^0 \rightarrow \mathfrak{g}^1 \rightarrow \mathfrak{g}^2$$

appears in formulas (3), (4). However, we want the whole of $\mathfrak{g}$ (for instance, the whole of the Hochschild complex) to be relevant to the deformation theory. Once we extend the functor $\mathrm{Del}_{\mathfrak{g}}$ to the nerve functor defined on $\mathtt{dgart}^{\leq 0}(k)$, the dg Lie algebra $\mathfrak{g}$ is defined uniquely up to a quasi-isomorphism.

1.6. **Content of the sections.** Throughout the paper we work a lot with simplicial categories and simplicial groupoids. We collect in the appendix the necessary information about this. In the beginning of Section 2 we recall the definition of a simplicial closed model category structure on the category of dg algebras over an operad taken from [H2].

In Section 2 we define the functor (2) describing deformations of an algebra A over a dg operad $\mathcal{O}$.

To describe the deformation functor as the nerve of the tangent dg Lie algebra, we provide in Section 3 a version of the nerve construction of [H3], Sect. 8, which assigns to a dg Lie algebra $\mathfrak{g}$ and to a dg artinian algebra R a simplicial groupoid.

In Section 4 we prove the main result. It follows easily from the model category structure on the category of simplicial categories.

Finally, in Section 5 we check that for algebras over linear operads our deformation functor (2) extends the classical one.

1.7. **Notation.** In what follows we use the following notation for different categories.

$\mathtt{Ens}, \mathtt{Grp}, \mathtt{Cat}$ are the categories of sets, small groupoids and small categories respectively.

Δ is the category of ordered sets $[n] = \{0, \dots, n\}$, $n \geq 0$ and order-preserving maps. For a category $\mathcal{C}$ we denote by $\Delta^0\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$.

Let $\mathcal{A}$ be a tensor (=symmetric monoidal) category (for instance, the category of modules mod (R) or of complexes $C(R)$). We denote by $\mathtt{Op}(\mathcal{A})$ the category of operads in $\mathcal{A}$. If $\mathcal{O} \in \mathtt{Op}(\mathcal{A})$, the category of $\mathcal{O}$-algebras is denoted by $\mathtt{Alg}(\mathcal{O})$.

A simplicial category (and, in particular, a simplicial groupoid) will be supposed to have a discrete set of objects, if it is not explicitly specified otherwise. The category of small simplicial categories is defined $\mathtt{sCat}$ and that of simplicial groupoids $\mathtt{sGrp}$.

For a simplicial category $\mathcal{C}$ and its objects $x, y \in \mathcal{C}$ we denote by $\mathcal{H}om_{\mathcal{C}}(x,y)$ the simplicial set of morphisms from x to y. The category $\pi_0(\mathcal{C})$ is defined as the one having the same objects as $\mathcal{C}$, with morphisms defined by the formula

$$\mathrm{Hom}_{\pi_0(\mathcal{C})}(x,y) = \pi_0(\mathcal{H}om_{\mathcal{C}}(x,y)).$$

The nerve $\mathcal{N}(\mathcal{C})$ of a simplicial category $\mathcal{C}$ is defined as the diagonal of the bisimplicial set corresponding to $\mathcal{C}$. The category $\mathtt{sCat}$ admits a closed model category structure. The description of this structure, as well as a detailed description of the nerve functor and its properties, are given in the appendix.

For a fixed field k of characteristic zero $\mathtt{dglie}(k)$ is the category of dg Lie algebras and $\mathtt{dgart}^{\leq 0}(k)$ is the category of non-positively graded commutative artinian dg algebras with residue field k.

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2. Homotopy algebras

In this section we define a deformation functor for algebras over a dg operad. In 2.1 we recall the structure of simplicial closed model category on the the category $\mathtt{Alg}(\mathcal{O})$ of algebras over a dg operad $\mathcal{O}$ over a field of characteristic zero. We use this structure to assign to any $A \in \mathtt{Alg}(\mathcal{O})$ and $R \in \mathtt{dgart}^{\leq 0}(k)$ a simplicial category $\mathcal{D}ef_A(R)$ of "higher" formal deformations of A with base R. Its nerve $\mathrm{Def}_A(R) = \mathcal{N}(\mathcal{D}ef_A(R))$ gives us the value of the deformation functor at R. It is always a Kan simplicial set.

2.1. Simplicial CMC structure on $\mathtt{Alg}(\mathcal{O})$ (see [H2]).

Let k be a field of characteristic zero. The category $\mathtt{Alg}(\mathcal{O})$ of algebras over a dg operad $\mathcal{O} \in \mathtt{Op}(C(k))$ admits a closed model category structure. Weak equivalences in $\mathtt{Alg}(\mathcal{O})$ are quasi-isomorphisms and fibrations are surjective maps. Recall the notion of a cofibration in $\mathtt{Alg}(\mathcal{O})$.

In the definition below we denote by $X^{\#}$ the graded module (algebra, operad) corresponding to a complex (dg algebra, dg operad) X. If $\mathcal{O}$ is an operad in $\mathcal{A}$ and $X \in \mathcal{A}$, $F_{\mathcal{O}}(X)$ denotes the free $\mathcal{O}$-algebra generated by X.

2.1.1. **Definition.** 1. A map $A \to B$ is called a standard cofibration if it can be presented as a direct limit of a sequence of maps

$$A = B_0 \to B_1 \to \dots \to B_n \to \dots$$

where for each n the algebra $B_n^{\#}$ is isomorphic to $B_{n-1}^{\#} \sqcup F_{\mathcal{O}^{\#}}(X)$ and the differential in B_n is given by a map $d : X \to B_{n-1}$.
2. A map $f : A \to B$ is called a cofibration if it can be presented as a retract of a standard cofibration.

2.1.2. The category $\mathtt{Alg}(\mathcal{O})$ admits a simplicial structure so that Quillen's axiom (SM7) is satisfied — see [H2], 4.8. The simplicial structure is defined by the simplicial path functor which assigns to an algebra $A \in \mathtt{Alg}(\mathcal{O})$ and to a finite simplicial set $S \in \Delta^0\mathtt{Ens}$ the algebra $A^S = \Omega(S) \otimes A$ where $\Omega(S)$ denotes the dg commutative algebra of polynomial differential forms on S.

More explicitly, n-simplices of $\mathcal{H}om(A, B)$ are maps

$$\phi : A \to \Omega_n \otimes B$$

where

$$\Omega_n = k[x_0, \dots, x_n, dx_0, \dots, dx_n]/(\sum x_i - 1, \sum dx_i) \tag{5}$$

is the algebra of polynomial k-valued differential forms on the standard n-simplex.

2.1.3. *Notation.* $\mathcal{W}_*^c(\mathcal{O})$ denotes the following simplicial category. Its objects are cofibrant $\mathcal{O}$-algebras. If A and B are two such algebras, n-simplex of $\mathcal{H}om(A, B)$ is given by a quasi-isomorphism

$$\phi : A \to \Omega_n \otimes B.$$

This simplicial category play the role of higher groupoid of $\mathcal{O}$-algebras.

2.2. Let k, $\mathcal{O}$ be as above.

Let R be a commutative dg k-algebra. We define an operad $R \otimes \mathcal{O}$ by the formula

$$(R \otimes \mathcal{O})(n) = R \otimes \mathcal{O}(n). \tag{6}$$

The operad $R \otimes \mathcal{O}$ can be equally considered as an object in $\mathtt{Op}(C(k))$ or in $\mathtt{Op}(\mathtt{mod}(R))$.

Note that the notion of $R \otimes \mathcal{O}$-algebra is the same for both base tensor categories $C(k)$ and $\mathtt{mod}(R)$.

In what follows we will write $\mathcal{W}_*^c(R, \mathcal{O})$ instead of $\mathcal{W}_*^c(R \otimes \mathcal{O})$, to stress the dependence of this simplicial category on R.

2.3. Two versions of the deformation functor.

Let k, $\mathcal{O}$ be as above and let $A \in \mathtt{Alg}(\mathcal{O})$. In this subsection we define two versions of the deformation functor of A — the first one with values in $\mathtt{sCat}$ and the second one with values in $\Delta^0\mathtt{Ens}$.

2.3.1. $\mathtt{sCat}$-*version.*

Definition. *Deformation functor*

$$\mathcal{D}ef_A : \mathtt{dgart}^{\leq 0}(k) \to \mathtt{sCat}$$

is defined as the homotopy fibre of the canonical map

$$\pi_* : \mathcal{W}_*^c(R, \mathcal{O}) \to \mathcal{W}_*^c(\mathcal{O}) \tag{7}$$

at a point $\widetilde{A} \in \mathcal{W}_*^c(\mathcal{O})$ where $\widetilde{A} \to A$ is a cofibrant resolution of A.

2.3.2. *Simplicial version.* Any simplicial category $\mathcal{C}$ is a simplicial object in $\mathtt{Cat}$. Applying componentwise the nerve functor $N : \mathtt{Cat} \to \Delta^0\mathtt{Ens}$, we obtain a bisimplicial set. Its diagonal is called *simplicial nerve* of $\mathcal{C}$, denoted by $\mathcal{N}(\mathcal{C})$ — see the appendix.

Definition. *Deformation functor*

$$\mathrm{Def}_A : \mathtt{dgart}^{\leq 0}(k) \to \Delta^0\mathtt{Ens} \tag{8}$$

is defined as the homotopy fibre of the canonical map

$$\mathcal{N}(\pi_*) : \mathcal{N}\left(\mathcal{W}_*^c(R, \mathcal{O})\right) \to \mathcal{N}\left(\mathcal{W}_*^c(\mathcal{O})\right). \tag{9}$$

Here as above, the homotopy fibre is taken at a point $\widetilde{A} \in \mathcal{W}_*^c(\mathcal{O}, k)$ where $\widetilde{A} \to A$ is a cofibrant resolution of A.

2.3.3. Recall (see [H2], Sect. 8) that for a $\mathcal{O}$-algebra A its tangent Lie algebra T_A is defined as

$$T_A = \mathrm{Der}(\widetilde{A}, \widetilde{A}).$$

This is a dg Lie algebra over k which does not depend, up to quasi-somorphism, on the choice of a cofibrant resolution $\widetilde{A} \to A$.

Now we are ready to formulate the main result of this paper.

2.3.4. **Theorem.** *Let $\mathcal{O}$ be a dg operad over a field k of characteristic zero and let A be an $\mathcal{O}$-algebra. Suppose that both $\mathcal{O}$ and A are non-positively graded. Then the deformation functor* $\mathrm{Def}_A : \mathtt{dgart}^{\leq 0}(k) \to \Delta^0\mathtt{Ens}$ *is equivalent to the nerve $\Sigma_{\mathfrak{g}}$ of the tangent dg Lie algebra $\mathfrak{g} := T_A$.*

Theorem 2.3.4 will be proven in Section 4.

3. Simplicial Deligne groupoid

3.1. **Definition.** Let k be a field of characteristic zero and $\mathfrak{g} \in \mathtt{dglie}(k)$ be a nilpotent dg Lie k-algebra. In this section we construct a simplicial groupoid $\mathcal{D}el(\mathfrak{g}) = \{\mathcal{D}el_n(\mathfrak{g})\}$ whose simplicial nerve (see 6.2) is naturally homotopically equivalent to the nerve $\Sigma(\mathfrak{g})$ defined in [H1].

The construction is a generalization (and a simplification) of the one we used in [H3], 9.7.6.

Recall (see [H1], [H3], 8.1.1) that the nerve $\Sigma(\mathfrak{g})$ of the nilpotent dg Lie algebra $\mathfrak{g}$ is defined as

$$\Sigma_n(\mathfrak{g}) = \mathrm{MC}(\Omega_n \otimes \mathfrak{g}), \tag{10}$$

Ω_n being defined as in 5.

Following [H3], Sect. 8, define a simplicial group $G = G(\mathfrak{g})$ by the formula

$$G_n = \exp(\Omega_n \otimes \mathfrak{g})^0. \tag{11}$$

Here $\Omega_n \otimes \mathfrak{g}$ is a nilpotent dg Lie algebra, so its zero component is an ordinary nilpotent Lie algebra, and therefore its exponent makes sense.

Define a simplicial groupoid $\Gamma := \mathcal{D}el(\mathfrak{g})$ (we will call it *simplicial Deligne groupoid* since its zero component is the conventional Deligne groupoid [GM]) as follows.

$\mathrm{Ob}\,\Gamma = \mathrm{MC}(\mathfrak{g})$;

$\mathrm{Hom}_\Gamma(x, y)_n = \{g \in G_n | g(x) = y\}$.

It is useful to have in mind the following easy

3.1.1. **Lemma.** *The simplicial group $G(\mathfrak{g})$ is always contractible.*

Proof. As a simplicial set, G is isomorphic to the simplicial vector space

$$n \mapsto (\Omega_n \otimes \mathfrak{g})^0.$$

The latter is a direct sum of simplicial vector spaces of form $\Omega^p_\bullet$ (each one $\dim \mathfrak{g}^{-p}$ times) which are all contractible — see [L], p. 44. □

3.2. **Equivalence.** Recall that any simplicial category (and more generally, any $\mathcal{C} \in \Delta^0\mathtt{Cat}$) defines a bisimplicial set whose diagonal is called *the nerve* of $\mathcal{C}$, denoted by $\mathcal{N}(\mathcal{C})$ — see 6.2.

3.2.1. **Proposition.** *The nerve $\Sigma(\mathfrak{g})$ of a nilpotent dg Lie algebra is naturally homotopically equivalent to $\mathcal{N}(\mathcal{D}el(\mathfrak{g}))$.*

Proof. Define $\Gamma' \in \Delta^0\mathtt{Grp}$ (a simplicial groupoid in the broad sense) by the following formulas.

$\mathrm{Ob}\,\Gamma'_n = \mathrm{MC}(\Omega_n \otimes \mathfrak{g})$;

$\mathrm{Hom}_{\Gamma'}(x, y)_n = \{g \in G_n(\mathfrak{g}) | g(x) = y\}$.

One has a natural fully faithful embedding $\mathcal{D}el(\mathfrak{g}) \to \Gamma'$. According to [H3], 8.2.5, the map $\mathcal{D}el_n(\mathfrak{g}) \to \Gamma'_n$ is an equivalence of groupoids for each n. This implies that the induced map of the nerves

$$\mathcal{N}(\mathcal{D}el(\mathfrak{g})) \to \mathcal{N}(\Gamma')$$

is a homotopy equivalence.

Let us compare $\mathcal{N}(\Gamma')$ to $\Sigma(\mathfrak{g})$. Look at Γ' as a bisimplicial set. One has

$$\Gamma'_{pq} = \Sigma_p(\mathfrak{g}) \times G_p(\mathfrak{g})^q.$$

This means that the simplicial set $\Gamma'_{\bullet q}$ is equal to $\Sigma(\mathfrak{g}) \times G(\mathfrak{g})^q$.

The simplicial set $G(\mathfrak{g})$ is contractible by Lemma 3.1.1. Therefore, $\Gamma'_{\bullet q}$ is canonically homotopy equivalent to $\Sigma(\mathfrak{g})$. This implies that the nerve $\mathcal{N}(\Gamma')$ is homotopy equivalent to $\Sigma(\mathfrak{g})$. □

3.2.2. *Remark.* Proposition 3.2.1 generalizes the claim used in the proof of 9.7.6 of [H3].

3.2.3. Let now $\mathfrak{g} \in \mathtt{dglie}(k)$. Following the well-known pattern, we define the functor

$$\mathcal{D}el_{\mathfrak{g}} : \mathtt{dgart}^{\leq 0}(k) \to \mathtt{sGrp}$$

by the formula

$$\mathcal{D}el_{\mathfrak{g}}(R) = \mathcal{D}el(\mathfrak{m} \otimes \mathfrak{g})$$

for $(R, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$.

The functor $\mathcal{D}el_{\mathfrak{g}}$ is also called the simplicial Deligne groupoid.

3.3. **Properties.** We wish to deduce now some properties of the simplicial Deligne groupoid functor which are similar to the properties of the nerve $\Sigma(\mathfrak{g})$ — see [H3], Sect. 8.

In what follows we use the closed model category structure on the category $\mathtt{sCat}$ — see the appendix.

3.3.1. **Proposition.** *Let $f : \mathfrak{g} \to \mathfrak{h}$ be surjective (resp., a surjective quasi-isomorphism). Then for each $(R, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$ the map*

$$f : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}el_{\mathfrak{h}}(R)$$

is a fibration (resp., an acyclic fibration) in $\mathtt{sCat}$.

Proof. Note first of all that the similar claim holds for the nerve functor: according to [H3], Prop. 7.2.1, the map $f : \Sigma_{\mathfrak{g}}(R) \to \Sigma_{\mathfrak{h}}(R)$ is a fibration (resp., acyclic fibration) provided f is a surjection (resp., a surjective quasi-isomorphism). This implies that the map

$$f : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}el_{\mathfrak{h}}(R)$$

satisfies the property (1) of fibrations (resp., of acyclic fibrations) — see 6.1.3, 6.1.5.

Let us check the property (2). It claims that for any $x, y \in \mathrm{Ob}\,\mathcal{D}el_{\mathfrak{g}}(R)$ the map of simplicial sets

$$f : \mathcal{H}om_{\mathfrak{g}}(x, y) \to \mathcal{H}om_{\mathfrak{h}}(fx, fy)$$

is a Kan fibration (resp., acyclic Kan fibration) — here we write for simplicity $\mathcal{H}om_{\mathfrak{g}}$ instead of $\mathcal{H}om_{\mathcal{D}el_{\mathfrak{g}}(R)}$.

Let $G = G(\mathfrak{g}),\ H = G(\mathfrak{h})$ be the simplicial groups corresponding to $\mathfrak{g}$, $\mathfrak{h}$ as in the formula (11).

A map from a simplicial set K to $\mathcal{H}om_{\mathfrak{g}}(x, y)$ is given by an element $g \in G(K) = \mathrm{Hom}(K, G)$ satisfying the condition $g(x) = y$.

Let a commutative diagram in $\Delta^0\mathtt{Ens}$

$$\begin{array}{ccc} K & \longrightarrow & \mathcal{H}om_{\mathfrak{g}}(x, y) \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ L & \longrightarrow & \mathcal{H}om_{\mathfrak{h}}(fx, fy) \end{array}$$

be given with $\alpha : K \to L$ being a cofibration of finite simplicial sets. We suppose also that either α or f is a weak equivalence. Our aim is to find a map $L \to \mathcal{H}om_{\mathfrak{g}}(x, y)$ commuting with the above diagram. Thus, we are given with a compatible pair of elements $g \in G(K),\ h \in H(L)$ satisfying the property

$$g(x) = y;\ h(fx) = fy.$$

Our aim is to lift this pair to an element $\widetilde{g} \in G(L)$ satisfying the property $\widetilde{g}(x) = y$.

We will do this in two steps. First of all, since f is surjective, the induced map of simplicial groups $f : G \to H$ is surjective, and, therefore, fibrant. Furthermore, since both G and H are contractible by Lemma 3.1.1, the map $f : G \to H$ is actually an acyclic fibration, and therefore the pair of compatible elements $g \in G(K),\ h \in H(L)$ lifts to an element $g' \in G(L)$. We can not, unfortunately, be sure that $g'(x) = y$. This is why we need the second step in which we correct g' to satisfy this property.

Suppose $g'(x) = y' \in \mathrm{MC}(\Omega(L) \otimes \mathfrak{m} \otimes \mathfrak{g})$. The elements y and y' of $\mathrm{MC}(\Omega(L) \otimes \mathfrak{m} \otimes \mathfrak{g})$ have the same images in both $\mathrm{MC}(\Omega(K) \otimes \mathfrak{m} \otimes \mathfrak{g})$ and $\mathrm{MC}(\Omega(L) \otimes \mathfrak{m} \otimes \mathfrak{h})$. Now, the commutative diagram

$$\begin{array}{ccc} \Omega(L) \otimes \mathfrak{g} & \longrightarrow & \Omega(K) \otimes \mathfrak{g} \\ \downarrow & & \downarrow \\ \Omega(L) \otimes \mathfrak{h} & \longrightarrow & \Omega(K) \otimes \mathfrak{h} \end{array}$$

induces an acyclic fibration

$$p : \mathfrak{g}_1 := \Omega(L) \otimes \mathfrak{g} \to \Omega(K) \otimes \mathfrak{g} \times_{\Omega(K)\otimes\mathfrak{h}} \Omega(L) \otimes \mathfrak{h} =: \mathfrak{g}_2$$

of dg Lie algebras. Then the map $\Sigma_p : \Sigma_{\mathfrak{g}_1}(R) \to \Sigma_{\mathfrak{g}_2}(R)$ is an acyclic fibration.

Now, we have two elements $y, y' \in \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}_1)$ satisfying $p(y) = p(y') \in \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}_2)$. Therefore, there exists an element $z \in \Sigma_{\mathfrak{g}_1}(R)_1$ such that $d_0 z = y, d_1 z = y'$ and $p(z) = s_0(p(y))$. Using the explicit description of $\Sigma_{\mathfrak{g}}(R)_1$ in [H3], 8.2.3, one obtains and element $\gamma \in \exp(\mathfrak{m} \otimes \mathfrak{g}_1)$ satisfying $p(\gamma) = 1 \in \exp(\mathfrak{m} \otimes \mathfrak{g}_2)$; $\gamma(y') = y$.

Then one immediately sees that the element $\widetilde{g} = \gamma g'$ is the one we need. $\square$

3.3.2. **Corollary.** *1. For any $\mathfrak{g} \in \mathtt{dglie}(k)$, $R \in \mathtt{dgart}^{\leq 0}(k)$, $x, y \in \mathrm{Ob}\,\mathcal{D}el_{\mathfrak{g}}(R)$ the simplicial set $\mathcal{H}om(x, y)$ is fibrant.*

2. Any quasi-isomorphism $f : \mathfrak{g} \to \mathfrak{h}$ induces a weak equivalence

$$f : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}el_{\mathfrak{h}}(R)$$

for each $R \in \mathtt{dgart}^{\leq 0}(k)$.

Proof. 1. Take $\mathfrak{h} = 0$ in Proposition 3.3.1.

2. The category $\mathtt{dglie}(k)$ admits a CMC structure with surjections as fibrations and quasi-isomorphisms as weak equivalences — see [H2], Sect. 4. Using this, present $f = p \circ i$ as a composition of an acyclic fibration p and an acyclic cofibration i. Any acyclic cofibration in

$\mathtt{dglie}(k)$ is left invertible: $q \circ i = \mathrm{id}$. The map q is obviously an acyclic fibration. Then by Proposition 3.3.1 the map $f : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}el_{\mathfrak{h}}(R)$ is a weak equivalence. $\square$

4. Proof of Theorem 2.3.4

In 4.1–4.2 we prove Theorem 2.3.4. In 4.3 we explain why the theorem needs not to be correct without the non-positivity conditions.

4.1. We start with an observation explaining the connection between T_A and the formal deformations of A. Let B be a cofibrant $R\otimes\mathcal{O}$-algebra with $(R, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$. Denote $A = k\otimes_R B$. The algebra B is isomorphic, as a graded $\mathcal{O}$-algebra, to $R \otimes A$. Choose a graded isomorphism

$$\theta : B \to R \otimes A$$

and put

$$z = \theta \circ d_B \circ \theta^{-1} - 1 \otimes d_A$$

where d_B (resp., d_A) is the differential in B (resp., in A).

The element z is a degree one derivation belonging to $\mathfrak{m} \otimes T_A$ satisfying the Maurer-Cartan equation. A different choice of isomorphism θ gives rise to a Maurer-Cartan element $z' \in \mathfrak{m}\otimes T_A$ equivalent to z: there exists $g \in \exp(\mathfrak{m} \otimes T_A)^0$ such that $z' = g(z)$.

In what follows we will use a (non-unique) presentation of a $R \otimes \mathcal{O}$-algebra B by an element $z \in \mathrm{MC}(\mathfrak{m} \otimes T_{k\otimes_R B})$.

4.2. To simplify the notation, denote $\mathcal{W} = \mathcal{W}^c_*(\mathcal{O}, R)$, $\quad \overline{\mathcal{W}} = \mathcal{W}^c_*(\mathcal{O}, k)$.

4.2.1. **Lemma.** *The natural map $\pi : \mathcal{W} \to \overline{\mathcal{W}}$ is a fibration in* $\mathtt{sCat}$.

Proof. 1. Let us check the condition (1) of Definition 6.1.3.

It means the following. Let $f : A \to B$ be a quasi-isomorphism of cofibrant $\mathcal{O}$-algebras over k. Let one of two elements $a \in \mathrm{MC}(\mathfrak{m} \otimes T_A)$ or $b \in \mathrm{MC}(\mathfrak{m} \otimes T_B)$ be given. We have to check that there exists a choice of the second element and a map

$$g : (R \otimes A, d + a) \to (R \otimes B, d + b)$$

of $R \otimes \mathcal{O}$-algebras which lifts $f : A \to B$.

We can consider separately the cases when f is an acyclic fibration or an acyclic cofibration.

In both cases we will be looking for the map g in the form

$$g = \gamma_B^{-1} \circ (\mathrm{id}_R \otimes f) \circ \gamma_A$$

where $\gamma_A \in \exp(\mathfrak{m} \otimes T_A)^0$ and similarly for γ_B. A map g as above should commute with the differentials $d + a$ and $d + b$. This amounts to the condition

$$f_*(\gamma_A(a)) = f^*(\gamma_B(b)),$$

where the natural maps

$$T_A \xrightarrow{f_*} \mathrm{Der}_f(A, B) \xleftarrow{f^*} T_B$$

are defined as in [H2], 8.1.

Recall that we are assuming that f is either acyclic cofibration or an acyclic fibration.

In both cases there exists a commutative square

$$\begin{tikzcd} T_f \arrow[r, "\alpha"] \arrow[d, "\beta"'] & T_A \arrow[d, "f_*"] \\ T_B \arrow[r, "f^*"] & \mathrm{Der}_f(A,B) \end{tikzcd}$$

where T_f is a dg Lie algebra and α, β are Lie algebra quasi-isomorphisms — see [H2], 8.2, 8.3. The maps α, β induce bijections

$$\pi_0(\Sigma_{T_A}(R)) \longleftarrow \pi_0(\Sigma_{T_f}(R)) \longrightarrow \pi_0(\Sigma_{T_A}(R))$$

which prove the assertion.

2. Let us check the condition (2) of 6.1.3. Let $\widetilde{A}, \widetilde{B} \in \mathcal{W}$ and let $A = k \otimes_R \widetilde{A}, B = k \otimes_R \widetilde{B}$. We have to check that the map

$$\mathcal{H}om(\widetilde{A}, \widetilde{B}) \to \mathcal{H}om(A, B) \tag{12}$$

is a Kan fibration. We claim that this results from the simplicial CMC structure on $\mathtt{Alg}(R \otimes \mathcal{O})$. In fact, $\widetilde{A}$ is cofibrant, and the reduction map $\widetilde{B} \to B$ can be considered as a fibration in $\mathtt{Alg}(R \otimes \mathcal{O})$. Therefore, the map

$$\mathcal{H}om(\widetilde{A}, \widetilde{B}) \to \mathcal{H}om(\widetilde{A}, B) \tag{13}$$

is a Kan fibration. But the maps (13) and (12) coincide, so the condition (2) of 6.1.3 is verified.

The lemma is proven. $\square$

4.2.2. Fix now a cofibrant $\mathcal{O}$-algebra A and denote $\mathfrak{g} = T_A$. Fix $(R, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$.

Define a map of simplicial categories

$$\alpha : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{W}$$

as follows. Let $z \in \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}) = \mathrm{Ob}\, \mathcal{D}el_{\mathfrak{g}}(R)$. Put

$$\alpha(z) = (R \otimes A, 1 \otimes d + z).$$

Now, any element $g \in G_n = \exp(\Omega_n \otimes \mathfrak{m} \otimes \mathfrak{g})^0$ defines a graded automorphism of $\Omega_n \otimes R \otimes A$. This obviously defines an isomorphism of $R \otimes \mathcal{O}$-algebras

$$(R \otimes A, 1 \otimes d + z) \longrightarrow (R \otimes A, 1 \otimes d + g(z)).$$

The observation 4.1 shows that the map $\alpha : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{W}$ identifies $\mathcal{D}el_{\mathfrak{g}}(R)$ with the fibre of $\pi : \mathcal{W} \to \overline{\mathcal{W}}$ at A. Since π is a fibration by Lemma 4.2.1, $\mathcal{D}el_{\mathfrak{g}}(R)$ is weakly equivalent to the homotopy fibre of π which is by definition $\mathcal{D}ef_A(R)$.

Now the theorem follows from Proposition 3.2.1 since the nerve functor $\mathcal{N}$ preserves fibrations and weak equivalences by 6.2.2.

Theorem is proven.

4.3. **Example.** Let $\mathfrak{g}$ be an arbitrary dg Lie algebra. The nerve functor

$$\Sigma_{\mathfrak{g}} : \mathtt{dgart}^{\leq 0} \to \Delta^0\mathtt{Ens}$$

commutes with inverse limits. In particular, for

$$R_\epsilon = k[\epsilon]/(\epsilon^2),\ R_\delta = k[\delta]/(\delta^2),\ R_{\epsilon\delta} = k[\epsilon,\delta]/(\epsilon^2,\epsilon\delta,\delta^2)$$

one has

$$\Sigma_{\mathfrak{g}}(R_{\epsilon\delta}) \xrightarrow{\sim} \Sigma_{\mathfrak{g}}(R_\epsilon) \times \Sigma_{\mathfrak{g}}(R_\delta).$$

This means that for a deformation problem governed by a dg Lie algebra, a two-parameter infinitesimal deformation is uniquely defined by a pair of one-parameter infinitesimal deformations.

Now we will show that this property is not fulfilled for our deformation problem if one does not impose a condition on the grading of A.

This means that the deformation functor Def_A in the example below can not be governed by a dg Lie algebra.

Let $\mathcal{O}$ be the trivial operad $\mathcal{O}(1) = k \cdot 1,\ \mathcal{O}(i) = 0$ for $i \neq 0$. $\mathcal{O}$-algebras are just complexes.

Let A be the complex with zero differential with $A^i = k$ for all $i \in \mathbb{Z}$.

Let $A_\epsilon = R_\epsilon \otimes A$ as a graded R_ϵ-module. Endow A_ϵ with the differential which vanishes in odd degrees and is a multiplication by ϵ in even degrees. Define the R_δ-module A_δ in a similar way, just interchanging even and odd degrees.

We claim there is no deformation of A over $R_{\epsilon\delta}$ which induces A_ϵ and A_δ over R_ϵ and R_δ respectively.

We need the following

4.3.1. **Lemma.** *Let $(R,\mathfrak{m})$ be an artinian local ring. Any complex of R-modules admits a minimal cofibrant resolution, i.e. a cofibrant resolution P satisfying the condition $dP \subseteq \mathfrak{m}P$.*

Proof. Since $(R,\mathfrak{m})$ is local artinian, any R-module M admits a projection $\phi : F \to M$ with F free and $\ker\phi \subseteq \mathfrak{m}F$. This easily implies the lemma. □

Lemma 4.3.1 implies that any deformation of A over $R_{\epsilon\delta}$ can be presented by a cofibrant minimal complex $A_{\epsilon\delta}$ having a free module with one generator in each degree. The conditions

$$A_{\epsilon\delta} \otimes_{R_{\epsilon\delta}} R_\epsilon = A_\epsilon; \quad A_{\epsilon\delta} \otimes_{R_{\epsilon\delta}} R_\delta = A_\delta$$

imply that all components of the differential in $A_{\epsilon\delta}$ are non-zero. This, however, is impossible since $A_{\epsilon\delta}$ is cofibrant.

4.4. **Remark.** Theorem 2.3.4 follows from the existence of a weak equivalence

$$\alpha : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}ef_A(R)$$

in $\mathtt{sCat}$. One might ask, therefore, is not it better to work with deformation functors taking values in $\mathtt{sCat}$ instead of pushing them to simplicial sets using the nerve functor. We do not have a good answer to this question. We chose simplicial sets because fibrant simplicial categories seem to be too general to describe an intuitive notion of higher groupoid.

One can probably use simplicial groupoids instead of simplicial sets — but the nerve functor $\mathcal{N}$ is conservative on $\mathtt{sGrp}$ (see Proposition 6.2.3), so the difference does not seem to be very big.

5. Compatibility with the classical definition

Suppose that $\mathcal{O}$ is an operad of vector spaces and A is an $\mathcal{O}$-algebra in $\mathtt{Vect}$. Then the classical deformation functor

$$\mathrm{Def}^{cl}_A : \mathtt{art} \to \mathtt{Grp} \tag{14}$$

assigns to each artinian local k-algebra $(R,\mathfrak{m})$ the groupoid of flat $R\otimes\mathcal{O}$-algebras $\widetilde{A}$ endowed with an isomorphism $\widetilde{A}\otimes_R k \xrightarrow{\sim} A$.

We claim that in this case the deformation functor Def_A defined in 2.3.2 is equivalent to the classical Def^{cl}_A.

5.1. Let $P\to A$ be a cofibrant resolution and let $\widetilde{P}\in\mathcal{W}^c_*(R\otimes\mathcal{O})$ be such that $\pi^*(\widetilde{P})$ is quasi-isomorphic to P. We claim that the homology of $\widetilde{P}$ is a deformation of A in the classical sense. This claim results from the following elementary lemmas.

5.1.1. **Lemma.** *Let $R\in\mathtt{art}(k)$ and let X be a complex of R-modules. The following conditions are equivalent*

1. *$H^i(X)=0$ for $i\neq 0$ and $H^0(X)$ is flat.*
2. *$X\otimes^{\mathbf{L}}_R k$ is concentrated at degree zero.*

Proof. By Lemma 4.3.1 it is enough to check the assertion of the lemma when X is cofibrant minimal. Then $X\otimes_R k$ has zero differential which implies that $X^i=0$ for $i\neq 0$. This proves the lemma. □

5.1.2. **Lemma.** *Any cofibrant $R\otimes\mathcal{O}$-algebra is cofibrant as a complex of R-modules.*

Proof. The proof presented below is very close to that of Theorem 4.7 from [H2].

One can easily reduce the assertion to the case of finitely generated standard cofibrant algebras. Let A be such algebra. Choose a set of graded free generators $\{x_i,\ i\in I\}$ for A where I is a totally ordered finite set, such that dx_i belongs to the subalgebra of A generated by x_j with $j<i$. The set of multi-indices $m:I\to\mathbb{N}$ is well-ordered with respect to the following lexicographic order:

$$m>m' \text{ if there exists } i\in I \text{ such that } m_j=m'_j \text{ for } j>i \text{ and } m_i>m'_i.$$

Then an increasing filtration on A by subcomplexes indexed by the set of multi-indices $m:I\to\mathbb{N}$ is defined by the formula

$$F_d(A)=\sum_{m<d} R\otimes\mathcal{O}(|m|)\otimes_{\Sigma_m}\bigotimes_{i\in I} x_i^{\otimes m_i}.$$

Here $|m|=\sum m_i$ and $\Sigma_m=\prod\Sigma_{m_i}\subseteq\Sigma_{|m|}$ is the product of symmetric groups. The associated graded complexes are (shifted) free R-modules. This proves the lemma. □

Lemmas 5.1.1 and 5.1.2 above imply that if $\widetilde{P} \in \mathcal{W}^c(\mathcal{O}, R)$ is a deformation of $P \in \mathcal{W}^c(\mathcal{O}, k)$ with $H(P) = A$ then $H(\widetilde{P})$ is a classical deformation of A with the base R.

This defines a map

$$h^{\mathtt{sCat}} : \mathcal{D}ef_A(R) \to \mathrm{Def}^{cl}_A(R) \tag{15}$$

of simplicial categories, where the right-hand side is considered as a discrete simplicial category and the left-hand side is realized as the fiber of $\pi : \mathcal{W}^c_*(\mathcal{O}, R) \to \mathcal{W}^c_*(\mathcal{O}, k)$. Applying the nerve functor $\mathcal{N}$ we get a map of simplicial sets

$$h : \mathrm{Def}_A(R) \to \mathrm{Def}^{cl}_A(R). \tag{16}$$

5.2. **Proposition.** *Let $\mathcal{O}$, A be as above. The map*

$$h : \mathrm{Def}_A(R) \to \mathrm{Def}^{cl}_A(R)$$

is a weak equivalence for each $R \in \mathtt{art}(k)$.

Proof. Let $\widetilde{A}$ be a classical deformation of A over R. Consider a cofibrant resolution $\widetilde{P}$ of $\widetilde{A}$ as of an $R \otimes \mathcal{O}$-algebra. Let $\pi : R \to k$ be the standard projection. There exists a quasi-isomorphism $\pi^*(\widetilde{P}) \to P$ which induces identity on the homology. Lemma 4.2.1 implies that the map $h^{\mathtt{sCat}}$ is essentially surjective.

Let now $\mathfrak{g} = \mathrm{Der}(P, P)$. One has

$$H^i(\mathfrak{g}) = H^i(\mathrm{Hom}_P(\Omega_P, P)) = H^i(\mathrm{Hom}_P(\Omega_P, A)) = 0 \text{ for } i < 0 \tag{17}$$

since A is concentrated in degree zero and the free generators of the graded algebra P can be chosen to have non-positive degrees. Therefore, $\mathrm{Def}_A(R) \cong \Sigma_{\mathfrak{g}}(R)$ is equivalent to a groupoid by [H1] or 1.4.2.

In order to prove that (16) is an equivalence, we have to check it induces an isomorphism of the fundamental groups.

Fix $z \in \mathfrak{m} \otimes \mathfrak{g}^1$ satisfying Maurer-Cartan equation. Let $\widetilde{P}_z$ be the object of $\mathcal{D}ef_P(R)$ corresponding to z and let $\widetilde{A}_z$ be its cohomology. The fundamental group of the right-hand side of (16) at the point z is the group $\mathrm{Aut}_A(\widetilde{A}_z)$ of automorphisms of $\widetilde{A}_z$ which are identical on A.

The fundamental group of the left-hand side of (16) at z can be calculated in $\mathcal{D}ef_A(R)$ as the group of homotopy classes of automorphisms of $\widetilde{P}_z$ inducing the identity on P.

The map from the left-hand side to the right-hand side in (16) assigns to each automorphism θ of $\widetilde{P}_z$ its homology $H(\theta) : \widetilde{A}_z \to \widetilde{A}$. To prove that (16) is an isomorphism, we have to check that any automorphism of $\widetilde{A}_z$ inducing the identity on A can be lifted, in a unique way up to homotopy, to an automorphism of $\widetilde{P}_z$ inducing identity on P. This follows from the standard properties of cofibrant resolutions in closed model categories. $\square$

6. Appendix: simplicial categories

In this section we recall some standard facts about simplicial categories. This includes the description of a model category structure on $\mathtt{sCat}$ and some properties of the nerve functor $\mathcal{N} : \mathtt{sCat} \to \Delta^0\mathtt{Ens}$.

The proofs of all claims, except for Proposition 6.2.3, can be found in [H3], Sect. 11.

6.1. **Closed model category structure on $\mathtt{sCat}$.**

6.1.1. Define the functor

$$\pi_0 : \mathtt{sCat} \to \mathtt{Cat}$$

as follows. For $\mathcal{C} \in \mathtt{sCat}$ the category $\pi_0(\mathcal{C})$ has the same objects as $\mathcal{C}$. For $x, y \in \operatorname{Ob}\pi_0(\mathcal{C})$

$$\operatorname{Hom}_{\pi_0(\mathcal{C})}(x, y) = \pi_0(\operatorname{Hom}_{\mathcal{C}}(x, y)).$$

6.1.2. **Definition.** A map $f : \mathcal{C} \to \mathcal{D}$ in $\mathtt{sCat}$ is called a weak equivalence if the following properties are satisfied.

(1) The map $\mathcal{N}(\pi_0(f))$ is a weak equivalence of simplicial sets.

(2) For all $x, x' \in \operatorname{Ob}\mathcal{C}$ the map $f : \mathcal{H}om(x, x') \to \mathcal{H}om(fx, fx')$ is a weak equivalence.

6.1.3. **Definition.** A map $f : \mathcal{C} \to \mathcal{D}$ in $\mathtt{sCat}$ is called a fibration if it satisfies the following properties

(1) the right lifting property (RLP) with respect to the maps

$$\partial^{0,1} : \Delta^0 \to \Delta^1$$

from the terminal category $\Delta^0 = *$ to the one-arrow category Δ^1.

(2) For all $x, x' \in \operatorname{Ob}\mathcal{C}$ the map $f : \mathcal{H}om(x, x') \to \mathcal{H}om(fx, fx')$ is a Kan fibration.

6.1.4. **Theorem.** *The category $\mathtt{sCat}$ admits a CMC structure with weak equivalences described in 6.1.2 and fibrations as in 6.1.3.*

Note the following

6.1.5. **Lemma.** *A map $f : \mathcal{C} \to \mathcal{D}$ is an acyclic fibration iff the following conditions are satisfied.*

(1) the map $\operatorname{Ob} f : \operatorname{Ob}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$ is surjective.

(2) For all $x, x' \in \operatorname{Ob}\mathcal{C}$ the map $f : \mathcal{H}om(x, x') \to \mathcal{H}om(fx, fx')$ is an acyclic Kan fibration.

6.2. **Simplicial nerve.**

6.2.1. In what follows we identify $\mathtt{Cat}$ with the full subcategory of $\Delta^0\mathtt{Ens}$. Then every simplicial category (and even every $\mathcal{C} \in \Delta^0\mathtt{Cat}$) can be seen as a bisimplicial set; its diagonal will be called *the nerve* of $\mathcal{C}$ and will be denoted $\mathcal{N}(\mathcal{C})$. If $\mathcal{C}$ is a "usual" category, $\mathcal{N}(\mathcal{C})$ is its "usual" nerve.

6.2.2. Proposition. *The nerve functor $\mathcal{N} : \mathtt{sCat} \to \Delta^0\mathtt{Ens}$ preserves weak equivalences, fibrations and cofibrations.*

Let $f : \mathcal{C} \to \mathcal{D}$ be a map in $\mathtt{sCat}$. It is not true in general that if $\mathcal{N}(f)$ is a weak equivalence then f is also a weak equivalence — take, for instance, any functor $f : \mathcal{C} \to \mathcal{D}$ between categories which is not fully faithful and add to $\mathcal{C}$ and to $\mathcal{D}$ a final object. This defines a functor which is not a weak equivalence between two categories whose nerves are contractible. This, however, cannot happen if $\mathcal{C}$ and $\mathcal{D}$ are simplicial groupoids.

6.2.3. Proposition. *Let $f : \mathcal{C} \to \mathcal{D}$ be a map of simplicial groupoids. If $\mathcal{N}(f)$ is a weak equivalence, f is also a weak equivalence.*

Proof. Let us check Property (1) of 6.1.2. In the case of simplicial groupoids it means that $\pi_0(f) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ is an equivalence of groupoids. It is clear that $\pi_0(f)$ induces a bijection on the set of connected components. The next thing is to check that $\pi_0(f)$ induces an isomorphism of authomorphism groups of objects of $\pi_0(\mathcal{C})$ and of $\pi_0(\mathcal{D})$. But this is a part of Property (2) proven below.

Let us check Property (2). We have to prove that for each object $x \in \mathcal{C}$ the map of the simplicial groups

$$\mathcal{H}om(x, x) \to \mathcal{H}om(y, y), \quad y = f(x),$$

is a homotopy equivalence. We will denote $x \in \mathcal{N}(\mathcal{C})$ and $y \in \mathcal{N}(\mathcal{D})$ the 0-simplices in the nerves corresponding to the objects $x \in \mathcal{C}$, $y \in \mathcal{D}$. We know that the maps $\pi_n(\mathcal{N}(\mathcal{C}), x) \to \pi_n(\mathcal{N}(\mathcal{D}), y)$ are isomorphisms. To prove the assertion, it is enough to check that

$$\pi_n(\mathcal{H}om(x, x)) \xrightarrow{\sim} \pi_{n+1}(\mathcal{N}(\mathcal{C}), x) \text{ for } n \geq 0.$$

It is enough to check this claim for simplicial groups (= simplicial groupoids having one object). Now it follows from the equivalence between $\mathcal{N}(\mathcal{C})$ and $\overline{W}(\mathcal{C})$ — see [H3], 11.5.1. □

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