DEFORMATIONS OF SHEAVES OF ALGEBRAS

VLADIMIR HINICH

ABSTRACT. A construction of the tangent dg Lie algebra of a sheaf of operad algebras on a site is presented. The requirements on the site are very mild; the requirements on the algebra are more substantial. A few applications including the description of deformations of a scheme and equivariant deformations are considered. The construction is based upon a model structure on the category of presheaves which should be of an independent interest.

0. Introduction

0.1. In this paper we study formal deformations of sheaves of algebras. The most obvious (and very important) example is that of deformations of a scheme X over a field k of characteristic zero. In two different cases, the first when X is smooth, and the second when X is affine, the description is well-known. In both cases there is a differential graded (dg) Lie algebra T_X over k such that formal deformations of X over the artinian local base $(R, \mathfrak{m})$ are described by the Maurer-Cartan elements of $\mathfrak{m} \otimes T_X$, modulo a gauge equivalence. We say that the dg Lie algebra T_X governs the formal deformations of X.

It is well-understood now that formal deformations over a field of characteristic zero are governed by a differential graded Lie algebra. One of possible explanations of this phenomenon was suggested in [H3]: we expect deformation problems to have formal moduli (which is expected to be a "commutative" formal dg scheme). Then the representing dg Lie algebra corresponds to the formal moduli by Koszul (or bar-cobar) duality. Thus, the existence of dg Lie algebra governing deformations is equivalent to the representability (in "higher", dg sense) of the deformation problem.

However, in the two cases mentioned above (X smooth and X affine) the governing dg Lie algebra T_X appears in seemingly different ways. This can be shortly described as follows.

0.1.1. *X is smooth.* Affine smooth scheme X has no formal deformations. Its trivial deformation U_R with an artinian local base $(R, \mathfrak{m})$ admits the automorphism group $\exp(\mathfrak{m} \otimes T_X)$ which is nothing but the value at R of the formal group corresponding to $T_X = \Gamma(X, \mathcal{T}_X)$. Descent theorem of [H1] asserts in this situation that for a general smooth scheme X the dg Lie algebra T_X governing the deformations of X can be calculated by the formula

$$T_X = \mathbf{R}\Gamma(X, \mathcal{T}_X).$$

0.1.2. *X is affine.* Let $X = \mathrm{Spec}(A)$ for a commutative k-algebra A. It is convenient to consider A as a dg commutative k-algebra concentrated at degree zero. Then the deformation theory of dg algebras [H4] suggests the following recipe of calculation of T_X. Let $P \to A$ be a cofibrant (some call it free or semi-free) resolution of A in the model category of commutative dg k-algebras (defined as in, for instance, [H2]). Deformations of A and of P are equivalent; deformations of P appear as perturbations of the differential which are described by the Maurer-Cartan elements of the Lie algebra of derivations of P. Thus, one has

$$T_X = \mathrm{Der}(P, P).$$

0.1.3. We wish to describe in a similar way deformations of a sheaf of algebras. The first problem seems to be the lack of cofibrant resolutions for sheaves of algebras. This turns out to have a very pleasant solution: the category of complexes of presheaves admits a model category structure describing the homotopy theory of complexes of sheaves.

A similar model category structure exists for sheaves of operad algebras in characteristic zero. This allows us to define deformation functor and to construct the corresponding dg Lie algebra in a way similar to the one described in 0.1.2. The construction is local, so, as a result, we obtain a presheaf of dg Lie algebras. We use the construction mentioned in 0.1.1 to get a global dg Lie algebra.

0.2. **Sheaves vs presheaves.** Let X be a site and $F : X^{\mathrm{op}} \to \mathcal{A}$ be a presheaf on X with values in a category $\mathcal{A}$ having a notion of weak equivalence (for instance, complexes, simplicial sets, categories or polycategories). The notion of sheaf is not very appropriate here: we know this well, for instance, in the case $\mathcal{A} = \mathtt{Cat}$. This was probably the reason Jardine [Ja] suggested a model category structure on the category of simplicial presheaves. The idea was to extend the notion of weak equivalence so that a presheaf will be weakly equivalent to its sheafification. Then the localization of the category of simplicial sheaves with respect to the weak equivalences can be described as the homotopy category of the category of presheaves.

We adopt a similar point of view. We need a model category structure on presheaves of algebras which would allow us to construct "semi-free resolutions". This model category structure is based upon a model category structure on the category $C(X_k^\frown)$ of complexes of presheaves of k-modules which is described by the following result.

0.2.1. **Theorem.** *Let X be a site, k a ring and let $C(X_k^\frown)$ denote the category of complexes of presheaves of k-modules on X.*

1. The category $C(X_k^\frown)$ admits a model category structure so that

- *weak equivalences are maps $f : M \to N$ inducing a quasi-isomorphisms $f^a : M^a \to N^a$ of sheafifications.*

- *cofibrations are generated by maps $f : M \to M\langle x; dx = z \in M(U)\rangle$ coresponding to adding a section to kill a cycle z over an object $U \in X$.*

2. A map $f : M \to N$ in the above model category structure is a fibration iff $f(U) : M(U) \to N(U)$ is surjective for any $U \in X$ and for any hypercover $\epsilon : V_\bullet \to U$ of $U \in X$ the corresponding commutative diagram

$$\begin{tikzcd} M(U) \arrow[r] \arrow[d] & \check{C}(V_\bullet, M) \arrow[d] \\ N(U) \arrow[r] & \check{C}(V_\bullet, N) \end{tikzcd}$$

is homotopy cartesian.

We remind the notion of hypercover in 1.2. Cech complex $\check{C}(V_\bullet, M)$ of M with respect to a hypercover $V_\bullet$ is defined as the total complex corresponding to the cosimplicial complex $n \mapsto M(V_n)$, see 1.2.3.

Notice that we do not require the existence of limits in the site X. This is important for us since we want to be able to apply this to the category of affine open subsets of a scheme which usually does not admit a final object.

0.2.2. The proof of Theorem 0.2.1 is given in 1.3.2. It is based on an explicit description of generating acyclic cofibrations.

Recently (see 0.9) we learned that the model structure described above (at least part 1 of Theorem 0.2.1) is known to specialists, see, for instance, [T], Appendix C. We decided, however, to present our proof since it is direct, general, and gives an explicit description of fibrations which we need in any case. A similar model category structure was used in [HiSi] for n-stacks.

We also present in Appendix B a version of Theorem 0.2.1 for simplicial presheaves. This model category structure on simplicial presheaves has the same weak equivalences as Jardine's [Ja] but the cofibrations are generated by gluing cells and fibrations have a similar description using hypercovers. The results of Appendix B are not used in the main body of the paper. They are, however, "ideologically related" to the framework of the paper, so we have found it appropriate to present them here.

Under some mild restrictions on the site X (any hypercover can be refined by a split hypercover) our model structure on simplicial presheaves coincides with a one recently defined in [DHI], Theorem 1.3. A similar CMC structure was defined by Toen-Vezzosi in [TV], 3.4. The main difference of our approach in Appendix B, as compared to the above references, is that we do not use the Bousfield localization techniques, presenting instead an explicit description of generating acyclic cofibrations which, in particular, provides a description of fibrations in terms of Čech complexes associated to hypercovers.

0.3. **Higher deformation functor.** Classical formal deformation functors can be usually described as follows. Let $\mathtt{art}(k)$ be the category of artinian local k-algebras $(R,\mathfrak{m})$ with residue field k. Let $\mathcal{C}(R)$ for each $R\in\mathtt{art}(k)$ denote a groupoid of "objects over R", so that for each map $f:R\to S$ in $\mathtt{art}(k)$ a base change functor $\alpha^*:\mathcal{C}(R)\to\mathcal{C}(S)$ is defined.

Then the groupoid of formal deformations of an object $A\in\mathcal{C}(k)$ over $R\in\mathtt{art}(k)$ is defined as the fiber of $\pi^*:\mathcal{C}(R)\to\mathcal{C}(k)$ at A.

In higher deformation theory one extends the category $\mathtt{art}(k)$ allowing artinian algebras which are not necessarily concentrated at degree zero. In this paper we work with the category $\mathtt{dgart}^{\leq 0}(k)$ of non-positively graded differential artinian algebras, see [H3] for the explanation. One cannot expect that the deformation functor extended in this way has values in the category of groupoids: one should expect a higher version of groupoid appearing here. We use simplicial groupoids (or simplicial weak groupoids which is the same from the homotopical point of view) as a higher version of groupoid.

Our higher formal deformation functors can be described as follows.

The role of the category of artinian local algebras is played by the following category $\mathtt{dgart}^{\leq 0}(k)$. Its objects are finite dimensional non-positively graded commutative local dg algebras over k with residue field k. Its morphisms are morphisms of dg algebras over k.

Let $\mathcal{C}(R)$ for each $R\in\mathtt{dgart}^{\leq 0}(k)$ denote a category of "objects over R". We suppose there is a subcategory $\mathcal{W}(R)$ of weak equivalences in $\mathcal{C}(R)$. Let $\widehat{\mathcal{W}}(R)$ be the full Dwyer-Kan (hammock) localization of $\mathcal{W}(R)$ (see A.2 for the details). This is a simplicial groupoid and we define the simplicial groupoid of formal deformations of $A\in\mathcal{C}(k)$ as the homotopy fiber of the map

$$\widehat{\mathcal{W}}(R)\to\widehat{\mathcal{W}}(k)$$

at A.

For the description of deformations of sheaves of algebras we take $\mathcal{C}(R)$ to be the category of sheaves of R-algebras flat over R. $\mathcal{W}(R)$ is the subcategory of quasi-isomorphisms of sheaves of algebras.

0.4. **Main result.** Let X be a site and let k be a field of characteristic zero. Let $\mathcal{O}$ be an operad in the category of complexes of sheaves of k-modules on X and let A be a sheaf of $\mathcal{O}$-algebras.

Our main result, Theorem 3.5.5, presents (under some restrictions on X, $\mathcal{O}$ and A) the dg Lie algebra governing formal deformations of A. The construction goes as follows. According to Theorem 2.2.1, the category of presheaves of $\mathcal{O}$-algebras admits a model structure generalizing the one defined in 0.2.1. Let P be a fibrant cofibrant $\mathcal{O}$-algebra weakly equivalent to A. Then the presheaf of derivations of P, $\mathcal{T}_A := \mathcal{D}er^\bullet(P, P)$, is a fibrant presheaf of dg Lie algebras on X. The dg Lie algebra governing deformations of A can be expressed then as $\mathbf{R}\Gamma(X, \mathcal{T}_A)$, where the functor $\mathbf{R}\Gamma$ is calculated via the Cech complex corresponding to a hypercover of X (see the details in 3.5).

Here are the assumptions for which the result is proven.

0.4.1. *Assumptions on X.*

- The topos $X^\sim$ admits enough points.
- The final presheaf in $X^\frown$ admits a finite hypercover.

The second condition is of course fulfilled for sites admitting a final object. However, in our main application X is the site of affine open subschemes of a scheme. In this case the condition is fulfilled for quasi-compact separated or finite dimensional schemes.

0.4.2. *Assumptions on $\mathcal{O}$.* Complexes $\mathcal{O}(n)$ are non-positively graded. This condition does not seem to be really restrictive. Operads one encounters are usually obtained by a tensor product of a sheaf of rings (e.g, the structure sheaf) with a constant operad.

0.4.3. *Restrictions on A.* For each $U \in X$ the cohomology $H^i(U, A)$ is supposed to vanish for $i > 0$. This condition means the following. Choose a fibrant resolution $A \to A'$ of A. Then for each $U \in X$ the complex $A'(U)$ has no positive cohomology. This is the most serious assumption. It is not in general fulfilled even in the case A is a sheaf of algebras. In fact, in this case the cohomology $H^i(U, A)$ has its usual meaning (cohomology of the sheaf $A|_U$) and it does not vanish in general for any $U \in X$.

The situation is, however, slightly better then one could think. The reason is that once we are given a sheaf of algebras A in a topos $X^\sim$, we have a freedom in the choice of X. If $X^\sim$ admits a generating family of sheaves U satisfying $H^i(U, A) = 0$, we can choose X to be the site generated by this family. For instance, if A is a quasi-coherent sheaf on a scheme, one chooses X to be the category of affine open subschemes of the scheme, with the Zariski topology.

0.5. **Applications.** Direct application of Theorem 3.5.5 gives the following result (see 4.1.1 and 4.1.2).

Let X be a scheme over a field of characteristic zero. Suppose X admits a finite dimensional hypercover by affine open subschemes. Then the functor of formal deformations of X (or, more generally, of a quasicoherent operad algebra on X) is represented by a dg Lie algebra.

The dg Lie algebra representing deformations of a quasicoherent sheaf of algebras, is usually difficult to determine. Its cohomology, however, can be easily identified with the Hochschild cohomology (for associative algebras), see 4.2. In a very special case of associative deformations of the structure sheaf of a smooth scheme, the tangent Lie algebra identifies with the (shifted and truncated) complex of Hochschild cochains given by polydifferential operators.

The last application we present in this paper is to the description of equivariant deformations. Let A be a sheaf of algebras on a site X satisfying the conditions of Theorem 3.5.5, and let T be the dg Lie algebra governing the deformations of A. Suppose now that a formal group G acts on X and on A in a compatible way. Then G acts in a natural way on T and the G-equivariant deformations of A are governed by a dg Lie algebra $\mathbf{R}\Gamma^G(T)$ whose i-th cohomology is $H^i(G,T)$. This is proven in 4.4.

0.6. **Structure of the sections.** In Section 1 we prove Theorem 0.2.1 describing the model category structure on the category of complexes of presheaves. We describe the functors $\mathbf{R}\mathcal{H}om^\bullet$ and $\mathbf{R}\Gamma$ using this model category structure. In Section 2 we present a model structure for the category of presheaves of operad algebras. In Section 3 we describe the deformation functor for sheaves of operad algebras on a site. Here the main Theorem 3.5.5 is proven. In Section 4 we present two examples: deformations of a scheme (or of a quasi-coherent algebra on a scheme) and equivariant deformations of a sheaf of algebras with respect to a discrete group.

In Appendix A we present the necessary information about simplicial categories, Dwyer-Kan localization and its different presentations for a model category.

Appendix B is not used in the main body of the paper. In it we present a model category structure on the category of simplicial presheaves and provide a description of fibrations similar to that of Theorem 0.2.1.

0.7. **Notation.** In this paper $\mathbb{N}$ denotes the set of non-negative integers, Δ the category of sets $[n]=\{0,\ldots,n\},\ n\in\mathbb{N}$ and of non-decreasing maps, and $\mathtt{Ens}$ denotes the category of sets. The category of simplicial sets is denoted $\Delta^{\mathrm{op}}\mathtt{Ens}$.

As well, we denote $\mathtt{Ab}$ the category of abelian groups, $\mathtt{Cat}$ the category of small categories, $\mathtt{Grp}$ the subcategory of groupoids.

If $\mathcal{A}$ is an abelian category, $C(\mathcal{A})$ is the category of complexes over $\mathcal{A}$; we write $C(k)$ for the category of complexes of k-modules. If $\mathcal{A}$ is a tensor category, $\mathtt{Op}(\mathcal{A})$ is the category of operads in $\mathcal{A}$. The notation $\mathtt{Mod}$ and $\mathtt{Alg}$ for the categories of modules and algebras is obvious. Σ_n is the symmetric group.

0.8. **Relation to other works.** This work extends the approach of [H4] to the sheaves of algebras. Both [H4] and the present work are based on an idea (which goes back to Halperin-Stasheff [HaSt], Schlessinger-Stashef [SchSt], Felix [F]) that deformations of an algebra can be described by perturbation of the differential in its free resolution. Since [HaSt, SchSt, F] a better understanding of the notion of deformation has been achieved, due to Drinfeld and Deligne, so that the language of obstructions is being substituted with the dg Lie algebra formulation of deformation theory.

One has to mention Illusie [I] and Laudal [La] who constructed the obstruction theory for deformation of schemes, and Gerstenhaber-Schack [GS] who studied obstruction theory for presheaves of algebras. Obstruction theory for deformations of sheaves of associative algebras was studied in [G] and [Lu].

0.9. **Acknowledgements.** A part of this work was made during my visits at MPIM and at IHES. I am grateful to these institutions for stimulating atmosphere and excellent working conditions. During the conference on polycategories at Nice (November, 2001) I knew that a part of the results on model category structures described here is known to specialists. I am very grateful to the organizers of the conference A. Hirschowitz, C. Simpson, B. Toen for the invitation. I am also grateful to the referee for numerous remarks.

1. Models for sheaves

Let X be a site, $X^{\frown}$ and $X^{\sim}$ be the categories of presheaves (resp., sheaves) on X. If k is a commutative ring, $X^{\frown}_k$ (resp., $X^{\sim}_k$) denotes the category or preseaves (resp., sheaves) of k-modules on X.

The categories $X^{\frown}_k$ and $X^{\sim}_k$ are tensor (=symmetric monoidal) categories. The sheafification functor $M \mapsto M^a$ is exact and preserves the tensor product, see [SGA4], IV.12.10.

In this section we provide a CMC (=closed model category) structure for the category $C(X^{\frown})$ of complexes of presheaves on X, see [Q1, Hir] for the notion of model category. This structure "remembers" the topology of X in a way that that the model category $C(X^{\frown})$ becomes a powerful tool in doing homological algebra of *sheaves* on X.

In the next section we will describe a similar model structure on the category of presheaves of algebras over a dg operad on X.

1.1. Coarse topology.

1.1.1. **Theorem.** *The category $C(X^{\frown}_k)$ of presheaves of k-modules admits a model structure with weak equivalences defined as objectwise quasi-isomorphisms and fibrations as objectwise surjections.*

Since the category $C(k)$ of complexes of k-modules is cofibrantly generated (see [Hir], 11.1.1, for the definition of cofibrantly generated model categories and [H2] for the model structure on $C(k)$), the result follows from the general observation of [Hir], 11.6.1. The CMC structure described above is also cofibrantly generated.

Let us recall the description of a generating collection of cofibrations. Let $U \in X$. We will identify U with the presheaf represented by U. If M is a k-module and $P \in X^{\frown}$, we denote by $M \cdot P$ the presheaf of k-modules $U \mapsto M \times P(U)$. In particular, $kx \cdot U$ is the presheaf of k-modules generated by a section x over U.

The generating cofibration given by a pair $(U \in X, n \in \mathbb{Z})$ is defined as

$$i: kx \cdot U \to ky \cdot U \oplus kz \cdot U$$

where $n = |x| = |y|,\ n-1 = |z|,\ i(x) = y,\ dx = dy = 0,\ dz = y.$

The generating acyclic cofibration is defined for each $(U \in X, n \in \mathbb{Z})$ as

$$j: 0 \to kx \cdot U \oplus ky \cdot U \tag{1}$$

where $n = |x|,\ n-1 = |y|,\ dx = 0,\ dy = x.$

The model category structure defined in 1.1.1 knows nothing about the topology of X. It corresponds to the coarse topology on X.

In the general case the notion of hypercover is of a great importance.

1.2. **Hypercovers.** Let us recall a few standard notions connected to hypercovers. The context presented here is slightly more general than that of [SGA4], Exposé V.

Let X be a site. Notice that we do not require that fiber products and finite products exist in X.

1.2.1. An object $K \in X^{\frown}$ is semi-representable if it is isomorphic to a coproduct of representable presheaves.

A simplicial presheaf $K_\bullet$ is called *hypercover* of X if

(HC0) For each $i \geq 0$ K_i is semi-representable.

(HC1) For each $n \geq 0$ the canonical map

$$K_{n+1} \to (\mathrm{cosk}_n(K))_{n+1}$$

is a cover of presheaves (i.e. its sheafification is surjective).

(HC2) The canonical map of presheaves $K_0 \to *$ is a cover.

Let $L \in X^{\frown}$. A simplicial presheaf $K_\bullet$ endowed with an augmentation $\epsilon : K_0 \to L$ is called a hypercover of L if it defines a hypercover of the site X/L.

1.2.2. Let $K_\bullet$ be a simplicial presheaf on X. We denote by $C_*(K, k)$, or simply $C_*(K)$, the complex of normalized chains of K. This is an object of $C(X^{\frown}_k)$. If L is a presheaf of sets considered as a discrete simplicial presheaf, $C_*(L) = kL$ is the free presheaf of vector spaces generated by L.

The following lemma is of crucial importance for us.

Lemma. (cf. [SGA4], V.7.3.2(3)). *Let $\epsilon : K_\bullet \to L$ be a hypercover. Then the induced map $C_*(K_\bullet) \to kL$ induces a quasi-isomorphism of sheafifications.*

1.2.3. Let $V_\bullet$ be a simplicial presheaf and let $M \in C(X^\frown_k)$.

The collection

$$n \mapsto \mathrm{Hom}(V_n, M) \tag{2}$$

is a cosimplicial object in $C(k)$. Čech complex of M, $\check{C}(V_\bullet, M)$, is defined as (the total complex of) the normalization of (2), so that

$$\check{C}^n(V_\bullet, M) = \{f \in \prod_{p+q=n} \mathrm{Hom}(V_p, M^q) | f \text{ vanishes on the degenerate simplices } \}. \tag{3}$$

In particular, any hypercover $\epsilon : V_\bullet \to U$ gives rise to a map

$$M(U) \to \check{C}(V_\bullet, M).$$

Note that one has an obvious isomorphism

$$\check{C}(V_\bullet, M) = \mathrm{Hom}^\bullet(C_*(V_\bullet), M). \tag{4}$$

1.3. **General case.** Now we will define another CMC structure on $C(X^\frown_k)$ which "remembers" about the topology on X.

1.3.1. **Theorem.** *1. The category $C(X^\frown_k)$ of presheaves of k-modules admits a CMC structure with cofibrations as in 1.1.1 and weak equivalences defined as maps $f : M \to N$ such that the sheafification f^a is a quasi-isomorphism of complexes of sheaves.*

2. A map $f : M \to N$ is a fibration iff $f(U) : M(U) \to N(U)$ is surjective for each $U \in X$ and for any hypercover $\epsilon : V_\bullet \to U$ of $U \in X$ the corresponding commutative diagram

$$\begin{tikzcd} M(U) \arrow[r] \arrow[d] & \check{C}(V_\bullet, M) \arrow[d] \\ N(U) \arrow[r] & \check{C}(V_\bullet, N) \end{tikzcd} \tag{5}$$

is homotopy cartesian.

The weak equivalences of the CMC structure defined above will be called *local equivalences.*

Note that fibrant object in our CMC structure are complexes M giving rise to a weak equivalence $M(U) \longrightarrow \check{C}(V_\bullet, M)$ for each hypercover $V_\bullet \longrightarrow U$. Thus, these are "higher homotopy generalizations of sheaves".

1.3.2. The first part of Theorem 1.3.1 is proven in 1.3.3–1.3.7. It is based on an expicit description of the collection of generating acyclic cofibration. The second part of the theorem is proven in 1.3.8.

1.3.3. The generating set of acyclic cofibrations is labelled by pairs

$$(\epsilon : V_\bullet \to U, n)$$

where ϵ is a hypercover and n an integer. An acyclic cofibration $j : K \to L$ corresponding to a pair (ϵ, n) as above is defined as follows.

The presheaf K is shifted by $-n$ cone of the mapping

$$C_*(\epsilon) : C_*(V_\bullet) \to k \cdot U.$$

This means that $K^n = k \cdot U$ and $K^{n-i-1} = k \cdot V_i / \sum_j k \cdot s_j V_{i-1}$ for $i \geq 0$, where $s_j : V_{i-1} \to V_i, \quad j = 0, \ldots, i-1$ are the degeneracies.

The presheaf L is defined as the cone of id_K, with the obvious canonical embedding $j : K \to L$.

We will write sometimes $K_{\epsilon,n}$ and $L_{\epsilon,n}$ for the complexes K and L corresponding to a pair $(\epsilon : V_\bullet \to U, n)$.

1.3.4. **Lemma.** *The map $j : K \to L$ of presheaves constructed above is a weak equivalence.*

Proof. The map j is obviously injective. The sheafification of its cokernel is contractible by [SGA4], V.7.3.2(3). □

1.3.5. *Note.* The sheafification of K is also contractible by [SGA4], V.7.3.2(3).

1.3.6. **Lemma.** *Let a map $f : M \to N$ of presheaves satisfy the right lifting property with respect to all generating acyclic cofibrations and let f^a be a (objectwise) quasi-isomorphism. Then f is objectwise surjective quasi-isomorphism.*

Proof. The map f is objectwise surjective by 1.1.1. Therefore, we have to prove that for any $U \in X$ the map $f(U)$ is a quasi-isomorphism. We can put $N = 0$ without loss of generality. Let $a \in M(U)^n$ be a cycle. We have to prove a is a boundary. For this we will construct a hypercover $\epsilon : V_\bullet \to U$ and a map

$$f : K_{\epsilon,n} \to M$$

whose restriction to the n-th component is the map $a : U \to M^n$. Then by the right lifting property f lifts to a map $L_{\epsilon,n} \to M$ which proves a is a boundary.

We proceed by induction: since M^a has zero cohomology and $da = 0$, there exists a cover $\epsilon : V_0 \to U$ such that $\epsilon^*(a)$ is a boundary. Fix $a_0 \in M(V_0)$ such that $da_0 = \epsilon^*(a)$. Suppose, by the induction hypothesis, there exist sections $a_i \in M(V_i), \quad i = 0, \ldots, n$ such that a_i

vanishes on the degeneracies $M(s_jV_{i-1})$ and $da_i=\sum(-1)^jd_j^*(a_{i-1})$. Then one can choose a cover $V_{n+1}\to\operatorname{cosk}_n(V_\bullet)_{n+1}$ for which the cycle $\sum(-1)^id_i^*(a_n)$ becomes a boundary. □

1.3.7. Let J be the collection of generating acyclic cofibrations. We define fibrations as the maps of presheaves satisfying the RLP with respect to the elements of J. We denote by $\overline{J}$ the collection of maps which can be obtained as a countable direct composition of pushouts of coproducts of maps in J. We call the maps from $\overline{J}$ standard acyclic cofibrations. They are cofibrations and weak equivalences by Lemma 1.3.4.

According to Lemma 1.3.6, fibrations which are local equivalences are precisely acyclic fibrations in the sense of 1.1.1 (i.e., objectwise surjective quasi-isomorphisms).

Let $f:A\to B$ be a map of presheaves. The existence of decomposition $f=pi$ where p is an acyclic fibration and i is a cofibration follows from Theorem 1.1.1. A small object argument, see [Hir], 10.5.14, implies the existence of a decomposition $f=qj$ where $j\in\overline{J}$ and q is a fibration. Suppose now f is a cofibration and a local equivalence and choose a decomposition $f=qj$ as above. The map q is therefore a fibration and a local equivalence, therefore, by Lemma 1.3.6, q is an acyclic fibration in the sense of 1.1.1. Therefore, q satisfies the RLP with respect to f. This implies that f is a retract of j.

We proved that any acyclic cofibration is a retract of a standard acyclic cofibration which yields the first part of the theorem. The second part of the theorem is explained in 1.3.8 below.

1.3.8. *Proof of Theorem 1.3.1(2).* Any fibration is objectwise surjective since the maps (1) are acyclic cofibrations. From now on we assume that f is objectwise surjective. Put $K=\operatorname{Ker}(f)$. Since the components V_n of $V_\bullet$ are semirepresentable, the diagram (5) is homotopy cartesian iff the map

$$K(U)\to\check{C}(V_\bullet,K)$$

is a weak equivalence. This is equivalent to the requirement that the complex $\operatorname{Hom}^\bullet(K_{\epsilon,0},K)$ has trivial cohomology or, equivalently, that the map

$$\operatorname{Hom}^\bullet(K_{\epsilon,0},M)\to\operatorname{Hom}^\bullet(K_{\epsilon,0},N)$$

is a quasi-isomorphism. This, in turn, implies the right lifting property of f with respect to the generating acyclic cofibrations $j:K_{\epsilon,n}\to L_{\epsilon,n}$.

In the other direction, the RLP of f with respect to the map $j:K_{\epsilon,n}\to L_{\epsilon,n}$ implies the RLP of the map $K\longrightarrow 0$ with respect to j. This implies acyclicity of $\operatorname{Hom}^\bullet(K_{\epsilon,0},K)$ which means that the diagram (5) is homotopy cartesian. □

1.3.9. *Note.* The model category structure in $C(X_k^\frown)$ described in the theorem, depends essentially on the site X (i.e., on the generating family of the topos $X^\sim$). For instance, a quasi-coherent sheaf is not fibrant in the Zariski site of a scheme. However, it is fibrant when considered as a presheaf on the site of *affine* open subschemes of the scheme.

The following observation will be useful in the sequel.

1.3.10. **Lemma.** *Let X be a site and let $U \in X$. Let $j : X/U \to X$ be the natural embedding. The restriction functor $j^* : C(X^\frown_k) \to C((X/U)^\frown_k)$ preserves fibrations and weak equivalences. If X admits finite products, j^* preserves cofibrations.*

Proof. Preservation of weak equivalences is immediate. Preservation of fibrations follows immediately from Theorem 1.3.1(2). To prove that j^* preserves cofibrations we will check that the right adjoint functor $j_* : C((X/U)^\frown_k) \to C(X^\frown_k)$ preserves acyclic fibrations. One has

$$j_*(M)(W) = M(U \times W)$$

for $M \in C((X/U)^\frown_k)$; acyclic fibrations are just objectwise surjective quasi-isomorphisms. This implies the lemma. $\square$

1.4. **Cohomology and RΓ.** In what follows X is a site and $C(X^\frown)$ is endowed with the CMC structure defined in 1.3.1.

1.4.1. Let $M \in C(X^\frown_k)$ and let $\widetilde{M}$ be a fibrant replacement of M. We define the i-th cohomology presheaf of M, $\mathcal{H}^i(M)$, to be the presheaf

$$\mathcal{H}^i(M)(U) = H^i(\widetilde{M}(U)).$$

The following lemma shows the result does not depend of the choice of the resolution.

1.4.2. **Lemma.** *Let $f : M \to N$ be a weak equivalence of fibrant objects in $C(X^\frown_k)$. Then for each $U \in X$ and $i \in \mathbb{Z}$ the map $H^i(M(U)) \to H^i(N(U))$ is bijective.*

Proof. Any acyclic fibration induces a objectwise quasi-isomorphism, so we can suppose that f is an acyclic cofibration. Since M is fibrant, f is split by a weak equivalence $g : N \to M$. Therefore, the map $H^i(f(U))$ is injective. This reasoning can be also applied to the map g instead of f, to conclude that $H^i(g(U))$ is injective as well. $\square$

1.4.3. *Global sections.* Suppose first that X admits a final object $* \in X$. Then the derived global sections $\mathbf{R}\Gamma(M)$ can be defined as $\widetilde{M}(*)$ where $\widetilde{M}$ is a fibrant resolution of M.

If X does not admit a final object, one defines the derived global sections functor as follows. Choose a hypercover $V_\bullet$ of X. We define

$$\mathbf{R}\Gamma(M) = \check{C}(V_\bullet, \widetilde{M}),$$

see 1.2.3. By formula (4) and 1.3.5, the result does not depend on the choice of hypercover $V_\bullet$ and of the fibrant resolution $\widetilde{M}$.

1.5. **Presheaves of modules.** Let $\mathcal{A}$ be a presheaf of dg associative k-algebras on X. We denote $\mathtt{Mod}(\mathcal{A}, X\hat{})$ the category of (presheaves of dg) $\mathcal{A}$-modules.

1.5.1. **Theorem.** *The category $\mathtt{Mod}(\mathcal{A}, X\hat{})$ admits a model structure for which a map $f : M \to N$ of presheaves of $\mathcal{A}$-modules is a weak equivalence (resp., a fibration) iff it is a weak equivalence (resp., a fibration) of presheaves of k-modules.*

The proof of the theorem is easily deduced from the following lemma.

1.5.2. **Lemma.** *Let $j : K \to L$ be a generating acyclic cofibration corresponding to a pair (ϵ, n) as in 1.3.3, M be a $\mathcal{A}$-module and $f : K \to M$ be a map of complexes of presheaves. Then the induced map*

$$M \to M \coprod^{\mathcal{A}\otimes K} \mathcal{A} \otimes L$$

is a weak equivalence.

Proof. The map in question being injective, it is enough to study the cokernel which is isomorphic (up to a shift) to $\mathcal{A} \otimes \mathrm{Coker}(j)$. Its sheafification is isomorphic to $\mathcal{A}^a \otimes \mathrm{Coker}(j)^a$. The complex $\mathrm{Coker}(j)^a$ is acyclic by Lemma 1.2.2. Since its components are flat, the tensor product is acyclic as well. $\square$

1.5.3. Let now an algebra homomorphism $f : \mathcal{A} \to \mathcal{A}'$ be given. One defines in a standard way a Quillen pair of adjoint functors

$$f^* : \mathtt{Mod}(\mathcal{A}, X\hat{}) \rightleftarrows \mathtt{Mod}(\mathcal{A}', X\hat{}) : f_*.$$

In particular, a pair of derived functors

$$\mathbf{R}f^* : D(\mathtt{Mod}(\mathcal{O}, X\hat{})) \rightleftarrows D(\mathtt{Mod}(\mathcal{O}', X\hat{})) : f_* = \mathbf{L}f_* \tag{6}$$

is induced.

The following lemma shows that weakly equivalent associative algebras give rise to equivalent derived categories of modules.

Lemma. *Let $f : \mathcal{A} \to \mathcal{A}'$ be a weak equivalence of presheaves of associative algebras. Then the adjoint pair (6) establishes an equivalence of the derived categories of modules.*

Proof. One has to check that of M is a cofibrant $\mathcal{A}$-module then the natural map

$$M \to f_*(f^*(M))$$

is a weak equivalence. The claim immediately reduces to the case $M = \mathcal{A} \otimes k \cdot U$ for $U \in X$. Then $f_*(f^*(M)) = \mathcal{A}' \otimes k \cdot U$ and the claim is obvious. $\square$

1.6. **Inner $\mathcal{H}om^\bullet$.** The model category $\mathtt{Mod}(\mathcal{A}, X^\frown)$ admits an extra structure similar to that of simplicial model category of Quillen [Q1].

1.6.1. **Definition.** Let M, $N \in \mathtt{Mod}(\mathcal{A}, X^\frown)$. The inner Hom

$$\mathcal{H}om^\bullet_{\mathcal{A}}(M, N) \in C(X_k^\frown)$$

assigns to each $U \in X$ the complex of k-modules defined as

$$\mathcal{H}om^\bullet_{\mathcal{A}}(M, N)(U) = \lim_{\substack{\longleftarrow \\ V \to V' \to U}} \mathrm{Hom}^\bullet_{\mathcal{A}(V')}(M(V'), N(V))$$

where the inverse limit is taken over the category whose objects are the diagrams $\delta : V \longrightarrow V' \longrightarrow U$ and the morphisms $\delta_1 \longrightarrow \delta_2$ are given by the commutative diagrams

$$\begin{tikzcd}
V_1 \arrow[r] \arrow[d] & V_1' \arrow[r] \arrow[d] & U \arrow[d, equal] \\
V_2 \arrow[r] & V_2' \arrow[r] & U
\end{tikzcd}$$

Here $\mathrm{Hom}^\bullet_{\mathcal{A}(V')}$ is the usual inner Hom in the category of complexes of $\mathcal{A}(V')$-modules. We will write as well $\mathrm{Hom}^\bullet_{\mathcal{A}}(M, N)$ for the complex of the global sections of $\mathcal{H}om^\bullet_{\mathcal{A}}(M, N)$, so that $\mathcal{H}om^\bullet_{\mathcal{A}}(M, N)(U) = \mathrm{Hom}^\bullet_{j^*\mathcal{A}}(j^*M, j^*N)$ where $j : X/U \longrightarrow X$ is the obvious forgetful functor.

1.6.2. **Lemma.** *Let $\alpha : M \to M'$ be a cofibration and $\beta : N \to N'$ be a fibration in* $\mathtt{Mod}(\mathcal{A})$*. Then the natural map*

$$\mathcal{H}om^\bullet(M', N) \to \mathcal{H}om^\bullet(M, N) \times_{\mathcal{H}om^\bullet(M,N')} \mathcal{H}om^\bullet(M', N')$$

is a fibration in $C(X_k^\frown)$. It is a weak equivalence if α or β is a weak equivalence.

Standard adjoint associativity isomorphism

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(X \otimes_k Y, Z) \xrightarrow{\sim} \mathrm{Hom}^\bullet_k(X, \mathcal{H}om^\bullet_{\mathcal{A}}(Y, Z))$$

reduces the claim to the following.

1.6.3. **Lemma.** *(Here $\otimes = \otimes_k$). Let $\alpha : A \to A'$ be a cofibration in $C(X_k^\frown)$ and let $\beta : M \to M'$ be a cofibration in* $\mathtt{Mod}(\mathcal{A}, X^\frown)$*. Then the induced map*

$$A \otimes M' \coprod^{A \otimes M} A' \otimes M \to A' \otimes M' \tag{7}$$

is a cofibration. It is an acyclic cofibration if α or β is.

Proof. For the first claim it is enough to check the case α and β are generating cofibrations. This is a very easy calculation. For the second claim note that the cokernel of (7) is isomorphic to $\mathrm{Coker}(\alpha) \otimes \mathrm{Coker}(\beta)$. Since the sheafification commutes with the tensor product and $\mathrm{Coker}(\alpha)$ is flat, the result is immediate.

□

1.7. **Comparing to sheaves.** Let $\mathcal{A}$ be a dg algebra in $X_k^{\sim}$.

Recall that one can define the functor $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{A}}$ on the category $\mathtt{Mod}(\mathcal{A}, X^{\sim})$ using Spaltenstein's notion of K-injective complex of sheaves [Sp]. A complex $I \in \mathtt{Mod}(\mathcal{A}, X^{\sim})$ is called K-injective if it satisfies the RLP with respect to injective quasi-isomorphisms of sheaves.

Thus, one defines $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{A}}(M, N)$ as $\mathcal{H}om^\bullet_{\mathcal{A}}(M, I)$ where $N \to I$ is a K-injective resolution.

1.7.1. **Lemma.** *Here $\mathcal{A}$, M, N are as above. Let $M' \to M$ be a cofibrant resolution of M in $\mathtt{Mod}(\mathcal{A}, X^{\frown})$ and $N \to N'$ be a fibrant resolution of N in $\mathtt{Mod}(\mathcal{A}, X^{\frown})$. Then $\mathcal{H}om^\bullet_{\mathcal{A}}(M', N')$ and $\mathbf{R}\mathcal{H}om^\bullet_{\mathcal{A}}(M, N)$ are equivalent.*

Proof. Let $N \to I$ be a K-injective resolution of N. Any K-injective complex is fibrant as a complex of presheaves. Therefore, $\mathcal{H}om^\bullet_{\mathcal{A}}(M', N')$ and $\mathcal{H}om^\bullet_{\mathcal{A}}(M', I)$ are weakly equivalent fibrant complexes of presheaves. On the other hand, $\mathcal{H}om^\bullet_{\mathcal{A}}(M', I) = \mathcal{H}om^\bullet_{\mathcal{A}}((M')^a, I) \xrightarrow{\sim} \mathcal{H}om^\bullet_{\mathcal{A}}(M, I)$ since I is K-injective. □

1.7.2. *Remark.* Subsection 1.4 and Lemma 1.7.1 show that the standard homological algebra of sheaves can be rephrased in the language of complexes of presheaves endowed with the model structure defined in Theorem 1.5.1. This has an advantage over the standard approach with sheaves since the analog of Theorem 1.5.1 takes place for presheaves of algebras as well.

1.7.3. We have to mention the following consequence of 1.7.1.

Let M be a cofibrant and N be a fibrant object in $\mathtt{Mod}(\mathcal{A}, X^{\frown})$. Let $U \in X$ and $j : X/U \to X$ be the natural embedding. According to Lemma 1.3.10 the functor j^* preserves fibrations and weak equivalences, but does not necessarily preserve cofibrations. However, the following takes place.

Proposition. *The complex $\mathcal{H}om^\bullet_{\mathcal{A}|_U}(j^*(M), j^*(N))$ "calculates the $\mathbf{R}\mathcal{H}om^\bullet$". More precisely, if $M' \to j^*(M)$ is a cofibrant resolution, then the induced map*

$$\mathcal{H}om^\bullet_{\mathcal{A}|_U}(j^*(M), j^*(N)) \to \mathcal{H}om^\bullet_{\mathcal{A}|_U}(M', j^*(N))$$

is a weak equivalence.

Proof. Let $N \to I$ be a K-injective resolution of N. Then $j^*N \to j^*I$ is a K-injective resolution of j^*N.

This implies that all morphisms in the composition below are equivalences.

$$\begin{aligned}(8)\quad \mathcal{H}om^{\bullet}_{\mathcal{A}|_U}(j^*(M), j^*(N)) = j^*\mathcal{H}om^{\bullet}_{\mathcal{A}}(M, N) \to j^*\mathcal{H}om^{\bullet}_{\mathcal{A}}(M, I) = \\ \mathcal{H}om^{\bullet}_{\mathcal{A}|_U}(j^*(M), j^*(I)) \to \mathcal{H}om^{\bullet}_{\mathcal{A}|_U}(M', j^*(I)) \leftarrow \mathcal{H}om^{\bullet}_{\mathcal{A}|_U}(M', j^*(N)).\end{aligned}$$

$\square$

2. (Pre)sheaves of operad algebras

In this section we describe a model structure on the category of presheaves of algebras over a Σ-split operad, in the case when the corresponding topos has enough points. The structure is based on the model structure on the category of complexes of presheaves described in Theorem 1.3.1.

We also discuss the category of modules over an operad algebra, derivations and modules of differentials.

Let $\mathcal{C}$ be a symmetric monoidal category. An operad in $\mathcal{C}$ is a collection of objects $\mathcal{O}(n)$, $n \in \mathbb{N}$, together with a right action of the symmetric group Σ_n, compositions

$$\mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \ldots \otimes \mathcal{O}(m_n) \longrightarrow \mathcal{O}(\sum m_i)$$

and a unit $\mathbb{1} \longrightarrow \mathcal{O}(1)$ satisfying well-known compatibilities, see, for instance, [May].

2.1. **Homotopical amenability.**

Mimicing [H5], Definition 2.2.1, we define homotopically amenable presheaves of operads.

2.1.1. *Notation.* Let X be a site and let k be a commutative ring. A presheaf of operads on X is, by definition, an operad in the tensor category $C(X^{\frown}_k)$ of complexes of presheaves of k-modules.

If $\mathcal{O} \in \mathtt{Op}(C(X^{\frown}_k))$ is such a presheaf, we denote by $\mathtt{Alg}(\mathcal{O}, X^{\frown})$ the category of (presheaves of) $\mathcal{O}$-algebras.

2.1.2. **Definition.** An operad $\mathcal{O} \in \mathtt{Op}(C(X^{\frown}_k))$ is called *homotopically amenable* if for each $A \in \mathtt{Alg}(\mathcal{O}, X^{\frown})$ and for each generating acyclic cofibration $j : K \to L$ in $C(X^{\frown}_k)$ with $K = K_{\epsilon,n}$ and $L = L_{\epsilon,n}$ where ϵ is a hypercover of $U \in X$ and $n \in \mathbb{Z}$, see 1.3.3, the map α defined by the cocartesian diagram

(9)
$$\begin{tikzcd} F(\mathcal{O},K) \arrow[r, "j"] \arrow[d] & F(\mathcal{O},L) \arrow[d] \\ A \arrow[r, "\alpha"] & A' \end{tikzcd}$$

is a weak equivalence.

Here $F(\mathcal{O},K)$ denotes the free $\mathcal{O}$-algebra generated by K.

2.1.3. The following result is fairly standard.

Theorem. *Let $\mathcal{O}$ be a homotopically amenable operad in $C(X^\frown_k)$. Then the category $\mathtt{Alg}(\mathcal{O},X^\frown)$ of $\mathcal{O}$-algebras admits a model structure with weak equivalences and fibrations defined as for $C(X^\frown_k)$ in 1.3.1.*

Proof. Consider the pair of adjoint functors

$$F : C(X^\frown_k) \rightleftarrows \mathtt{Alg}(\mathcal{O},X^\frown) : \#$$

where F is the free $\mathcal{O}$-algebra functor and $\#$ is the forgetful functor. Under the homotopy amenability condition a model structure on $C(X^\frown_k)$ can be transferred to $\mathtt{Alg}(\mathcal{O},X^\frown)$, see a similar result at [H2], Thm. 2.2.1. $\square$

2.2. **Σ-split operads.** In the case $X^\frown = \mathtt{Ens}$ Σ-split operads defined in [H2], 4.2, are homotopically amenable. Recall that a Σ-splitting of an operad $\mathcal{O}$ is defined as a collection of Σ_n-equivariant splittings of the canonical maps

$$\mathcal{O}(n)\otimes k\Sigma_n \to \mathcal{O}(n), \quad o\otimes\sigma \mapsto o\sigma$$

satisfying some extra compatibility properties, see [H2], 4.2.4 for the precise definition.

The definition of Σ-split operad makes sense in any tensor category. Thus, we can speak about Σ-split presheaves of operads on X.

It is worthwhile to mention two big classes of Σ-split operads.

- If $k \supseteq \mathbb{Q}$ then all operads in $C(X^\frown_k)$ are Σ-split.
- If $\mathcal{A}$ is an asymmetric operad then $n \mapsto \mathcal{A}(n)\otimes\Sigma_n$ is a Σ-split operad. In particular, the operad for associative algebras is Σ-split over $\mathbb{Z}$.

The operad $\mathtt{COM}$ for commutative algebras in not homotopically amenable (and, therefore, is not Σ-split) if k is a field of positive characteristic.

2.2.1. **Theorem.** *Let X be a site having enough points. Then any Σ-split operad $\mathcal{O} \in \mathtt{Op}(C(X^{\frown}_k))$ is homotopically amenable.*

Proof. It is convenient to sheafify all the picture. If $\mathcal{O}$ is a presheaf of Σ-split dg operads, $\mathcal{O}^a$ is a sheaf of Σ-split dg operads and A^a is an $\mathcal{O}^a$-algebra. Sheafification also commutes with the free algebra functor and with the coproducts.

We have to check that any generating acyclic cofibration $j : K \to L$ gives rise to a weak equivalence α in the diagram (9). To check this it is sufficient to check that any fiber functor $\phi : X^{\sim} \to \mathtt{Ens}$ transforms the sheafification α^a into a quasi-isomorphism.

A fiber functor transforms the diagram (9) into a cocartesian diagram over a ring k with an acyclic complex K and with L being the cone of id_K. Note that $K = \phi(K_{\epsilon,n})$ is a non-positively graded acyclic complex of k-modules. Moreover, K admits an explicit presentation, see [SGA4], IV.6.8.3, as a filtered direct limit of non-positively graded acyclic complexes of free k-modules.

Since the free algebra functor commutes with filtered colimits, it is enough for us to check that the map α in the diagram (9) is a weak equivalence provided K is a non-negatively graded complex of free k-modules, $L = \mathrm{cone}(\mathrm{id}_K)$ and $\mathcal{O}$ is Σ-split operad in $C(k)$. But this latter claim follows from the homotopical amenability of Σ-split operads over k. $\square$

In what follows the following definition will be used.

2.2.2. **Definition.** A map $f : A \to B$ in a cofibrantly generated category is called *a standard cofibration* (resp., *a standard acyclic cofibration*) if it can be presented as a direct limit $B = \lim\limits_{\substack{\to \\ i\in\mathbb{N}}} A_i$ with $A_0 = A$ and A_{i+1} being a coproduct of generating cofibrations (resp., of generating acyclic cofibrations) over A_i.

We have the following

2.2.3. **Proposition.** *Any cofibration is a retract of a standard cofibration. Any acyclic cofibration is a retract of a standard acyclic cofibration.*

This, in fact, is true for any cofibrantly generated closed model category, see [Hir], 11.2.

In the rest of this section we suppose that the operad $\mathcal{O}$ is homotopically amenable.

2.3. **Modules.** In this subsection we sketch a presheaf version of [H2], Section 5.

2.3.1. *Enveloping algebra.* The enveloping algebra $U(A)$ of an operad algebra $A \in \mathtt{Alg}(\mathcal{O}, X^{\frown})$ is defined in a usual way. This is an associative algebra in the category of complexes $C(X^{\frown}_k)$, such that $U(A)$-modules are just the modules over the operad algebra A.

In particular, the category of modules $\mathtt{Mod}(\mathcal{O},A)$ admits a model structure as in 1.5.1. Moreover, a presheaf $\mathcal{H}om^\bullet_A(M,N)$ is defined for a pair (M,N) of A-modules.

2.3.2. Weakly equivalent operad algebras have sometimes non-equivalent derived categories of modules even for $X^\frown=\mathtt{Ens}$, see [H2], Section 5.

To get a "correct" derived category of modules, one has to work with cofibrant algebras (or with cofibrant operads and flat algebras, see [H2], 6.8).

2.3.3. **Lemma.** *Suppose $\mathcal{O}$ is Σ-split. Let $f:A\to B$ be an acyclic cofibration of cofibrant $\mathcal{O}$-algebras. Then the induced map $U(f):\ U(A)\to U(B)$ is a weak equivalence.*

Proof. The proof is basically the same as that of Corollary 5.3.2, [H2]. Everything reduces to the case A is generated by a finite number of sections x_i, $i=1,\ldots,n$, over $U_i\in X$ and B is the colimit of a diagram

$$A\leftarrow F(M)\to F(\mathrm{id}_M),$$

where M is a coproduct of shifts of representable presheaves and has acyclic sheafification. The enveloping algebra $U(A)$ admits a filtration numbered by multi-indices $d:\{1,\ldots,n\}\to\mathbb{N}$, with associated graded pieces given by the formula

$$\mathrm{gr}_d(U(A))=\mathcal{O}(|d|+1)\otimes_{\Sigma_d}\bigotimes U_i^{\otimes d_i},$$

where, as usual, $|d|=\sum d_i$, $\Sigma_d=\prod\Sigma_{d_i}\subseteq\Sigma_{|d|}$.

The enveloping algebra $U(B)$ admits a filtration indexed by pairs (d,k) with d as above and $k\in\mathbb{N}$. The associated graded piece corresponding to (d,k) takes form

$$\mathrm{gr}_{d,k}(U(B))=\mathcal{O}(|d|+k+1)\otimes_{\Sigma_k\times\Sigma_d}M[1]^{\otimes k}\otimes\bigotimes U_i^{\otimes d_i}.$$

Since $\mathcal{O}$ is Σ-split and M^a is flat and acyclic, the sheafifications of $\mathrm{gr}_{d,k}(U(B))$ are acyclic for $k>0$. This proves the lemma. $\square$

2.3.4. **Corollary.** *Let $\mathcal{O}$ be Σ-split. Any weak equivalence $f:A\to B$ of cofibrant $\mathcal{O}$-algebras induces a weak equivalence $U(f)$ of the enveloping algebras.*

Proof. We already know the claim in the case f is an acyclic cofibration. Therefore, it suffices to prove it for acyclic fibrations.

Let $f:A\to B$ be an acyclic fibration of cofibrant algebras. There exist a map $g:B\to A$ splitting f: $fg=\mathrm{id}_B$. This implies that for any weak equivalence $f:A\to B$ the induced map $H(U(A)^a)\to H(U(B)^a)$ of the homologies of the sheafifications splits. Applying this to $g:B\to A$ we deduce that it is in fact invertible. $\square$

2.4. **Differentials and derivations.** In this subsection we present a presheaf version of parts of [H2], 7.2, 7.3.

2.4.1. **Definition.** Let $\mathcal{O} \in \mathtt{Op}(C(X_k^\frown))$ be a presheaf of operads on X, $\alpha : B \to A$ be a map in $\mathtt{Alg}(\mathcal{O}, X^\frown)$ and let M be an A-module. The presheaf $\mathcal{D}er^{\bullet}{}^{\mathcal{O}}_B(A, M)$ of $\mathcal{O}$-derivations over B from A to M is defined as the subpresheaf of $\mathcal{H}om_k^{\bullet}(A, M)$ consisting of local sections which are $\mathcal{O}$-derivations (see [H2], 7.2) from A to M vanishing at B.

By definition $\mathcal{D}er^{\bullet}{}^{\mathcal{O}}_B(A, M)$ is a subcomplex of $\mathcal{H}om_k^{\bullet}(A, M)$. The functor $M \mapsto \mathcal{D}er^{\bullet}{}^{\mathcal{O}}_B(A, M)$ is representable in the following sense.

2.4.2. **Lemma.** *There exists a (unique up to a unique isomorphism) A-module $\Omega_{A/B}$ together with a global derivation $\partial : A \to \Omega_{A/B}$ inducing a natural isomorphism in $C(X^\frown)$*

$$\mathcal{H}om_A^{\bullet}(\Omega_{A/B}, M) \xrightarrow{\sim} \mathcal{D}er^{\bullet}{}^{\mathcal{O}}_B(A, M).$$

The proof of Lemma 2.4.2 is standard, see Proposition 7.2.2 of [H2].

The following lemma is the key to the calculation of $\Omega_{A/B}$.

2.4.3. **Lemma.** *Let $\alpha : B \to A$ be a map of $\mathcal{O}$-algebras, $M \in C(X_k^\frown)$. Let $f : M \to A$ be a map in $C(X_k^\frown)$ and let $A' = A\langle M, f\rangle$ be defined by the cocartesian diagram*

$$\begin{tikzcd} F(\mathcal{O}, M) \arrow[r] \arrow[d] & F(\mathcal{O}, \mathrm{cone}(\mathrm{id}_M)) \arrow[d] \\ A \arrow[r] & A' \end{tikzcd}.$$

Put $U = U(\mathcal{O}, A)$ and $U' = U(\mathcal{O}, A')$. The map $\partial \circ f : M \to \Omega_{A/B}$ defines $f' : U' \otimes M \to U' \otimes_U \Omega_{A/B}$.

Then the module of differentials $\Omega_{A'/B}$ is naturally isomorphic to the cone of f'.

For the proof see Lemma 7.3.2 of [H2]. $\square$

2.4.4. **Proposition.** *Let $\mathcal{O}$ be homotopically amenable. Let $\alpha : B \to A$ be a cofibration in $\mathtt{Alg}(\mathcal{O}, X^\frown)$. Then $\Omega_{A/B}$ is cofibrant in $\mathtt{Mod}(\mathcal{O}, X^\frown)$.*

Proof. One can easily reduce the claim to the case when A is generated over B by a section a over $U \in X$ subject to a condition $da = b \in B(U)$. In this case $\mathcal{D}er^{\bullet}{}^{\mathcal{O}}_B(A, M) = \mathcal{H}om_k^{\bullet}(U, M)$ so that $\Omega_{A/B}$ is isomorphic to $U(A) \otimes U$. $\square$

2.4.5. **Corollary.** *Let $\mathcal{O}$ be homotopically amenable. Let $\alpha : B \to A$ is a cofibration in $\mathtt{Alg}(\mathcal{O}, X^\frown)$ and let M be a fibrant A-module. Then $\mathcal{D}er^\bullet{}_B^{\mathcal{O}}(A, M) \in C(X^\frown_k)$ is fibrant.*

Proof. Follos from Lemma 2.4.2 and Lemma 1.6.2 with $\mathcal{A} := U(\mathcal{O}, A)$. $\square$

2.4.6. Let $C \xrightarrow{\alpha} B \xrightarrow{f} A$ be a pair of morphisms in $\mathtt{Alg}(\mathcal{O}, X^\frown_k)$. For each A-module M every derivation $\partial : A \to M$ over C defines a derivation $\partial \circ f : B \to M$ over C. This defines a canonical map

$$\Omega^f : U(B) \otimes_{U(A)} \Omega_{B/C} \to \Omega_{A/C}.$$

The following proposition shows that the module of differentials $\Omega_{A/B}$ has a correct homotopy meaning for cofibrant morphisms $B \to A$. It generalizes Proposition 7.3.6 of [H2] where the case $X^\frown = \mathtt{Ens}$ is considered.

2.4.7. **Proposition.** *Let $C \xrightarrow{\alpha} B \xrightarrow{f} A$ be a pair of morphisms in $\mathtt{Alg}(\mathcal{O}, X^\frown_k)$. If f is a weak equivalence and $\alpha, f \circ \alpha$ are cofibrations than the map Ω^f is a weak equivalence.*

Proof. First of all one proves the claim in the case f is an acyclic cofibration. This easily follows from Lemma 2.4.3. Then one proves the assertion in the case the algebras A and B are fibrant. Here the proof of Proposition 7.3.6 of [H2] can be repeated verbatim. Finally, the general case reduces to the case A and B are fibrant passing to fibrant resolutions of A and B. $\square$

2.4.8. Let $\mathcal{O}$ be homotopically amenable. Let A be a cofibrant $\mathcal{O}$-algebra on X and let M be a fibrant A-module. According to Corollary 2.4.5, $\mathcal{D}er^\bullet(A, M) \in C(X^\frown_k)$ is fibrant. We want to show that this object behaves well under localizations.

Let $U \in X$ and let $j : X/U \to X$ be the obvious embedding. We claim that the presheaf $\mathcal{D}er^\bullet(j^*(A), j^*(M)) = j^*\mathcal{D}er^\bullet(A, M) \in C((X/U)^\frown_k)$ has "the correct homotopy meaning". As in 1.7.3, the only problem is that the functor j^* does not always preserve cofibrations. Therefore, we claim the following.

Lemma. *Let $f : A' \to j^*(A)$ be a cofibrant resolution of $j^*(A)$. Then the restriction map*

$$\mathcal{D}er^\bullet(j^*(A), j^*(M)) \to \mathcal{D}er^\bullet(A', j^*(M))$$

is a weak equivalence.

Proof. It is enough by Lemma 1.7.3 to check that the map

$$\Omega^f : U(j^*(A)) \otimes_{U(A')} \Omega_{A'} \to \Omega_{j^*(A)} \tag{10}$$

is a weak equivalence.

Recall that A is cofibrant. One easily reduces the claim to the case A is standard cofibrant. This means that A is presented as a colimit of $A_n,\ n \in \mathbb{N}$ with $A_0 = \mathcal{O}(0)$ (the initial $\mathcal{O}$-algebra) and A_{n+1} defined as the colimit of a diagram

$$A_n \leftarrow F(\mathcal{O}, M_n) \to F(\mathcal{O}, \mathrm{cone}(\mathrm{id}_{M_n})),$$

M_n being a direct sum of representable presheaves and their shifts.

Then $j^*(A)$ is the colimit of $j^*(A_n)$, with $j^*(A_{n+1})$ isomorphic to the colimit of the diagram

$$j^*(A_n) \leftarrow F(\mathcal{O}, j^*(M_n)) \to F(\mathcal{O}, \mathrm{cone}(\mathrm{id}_{j^*(M_n)})).$$

If we choose cofibrant resolutions $M'_n \to j^*(M_n)$, one defines recursively the collection of cofibrant algebras A'_{n+1} as colimits of the diagram

$$A'_n \leftarrow F(\mathcal{O}, M'_n) \to F(\mathcal{O}, \mathrm{cone}(\mathrm{id}_{M'_n})).$$

Then by induction on n one checks using Lemma 2.4.3 that the map

$$\Omega^{f_n} : U(j^*(A_n)) \otimes_{U(A'_n)} \Omega_{A'_n} \to \Omega_{j^*(A_n)}$$

is a weak equivalence for each n. Choosing $A' = \varinjlim A'_n$, the map Ω^f at (10) is a weak equivalence as a filtered colimit of the weak equivalences Ω^{f_n}. $\square$

2.5. **Simplicial structure.** Similarly to the case $X = *$ described in [H2], sect. 4.8, we can define a simplicial structure on $\mathtt{Alg}(\mathcal{O}, X^{\widehat{}}_k)$ provided $k \supseteq \mathbb{Q}$.

2.5.1. Let S be a finite simplicial set and let $A \in \mathtt{Alg}(\mathcal{O}, X^{\widehat{}}_k)$. We define the presheaf A^S by the formula

$$A^S(U) = \Omega(S) \otimes A(U)$$

where $\Omega(S)$ denotes the commutative dg k-algebra of polynomial differential forms on S.

2.5.2. For $A, B \in \mathtt{Alg}(\mathcal{O}, X^{\widehat{}}_k)$ define $\mathrm{Hom}_\bullet(A, B) \in \Delta^{\mathrm{op}}\mathtt{Ens}$ by the formula

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}\mathtt{Ens}}(S, \mathrm{Hom}_\bullet(A, B)) = \mathrm{Hom}(A, B^S)$$

where $S \in \Delta^{\mathrm{op}}\mathtt{Ens}$ is finite.

The following theorem says that the simplicial structure defined satisfies Quillen's axiom (SM7).

2.5.3. **Theorem.** *Let $\alpha : A \to B$ be a cofibration and $\beta : C \to D$ is a fibration in $\mathtt{Alg}(\mathcal{O}, X^{\widehat{}}_k)$. Then the natural map*

$$\mathrm{Hom}_\bullet(B, C) \to \mathrm{Hom}_\bullet(A, C) \times_{\mathrm{Hom}_\bullet(A,D)} \mathrm{Hom}_\bullet(B, D)$$

is a Kan fibration. It is a weak equivalence if α or β is a weak equivalence.

Theorem 2.5.3 results from the following

2.5.4. **Lemma.** *Let $\alpha : K \to L$ be an injective map of finite simplicial sets and let $\beta : C \to D$ be a fibration in $\mathtt{Alg}(\mathcal{O}, X_k^{\frown})$. Then the natural map*

$$C^L \to C^K \times_{D^K} D^L \tag{11}$$

is a fibration. It is weak equivalence if α or β is a weak equivalence.

Proof. The map (11) can be rewritten as

$$\Omega(L) \otimes C \to \Omega(K) \otimes C \times_{\Omega(K)\otimes D} \Omega(L) \otimes D.$$

it is objectwise surjective since β is objectwise surjective. This also implies that (11) is an acyclic fibration provided α or β is a weak eqivalence.

Since (11) is objectwise surjective, it is enough to check that its kernel is fibrant. The kernel easily identifies with the tensor product $K_\alpha \otimes K_\beta$ where $K_\alpha = \mathrm{Ker}\{\alpha : \Omega(L) \to \Omega(K)\}$ and $K_\beta = \mathrm{Ker}(\beta)$.

Presheaf K_β is fibrant. Complex of k-modules K_α has form $k \otimes_{\mathbb{Q}} K_\alpha^{\mathbb{Q}}$ where the complex $K_\alpha^{\mathbb{Q}}$ has finite dimensional cohomology. One can therefore write $K_\alpha = H \oplus C$ where H is the finite dimensional cohomology of K_α and C is contractible. Then $H \otimes K_\beta$ is fibrant as a finite direct sum of (shifts of) K_α; $C \otimes K_\beta$ is objectwise contractible, and therefore fibrant. □

2.5.5. *Functor* Tot. Let $A^\bullet$ be a cosimplicial object in $\mathtt{Alg}(\mathcal{O}, X^{\frown})$. The algebra $\mathrm{Tot}(A^\bullet)$ is defined by the standard formula

$$\mathrm{Tot}(A^\bullet) = \varprojlim_{p \to q} (A^q)^{\Delta^p}.$$

2.6. **Descent.**

2.6.1. Let $V_\bullet$ be a hypercover of X and let $\mathcal{O}$ be an operad in $X_k^{\sim}$. Put $\mathcal{O}_n = \mathcal{O}|_{V_n}$. Let $\mathtt{Alg}^{\mathrm{f}}$ (resp., $\mathtt{Alg}_n^{\mathrm{f}}$) denote the category of fibrant $\mathcal{O}$-algebras on X (resp., fibrant $\mathcal{O}_n$-algebras on X/V_n).

The assignment $n \longmapsto \mathtt{Alg}_n^{\mathrm{f}}$ defines a cosimplicial object in $\mathtt{Cat}$ (more precisely, a category cofibered over Δ). We denote by $\mathtt{Alg}^{\mathrm{f}}(V_\bullet)$ the following category. The objects of $\mathtt{Alg}^{\mathrm{f}}(V_\bullet)$ are collections $\{A_n \in \mathtt{Alg}_n^{\mathrm{f}}\}$ together with weak equivalences $A_n \to \phi^*(A_m)$ corresponding to each $\phi : m \to n$ in Δ, satisfying the standard cocycle condition.

The functor $\epsilon^* : \mathtt{Alg}^{\mathrm{f}} \to \mathtt{Alg}^{\mathrm{f}}(V_\bullet)$ assigns to each algebra on X the collection of its restrictions. This functor preserves weak equivalences. We define the functor

$$\alpha_* : \mathtt{Alg}^{\mathrm{f}}(V_\bullet) \to \mathtt{Alg}^{\mathrm{f}}$$

by the formula

$$\alpha_*(A_n, \phi) = \mathrm{Tot}\{n \mapsto \iota_*^n(A_n)\}.$$

Here the functor Tot is defined as in 2.5 and ι^n_* is the direct image of the localization functor $\iota^n : X/V_n \to X$, [SGA4], III.5.

Recall that the Dwyer-Kan construction [DK1, DK2, DK3] assigns to a pair $(\mathcal{C}, \mathcal{W})$ where $\mathcal{W}$ is a subcategory of $\mathcal{C}$, a *hammock localization* $L^H(\mathcal{C}, \mathcal{W})$ which is a simplicial category.

2.6.2. **Proposition.** *The functors α_* and α^* induce an adjoint pair of equivalences on the corresponding hammock localizations of $\mathtt{Alg}^{\mathrm{f}}(V_\bullet)$ and $\mathtt{Alg}^{\mathrm{f}}$ with respect to quasi-isomorphisms.*

Proof. The functors involved do not depend on the operad $\mathcal{O}$. Therefore, it is enough to prove the claim for the trivial operad so that $\mathtt{Alg}(\mathcal{O})$ is just the category of complexes. This is done in [SGA4], Exposé Vbis, for $C^+(X^\sim_k)$ and in [HiSi], Sect. 21, for unbounded complexes.

$\square$

3. Deformations

Let k be a field. In this section we define a functor of formal deformations of a sheaf of operad algebras over k. In the case when X has enough points and $k \supseteq \mathbb{Q}$ we present a dg Lie algebra which governs (under some extra restrictions on the sheaf of algebras A) the described deformation problem.

3.1. **Deformation functor.**

3.1.1. Let X be a site, $\mathcal{O} \in \mathtt{Op}(C(X^\sim_k))$ be a sheaf of dg operads on X and let $A \in \mathtt{Alg}(\mathcal{O}, X^\sim)$ be a sheaf of $\mathcal{O}$-algebras. The functor we define below describes formal deformations of the sheaf of $\mathcal{O}$-algebras A.

3.1.2. *Bases of deformations.* Fix a commutative algebra homomorphism $\pi : K \to k$ (it will usually be the identity $\mathrm{id} : k \to k$ of a field of characteristic zero).

Let $\mathtt{dgart}^{\leq 0}(\pi)$ denote the category of commutative non-positively graded dg algebras R endowed with morphisms $\alpha : K \to R$, $\beta : R \to k$, such that $\beta \circ \alpha = \pi$ and $\mathrm{Ker}(\beta)$ is a nilpotent ideal of finite length. The category $\mathtt{dgart}^{\leq 0}(\pi)$ is the category of allowable bases for formal deformations of A.

We assume that the operad $\mathcal{O}$ is obtained by base change from a fixed operad $\mathcal{O}_K \in \mathtt{Op}(C(X^\sim_K))$, so that $\mathcal{O} = \mathcal{O}_K \otimes k$. For each $R \in \mathtt{dgart}^{\leq 0}(\pi)$ we put $\mathcal{O}_R = \mathcal{O}_K \otimes R$.

3.1.3. Fix $R \in \mathtt{dgart}^{\le 0}(\pi)$. Let $\mathtt{Alg}^{f\ell}(\mathcal{O}_R, X^\sim)$ denote the category of sheaves of $\mathcal{O}_R$-algebras *flat as sheaves of R-modules.* Let $\mathcal{W}^{f\ell}_\sim(R, X)$ be the subcategory of weak equivalences in $\mathtt{Alg}^{f\ell}(\mathcal{O}_K \otimes R, X^\sim)$ (i.e. quasi-isomorphisms of complexes of sheaves). In what follows we will usually omit X from the notation.

Let $\mathcal{W}$ be a category. The Dwyer-Kan localization of the pair $(\mathcal{W}, \mathcal{W})$ gives a simplicial groupoid. We will denote it $\widehat{\mathcal{W}}$ and call *a weak groupoid completion*, see Appendix A for details.

The base change functor

$$M \mapsto M \otimes_R k$$

induces a functor between the weak groupoid completions

$$\beta^* : \widehat{\mathcal{W}}^{f\ell}_\sim(R) \to \widehat{\mathcal{W}}_\sim(k). \tag{12}$$

3.1.4. **Definition.** The deformation functor of the $\mathcal{O}$-algebra A is the functor

$$\mathcal{D}ef_A : \mathtt{dgart}^{\le 0}(\pi) \to \mathtt{sGrp}$$

from the category $\mathtt{dgart}^{\le 0}(\pi)$ to the category of simplicial groupoids $\mathtt{sGrp}$ defined as the homotopy fiber of (12) at $A \in \widehat{\mathcal{W}}_\sim(k)$.

Since (12) is a functor between fibrant simplicial categories, the homotopy fiber is represented by the fiber product

$$* \times_{\widehat{\mathcal{W}}_\sim(k)} \widehat{\mathcal{W}}_\sim(k)^{\Delta^1} \times_{\widehat{\mathcal{W}}_\sim(k)} \widehat{\mathcal{W}}^{f\ell}_\sim(R),$$

see A.1.13. This allows one to consider $\mathcal{D}ef_A$ as a functor with values in $\mathtt{sGrp}$ and not in the corresponding homotopy category.

3.2. Properties of $\mathcal{D}ef_A$.

We list below some properties of the functor $\mathcal{D}ef_A$ which justify the above defintion.

3.2.1. The functor $\mathcal{D}ef_A$ does not depend on the specific choice of the site X; only the topos $X^\sim$ counts.

3.2.2. Suppose that X is a one-point site, so that we are dealing with deformations of an operad algebra A. Suppose also that k is a field of characteristic zero and $\pi = \mathrm{id}_k : K = k \to k$.

Recall that in [H4] a deformation functor is defined for this setup. In the definition the simplicial category of cofibrant algebras and weak equivalences is used instead of the hammock localization. We prove in Proposition A.3.4 that these two functors are equivalent.

3.2.3. *Descent.* Let $V_\bullet$ be a hypercover of X and let A be a sheaf of operad algebras on X. Put $A_n = A|_{V_n}$. The assignment $n \longmapsto \mathcal{D}ef_{A_n}$ defines a pseudofunctor $\Delta \longrightarrow \mathtt{sCat}$ (see A.1.10, A.2.4). One has a natural descent functor

$$\mathcal{D}ef_A \to \operatorname{holim}\{n \mapsto \mathcal{D}ef_{A_n}\},$$

see A.1.11. We claim the functor described is an equivalence. In fact, since $\mathcal{D}ef_A$ is defined as a homotopy fiber, the right-hand side of (3.2.3) is the homotopy fiber of the map

$$\operatorname{holim}\{n \mapsto \widehat{\mathcal{W}}^{f\ell}_{\sim}(R, V_n)\} \to \operatorname{holim}\{n \mapsto \widehat{\mathcal{W}}^{f\ell}_{\sim}(k, V_n)\}.$$

It is enough therefore to check that the functor

$$\widehat{\mathcal{W}}^{f\ell}_{\sim}(R, X) \to \operatorname{holim}\{n \mapsto \widehat{\mathcal{W}}^{f\ell}_{\sim}(R, V_n)\}$$

is an equivalence. Since all simplicial categories involved are simplicial groupoids, it is enough by A.2.5 to check that this functor induces an equivalence of the nerves. The functor holim commutes with the nerve functor, see [H3], Prop. A.5.2, or A.1.12. Moreover, the nerve of a Dwyer-Kan localization is equivalent to the nerve of the original category. Therefore, the claim follows from Proposition 2.6.2 and A.3.4.

3.2.4. *Connected components.* Here we assume that the topos $X^\sim$ admits enough points. We assume as well that k is a field of characteristic zero and $\pi = \mathrm{id}_k$.

Suppose that R, A and $\mathcal{O}$ are concentrated at degree zero, so that we have a classical deformation problem. We claim that $\mathrm{Def}_A(R)$ is equivalent to the groupoid $\mathrm{Def}^{cl}_A(R)$ of (classical) flat R-deformations of A.

Let B be a R-flat sheaf of (dg) $R \otimes \mathcal{O}$-algebras such that the reduction $\beta^*(B)$ is quasi-isomorphic to A. We claim that B is concentrated in degree zero and $H^0(B)$ is a flat deformation of A. In fact, since $X^\sim$ has enough points, the claim can be verified fiberwise. The case $X^\sim = \mathtt{Ens}$ is explained in [H4], 5.1.

The assignment $B \mapsto H^0(B)$ defines, therefore, a simplicial functor $\mathrm{Def}_A(R) \to \mathrm{Def}^{cl}_A(R)$. We claim this functor is an equivalence. In fact, it is enough by A.2.5 to check the functor induces an equivalence of the nerves. Let $\mathcal{W}^{f\ell}_{\sim}(R)^0$ denote the full subcategory of $\mathcal{W}^{f\ell}_{\sim}(R)$ consisting of algebras whose cohomology is flat and concentrated in degree zero , and let $\mathtt{Alg}^{\mathrm{iso}}(R)$ denote the groupoid of flat $R \otimes \mathcal{O}$-algebras concentrated in degree zero.

It is enough to check that the functor

$$H^0 : \mathcal{W}^{f\ell}_{\sim}(R)^0 \to \mathtt{Alg}^{\mathrm{iso}}(R)$$

induces an equivalence of the nerves.

To prove this, consider a third category $\mathcal{W}^{\mathrm{c}}(R)$ consisting of cofibrant *presheaves* P of $R \otimes \mathcal{O}$-algebras. Sheafification defines a functor

$$a : \mathcal{W}^{\mathrm{c}}(R) \to \mathcal{W}^{f\ell}_{\sim}(R).$$

We denote by $\mathcal{W}^{\mathrm{c}}(R)^{0,\mathit{fl}}$ the full subcategory of $\mathcal{W}^{\mathrm{c}}(R)$ consisting of complexes of presheaves whose sheafification belongs to $\mathcal{W}^{\mathit{fl}}_{\sim}(R)^0$. The restriction defines the finctors

$$a : \mathcal{W}^{\mathrm{c}}(R)^{0,\mathit{fl}} \to \mathcal{W}^{\mathit{fl}}_{\sim}(R).$$

and

$$H^0 \circ a : \mathcal{W}^{\mathrm{c}}(R)^{0,\mathit{fl}} \to \mathtt{Alg}^{\mathrm{iso}}(R).$$

Both functors a and $H^0 \circ a$ induce an equivalence of nerves by Quillen's Theorem A [Q2] and Theorem A.3.2.

This proves the assertion.

3.3. **Reformulation in terms of presheaves.** From now on we assume that k is a field of characteristic zero and $\pi = \mathrm{id}_k$. We assume as well that the topos $X^{\sim}$ admits enough points.

3.3.1. *Notation.* The category $\mathtt{Alg}(\mathcal{O} \otimes R, X^{\frown})$ admits a model structure defined in 2.1.3. We denote the subcategory of weak equivalences by $\mathcal{W}(R)$. The notation $\mathcal{W}^{\mathrm{c}}(R)$, $\mathcal{W}^{\mathrm{f}}(R)$, $\mathcal{W}^{\mathrm{cf}}(R)$ and $\mathcal{W}^{\mathrm{cf}}_{*}(R)$ has the same meaning as in A.3.4.

Proposition 3.3.5 below claims that the weak groupoid $\widehat{\mathcal{W}^{\mathit{fl}}_{\sim}}(R)$ admits an equivalent description in terms of the weak groupoid of the category of presheaves. This description is functorial in R. This allows one to conveniently describe the homotopy fibre of (12).

Note the following technical lemma.

3.3.2. **Lemma.** *Let A be a cofibrant $\mathcal{O} \otimes R$-algebra and let $A^{\#}$ be the corresponding presheaf of R-modules. There is an increasing filtration $\{F^i A, i \in I\}$ of $A^{\#}$ indexed by a well-ordered set I such that the associated graded factors $\mathrm{gr}^i(A) = F^i(A)/\sum_{j<i} F^j A$ are isomorphic to*

$$R \otimes X^i$$

with $X^i \in C(X^{\frown}_k)$.

Proof. We can assume that A is a direct limit over a well ordered set of generating acyclic cofibrations. In fact, a general cofibrant algebra is a retract of an algebra of this type, and the property claimed in the lemma is closed under retractions.

Let, therefore, the algebra A be freely generated by a collection of sections x_j over $U_j \in X$ of degree d_j numbered by a well-ordered set J. We define I to be the collection of functions $m : J \to \mathbb{N}$ nonzero at a finite number of elements of J. The set I is endowed with the lexicographic order:

$$m < m' \text{ iff } \exists k \in J : m(k) < m'(k) \text{ and } m(j) = m'(j) \text{ for } j > k.$$

The increasing filtration is numbered by elements of I and the associated graded factors have the form

$$R\otimes\mathcal{O}(n)\otimes_{\Sigma_m}k[-d_1]\cdot U_1^{\otimes m_1}\otimes\ldots\otimes k[-d_k]\cdot U_k^{\otimes m_k}$$

where $n=\sum m_i$, $\quad\Sigma_m=\prod\Sigma_{m_i}\subseteq\Sigma_n$, Σ_n being the symmetric group. $\square$

3.3.3. **Lemma.** *1. The functor $\beta^*:\mathtt{Alg}(\mathcal{O}_R,X\hat{})\to\mathtt{Alg}(\mathcal{O},X\hat{})$ preserves cofibrations.*

2. Let $A\in\mathtt{Alg}^c(\mathcal{O}_R,X\hat{})$. Then A is fibrant if and only if $\beta^*(A)$is fibrant.

Proof. 1. The first claim is obvious for generating cofibrations. The general case follows from the fact that β^* commutes with colimits and retracts.

2. According to Lemma 3.3.2, $A^\#$ admits an increasing filtration $\{F^iA,i\in I\}$ with associated graded factors having a form

$$\mathrm{gr}^i A=R\otimes X^i.$$

This implies that $A(U)$ is a cofibrant R-module for each $U\in X$.

A is fibrant iff for any hypercover $\epsilon:V_\bullet\to U$ in X the natural map

$$A(U)\to\check{C}(V_\bullet,A)\tag{13}$$

is a weak equivalence. In Lemma 3.3.4 below we prove that $\check{C}(V_\bullet,A)$ is a cofibrant R-module. The Cech complex of $\beta^*(A)$ is just the reduction of the Cech complex of A modulo the maximal ideal of R. Therefore, if (13) is a weak equivalence, the reduction modulo the maximal ideal of R is also a weak equivalence. In the other direction, there exists a finite filtration of R by k-subcomplexes such that the associated graded factors are isomorphic to k up to shift. Therefore, the cone of (13) admits a finite filtration with acyclic associated graded factors. $\square$

3.3.4. **Lemma.** *Let $A\in\mathtt{Mod}(R,\widehat{X_k})$ admit an increasing filtration $\{F^iA,i\in I\}$ with associated graded factors of form*

$$\mathrm{gr}^i(A)=R\otimes X^i,$$

for some $X^i\in C(\widehat{X_k})$. Then for each hypercover $\epsilon:V_\bullet\to U$ the Cech complex $\check{C}(V_\bullet,A)$ is a cofibrant R-module.

Proof. Cech complex of A is the total complex corresponding to the bicomplex

$$A_0\xrightarrow{\delta^0}A_1\xrightarrow{\delta^1}\ldots$$

where A_n is the collection of sections of $A(V_n)$ vanishing on the degenerate part of V_n (see (3) for the precise definition).

Each A_n is filtered and δ^n preserves the filtration. Define $B_n = \ker(\delta^n)$. B_n are filtered R-modules with associated graded factor having the form

$$R \otimes Y_n^i, \quad Y_n^i = \ker(\delta^n : X_n^i \to X_{n+1}^i).$$

The restriction of the differential δ_n on B_n vanishing, the corresponding total subcomplex is isomorphic to $\prod_n B_n[-n]$. Let us show it is a cofibrant R-module. In fact, each B_n admits an increasing filtration with R-free associated graded factors. Since R is artinian, a product of free R-modules is free. Therefore, $\prod_n B_n[-n]$ admits as well an increasing filtration with R-free associated graded factors. Therefore, it is R-cofibrant.

The quotient is the total complex of the bicomplex

$$A_0/B_0 \to A_1/B_1 \to \dots$$

which has also a vanishing horizontal differential. Each quotient A_n/B_n admits a filtration with the associated graded factors having form $R \otimes X_n^i/Y_n^i$ which is also R-cofibrant. Lemma is proven.

□

The following proposition claims that the simplicial categories $\mathcal{W}_*^{\mathrm{cf}}(R)$ and of $\widehat{\mathcal{W}}_\sim^{\mathit{f\ell}}(R)$ are canonically equivalent.

3.3.5. **Proposition.** *There is a canonical in R collection of equivalences of weak groupoids*

$$\mathcal{W}_*^{\mathrm{cf}}(R) \xrightarrow{c} \widehat{\mathcal{W}}_*^{\mathrm{cf}}(R) \xleftarrow{b} \widehat{\mathcal{W}}^{\mathrm{cf}}(R) \xrightarrow{a} \widehat{\mathcal{W}}_\sim^{\mathit{f\ell}}(R).$$

Proof. The map c is the canonical map from a simplicial category to its hammock localization. It is an equivalence since $\mathcal{W}_*^{\mathrm{cf}}(R)$ is a weak groupoid, see A.2.3. The map b is equivalence by part (ii) of Proposition A.3.4. The map a is induced by the sheafification.

First of all, by Lemma 3.3.2 the sheafification of $A \in \mathcal{W}^{\mathrm{c}}(R)$ is R-flat. Therefore, the functor a is defined. To prove a is an equivalence, it suffices, by Corollary A.2.5, to check that the functor

$$a : \mathcal{W}^{\mathrm{cf}}(R) \to \mathcal{W}_\sim^{\mathit{f\ell}}(R)$$

induces an equivalence of the nerves.

Proposition A.3.4 (2) asserts that the map

$$\mathcal{W}^{\mathrm{cf}}(R) \to \mathcal{W}^{\mathrm{c}}(R)$$

induces an equivalence of the nerves.

The sheafification functor $a : \mathcal{W}^{\mathrm{c}}(R) \to \mathcal{W}_\sim(R)$ has its image in $\mathcal{W}_\sim^{\mathit{f\ell}}(R)$. The resulting functor

$$a : \mathcal{W}^{\mathrm{c}}(R) \to \mathcal{W}_\sim^{\mathit{f\ell}}(R)$$

induces an equivalence of the nerves by Theorem A.3.2 and Quillen's Theorem A, see [Q2].

□

3.4. **Fibration lemma.**

3.4.1. In this subsection we assume the following properties.

- For each $n \in \mathbb{N}$ the complex of sheaves $\mathcal{O}(n)$ is non-positively graded, $\mathcal{O}(n) \in C^{\leq 0}(X^\sim)$.
- The site X admits a final object.

Let $\mathcal{W}^{\mathrm{cf}}_*(R)_{\leq 0}$ denote the full simplicial subcategory of $\mathcal{W}^{\mathrm{cf}}_*(R)$ consisting of algebras A satisfying the extra condition $\mathcal{H}^i(A) = 0$ for $i > 0$.

3.4.2. **Lemma.** *Suppose that the condition of 3.4.1 on $\mathcal{O}$ and X are fulfilled. Then the functor*

$$\beta^* : \mathcal{W}^{\mathrm{cf}}_*(R)_{\leq 0} \to \mathcal{W}^{\mathrm{cf}}_*(k)_{\leq 0} \tag{14}$$

is a fibration in `sCat`.

We recall the model category structure on `sCat` in A.1. The proof of Lemma 3.4.2 is similar to that of Lemma 4.2.1 of [H4]. It is presented in 3.4.3 – 3.4.6 below. The properties of derivations studied in 2.4 play an important role in the proof.

3.4.3. Let B be a cofibrant presheaf of $\mathcal{O}$-algebras. We claim that B and $\alpha^*\beta^*(B)$ are isomorphic as presheaves of graded (without differential) algebras. In fact, any cofibrant presheaf is a retract of a standard cofibrant (see 2.2.2) presheaf of algebras, so we can assume that B is standard. This means, in particular, that B is freely generated, as a graded presheaf of algebras, by a collection of generators b_i which are sections of B over some objects $U_i \in X$. In this case the claim is obvious.

Therefore, any cofibrant presheaf B of $\mathcal{O}$-algebras is isomorphic to a presheaf of form $(\alpha^*(A), d + z)$ where $(A, d) = \beta^*(B)$ and $z \in \mathfrak{m} \otimes \mathrm{Der}(A, A)$ satisfies the Maurer-Cartan equation. Here $\mathfrak{m} = \mathrm{Ker}(\beta : R \to k)$ is the maximal ideal of R and Der denotes the global sections of the presheaf $\mathcal{D}er^\bullet$.

3.4.4. A morphism of simplicial categories is a fibration if it satisfies the conditions (1), (2) of Definition A.1.8. Let us check the condition (1).

Let $f : A \to B$ be a weak equivalence of fibrant cofibrant algebras in $\mathtt{Alg}(\mathcal{O}, X^\frown)$. Let $\mathcal{T}_A = \mathcal{D}er^\bullet(A, A)$ and similarly for $\mathcal{T}_B$. The global sections of the derivation algebras are denoted T_A and T_B correspondingly. Let one of two elements $a \in \mathrm{MC}(\mathfrak{m} \otimes T_A)$, $b \in \mathrm{MC}(\mathfrak{m} \otimes T_B)$ be given. We have to check that there exists a choice of the second element and a map

$$g : (\alpha^*(A), d + a) \to (\alpha^*(B), d + b) \tag{15}$$

in $\mathtt{Alg}(R\otimes\mathcal{O},X^\frown)$ lifting f.

Note that under the restrictions of 3.4.1 any algebra $(\alpha^*(A),d+a)$ belongs to $\mathcal{W}^{\mathrm{cf}}(R)$.

We can consider separately the cases when f is an acyclic fibration or an acyclic cofibration. In both cases we will be looking for the map (15) in the form

$$g=\gamma_B^{-1}\circ\alpha^*(f)\circ\gamma_A$$

where $\gamma_A\in\exp(\mathfrak{m}\otimes T_A)^0$ and similarly for γ_B. A map (15) should commute with the differentials $d+a$ and $d+b$. This amounts to the condition

$$f_*(\gamma_A(a))=f^*(\gamma_B(b)),$$

where the natural maps

$$T_A\xrightarrow{f_*}\mathrm{Der}_f(A,B)\xleftarrow{f^*}T_B \tag{16}$$

are defined as the global sections of the standard maps

$$\mathcal{T}_A=\mathcal{H}om^\bullet(\Omega_A,A)\xrightarrow{f_*}\mathcal{H}om^\bullet(\Omega_A,B)\xleftarrow{f^*}\mathcal{H}om^\bullet(\Omega_B,B)=\mathcal{T}_B.$$

The maps f_* and f^* in (16) are weak equivalences as global sections of weak equivalences between the presheaves $\mathcal{T}_A$, $\mathcal{T}_B$, $\mathcal{D}er^\bullet{}_f(A,B)$ which are fibrant by Lemma 1.6.2.

Recall that we assume that f is either an acyclic cofibration or an acyclic fibration.

3.4.5. **Lemma.** *Let $f:A\to B$ be a weak equivalence of fibrant cofibrant algebras. Suppose that X admits a final object. Suppose that either*

(af) *f is an acyclic fibration*

or

(ac) *f is a standard acyclic cofibration.*

Then there exists a commutative square

$$\begin{tikzcd} & T_f \arrow[dl, "g"'] \arrow[dr, "h"] & \\ T_A \arrow[r, "f_*"] & \mathrm{Der}_f(A,B) & T_B \arrow[l, "f^*"'] \end{tikzcd}$$

where T_f is a dg Lie algebra and g,h are Lie algebra quasi-isomorphisms.

Proof. Note first of all that the maps f^*, f_* are weak equivalences. This follows, as for the "absolute" case of [H2], 8.1, from the presentation

$$T_A=\mathrm{Hom}^\bullet(\Omega_A,A);\quad T_B=\mathrm{Hom}^\bullet(\Omega_B,B);\quad \mathrm{Der}_f(A,B)=\mathrm{Hom}^\bullet(\Omega_A,B).$$

We construct the dg Lie algebra T_f as follows.

Case 1. f is acyclic fibration. Let $I = \mathrm{Ker}(f)$. Define T_f to be the subalgebra of T_A consisting of (global) derivations preserving I. Since A is cofibrant, any global derivation of B can be lifted to A. Therefore, the map $h : T_f \to T_B$ is surjective. The kernel of h consists of global derivations on A with values in I. The presheaf $\mathcal{D}er^\bullet(A, I) = \mathrm{Hom}_\bullet(\Omega_A, I)$ is acyclic fibrant. Therefore, its global sections are acyclic since X admits a final object.

Case 2. f is a standard acyclic cofibration. We define T_f as the collection of (global) derivations of B preserving $f(A)$. Let us check that the map $g : T_f \to T_A$ is a surjective quasi-isomorphism. The algebra B is obtained from A by a sequence of generating acyclic cofibrations corresponding to hypercovers of X.

Let $B = \varinjlim B_i$ with $B_0 = A$ and B_i obtained from the cocartesian diagram

$$\begin{tikzcd}
F(K_i) \arrow[r, "F(j_i)"] \arrow[d, "F(\phi_i)"'] & F(L_i) \arrow[d] \\
B_{i-1} \arrow[r] & B_i,
\end{tikzcd}$$

where $j_i : K_i \to L_i$ are acyclic cofibrations of presheaves and $F(\quad)$ is the free algebra functor.

Put $T_i = \{\delta \in \mathrm{Der}(B_i, B) | \delta(f(A)) \subseteq f(A)\}$. Then $T_0 = T_A$ and $T_f = \varprojlim T_i$. We claim that the maps $g_i : T_i \to T_{i-1}$ are acyclic fibrations. Then $g : T_f \to T_A$ is also an acyclic fibration.

To prove surjectivity of g_i fix a derivation $\delta : B_{i-1} \to B$. Derivations of B_i extending δ correspond to commutative diagrams of presheaves

$$\begin{tikzcd}
K_i \arrow[r, "j_i"] \arrow[d, "\phi_i"'] & L_i \arrow[d] \\
B_{i-1} \arrow[r, "\delta"] & B
\end{tikzcd}$$

Since the map j_i is an acyclic cofibration and B is fibrant, there is no obstruction to extending δ. Surjectivity of g_i is proven.

Now the kernel of g_i identifies with $\mathrm{Hom}^\bullet(L_i/K_i, B)$. The presheaf $\mathcal{H}om^\bullet(L_i/K_i, B)$ being acyclic and fibrant, its global sections $\mathrm{Hom}^\bullet(N/M, B)$ are acyclic. Lemma 3.4.5 is proven. $\square$

Recall now (see A.4) that for a dg Lie algebra $\mathfrak{g}$ the formal groupoid

$$\mathrm{Del}_\mathfrak{g} : \mathtt{dgart}^{\leq 0}(k) \to \mathtt{Grp}$$

is defined as the transformation groupoid of the group $\exp(\mathfrak{m} \otimes \mathfrak{g})^0$ acting on the set of Maurer-Cartan elements of $(\mathfrak{m} \otimes \mathfrak{g})^1$.

The set of connected components $\pi_0(\mathrm{Del}_\mathfrak{g}(R))$ is a weak homotopy invariant of $\mathfrak{g}$. Thus, the maps g and h induce bijections

$$\pi_0(\mathrm{Del}_{T_A}(R)) \longleftarrow \pi_0(\mathrm{Del}_{T_f}(R)) \longrightarrow \pi_0(\mathrm{Del}_{T_B}(R)).$$

of the sets of components. This proves the condition (1) of A.1.8.

3.4.6. Let us check the condition (2) of A.1.8.

Let $\widetilde{A}$, $\widetilde{B} \in \mathcal{W}^{\mathrm{cf}}(R)$ and let $A = \beta^*(\widetilde{A})$, $B = \beta^*(\widetilde{B})$. We have to check that the map

$$\mathrm{Hom}_\bullet(\widetilde{A}, \widetilde{B}) \to \mathrm{Hom}_\bullet(A, B) \tag{17}$$

is a Kan fibration. The algebra B can be considered as $\mathcal{O}_R$-algebra with R acting on B through $\beta : R \to k$. The canonical map $\widetilde{B} \to B$ is objectwise surjective and both B and $\widetilde{B}$ are fibrant. Therefore, it is a fibration by Theorem 1.3.1 (2). Since $\widetilde{A}$ is cofibrant, the map

$$\mathrm{Hom}_\bullet(\widetilde{A}, \widetilde{B}) \to \mathrm{Hom}_\bullet(\widetilde{A}, B) \tag{18}$$

is a Kan fibration. But the maps (17) and (18) coincide. This proves the condition (2) of A.1.8.

Fibration Lemma 3.4.2 is proven. $\square$

3.5. **The main theorem.**

3.5.1. *Assumptions.* In this subsection we assume that the following conditions hold.

- For each $n \in \mathbb{N}$ the complex $\mathcal{O}(n)$ of sheaves is non-positively graded, $\mathcal{O}(n) \in C^{\leq 0}(X^\sim)$.
- The $\mathcal{O}$-algebra A satisfies the property $\mathcal{H}^i(A) = 0$ for $i > 0$.

3.5.2. **Lemma.** *Under the assumptions of 3.5.1, there exists an algebra A' weakly equivalent to A such that*

- *A' is fibrant and cofibrant;*
- *$A' \in C^{\leq 0}(X^{\frown}_k)$.*

Proof. Let $A \to B$ be a fibrant resolution of A. One has $H^i(B) = 0$ for $i > 0$. Let $C = \tau^{\leq 0}(B)$. The canonical map $C \to B$ is therefore objectwise quasi-isomorphism. By Theorem 1.3.1 (2) fibrantness of B means that the natural map $B(U) \longrightarrow \check{C}(V_\bullet, B)$ is a weak equivalence for any hypercover $\epsilon : V_\bullet \longrightarrow U$. Then the same condition is fulfilled for C which asserts that C is fibrant as well. One can choose a cofibrant resolution A' of C having generators only in non-positive degrees. The algebra A' satisfies all necessary properties. $\square$

3.5.3. Let A be as above. Let A' be an algebra whose existence is guaranteed by Lemma 3.5.2. We define the *local* tangent Lie algebra of A by the formula

$$\mathcal{T}_A = \mathcal{D}er^\bullet(A', A').$$

This is a fibrant presheaf of dg Lie algebras. Define, finally, *global* tangent Lie algebra of A by the formula

$$T_A = \mathbf{R}\Gamma\mathcal{T}_A$$

where the functor $\mathbf{R}\Gamma$ assigning a dg Lie algebra to a presheaf of dg Lie algebras, is understood in the following sense (compare to 1.4.3). Choose a hypercover $V_\bullet$ of X. The collection

$$n \mapsto \operatorname{Hom}(V_n, \mathcal{T}_A)$$

is a cosimplicial dg Lie algebra. The corresponding Tot functor produces a dg Lie algebra which is denoted $\mathbf{R}\Gamma\mathcal{T}_A$. The result does not depend on the choice of $V_\bullet$.

Note that if the site X admits a final object $*$, T_A is equivalent to $\mathcal{T}_A(*)$.

3.5.4. A simplicial presheaf $K_\bullet$ is called *finite dimensional* if it coincides with its n-th skeleton for some n.

In Theorem 3.5.5 below we require the site X admit a finite dimensional hypercover. This condition is void if X admits a final object. In the case X is the site of affine open subschemes of a scheme S, the requirement is fulfilled if S is quasi-compact separated or finite dimensional scheme.

3.5.5. **Theorem.** *Suppose that the conditions of 3.5.1 on $\mathcal{O}$ and A are fulfilled. Suppose also that the site X admits a finite dimensional hypercover.*

Then the deformation functor $\mathcal{D}ef_A$ is equivalent to $\mathcal{D}el_{T_A}$ where T_A is the (global) tangent Lie algebra of A.

3.5.6. *Remark.* Fibration Lemma 3.4.2 implies Theorem 3.5.5 if X admits a final object. In fact, let A be a fibrant cofibrant $\mathcal{O}$-algebra with $A^i = 0$ for $i > 0$. By Lemma 3.4.2 the homotopy fiber of (14) is equivalent to its usual fiber. Fix $(R, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$. The fiber of (14) at A is the simplicial groupoid defined by the perturbations of the differential in $R \otimes A$. This is precisely the simplicial Deligne groupoid $\mathcal{D}el_T(R)$ defined in [H4] (see A.4), with $T = \mathrm{Der}(A, A)$.

Proof of the theorem. In a few words, the proof is the following. By Remark 3.5.6 the result is proven in the case X admits a final object. To prove the result in general, one uses the descent properties of the objects involved: that of the deformation functor according to 3.2.3, and that of the Deligne groupoid by [H1]. The requirement on the existence of a finite dimensional hypercover is due to a similar requirement in the proof of [H1], Theorem 4.1.

Here are the details.

Choose A to be a fibrant cofibrant algebra such that $A^i = 0$ for $i > 0$. Choose a finite dimensional hypercover $\epsilon : V_\bullet \to X$.

Let $A_n = A|_{V_n}$. This is a fibrant algebra on X/V_n; it is not necessarily cofibrant but it "behaves as if it were cofibrant".

We have $T_A = \mathbf{R}\Gamma(\mathcal{T}_A)$. Let $T_n = \mathcal{T}_A(V_n) = \mathrm{Der}(A_n, A_n)$.

According to [H1] (more precisely, according to its simplicial version Proposition A.4.5), there is an equivalence

$$\mathcal{D}el_{T_A} \to \mathrm{holim}\{n \mapsto \mathcal{D}el_{T_n}\}.$$

For each n we define canonically a functor

$$\rho_n : \mathcal{D}el_{T_n} \to \mathcal{D}ef_{A_n}$$

which will turn out to be an equivalence.

It is convenient to interpret here $\mathcal{D}ef_{A_n}$ as the homotopy fiber of the map

$$\mathcal{W} \to \overline{\mathcal{W}}$$

where $\mathcal{W}$ denotes the simplicial groupoid $\widehat{\mathcal{W}_*(R)}$ obtained by Dwyer-Kan localization from the simplical category $\mathcal{W}_*(R)$ of $R \otimes \mathcal{O}|_{V_n}$-algebras, and $\overline{\mathcal{W}} = \widehat{\mathcal{W}_*(k)}$.

This means, according to A.1.13, that

$$\mathcal{D}ef_{A_n}(R) = * \times_{\overline{\mathcal{W}}} \overline{\mathcal{W}}^{\Delta^1} \times_{\overline{\mathcal{W}}} \mathcal{W}.$$

The functor ρ_n perturbs the differential. Here is its description. Let $(R, \mathfrak{m}) \in \mathtt{dgart}^{\leq 0}(k)$. An object z of $\mathcal{D}el_{T_n}(R)$ is a Maurer-Cartan element of $\mathfrak{m} \otimes T_n$. Then $\rho_n(z) = (R \otimes A_n, d + z)$ where d denotes the differential of the algebra A. This defines ρ_n on objects.

Let now $\gamma : z \to z'$ be an m-morphism in $\mathcal{D}el_{T_n}$. This means that $\gamma \in \exp(\Omega_m \otimes \mathfrak{m} \otimes T_n)$ and $\gamma(z) = z'$.

Thus, γ induces a map $\gamma : \rho_n(z) \to \Omega_m \otimes \rho_n(z')$ which gives an m-morphism in $\mathcal{W}_*(R)$ and, therefore, in $\mathcal{W} = \widehat{\mathcal{W}_*(R)}$.

Let us check now that ρ_n is an equivalence. Choose a cofibrant resolution $\pi : A' \to A_n$. Let $T' = \mathrm{Der}(A', A')$. According to Lemma 2.4.8, the maps

$$T' \xrightarrow{\phi} \mathrm{Der}(A', A_n) \xleftarrow{\psi} T_n$$

induced by π are weak equivalences and, moreover, ϕ is surjective. Define a dg Lie algebra $\mathfrak{g}$ by the cartesian diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi'"] \arrow[d, "\psi'"'] & T_n \arrow[d, "\psi"] \\
T' \arrow[r, "\phi"] & \mathrm{Der}(A', A_n)
\end{tikzcd}.$$

The maps ϕ' and ψ' are, therefore, quasi-isomorphisms of dg Lie algebras.

We have the following functors:

- $\Psi : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}el_{T'}(R)$ induced by ψ';
- $\Phi : \mathcal{D}el_{\mathfrak{g}}(R) \to \mathcal{D}el_{T_n}(R)$ induced by ϕ';
- $\rho_n : \mathcal{D}el_{T_n}(R) \to \mathcal{D}ef_{A_n}(R)$ defined in (3.5.6).

The functors Ψ and Φ are equivalences. Define an arrow $r : \mathcal{D}el_{T'}(R) \to \mathcal{D}ef_{A_n}(R)$ as follows.

Recall that $\mathcal{D}ef_{A_n}(R)$ is the fiber of the map

$$\overline{\mathcal{W}}^{\Delta^1} \times_{\overline{\mathcal{W}}} \mathcal{W} \longrightarrow \overline{\mathcal{W}}.$$

Each object $z \in \mathcal{D}el_{T'}(R)$ gives rise to an algebra $(R \otimes A', d + z)$ whose reduction is A'. This gives an object in $\mathcal{D}ef_{A_n}(R)$ presented by the morphism $A' \to A_n$. The action of r on morphisms is obvious.

Look at the diagram

$$\begin{tikzcd}
\mathcal{D}el_{\mathfrak{g}}(R) \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & \mathcal{D}el_{T'}(R) \arrow[d, "r"] & = \mathcal{D}ef_{A'}(R) \\
\mathcal{D}el_{T_n}(R) \arrow[r, "\rho_n"] & \mathcal{D}ef_{A_n}(R) &
\end{tikzcd}$$

The diagram is not commutative. However, there is a homotopy connecting $r\Psi$ with $\rho_n\Phi$ assigning to each object $z \in \mathcal{D}el_{\mathfrak{g}}(R)$ a morphism defined by the quasi-isomorphism

$$(R \otimes A', d + \pi(z)) \longrightarrow (R \otimes A_n, d + \chi(z))$$

induced by $\pi : A' \to A_n$.

The map r is an equivalence (of homotopy fibers at A_n and at A'). Therefore, ρ_n is also an equivalence.

Since simplicial groupoids are fibrant objects in `sCat`, the collection of functors ρ_n sums up to an equivalence

$$\mathrm{holim}(\rho) : \mathrm{holim}\{n \mapsto \mathcal{D}el_{\mathcal{T}_A(V_n)}\} \to \mathrm{holim}\{n \mapsto \mathcal{D}ef_{A_n}\}.$$

We have already mentioned that the left hand side is equivalent to $\mathcal{D}el_{T_A}$. The right-hand side is equivalent to $\mathcal{D}ef_A$ by the descent property 3.2.3. □

4. Examples

4.1. **Deformations of schemes.** Let X be a scheme over a field k of characteristic zero. Denote by $X^{\mathrm{aff}}_{\mathrm{Zar}}$ the site of affine open subschemes of X with the Zariski topology. The topoi corresponding to the sites X_{Zar} and $X^{\mathrm{aff}}_{\mathrm{Zar}}$ being equivalent, we can use $X^{\mathrm{aff}}_{\mathrm{Zar}}$ to describe deformations of X. The structure sheaf $\mathcal{O}$ considered as an object in $C((X^{\mathrm{aff}}_{\mathrm{Zar}})\widehat{_k})$, is fibrant. Therefore, our general result Theorem 3.5.5 is applicable under some finiteness conditions. We get the following.

4.1.1. **Corollary.** *Let X be a scheme over a field k of characteristic zero. Suppose that X admits a finite dimensional hypercover by affine open subschemes. Then the functor of formal deformations of X, $\mathcal{D}ef_X$, is equivalent to the simplicial Deligne groupoid $\mathcal{D}el_T$ where T is the dg Lie algebra of global derivations of a cofibrant resolution of $\mathcal{O}$.*

Proof. Let $p : A \longrightarrow \mathcal{O}$ be a cofibrant resolution of $\mathcal{O}$. Since $\mathcal{O}$ is fibrant and p is an acyclic fibration, A is fibrant as well. The conditions 3.5.1 are fulfilled, so Theorem 3.5.5 gives the claim in question. □

The same reasoning provides a similar description of formal deformations of a quasi-coherent sheaf of algebras.

4.1.2. **Corollary.** *Let X be a scheme over a field k of characteristic zero. Suppose that X admits a finite dimensional hypercover by affine open subschemes. Let A be a quasi-coherent sheaf of algebras over a linear operad P. Then the functor of formal deformations of P-algebra A is equivalent to the simplicial Deligne groupoid $\mathcal{D}el_T$ where T is the dg Lie algebra of global derivations of a cofibrant resolution of A.*

Note that here there is no connection between the P-algebra structure and the $\mathcal{O}_X$-module structure on A: we use the fact that A is quasi-coherent in order to deduce that $\mathcal{H}^i(A)=0$ for positive i.

4.2. **Obstruction theory.** The tangent Lie algebra T in Corollary 4.1.2 is not easy to determine. The classical obstruction theory task, the determination of the cohomology of T is a much easier problem.

Assume we are dealing with deformations of associative algebras.

Let A be a quasicoherent sheaf of associative $\mathcal{O}_X$-algebras. We wish to describe deformations of A as a k-algebra.

Consider A as a presheaf on the site $X^{\mathrm{aff}}_{\mathrm{Zar}}$ of affine open subschemes of X and let $\phi: P\longrightarrow A$ be a cofibrant resolution. According to our definition, $\mathcal{T}_A=\mathcal{D}er^\bullet(P,P)$ governs local deformations of A. Let $C^*(P,P)$ be the Hochschild cochain complex for P. Let $\pi: C^*(P,P)\longrightarrow P$ be the obvious projection to $P=C^0(P,P)$. The natural map

$$i_P:\mathcal{D}er^\bullet(P,P)\to \mathrm{cone}(\pi: C^*(P,P)\to P)$$

from the presheaf of derivations of P to the (shifted and truncated) Hochshild cochains presheaf, is a weak equivalence.

According to Lemma 1.7.1, $C^*(P,P)$ represents $\mathbf{R}\mathcal{H}om^\bullet_{A\otimes A^{\mathrm{op}}}(A,A)$, the (local) Hochschild cohomology of A. This gives the exact sequence

$$\ldots\longrightarrow HH^i(A,A)\longrightarrow H^i(X,A)\longrightarrow H^i(T)\longrightarrow HH^{i+1}(A,A)\longrightarrow\ldots$$

where the global Hochschild cohomology $HH^i(A,A)$ is defined as

$$H^i(\mathbf{R}\,\mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A,A))$$

(compare to Lunts' [Lu], Cor. 5.4).

4.3. **Standard complex.** It seems too naive to expect that the standard complex

$$\mathrm{cone}(C^*(A,A)\longrightarrow A))$$

represent the tangent Lie algebra $\mathcal{T}_A$. However, in a very special case of associative deformations of the structure sheaf of a smooth variety, the version of the standard complex based on cochains which are differential operators in each argument, gives a correct result.

In more details, let A be the structure sheaf of a smooth algebraic variety X over k, and let $C^*_{do}(A,A)$ denote the subcomplex of $C^*(A,A)$ consisting of cochains given by differential operators in each argument.

Let us compare the Hochschild cochains $C^*(P,P)$ and $C^*_{do}(A,A)$. One has the following commutative diagram

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr, "\phi''"] & \\ C^*(P,P) \arrow[dr, "\phi'"] & & C^*_{do}(A,A) \arrow[dl, "\psi"'] \\ & C^*(P,A) & \end{tikzcd}$$

where T is defined to make the diagram cartesian. The map ϕ' is an acyclic fibration since $P^{\otimes n}$ are cofibrant complexes of presheaves and ϕ is an acyclic fibration. Therefore, ϕ'' is an acyclic fibration of dg Lie algebras. By a version of Hochschild-Kostant-Rosenberg theorem proven in [Y], ψ is a weak equivalence of complexes of presheaves. This gives a weak equivalence of Lie dg algebras $C^*(P,P)[1]$ and $C^*_{do}(A,A)[1]$.

4.4. **Equivariant deformations.** Let G be a group and let $\mathfrak{g}$ be a dg Lie algebra governing formal deformations of some object A (we are informal at the moment). If G acts on A, one should expect a G-action to be induced on $\mathfrak{g}$. One can expect that the equivariant deformations of A are governed by the dg Lie algebra $\mathbf{R}\Gamma^G(\mathfrak{g})$ whose i-th cohomology is $H^i(G,\mathfrak{g})$.

We are able to prove this when G is a formal group, "an object A" meaning "a sheaf of operad algebras A", under the restrictions of Theorem 3.5.5.

In 4.4.1–4.4.5 we discuss the action of formal groups on sites. In 4.4.6 we describe the equivariant deformation functor. The formula for the equivariant tangent Lie algebra is deduced in 4.4.9.

4.4.1. In this subsection *a formal group* is a functor

$$G : \mathtt{art}(k) \longrightarrow \mathtt{Groups}$$

from the category of artinian local k-algebras to the category of groups, commuting with the fiber products.

According to formal Lie theory, the fiber at 1 functor

$$\overline{G}(R) = \mathrm{Ker}(G(R) \longrightarrow G(k))$$

is uniquely defined by the corresponding Lie algebra $\mathfrak{g}$ (possibly, infinite-dimensional). The group $G(k)$ of k-points of G acts on $\mathfrak{g}$ (adjoint action).

Thus, a formal group G in our sense is described by a pair $(G(k), \mathfrak{g})$ consisting of a discrete group $G(k)$, a Lie algebra $\mathfrak{g}$ and an action of $G(k)$ on $\mathfrak{g}$.

A representation of a formal group G is a k-vector space V together with a collection of compatible operations

$$G(R) \longrightarrow GL_R(R \otimes V).$$

This can be rephrased in terms of the corresponding pair $(G(k), \mathfrak{g})$ as follows. To define a representation of the formal group corresponding to $(G(k), \mathfrak{g})$ on a k-vector space V, one has to define the action of $G(k)$ and of $\mathfrak{g}$ on V satisfying the compatibility condition

$$\gamma(x)(v) = \gamma(x(\gamma^{-1}v)), \quad \gamma \in G(k),\ x \in \mathfrak{g},\ v \in V.$$

4.4.2. Let X be a site and $\mathcal{O}$ be a sheaf of commutative k-algebras on X. An action of a formal group G on the ringed site $(X, \mathcal{O})$ is a collection of compatible actions of groups $G(R)$ on $(X, R \otimes \mathcal{O})$ for $R \in \mathtt{art}(k)$.

If a formal group G is described by a pair $(G(k), \mathfrak{g})$ as in 4.4.1, its action on $(X, \mathcal{O})$ is given by an action of the discrete group $G(k)$ on $(X, \mathcal{O})$, action of $\mathfrak{g}$ by vector fields on $\mathcal{O}(U)$ for each $U \in X$, subject to the compatibility

$$\gamma(x)(\gamma(f)) = \gamma(x(f))$$

with $f \in \mathcal{O}(U),\ x \in \mathfrak{g},\ \gamma \in G(k)$.

In the case $\mathcal{O} = k$ (our assumption below) we will assume that the action of $\mathfrak{g}$ on k is trivial, so that the action of G on X is reduced to the action of the discrete group $G(k)$.

4.4.3. Now we are able to define G-equivariant (pre)sheaves on X. Suppose a formal group G acts on X as above. A G-(pre)sheaf M is given by a (pre)sheaf M of k-vector spaces on X together with a compatible collection of structures of $G(R)$-module on R-module $R \otimes M$, for $R \in \mathtt{art}(k)$.

If G is presented by a pair $(G(k), \mathfrak{g})$ as above, a G-module structure on M amounts to a compatible collection of maps $\gamma : M(U) \longrightarrow M({}^{\gamma}U) \quad (\gamma \in G(k))$, a collection of actions $\mathfrak{g} \longrightarrow \mathrm{End}(M(U))$ satisfying the condition

$$\gamma(xm) = \gamma(x)\gamma(m).$$

This implies the following

Proposition. *There is an explicitly defined ringed site $(X/G, \mathcal{U})$ such that the category of G-(pre)sheaves of k-modules on X is equivalent to that of $(X/G, \mathcal{U})$-modules.*

Proof. The category X/G has the same objects as X; for $U, V \in X$ one has

$$\mathrm{Hom}_{X/G}(U, V) = \{(\gamma, f) | \gamma \in G(k),\ f \in \mathrm{Hom}_X(U, {}^{\gamma}V)\}.$$

The composition of morphisms in X/G is defined in a standard way, so that the morphism defined by a pair (γ, f) is denoted as γf. One has the identity

$$f\gamma = \gamma\, {}^{\gamma}f.$$

The topology on X/G is generated by that on X, so that a family $\{\phi_i\gamma_i :^{\gamma_i} V_i \longrightarrow U\}$ in X/G is a covering iff the collection $\{\phi_i : V \longrightarrow U\}$ is a covering in X.

We define the sheaf of rings $\mathcal{U}$ as the sheafification of a presheaf $\mathcal{U}'$ defined below. Let $U \in X$. We set $\mathcal{U}'(U)$ to be the enveloping algebra of $\mathfrak{g}$.

If $f : U \longrightarrow V$ is a map in X, the corresponding map

$$\mathcal{U}'(V) \longrightarrow \mathcal{U}'(U)$$

is identity. For $\gamma :^{\gamma} U \longrightarrow U$ the corresponding map

$$\mathcal{U}'(U) \longrightarrow \mathcal{U}'({}^{\gamma}U)$$

is induced by the automorphism of the Lie algebra $\mathfrak{g}$ sending $x \in \mathfrak{g}$ to $\gamma(x)$.

A straightforward check shows the pair $(X/G, \mathcal{U})$ satisfies the required property. $\square$

4.4.4. A special case $X = *$ of the above construction gives the ringed site $BG = (*/G, \mathcal{U})$. As a category, this is the classifying groupoid of $G(k)$; the sheaf of rings is defined by the enveloping algebra $U\mathfrak{g}$ endowed with the adjoint $G(k)$-action. Sheaves on this site are G-modules.

4.4.5. If M is a presheaf of complexes on X/G, we denote by $M^{\#}$ the presheaf on X obtained by forgetting the G-structure. If M is fibrant, $M^{\#}$ is fibrant as well; the global sections $\mathbf{R}\Gamma(X, M^{\#})$ admit a natural G-structure, that is define a sheaf on BG (this is just the higher direct image of the morphism $X/G \longrightarrow BG$ induced by the morphism $X \longrightarrow *$). We denote this sheaf $\mathbf{R}\Gamma(X, M)$. One has

$$\mathbf{R}\Gamma(X/G, M) = \mathbf{R}\Gamma^{G}(\mathbf{R}\Gamma(X, M)).$$

4.4.6. *Equivariant deformation functor.* Let X be a site and let a formal group G act on X (through $G(k)$). Let $\mathcal{O}$ be a G-equivariant operad. Let $\widehat{\mathcal{W}}^{f\ell}_{\sim,G}(R)$, $R \in \mathtt{dgart}^{\leq 0}(k)$, be the weak groupoid of R-flat sheaves of equivariant $R\otimes\mathcal{O}$-algebras. Let A be an $\mathcal{O}$-algebra in $C(X^{\sim}, k)$ endowed with a G-action.

Similarly to 3.1.4, we define the equivariant deformation functor

$$\mathcal{D}ef_{A,G} : \mathtt{dgart}^{\leq 0}(k) \longrightarrow \mathtt{sGrp}$$

as the homotopy fiber at A of the functor

$$\widehat{\mathcal{W}}^{f\ell}_{\sim,G}(R) \longrightarrow \widehat{\mathcal{W}}^{f\ell}_{\sim,G}(k)$$

induced by the projection $R \to k$.

4.4.7. ... *and its tangent Lie algebra.* We assume that the conditions of Theorem 3.5.5 are satisfied for the $\mathcal{O}$-algebra A. In particular, deformations of A are governed by the global tangent Lie algebra T_A which can be calculated using a cofibrant resolution P of A in the category $\mathtt{Alg}(\mathcal{O}, X^{\widehat{}}_k)$ by the formula

$$T_A = \mathbf{R}\Gamma(X, \mathcal{D}er^{\bullet}(P, P)).$$

We wish to express the functor of equivariant deformations through T_A.

This can be done as follows. Consider the ringed site $(X/G, \mathcal{U})$ described in 4.4.3.

Sheaves (resp., presheaves) on $(X/G, \mathcal{U})$ are precisely equivariant sheaves (resp., presheaves) on (X, k).

We define a new operad $\mathcal{O}\#G$ in $C((X/G)^{\widehat{}}_k)$ as the one governing equivariant $\mathcal{O}$-algebras on X. It is explicitly given by the formula

$$\mathcal{O}\#G(n) = \mathcal{O}(n) \otimes \mathcal{U}^{\otimes n},$$

with the operations uniquely defined by the G-action on $\mathcal{O}$

$$\mathcal{U} \otimes \mathcal{O}(n) \xrightarrow{\Delta^n \otimes \mathrm{id}} \mathcal{U}^{n+1} \otimes \mathcal{O}(n) \longrightarrow \mathcal{U} \otimes \mathcal{O}(n) \otimes \mathcal{U}^{\otimes n} \longrightarrow \mathcal{O}(n) \otimes \mathcal{U}^{\otimes n},$$

where the second arrow swaps the arguments and the third one is defined by the G-action on $\mathcal{O}(n)$.

This is a generalization of the twisted group ring construction.

4.4.8. **Lemma.** *1. The forgetful functor $\# : (X/G)^{\widehat{}}_{\mathcal{U}} \to X^{\widehat{}}_k$ preserves weak equivalences, fibrations and cofibrations.*

2. The same is true for the forgetful functor

$$\# : \mathtt{Alg}(\mathcal{O}\#G, (X/G)^{\widehat{}}) \longrightarrow \mathtt{Alg}(\mathcal{O}, X^{\widehat{}}).$$

Proof. The proofs of both claims are identical. Since sheafification commutes with $\#$, weak equivalences are preserved. Cofibrations are preserved since joining a section over $U \in X$ corresponds, after application of $\#$, to joining sections corresponding to a chosen basis of $U\mathfrak{g}$ over all $\gamma(U)$, for $\gamma \in G$. The functor $F \mapsto F_!$ left adjoint to the forgetful functor $\#$, carries representable presheaves on X to representable (by the same object) presheaves on X/G. Since any hypercover in X/G is of form $V_{\bullet !} \longrightarrow U_!$ where $V_\bullet \longrightarrow U$ is a hypercover in X, the functor $\#$ preserves the fibrations. □

4.4.9. Thus, A can be considered as a sheaf of algebras on X/G. By Lemma 4.4.8 the condition $\mathcal{H}^i(A) = 0$ for $i > 0$ is valid also when A is considered as a sheaf on X/G. Therefore, Theorem 3.5.5 is applicable to A in the equivariant setting. Let us calculate the equivariant global tangent Lie algebra. Choose an equivariant fibrant cofibrant representative P of A. We can calculate both equivariant and non-equivariant tangent Lie algebras using P.

Thus, the equivariant local tangent Lie algebra is

$$\mathcal{T}_{A,G} = \mathcal{D}er^{\bullet}{}_{X/G}(P,P)$$

and the one in X is

$$\mathcal{T}_A = \mathcal{D}er^{\bullet}{}_X(P^{\#},P^{\#}).$$

Note that $\mathcal{T}_A = \mathcal{T}^{\#}_{A,G}$. Since $\mathcal{T}_{A,G}$ is fibrant by 2.4.5, one has

$$\mathbf{R}\Gamma(X/G,\mathcal{T}_{A,G}) = \mathbf{R}\Gamma^G \circ \mathbf{R}\Gamma(X,\mathcal{T}_{A,G}).$$

Appendix A. Simplicial categories and all that

A.1. **Simplicial categories.** Throughout the paper simplicial category means a simplicial object in the category $\mathtt{Cat}$ of small categories having a discrete simplicial set of objects. The category of simplicial categories is denoted by $\mathtt{sCat}$.

A.1.1. The functor

$$\pi_0 : \mathtt{sCat} \to \mathtt{Cat}$$

is defined by the formulas

$$\begin{array}{rcl} \mathrm{Ob}\,\pi_0(X) & = & \mathrm{Ob}\,X \\ \mathrm{Hom}_{\pi_0(X)}(x,y) & = & \pi_0(\mathrm{Hom}_X(x,y)). \end{array}$$

A.1.2. As a simplicial category is as a simplicial object in $\mathtt{Cat}$, by applying the nerve functor

$$\mathcal{N} : \mathtt{Cat} \to \Delta^{\mathrm{op}}\mathtt{Ens}$$

layer by layer and taking the diagonal, we get a simplicial set called *simplicial nerve* (or just nerve) $\mathcal{N}(\mathcal{C})$ of a simplicial category $\mathcal{C}$.

A.1.3. *Model structure.* In this paper we use a model category structure on $\mathtt{sCat}$ defined in [H3].

A.1.4. **Definition.** A map $f : \mathcal{C} \to \mathcal{D}$ in $\mathtt{sCat}$ is called a *weak equivalence* if the map $\pi_0(f) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ induces a weak homotopy equivalence of the nerves and for each $x,y \in \mathrm{Ob}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\bullet}(x,y) \to \mathrm{Hom}_{\bullet}(f(x),f(y))$$

of the simplicial Hom-sets is a weak equivalence.

Sometimes the following notion of *strong equivalence* is useful.

A.1.5. **Definition.** A weak equivalence $f : \mathcal{C} \to \mathcal{D}$ is called *strong equivalence* if the functor $\pi_0(f) : \pi_0(\mathcal{C}) \to \pi_0(\mathcal{D})$ is an equivalence of categories.

A.1.6. Cofibrations in `sCat` are generated by the following maps:

(cof-1) $\emptyset \to *$, the functor from the empty simplicial category to a one-point category.

(cof-2) For each cofibration $K \to L$ in $\Delta^{\mathrm{op}}\mathtt{Ens}$ the induced map from K_{01} to L_{01}. Here K_{01} denotes the simplicial category having two objects 0 and 1 with the only nontrivial maps $K = \mathrm{Hom}_\bullet(0, 1)$.

A.1.7. **Theorem.** *(*[H3]*) The collections of cofibrations and of weak equivalences define a CMC structure on* `sCat`*.*

The maps in `sCat` satisfying RLP with respect to acyclic cofibrations (=cofibrations + weak equivalences) will be called fibrations.

Recall for the sake of completeness the explicit definition of fibration.

A.1.8. **Definition.** A map $f : \mathcal{C} \to \mathcal{D}$ in `sCat` is called a fibration if it satisfies the following properties

(1) the right lifting property (RLP) with respect to the maps

$$\partial^{0,1} : \Delta^0 \to \Delta^1$$

from the terminal category $\Delta^0 = *$ to the one-arrow category Δ^1.

(2) For all $x, x' \in \mathrm{Ob}\,\mathcal{C}$ the map $f : \mathrm{Hom}_\bullet(x, x') \to \mathrm{Hom}_\bullet(fx, fx')$ is a Kan fibration.

Note that our CMC structure on `sCat` is not standard. In [DHK] another defintion is given based on the same cofibrations and strong equivalences. The definitions coincide for simplicial groupoids.

A.1.9. *Simplicial structure and* Tot*.* The category `sCat` admits a structure of a simplicial category.

Let S be a simplicial set, $\mathcal{C}$ be a category and let $\mathcal{N}\mathcal{C}$ be the nerve of $\mathcal{C}$. The simplicial set $\mathrm{Hom}_\bullet(S, \mathcal{N}\mathcal{C})$ is the nerve of a category which will be denoted by $\mathcal{C}^S$.

Let $X = \{X_n\} \in \mathtt{sCat}$, $\quad S \in \Delta^{\mathrm{op}}\mathtt{Ens}$. The collection

$$n \mapsto X_n^S$$

forms a simplicial object in `Cat`. We define X^S to be the simplicial category given by the formulas

$$\mathrm{Ob}\, X^S = \mathrm{Ob}\, X_0^S \tag{19}$$

$$\mathrm{Hom}_{X^S}(x, y)_n = \mathrm{Hom}_{X_n^S}(x_n, y_n) \tag{20}$$

where x, y are objects of X_0^S and x_n, y_n are their degeneracies in X_n^S.

For a pair $\mathcal{C}, \mathcal{D}$ of simplicial categories the simplicial set $\mathrm{Hom}_\bullet(\mathcal{C}, \mathcal{D})$ defined by the equality

$$\mathrm{Hom}_\bullet(\mathcal{C}, \mathcal{D})_n = \mathrm{Hom}(\mathcal{C}, \mathcal{D}^{\Delta^n}).$$

The above defined simplicial structure provides on $\mathtt{sCat}$ a simplicial framing (see [Hir], 16.6). This is proven in [H3], A.4.3.

Given a cosimplicial object $X^\bullet$ in $\mathtt{sCat}$, we can now define $\mathrm{Tot}(X^\bullet)$ by the usual formula

$$\mathrm{Tot}(X^\bullet) = \lim_{\leftarrow}{}_{p \to q} (X^q)^{\Delta^p}. \tag{21}$$

A.1.10. *Pseudofunctors.* In this paper we encounter weak functors to $\mathtt{Cat}$ and $\mathtt{sCat}$. Here is a convenient way of dealing with them borrowed from [HS], 1.5.

Let D be a category. A coefficient system (c.s.) E for D is an assignment

- for each morphism ϕ of D a contractible simplicial set $E(\phi)$.
- for each object $x \in D$ a 0-simplex $e_x \in E(\mathrm{id}_x)$.
- for each pair (ϕ, ψ) of composable morphisms a composition $E(\phi) \times E(\psi) \longrightarrow E(\phi\psi)$.

The composition is assumed to satisfy obvious associativity and unit conditions.

We define a pseudofunctor $F : D \longrightarrow \mathcal{C}$ to a simplicial category $\mathcal{C}$ (over a c.s. E) as collection of the following data

- An object $F(x) \in \mathcal{C}$ for each object x of D.
- A composition map $E(\phi) \longrightarrow \mathrm{Hom}_\bullet(F(x), F(y))$ of simplicial sets for each $\phi : x \to y$ in D.

The associativity and the unit conditions are assumed for the composition map.

A special choice of c.s. (this is basically the "universal c.s." Q defined in [HS], A.2.4) gives rise to the notion of pseudofunctor which coincides with the standard one [SGA1], VI.8, when $\mathcal{C} = \mathtt{Cat}$. In particular, a cofibered category over D can be described by a pseudofunctor $D \longrightarrow \mathtt{Cat}$.

A.1.11. *Homotopy limits.* The functor Tot defined in (21) allows one to define homotopy limits of pseudofunctors similarly to [HS], 1.5.3.

Assume we are given a pseudofuctor $F : D \to \mathtt{sCat}$ over a coefficient system E. The homotopy limit $\mathrm{holim}\, F$ is defined as $\mathrm{Tot}(\widetilde{F})$ where the cosimplicial object $\widetilde{F}$ in $\mathtt{sCat}$ is defined by the formula

$$\widetilde{F}^n = \prod_{\sigma \in \mathcal{N}_n(D)} F(t\sigma)^{E(\sigma)},$$

where for

$$\sigma = x_0 \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_n} x_n$$

one defines $t\sigma = x_n$ and $E(\sigma) = E(\phi_1) \times \ldots \times E(\phi_n)$.

A.1.12. **Proposition.** *(see* [H3]*) Let $F : D \to \mathtt{sCat}$ be a pseudofunctor such that $F(d)$ are fibrant for all $d \in D$. Then the natural map of simplicial sets*

$$\mathcal{N}(\operatorname{holim} F(d)) \longrightarrow \operatorname{holim}(\mathcal{N}(F(d)))$$

is a weak equivalence.

A.1.13. *Homotopy fibers.* In this paper we are particularly interested in homotopy fibers. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor in $\mathtt{sCat}$ and let $d \in \mathcal{D}$. We assume also that the simplicial categories $\mathcal{C}$ and $\mathcal{D}$ are fibrant (for instance, simplicial groupoids), so that the homotopy fiber represents the right derived functor of the usual fiber product. In this case homotopy fiber of f at d can be represented by the fiber product

$$* \times_{\mathcal{D}} \mathcal{D}^{\Delta^1} \times_{\mathcal{D}} \mathcal{C}$$

where the map $* \to \mathcal{D}$ is given by $d \in \mathcal{D}$ and the maps from $\mathcal{D}^{\Delta^1}$ to $\mathcal{D}$ are given by the ends of the segment Δ^1.

This immediately follows from [H3], Prop. A.4.3 claiming, in particular, that the map $\mathcal{D}^{\Delta^1} \to \mathcal{D} \times \mathcal{D}$ is a fibration for fibrant $\mathcal{D}$.

A.2. Weak groupoids.

A.2.1. **Definition.** A simplicial category $\mathcal{G}$ is called a *weak groupoid* if $\pi_0(\mathcal{G})$ is a groupoid.

The following fact justifies the above definition.

A.2.2. **Proposition.** *A simplicial category $\mathcal{C}$ is a weak groupoid if and only if it is strongly equivalent to a simplicial groupoid.*

Proof. The "if" part is obvious. According to [DK2], if $\mathcal{C}$ is a weak groupoid, the map $\mathcal{C} \to L^H(\mathcal{C}, \mathcal{C})$ from $\mathcal{C}$ to the hammock localization of $\mathcal{C}$ is a strong equivalence. ☐

A.2.3. **Corollary.** *A map $f : \mathcal{V} \to \mathcal{W}$ of weak groupoids is a weak equivalence iff its nerve $\mathcal{N}(f) : \mathcal{N}(\mathcal{V}) \to \mathcal{N}(\mathcal{W})$ is a weak homotopy equivalence.*

Proof. Proposition A.2.2 reduces the claim to simplicial groupoids. In this case the result is proven, for instance, in [H4], 6.2.2, 6.2.3. ☐

Note that the notions of weak and strong equivalence coincide for weak groupoids.

A.2.4. **Definition.** Let $\mathcal{W}$ be a simplicial category. Its weak groupoid completion $\widehat{\mathcal{W}}$ is the hammock localizaton $L^H(\mathcal{W},\mathcal{W})$, see [DK2].

One has a canonical map $\mathcal{W}\to\widehat{\mathcal{W}}$ from a simplicial category to its weak groupoid completion.

The formulas for the simplicial path functor A.1.9 on `sCat` show that one has a canonical functor

$$\widehat{W^S}\longrightarrow\widehat{W}^S.$$

This implies that the functor of groupoid completion preserves the simplicial structure. In particular, given a pseudofunctor $D\longrightarrow$ `Cat` (see A.1.10), one can compose it with the groupoid completion to get a pseudofunctor $D\longrightarrow$ `sGrp`.

Since simplicial localizations preserve the homotopy type of the nerve, see [DK1], 4.3, we obtain immediately the following

A.2.5. **Corollary.** *A map $f:\mathcal{C}\to\mathcal{D}$ of simplicial categories induces a weak equivalence of the weak groupoid completions $\widehat{\mathcal{C}}\to\widehat{\mathcal{D}}$ iff the nerve $\mathcal{N}(f):\mathcal{N}(\mathcal{C})\to\mathcal{N}(\mathcal{D})$ is a weak homotopy equivalence.*

A.3. **Weak groupoid of a model category.** Let $\mathcal{C}$ be a closed model category and let $\mathcal{W}$ be the subcategory of weak equivalences in $\mathcal{C}$.

One can assign to $\mathcal{C}$ a few different weak groupoids. These are weak groupoid completions of the categories $\mathcal{W}$, $\mathcal{W}^{\mathrm{c}}$ (the objects are cofibrant objects of $\mathcal{C}$, the morphisms are weak equivalences), and also $\mathcal{W}^{\mathrm{f}}$ and $\mathcal{W}^{\mathrm{cf}}$. We denote them $\widehat{\mathcal{W}}$, $\widehat{\mathcal{W}^{\mathrm{c}}}$, $\widehat{\mathcal{W}^{\mathrm{f}}}$ and $\widehat{\mathcal{W}^{\mathrm{cf}}}$.

In the case $\mathcal{C}$ admits a simplicial structure so that the axiom (SM7) of [Q1] is satisfied, one defines another weak groupoid $\mathcal{W}^{\mathrm{cf}}_*$ whose objects are cofibrant fibrant objects of $\mathcal{C}$ and n-morphisms are the ones lying in the components of weak equivalences. Similarly the weak groupoid $\widehat{W_*}$ is defined.

In Proposition A.3.4 below we show that all these weak groupoids are strongly equivalent.

A.3.1. *Contractibility of resolutions.* Homological algebra starts with an observation that resolutions are usually unique up to a homotopy which is itself unique up to homotopy. In this section we prove a generalization of this fact: the category of resolutions has a contractible nerve. This result will be used in the proof of equivalences A.3.4 below.

Let $\mathcal{C}$ be a closed model category, $M\in\mathcal{C}$. Let $\mathcal{C}^{\mathrm{c}}_M$ denote the full subcategory of the category $\mathcal{C}/M$ whose objects are the weak equivalences $f:P\to M$ with cofibrant P. Similarly, $\mathcal{C}^{\mathrm{cf}}_M$ consists of weak equivalences $P\to M$ with P cofibrant and fibrant.

The first part of the following theorem can be found in [Hir], Thm 14.6.2.

A.3.2. **Theorem.** *1. The nerve of the category $\mathcal{C}^{\mathrm{c}}_M$ is contractible.*

2. If M is fibrant, the nerve of $\mathcal{C}^{\mathrm{cf}}_M$ is contractible.

Proof. Step A. First of all, we check that the nerve of $\mathcal{C}^{\mathrm{c}}_M$ is simply connected. Suppose $f : P \to M$ and $g : Q \to M$ are two objects of $\mathcal{C}^{\mathrm{c}}_M$. Present the map $f \coprod g : P \coprod Q \to M$ as a composition of a cofibration ι and an acyclic fibration π

$$P \coprod Q \xrightarrow{\iota} R \xrightarrow{\pi} M.$$

The map $\pi : R \to M$ presents an object of $\mathcal{C}^{\mathrm{c}}_M$. This proves that the nerve of $\mathcal{C}^{\mathrm{c}}_M$ is connected. Note that the same construction proves that the nerve of $\mathcal{C}^{\mathrm{cf}}_M$ is connected if M is fibrant.

To prove that the nerves in question are simply connected, one can pass to groupoid completions and calculate the automorphism group of any object of the obtained groupoid. A standard reasoning shows that an acyclic fibration $f : P \to M$ with cofibrant P has no nontrivial automorphisms in the groupoid completion.

Step B. Choose an acyclic fibration $f : P \to M$ with cofibrant P. This defines a functor

$$F : \mathcal{C}^{\mathrm{c}}_P \to \mathcal{C}^{\mathrm{c}}_M$$

which carries a weak equivalence $g : Q \to P$ to $fg : Q \to M$. Fix an object $g : R \to M$ of $\mathcal{C}^{\mathrm{c}}(M)$. The fibre category F/g can be easily identified with the category $\mathcal{C}^{\mathrm{c}}_{P \times_M R}$. According to Step A, all fibres of F are simply connected. Note that the nerve of $\mathcal{C}^{\mathrm{c}}(P)$ is contractible since the category admits a final object.

Step C. We have to check that the reduced homology of the nerve of $\mathcal{C}^{\mathrm{c}}_M$ vanishes. The construction of Step B allows to prove this by induction. In fact, let $\widetilde{H}_i(\mathcal{C}^{\mathrm{c}}_M) = 0$ for all $i < n$ and for all objects M. Then Proposition A.3.3 below shows that $\widetilde{H}_n(\mathcal{C}^{\mathrm{c}}_M) = 0$. This proves the theorem modulo Proposition A.3.3 below. $\square$

The following result is very much in the spirit of [Q2], Theorems A and B.

A.3.3. **Proposition.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose that*

(a) the nerve of $\mathcal{C}$ is contractible.

(b) $\widetilde{H}_i(F/d) = 0$ for all $i < n$, $d \in \mathcal{D}$.

Then $\widetilde{H}_n(\mathcal{D}) = 0$.

Proof. Consider the bisimplicial set $T_{\bullet\bullet}$ (used by Quillen in the proof of Theorem A, cf. [Q2], p. 95) defined by the formula

$$T_{pq} = \{c_q \to \dots c_0;\ F(c_0) \to d_0 \to \dots \to d_p\},$$

with the obvious faces and degeneracy maps.

The diagonal of this bisimplicial set, diag T, is homotopy equivalent to the nerve of $\mathcal{C}$ (see [Q2], p. 95). Therefore, it has trivial homology.

Recall that the homology of a simplicial set X can be calculated as follows. First, one considers $\mathbb{Z}X \in \Delta^{\mathrm{op}}\mathtt{Ab}$. Then one applies Dold-Puppe equivalence of categories

$$\mathrm{Norm} : \Delta^{\mathrm{op}}\mathtt{Ab} \to C^{\leq 0}(\mathbb{Z}).$$

Finally, one has $H_i(X) = H^{-i}(\mathrm{Norm}(\mathbb{Z}X))$.

If $Y \in (\Delta^{\mathrm{op}})^2\mathtt{Ab}$, we denote $\mathrm{Norm}^2(Y)$ the bicomplex obtained from Y by normalization in both directions.

Lemma.

$$\mathrm{Norm}(\mathrm{diag}(Y)) = \mathrm{Tot}(\mathrm{Norm}^2(Y)).$$

The lemma is similar to Quillen's lemma at p. 94, [Q2]. One checks it for representable $Y = \mathbb{Z}h^{pq}$ where $h^{pq}_{rs} = \Delta^p_r \times \Delta^q_s$, and then checks that both sides of the equality commute with the direct limits.

Now consider the map

$$T_{pq} = \coprod_{d_0 \to \ldots \to d_p} \mathcal{N}(F/d_0) \to \coprod_{d_0 \to \ldots \to d_p} \mathrm{pt} = \mathcal{N}_p(\mathcal{D}).$$

Let Z be the bicomplex corresponding to the bisimplicial abelian group $\mathbb{Z}T_{\bullet\bullet}$. Denote by H^{vert} and H^{hor} the homology with respect to the vertical (of degree $(0,1)$) and the horizontal (of degree $(1,0)$) differential. By the assumptions of the proposition, one has $H^{\mathrm{vert}}_q(Z) = 0$ for $q = 1, \ldots, n-1$ and $H^{\mathrm{vert}}_0(Z) = \mathrm{Norm}_p(\mathcal{N}(\mathcal{D}))$. Look at the spectral sequence

$$E^2_{pq} = H^{\mathrm{hor}}_p H^{\mathrm{vert}}_q(Z) \Rightarrow H_n(\mathrm{Tot}(Z)).$$

According to the above lemma, the spectral sequence converges to zero. Our calculation shows that $E^2_{pq} = 0$ for $q = 1, \ldots, n-1$ and $E^2_{p0} = H_p(\mathcal{N}(\mathcal{D}))$. This implies Proposition A.3.3 since the map $H_n(\mathcal{N}(\mathcal{D})) = E^2_{n0} \to E^\infty_n$ should be injective. $\square$

A.3.4. **Proposition.** *(i) The weak groupoids $\widehat{\mathcal{W}}$, $\widehat{\mathcal{W}^{\mathrm{c}}}$, $\widehat{\mathcal{W}^{\mathrm{f}}}$ and $\widehat{\mathcal{W}^{\mathrm{cf}}}$ are equivalent.*

(ii) Suppose $\mathcal{C}$ admits a simplicial structure and suppose that (SM7) and half the axiom (SM0) of [Q1]*, (existence of simplicial cylinder or path spaces), is fulfilled. Then the weak groupoids $\mathcal{W}^{\mathrm{cf}}_*$ and $\widehat{W}_*$ are also equivalent to the above.*

Proof. (i) According to A.2.3 it is enough to prove the nerves of the four categories are weakly homotopically equivalent. This follows from Theorem A.3.2 by Quillen's Theorem A, see [Q2].

(ii) Since $\mathcal{W}^{\mathrm{cf}}_*$ is a weak groupoid, it is equivalent to its weak groupoid completion. In order to prove that the latter is equivalent to the weak groupoid completion of $\mathcal{W}^{\mathrm{cf}}$, one has to compare the (simplicial) nerves of $\mathcal{W}^{\mathrm{cf}}_*$ and of $\mathcal{W}^{\mathrm{cf}} = \mathcal{W}^{\mathrm{cf}}_0$. This will immediately follow once we prove that the multiple degeneracy $s:\mathcal{W}_0^{\rm cf}\to\mathcal{W}_n^{\rm cf}$ induces a weak equivalence $\mathcal{N}(s):\mathcal{N}(\mathcal{W}_0^{\rm cf})\to\mathcal{N}(\mathcal{W}_n^{\rm cf})$.

Let us suppose that the simplicial path functor exists. Then $\mathrm{Hom}_n(x,y)=\mathrm{Hom}(x,y^{\Delta^n})$. Therefore, the fibre s/y is equivalent to the category of fibrant cofibrant resolutions of the object y^{Δ^n}. This object being fibrant, Theorem A.3.2 asserts that s/y is contractible. Once more Quillen's Theorem A accomplishes the proof. The case when the simplicial cylinder functor exists, as well as the weak groupoid $\widehat{W}_*$ are treated similarly. $\square$

A.4. **Simplicial Deligne groupoid.** In this paper weak groupoids appear as values of a formal deformation functor on artinian algebras. One looks for a presentation of such a functor with a dg Lie algebra. We recall below three functors assigned to a dg Lie algebra, of which the last one is used in this paper.

A.4.1. *Deligne groupoid* (see [GM]). Let $\mathfrak{g}$ be a Lie dg algebra over a field k of characteristic zero and $(R,\mathfrak{m})\in\mathtt{dgart}^{\leq 0}(k)$. The Deligne groupoid $\mathrm{Del}_{\mathfrak{g}}(R)$ has as objects the Maurer-Cartan elements of $\mathfrak{m}\otimes\mathfrak{g}$,

$$\mathrm{Ob}\,\mathrm{Del}_{\mathfrak{g}}(R)=\mathrm{MC}(\mathfrak{m}\otimes\mathfrak{g}):=\{z\in(\mathfrak{m}\otimes\mathfrak{g})^1|dz+\frac{1}{2}[z,z]=0\}.$$

The group $\exp(\mathfrak{m}\otimes\mathfrak{g})^0$ acts in a natural way on the set $\mathrm{MC}(\mathfrak{m}\otimes\mathfrak{g})$ and one defines

$$\mathrm{Hom}_{\mathrm{Del}_{\mathfrak{g}}(R)}(z,z')=\{\gamma\in\exp(\mathfrak{m}\otimes\mathfrak{g})^0|z'=\gamma(z)\}.$$

This definition is homotopy invariant if one requires $(\mathfrak{m}\otimes\mathfrak{g})^i=0$ for $i<0$.

A.4.2. *Nerve of a dg Lie algebra* (see [H1, H3]). Let $\mathfrak{g}$ and R be as above. Define for $n\geq 0$

$$\Sigma_{\mathfrak{g}}(R)_n=\mathrm{MC}(\Omega_n\otimes\mathfrak{m}\otimes\mathfrak{g})$$

where Ω_n is the algebra of polynomial differential forms on the standard n-simplex. Then $\Sigma_{\mathfrak{g}}(R)$ is a Kan simplicial set, its fundamental groupoid is canonically identified with $\mathrm{Del}_{\mathfrak{g}}(R)$ and the two are homotopically equivalent if $\mathfrak{m}\otimes\mathfrak{g}$ is non-negatively graded.

A.4.3. *Simplicial Deligne groupoid* (see [H4]). Here $\mathfrak{g}$, R are as before. Define simplicial groupoid $\mathcal{D}el_{\mathfrak{g}}(R)$ as follows. Its objects are the objects of $\mathrm{Del}_{\mathfrak{g}}(R)$. The collection of n-morphisms from z to z' coincides with the collection of such morphisms in the Deligne groupoid corresponding to the Lie dg algebra $\Omega_n\otimes\mathfrak{g}$. Here we identify $\mathfrak{g}$ with the subset $1\otimes\mathfrak{g}$ of $\Omega_n\otimes\mathfrak{g}$.

The following lemma connects between the different constructions.

A.4.4. **Lemma.** *1. There is a natural weak equivalence of simplicial sets*

$$\mathcal{N}(\mathcal{D}el_{\mathfrak{g}}(R)) \xrightarrow{\sim} \Sigma_{\mathfrak{g}}(R).$$

2. The Deligne groupoid $\mathrm{Del}_{\mathfrak{g}}$ *naturally identifies with the fundamental groupoid* $\Pi(\Sigma_{\mathfrak{g}})$ *of the nerve* $\Sigma_{\mathfrak{g}}$*. The composition*

$$\Sigma_{\mathfrak{g}}(R) \to \Pi(\Sigma_{\mathfrak{g}}(R)) = \mathrm{Del}_{\mathfrak{g}}(R)$$

is a weak equivalence if $\mathfrak{m} \otimes \mathfrak{g}$ *is non-negatively graded.* Part 1 of Lemma A.4.4 is proven in [H4], 3.2.1; part 2 is [H1], 2.2.3.

A.4.5. *Descent.* All three functors mentioned above are defined by a nilpotent dg Lie algebra $\mathfrak{g}_R := \mathfrak{m} \otimes \mathfrak{g}$: one has $\mathrm{Del}_{\mathfrak{g}}(R) = \mathrm{Del}(\mathfrak{g}_R)$, $\Sigma_{\mathfrak{g}}(R) = \Sigma(\mathfrak{g}_R)$ and $\mathcal{D}el_{\mathfrak{g}}(R) = \mathcal{D}el(\mathfrak{g}_R)$ in an obvious notation. In what follows we will fix once and forever the commutative dg algebra R and we will erase the subscript R from the notation.

Let $\mathfrak{g}^{\bullet}$ be a nilpotent cosimplicial dg Lie algebra. A natural morphism of simplicial sets

$$\Sigma(\mathrm{Tot}(\mathfrak{g}^{\bullet})) \to \mathrm{Tot}(\Sigma(\mathfrak{g}^{\bullet}))$$

can be easily constructed. The main result of [H1] claims that this map is a homotopy equivalence provided $\mathfrak{g}^{\bullet}$ is finitely dimensional in the cosimplicial direction.

In the main body of the paper we need a similar result in the context of simplicial Deligne groupoids. Let us show it easily follows from the result of [H1].

Let us construct a map of simplicial groupoids

$$\mathcal{D}el(\mathrm{Tot}(\mathfrak{g}^{\bullet})) \to \mathrm{Tot}(\mathcal{D}el(\mathfrak{g}^{\bullet})). \tag{22}$$

On the level of objects the map is constructed as follows. An object of the left-hand side is an element of

$$\varprojlim_{p\to q} \mathrm{MC}(\Omega_p \otimes \mathfrak{g}^q).$$

An element of $\mathrm{MC}(\Omega_p \otimes \mathfrak{g}^q) = \Sigma_p(\mathfrak{g}^q)$ gives rise to a sequence of p 1-simplices in $\Sigma(\mathfrak{g}^q)$; passing to the fundamental groupoid we get a p-simplex in the groupoid $\mathrm{Del}(\mathfrak{g}^q)$. Since an object of the right-hand side of (22) is an element of

$$\varprojlim_{p\to q} \mathrm{Ob}\ \mathrm{Del}(\mathfrak{g}^q)^{\Delta^p},$$

the morphism (22) is defined on the level of objects.

Fix $z, z' \in \mathrm{MC}(\mathrm{Tot}(\mathfrak{g}^{\bullet}))$. A n-map from z to z' on the left-hand side of (22) is given by an element $\gamma \in \exp(\Omega_n \otimes \mathrm{Tot}(\mathfrak{g}^{\bullet}))^0$ satisfying the equation $z' = \gamma(z)$.

The composition

$$\exp(\Omega_n \otimes \mathrm{Tot}(\mathfrak{g}^{\bullet}))^0 \to \exp(\mathrm{Tot}(\Omega_n \otimes \mathfrak{g}^{\bullet})) = \varprojlim_{p\to q} \exp(\Omega_p \otimes \Omega_n \otimes \mathfrak{g}^q)^0 \to$$
$$\to \varprojlim_{p\to q} \mathcal{N}_p(\mathrm{Del}(\Omega_n \otimes \mathfrak{g}^q))$$

defines the map (22) for the n-morphisms.

Proposition. *Suppose $\mathfrak{g}^\bullet$ is finite dimensional in the cosimplicial direction, i.e. there exists n such that the intersection of kernels of all codegeneracies vanishes in degrees $> n$. Then the morphism (22) defined above is an equivalence.*

To prove a map of simplicial groupoids is an equivalence, it is enough to check it induces a homotopy equivalence of the nerves. Applying the nerve functor to the both sides of (22), we get the morphism (A.4.5) which is an equivalence by [H1]. This proves the proposition.

Appendix B. Simplicial presheaves

The material of this appendix is not formally used in the main body of the paper. It is, however, "ideologically connected" to Section 1 where a CMC structure on complexes of presheaves is given.

We present here a CMC structure on the category of simplicial presheaves on an arbitrary site.

This model category structure differs from the one defined in [Ja]. More precisely, weak equivalences are the same; we have much less cofibrations and, consequently, much more fibrations.

B.1. Let X be a site. Since the category $\Delta^{\mathrm{op}}\mathtt{Ens}$ of simplicial sets is cofibrantly generated, the category of simplicial presheaves $\Delta^{\mathrm{op}}(X\hat{\ })$ admits a CMC structure in which cofibrations are generated by cofibration in $\Delta^{\mathrm{op}}\mathtt{Ens}$, see [Hir], 11.6.1.

In this model structure a map $f : A \to B$ is a weak equivalence (resp., a fibration) iff for each $U \in X$ $f(U) : A(U) \to B(U)$ is a weak equivalence (resp., a fibration). The collection of cofibration is generated by gluing simplices along a boundary over some $U \in X$.

This is the model structure we have in mind in the case X is endowed with the coarse topology. In order to describe the model structure on $\Delta^{\mathrm{op}}(X\hat{\ })$ which remembers the topology of X, we redefine the notion of weak equivalence as in [Ja], leaving the notion of cofibration unchanged. Recall the following definition due to Joyal [Jo] and Jardine, [Ja].

For $A \in \Delta^{\mathrm{op}}X\hat{\ }$ one defines $\pi_0(A)$ as the sheafification of the presheaf $U \mapsto \pi_0(A(U))$. Similarly, for $n > 0$ and $a \in A(U)_0$ one defines $\pi_n(A, a)$ as the sheafification of the presheaf $V \mapsto \pi_n(A(V); a)$ defined on X/U.

B.2. **Definition.** A map of simplicial presheaves $f : A \to B$ is a weak equivalence if

- $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is an isomorphism;
- for each $a \in A(U)_0$ the map $\pi_n(f) : \pi_n(A; a) \to \pi_n(B; f(a))$ is an isomorphism.

B.3. **Theorem.** *The category $\Delta^{\mathrm{op}}(X\hat{\ })$ endowed with the classes of weak equivalences described above and cofibrations as for the coarse topology, is a closed model category.*

A nice feature of this model structure is the following description of fibrations (see Proposition B.3.12 below).

Proposition. *A map $f : M \to N \in \Delta^{\mathrm{op}}(X^\frown)$ is a fibration if and only if the following conditions are satisfied.*

- *$f(U) : M(U) \to N(U)$ is a Kan fibration for each $U \in X$.*
- *For each hypercover $\epsilon : V_\bullet \to U$ the commutative diagram*

$$\begin{tikzcd} M(U) \arrow[r] \arrow[d] & \check{C}(V_\bullet, M) := \operatorname{Tot} M(V_\bullet) \arrow[d] \\ N(U) \arrow[r] & \check{C}(V_\bullet, N) \end{tikzcd}$$

  is homotopy cartesian.

Here Tot $M(V_\bullet)$ *is defined by the usual formula*

$$\operatorname{Tot}\ M(V_\bullet) = \varprojlim_{p\to q} M(V_q)^{\Delta^p}.$$

B.3.1. *Remark.* The Cech complex appearing in the description of fibrations, is not homotopy invariant, even for objectwise fibrant presheaves. This means that that fibrantness is not necessarily preserved under objectwise weak equivalence of objectwise fibrant presheaves.

An objectwise fibrant presheaf F is called to satisfy *descent property* with respect to a hypercover $V_\bullet \to U$ if $F(U)$ is homotopy equivalent to $\operatorname{holim} F(V_\bullet)$.

In general, our fibrant presheaves do not satisfy the descent property. However, assume $V_\bullet$ is *split*, see [SGA4], Éxp. Vbis, 5.1.1. This means that for each n the map $D_n \to V_n$ from the subpresheaf of degenerate n-simplices to the presheaf of all n-simplices, can be presented as a composition

$$D_n \to D_n \sqcup N_n \xrightarrow{\sim} V_n.$$

Then, for each acyclic cofibration $S \to T$ of simplicial sets the induced map of simplicial presheaves

$$(S \times V_n) \coprod^{S\times D_n} (T \times D_n) \to T \times V_n$$

is an acyclic cofibration. This implies that if F is a fibrant presheaf and $V_\bullet$ is a split hypercover, then $F(V_\bullet)$ is a fibrant cosimplicial simplicial set in the sense of [BK], X.4. This implies that $\operatorname{Tot} F(V_\bullet)$ is homotopy equivalent to $\operatorname{holim} F(V_\bullet)$ and, therefore, F satisfies the descent property with respect to $V_\bullet$. This implies that if any hypercover in X

is refined by a split hypercover, our model structure coincides with the one named $U\mathcal{C}/S$ in [DHI], Theorem 1.3.

B.3.2. To prove Theorem B.3, we describe a collection of morphisms which are simultaneously weak equivalences and cofibrations. These morphisms are called *generating acyclic cofibrations*. Theorem B.3 then follows from Lemma B.3.10 below claiming that weak equivalences satisfying the RLP with respect to all generating acyclic cofibrations, are objectwise acyclic Kan fibrations.

B.3.3. The following version of [SGA4], V.7.3.2, plays a very important role here.

Let E be a topos, $M, N \in \Delta^{\mathrm{op}}E$. We will use the notion of weak equivalence in $\Delta^{\mathrm{op}}E$ defined in B.2.

Proposition. *Let $f : M \to N$ be a morphism of simplicial objects in E. Suppose that*

- *f_p is an isomorphism for $p < n$.*
- *f_n is an epimorphism.*
- *morphisms $M \to \mathrm{cosk}_n(M)$, $N \to \mathrm{cosk}_n(N)$, are isomorphisms.*

Then f is a weak equivalence.

Proof. The proof is essentially the same as in [SGA4], V.7.3.2. Embed E into a topos E' having enough points. If $E = X^{\sim}$, E' can be taken to be $X^{\frown}$. Let

$$a^* : E' \rightleftarrows E : a_*$$

be the corresponding inverse and direct image functors. We construct $f' : M' \to N'$ as follows.

$$\begin{aligned} M' &= a_*(M). \\ N'_i &= a_*(N_i) \text{ for } i < n;\ N'_n = a_*(f)(M'_n);\ N' = \mathrm{cosk}_n(N'). \\ f' &= a_*(f) : M' \to N'. \end{aligned}$$

Then $f = a^*(f')$.

The functor a^* preserves weak equivalences. In fact, weak equivalences can be described using finite limits and arbitrary colimits which are preserved by the inverse image functor.

This reduces the claim to the case E has enough points. In this case f is a weak equivalence iff for each point x f_x is a weak equivalence. Inverse image functor preserves the properties listed in the proposition. Thus everything is reduced to the case $E = \mathtt{Ens}$ where the claim is well-known.

□

B.3.4. We denote by Δ_+ the category obtained from Δ by attaching an initial object $\emptyset = [-1]$. In particular, any hypercover $\epsilon : V_\bullet \to U$ defines an object $V_\circ \in \Delta_+^{\mathrm{op}} X\hat{}$ with $V_{-1} = U$. In what follows we will use the subscript (resp., the superscript) $\circ$ to denote the augmented simplicial (resp., cosimplicial) object.

Let $A_\bullet^\circ \in \Delta_+(\Delta^{\mathrm{op}}\mathtt{Ens})$ and let $V_\circ \in \Delta_+^{\mathrm{op}} X\hat{}$.

We define a simplicial presheaf $A_\bullet^\circ \otimes V_\circ$ by the formula

$$(A \otimes V)_n = \lim_{\rightarrow}{}_{p\to q\in\Delta_+} A_n^p \times V_q,$$

where the direct limit is taken along the category whose objects are morphisms $\alpha : p \to q$ in Δ_+, the morphisms from $\alpha_1 : p_1 \to q_1$ to $\alpha_2 : p_2 \to q_2$ being given by the commutative diagrams

$$\begin{tikzcd}
p_2 \arrow[r, "\alpha_2"] & q_2 \arrow[d] \\
p_1 \arrow[u] \arrow[r, "\alpha_1"] & q_1
\end{tikzcd}$$

B.3.5. Let us make a few calculations. Let, for instance, $A_n^p = K_n$ for some $K \in \Delta^{\mathrm{op}}\mathtt{Ens}$ for all $p \geq -1$. Then $A \otimes V = K \times V_{-1}$.

B.3.6. Another important example is given by the formulas

$$A_n^p = K_n \times \Delta_n^p,$$

where Δ^p is the standard p-simplex for $p \geq 0$ and $\emptyset$ for $p = -1$. One can easily see that $\Delta_\bullet^\circ \otimes V_\circ = V_\bullet$. Also, if $K \in \Delta^{\mathrm{op}}\mathtt{Ens}$ then for any $A_\bullet^\circ$ one has

$$(K \times A_\bullet^\circ) \otimes V_\circ = K \times (A_\bullet^\circ \otimes V_\circ).$$

Thus we finally get

$$(K \times \Delta_\bullet^\circ) \otimes V_\circ = K \times V_\bullet,$$

and an easy calculation shows that for a simplicial presheaf M one has

$$\mathrm{Hom}(A_\bullet^\circ \otimes V_\circ, M) = \mathrm{Hom}(K, \check{C}(V_\bullet, M)).$$

B.3.7. Generating acyclic cofibrations for our model structure consist of two collections.

The first collection is labelled by $U \in X$ and a pair of integers (i, n) such that $0 \leq i \leq n$. It consists of maps

$$\Lambda_i^n \times U \to \Delta^n \times U.$$

This collection defines the model category structure on the simplicial presheaves on X corresponding to the coarse topology.

The second collection is labelled by hypercovers $V_\circ$ in X and integers $n \geq 0$. It is defined as follows.

Let Δ^+ (don't confuse with Δ_+!) denote the cosimplicial simplicial space with

$$\Delta^{+n} = \Delta^{n+1}$$
$$\delta^{+i} = \delta^i$$
$$\sigma^{+i} = \sigma^i.$$

The map $i : \Delta \to \Delta^+$ is defined by the "last face":

$$i^n = \delta^{n+1} : \Delta^n \to \Delta^{n+1}.$$

Define $A(n)$ by the cocartesian diagram

$$\begin{tikzcd} \partial\Delta^n \times \Delta \arrow[r, "i"] \arrow[d] & \partial\Delta^n \times \Delta^+ \arrow[d] \\ \Delta^n \times \Delta \arrow[r] & A(n) \end{tikzcd} \tag{23}$$

and put $B(n) = \Delta^n \times \Delta^+$. Then the natural map $A(n) \otimes V \to B(n) \otimes V$ is the basic acyclic cofibration corresponding to the pair (V, n).

B.3.8. The object $\Delta^+ \in \Delta_+\Delta^{\mathrm{op}}\mathtt{Ens}$ identifies with the pushout of the diagram

$$\begin{tikzcd} \Delta \arrow[r, "\delta^0 \times \mathrm{id}_\Delta"] \arrow[d] & \Delta^1 \otimes \Delta \arrow[d] \\ * \arrow[r] & \Delta^+ \end{tikzcd}$$

Under this identification the map $i : \Delta \to \Delta^+$ is induced by

$$\delta^1 \times \mathrm{id}_\Delta : \Delta \to \Delta^1 \times \Delta.$$

Therefore, $\Delta^+ \otimes V_\circ$ can be calculated from the cocartesian diagram

$$\begin{tikzcd} V_\bullet \arrow[r, "\delta^0 \times \mathrm{id}_{V_\bullet}"] \arrow[d] & \Delta^1 \times V_\bullet \arrow[d] \\ V_{-1} \arrow[r, "v"] & \Delta^+ \otimes V_\circ \end{tikzcd}$$

Lemma. *The morphism $v : V^{-1} \to \Delta^+ \otimes V_\circ$ is a weak equivalence.*

Proof. The simplicial set Δ^1 being contractible, the map $\delta^0 \times \mathrm{id}_{V_\bullet}$ is a weak equivalence. It is also objectwise injective. This means that it is a trivial cofibration in the sense of Jardine [Ja]. Then Proposition 2.2 of [Ja] asserts that v is a weak equivalence. □

B.3.9. **Lemma.** *Let $V_\circ$ be a hypercover and $n \in \mathbb{N}$. The map $A(n) \otimes V_\circ \to B(n) \otimes V_\circ$ of simplicial presheaves is a cofibration and a weak equivalence.*

Proof. The map is cofibration since V_i are coproducts of representable presheaves. Let us check the acyclicity.

Tensoring diagram (23) by $V_\circ$ and using Proposition 2.2 of [Ja] we get that $A(n) \otimes V_\circ$ is equivalent to $V_\bullet$. The tensor product $B(n) \otimes V_\circ$ is obviously equivalent to V_{-1}.

By Proposition B.3.3 the map $\epsilon : V_\bullet \to V_{-1}$ is a weak equivalence. This implies the lemma. □

The following lemma is the analog of Lemma 1.3.6.

B.3.10. **Lemma.** *Let $f : M \to N$ be a weak equivalence. Suppose f satisfies the RLP with respect to all generating acyclic cofibrations. Then $f(U) : M(U) \to N(U)$ is an acyclic Kan fibration for all $U \in X$.*

Proof. The map $f(U)$ is a fibration for any $U \in X$. Fix U and fix a section of N over U. Let F be the fiber of f at the chosen section. We wish to prove that $F(U)$ is acyclic. To prove this we have to check that any commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & F(U) \\
\Delta^n \arrow[ur, dotted] &
\end{tikzcd}$$

can be completed with a dotted arrow. Below we denote $A(n)$ by A.

Since f is a weak equivalence, there exists a covering $\epsilon : V_0 \to U$ and a dotted arrow $A^0 \to F(V_0)$ making the diagram

$$\begin{tikzcd}
A^{-1} = \partial\Delta^n \arrow[r] \arrow[d] & F(U) \arrow[r] & F(V_0) \\
A^0 \arrow[urr, dotted] & &
\end{tikzcd}$$

commutative.

Suppose, by induction, a k-hypercover $\epsilon : V_{\leq k} \to U$ and a collection of compatible maps $A^i \to F(V_i)$, $i \leq k$, has been constructed. This induces a map $\mathrm{sk}^k(A^\bullet)^{k+1} \to F(V_k)$. Since the sheafification of the homotopy groups $\pi_i(F)$ vanishes, there exists a covering $V_{k+1} \to \mathrm{cosk}_n(V)_{n+1}$ and a map $A^{k+1} \to F(V_{k+1})$ compatible with he above.

Therefore, a hypercover $\epsilon : V_\bullet \to U$ and a map $A(n) \otimes V \to F$ is constructed. According to hypothesis of the lemma, this latter can be extended to a map $B(n) \otimes V \to F$ which includes a map $B^{-1} = \Delta^n \to F(U)$. Lemma is proven. $\square$

B.3.11. Let J be the collection of generating acyclic cofibrations. Let, furthermore, $\overline{J}$ denote the collection of maps which can be obtained as a countable direct composition of pushouts of coproducts of maps in J. Fibrations are defined as the maps satisfying RLP with respect to J.

Repeating the reasoning of 1.3.7, we prove this defines a model structure on $\Delta^{\mathrm{op}}X\hat{}$. As in 1.3.7, acyclic cofibrations in this model structure are retracts of elements of $\overline{J}$.

Theorem B.3 is proven.

Note the following description of fibrations.

B.3.12. **Proposition.** *A map $f : M \to N \in \Delta^{\mathrm{op}}(X\hat{})$ is a fibration if and only if the following conditions are satisfied.*

- *$f(U) : M(U) \to N(U)$ is a Kan fibration for each $U \in X$.*
- *For each hypercover $\epsilon : V_\bullet \to U$ the comutative diagram*

$$\begin{tikzcd}
M(U) \arrow[r] \arrow[d] & \check{C}(V_\bullet, M) := \mathrm{Tot}\, M(V_\bullet) \arrow[d] \\
N(U) \arrow[r] & \check{C}(V_\bullet, N)
\end{tikzcd}$$

is homotopy cartesian.

Proof. A map $f : X \to Y$ of Kan simplicial sets is a weak equivalence iff the space $X \times_Y P(Y) \times_Y *$ is acyclic Kan for each point $* \to Y$. Here $P(Y)$ is the path space of Y. The rest of the proof is a direct calculation. $\square$

References

[BK] Bousfield, Kan, *Homotopy limits, completions and localizations,* Lecture notes in mathematics, **304**.

[DHI] D. Dugger, S. Hollander, D. Isaksen, Hypercovers and simplicial presheaves, preprint `math.AT/0205027`

[DHK] W. Dwyer, P. Hirschhorn, D. Kan, Model categories and more general abstract homotopy theory, preprint available at `www-math.mit.edu/~psh/#Mom`.

[DK1] W. Dwyer, D. Kan, Simplicial localizations of categories, J. Pure Appl. Algebra **17**(1980), 267–284.

[DK2] W. Dwyer, D. Kan, Calculating simplicial localizations, J. Pure Appl. Algebra **18**(1980), 17–35.

[DK3] W. Dwyer, D. Kan, Function complexes in homotopical algebra, Topology **19**(1980), 427–440.

[F] Y. Felix, Classification homotopique des espaces rationnels de cohomologie donnée, Bull. Soc. Math. Belg. Sér. B 31 (1979), no. 1, 75–86.

[G] D. Gaitsgory, Grothendieck topologies and deformation theory, II, Compositio Math., **106**(1997), 321–348.

[GS] M. Gerstenhaber, Schack, Algebraic cohomology and deformatioin theory, *Deformation theory of algebras and structures and applications*, 11–264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 247, Kluwer, 1988.

[GM] W. Goldman, J. Millson, The deformation theory of representations of fundamental groups of compact Kähler manifolds, Publ. Math. IHES, **67** (1988), 43–96.

[HaSt] S. Halperin, J. Stasheff, Obstructions to homotopy equivalences, Adv. in Math. 32 (1979), no. 3, 233–279.

[H1] V. Hinich, Descent of Deligne groupoids, Intern. Math. Res. Notices, 1997, 223-239.

[H2] V. Hinich, Homological algebra of homotopy algebras, Comm. in Algebra, **25(10)**(1997), 3291–3323.

[H3] V. Hinich, DG coalgebras as formal stacks, J. Pure Appl. Algebra, **162**(2001), no. 2-3, 209–250.

[H4] V. Hinich, Deformations of homotopy algebras, preprint `math.AG/9904145`, to appear in Communications in Algebra.

[H5] V. Hinich, Virtual operad algebras and realization of homotopy types, J. Pure Appl. Algebra, **159**(2001), 173–185.

[HS] V. Hinich, V. Schechtman, On homotopy limit of homotopy algebras, Lecture Notes in mathematics, **1289** (1987), 240–264.

[HiSi] A. Hirschowitz, C. Simpson, Descente pour les n-champs, preprint `math.AG/9807049`

[Hir] P. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, AMS, Providence 2003.

[I] L. Illusie, Complexe cotangent et déformations, Lecture notes in mathematics, **239**(1971) and **283**(1972).

[Ja] J. F. Jardine, Simplicial presheaves, J. Pure Appl. Algebra, **47**(1987), 35–87.

[Jo] A. Joyal, lettre à Grothendieck, le 11/4/84.

[La] Laudal, Formal moduli of algebraic structures, Lecture notes in mathematics, **754**, 1979.

[Lu] V. Lunts, Deformations of quasi-coherent sheaves of algebras, J. Algebra **259**(2003), 59–86.

[May] J.P. May, The geometry of iterated loop spaces, Lecture Notes in Mathematics, 271, Springer, 1972.

[Q1] D. Quillen, Homotopy algebra, Lecture notes in mathematics., **43**(1967).

[Q2] D. Quillen, Higher algebraic K-theory, I, Lecture notes in mathematics., **341**(1973), 85–147.
[Sch] M. Schlessinger, Functors on Artin rings, Trans. A.M.S., **130**(1968), 208–222.
[SchSt] M. Schlessinger, J. Stasheff, Deformation theory and rational homotopy type, preprint.
[SGA1] Revêtements étales et groupe fondamental, Séminaire de la géométrie algébrique du Bois Marie, SGA1, dirigé par A. Grothendieck, Lecture notes in mathematics, **224**.
[SGA4] Théorie des topos et cohomologie étale des schémas, Séminaire de la géométrie algébrique du Bois Marie, 63/64, SGA4, dirigé par M. Artin, A. Grothendieck, J.L. Verdier. Tome 1: Lecture notes in mathematics, **269**; Tome 2: Lecture notes in mathematics, **270**(1972).
[Sp] M. Spaltenstein, Resolutions of unbounded complexes. Compositio Math. 65 (1988), no. 2, 121–154.
[T] B. Toen, Dualité de Tannaka supérieure I: Structures monoïdales, Preprint MPIM 2000 (57).
[TV] B. Toen, G. Vezzosi, Homotopical algebraic geometry, 1: Topos theory, preprint `math.AG/0207028`, to appear in Advances in Mathematics.
[Y] A. Yekutieli, The continuous Hochschild complex of a scheme, Canad. J. Math., **54**(6), 2002, 1319–1337.

Department of Mathematics, University of Haifa, Mount Carmel, Haifa 31905, Israel

E-mail address: `hinich@math.haifa.ac.il`