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James A. Flaten and Kevin A. Parendo

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Pendulum waves: A lesson in aliasing

James A. Flaten[a)] and Kevin A. Parendo
Division of Science and Mathematics, University of Minnesota–Morris, Morris, Minnesota 56267

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A set of uncoupled pendula may be used to exhibit ''pendulum waves,'' patterns that alternately look like traveling waves, standing waves, and chaos. The pendulum patterns cycle spectacularly in a time that is large compared to the oscillation period of the individual pendula. In this article we derive a continuous function to explain the pendulum patterns using a simple extension to the equation for traveling waves in one dimension. We show that the cycling of the pendulum patterns arises from aliasing of this underlying continuous function, a function that does not cycle in time.

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I. INTRODUCTION

In 1991 Richard Berg from the University of Maryland described in this journal how to construct a set of uncoupled pendula that exhibit ''pendulum waves.''[1] The patterns this demonstration exhibits are very beautiful and the way in which the patterns cycle is nothing short of stunning. This demonstration appears in *The Video Encyclopedia of Physics Demonstrations*[2] and it is also fairly easy to build from scratch. A video clip of our version of this demonstration in action, plus animations of some of the functions discussed in this article, may be viewed at *http://www.mrs.umn.edu/~flatenja/pendulumwaves.shtml*.[3]

The purpose of this article is to discuss how the wave-like patterns formed by the swinging pendula can be described by a simple extension to the standard description of transverse traveling waves in one dimension. Not only is the math quite elegant in its own right, but it is instructive to realize that the recurring patterns seen in the pendula actually arise from aliasing of the underlying continuous function, a function that does not cycle but gets more and more complicated as time elapses.

II. THE DEMONSTRATION

The apparatus consists of a set of equally spaced uncoupled pendula of decreasing lengths, shown schematically in Fig. 1. The lengths of the pendula are tuned so that in the time, Γ, the longest pendulum takes to go through some integer number of full cycles, N, the next-longest pendulum goes through $N+1$ cycles, the next-longest through $N+2$ cycles, and so on. In our classroom demonstration we arbitrarily selected $\Gamma=20\,\mathrm{s}$, $N=20$, and there are 15 pendula in the set. Thus the longest pendulum has a period of $\Gamma/N=20\,\mathrm{s}/20=1\,\mathrm{s}$, the next longest has a period of $20\,\mathrm{s}/21=0.952\,\mathrm{s}$, and so on.

If all the pendula in the set are equally displaced from equilibrium perpendicular to the plane of the apparatus and then released at time $t=0$, their relative phases will continuously drift as they swing because of their different periods. At any moment the phase difference between adjacent pendula is fixed across the entire set, but the value of this phase shift grows as time goes by. This results in sinusoidal wave-like patterns that move up and down the line of pendula. The patterns in the pendula at a few specific times are shown in Fig. 2. For example, at time $t=\Gamma/2$ every pendulum is exactly out of phase with its nearest neighbors [Fig. 2(e)], so the phase shift between adjacent pendula is π rad. At time $t=\Gamma$ the pendula are all back in phase again [Fig. 2(i)], so the phase shift between adjacent pendula is 2π rad, just as they were all in phase at time $t=0$ [Fig. 2(a)], with zero phase shift between adjacent pendula.

Interestingly enough, the pendula set evolves through exactly the same patterns from $t=\Gamma/2$ to Γ as it did from $t=0$ to Γ, only in reverse order [compare Figs. 2(b) and 2(h), 2(c) and 2(g), and 2(d) and 2(f)]. Starting at time $t=0$, traveling wave-like patterns move toward the long-pendulum end of the apparatus. These traveling patterns reverse direction at $t=\Gamma/2$, though this is hard to observe because other distracting patterns are also present near this time. As time approaches $t=\Gamma$, the traveling patterns reappear, now moving toward the short-pendulum end of the apparatus.

III. TRAVELING WAVES (WITH A TWIST)

Let us deduce the continuous mathematical function $y[x,t]$ described by the pendula when viewed from above or below, where y is the displacement (measured perpendicular to the plane of the apparatus) at position x and time t.[4] The patterns look strikingly similar to sinusoidal transverse traveling waves in one dimension, if the slight curvature of the line of pendulum bobs is ignored. Thus we begin with the familiar equation to describe such waves moving in the minus x direction

$$y[x,t]=A\cos[kx+\omega t+\phi]. \tag{1}$$

Here A is the amplitude, $k=(2\pi\,\mathrm{rad})/\lambda$ is the wave number that characterizes the wave repetition in space (using the wavelength λ), and $\omega=2\pi\,\mathrm{rad}/T$ is the angular frequency that characterizes the wave repetition in time (using the period T). If all the pendula are started in phase at the maximum amplitude when $t=0$, the initial phase ϕ will be 0 and henceforth it will be omitted.

Typically when Eq. (1) is used, λ and T are fixed, so k and ω are also constants. However it is clear from Fig. 2 that the wavelength of the pendulum patterns varies with time, so k is actually $k[t]$. Similarly, the apparatus is constructed so that the periods of the pendula vary with their location along the x axis, so ω is really $\omega[x]$. In fact, the time dependence of k is not imposed independently, but is a physical consequence of the tuning of ω with x. Thus the full x and t dependence can be described either by

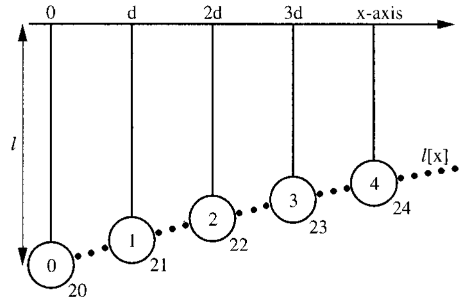


Fig. 1. Equally spaced pendula hanging from the x axis. The pendula swing perpendicular to the plane of the page. Numbers on the pendulum bobs are the pendulum index n. Values below the numbered pendulum bobs indicate how many cycles each pendulum completes during the overall pattern cycling time Γ.

$$y[x,t]=A\cos[k_0 x+\omega[x]t], \quad (2)$$

where the constant k_0 is used to describe the shape of the pattern at $t=0$, or by

$$y[x,t]=A\cos[k[t]x+\omega_0 t], \quad (3)$$

where the constant ω_0 is used to describe the time evolution of the pattern at $x=0$.

For starters, let us pursue the form suggested in Eq. (2). Since $y[x,0]$ equals the maximum amplitude A for all values of x, the constant k_0 is 0. The angular frequency ω of a simple pendulum of length l (in the small angle approximation) is

$$\omega=(g/l)^{1/2} \quad (4)$$

so in this case

$$\omega[x]=(g/l[x])^{1/2}. \quad (5)$$

Here $l[x]$ is the continuous curve along which the pendulum bobs hang below the x axis, shown as a dotted curve in Fig. 1.

In actuality there are pendula only at discrete x values, so we index the pendula with an integer n, using $n=0$ to label the longest pendulum hanging from $x=0$. Since the pendulum spacing is d, the location x_n of the nth pendulum is simply

$$x_n=nd. \quad (6)$$

This pendulum must go through $N+n$ full cycles in time Γ, so its period T_n is given by

$$T_n=\Gamma/(N+n) \quad (7)$$

and its corresponding angular frequency ω_n is

$$\omega_n=(2\pi\,\text{rad})/T_n=(2\pi\,\text{rad})(N+n)/\Gamma. \quad (8)$$

Using Eq. (6) to replace n by x_n/d in Eq. (8), this becomes

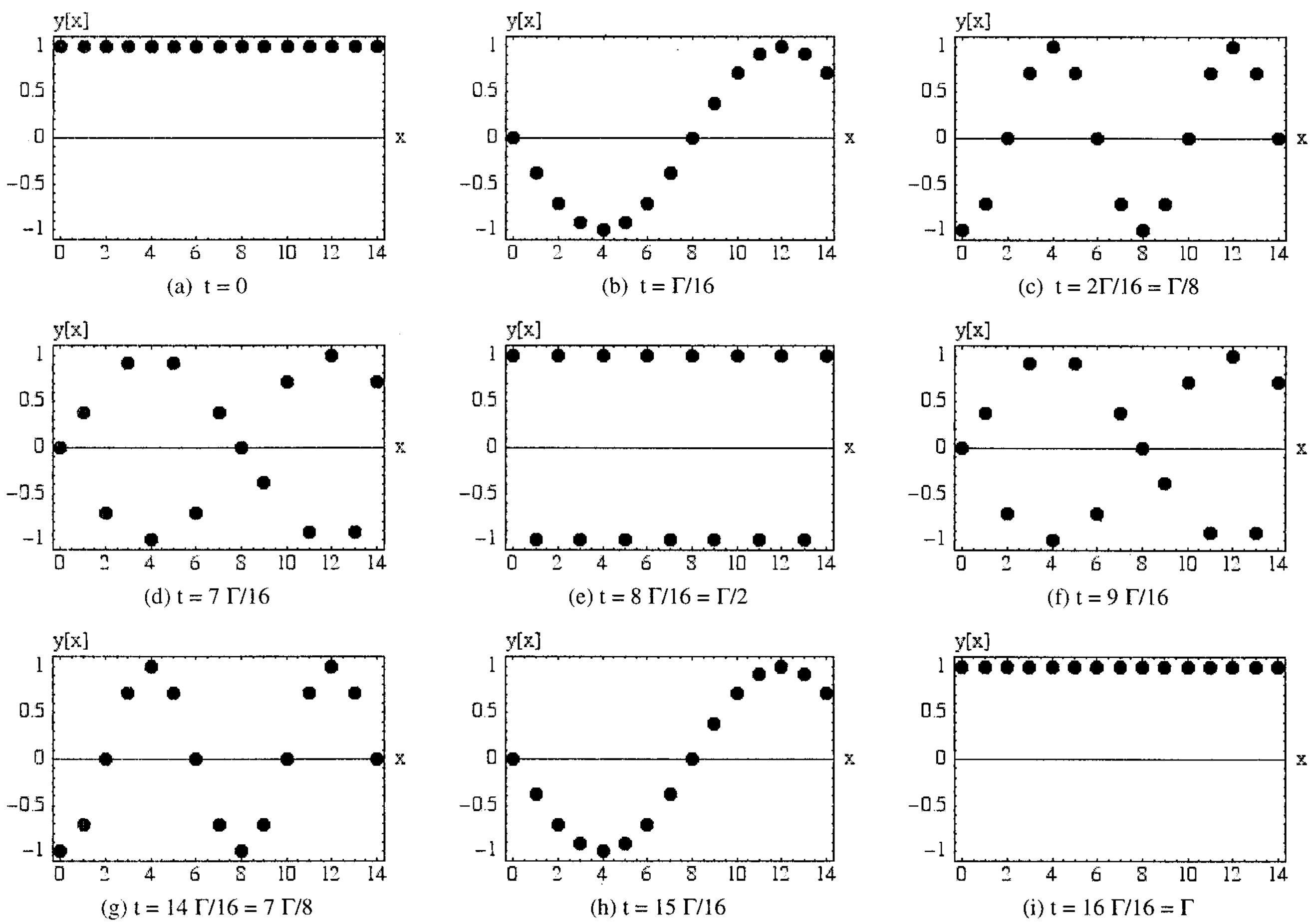


Fig. 2. Displacements of 15 pendula started in phase at $t=0$, shown at nine instants during the cycling time Γ. Here $N=20$, $d=1$, and $A=1$. The series of patterns is symmetric in time with respect to the out-of-phase pattern at $t=\Gamma/2$.

$$\omega_n = (2\pi\,\text{rad})(N + (x_n/d))/\Gamma \tag{9}$$

or, as a continuous function of x,

$$\omega[x] = (2\pi\,\text{rad})(N + (x/d))/\Gamma = (2\pi\,\text{rad})(x + Nd)/(\Gamma d). \tag{10}$$

Now it is easy to show, using Eqs. (5) and (10), that the lengths of the pendula are described by

$$l[x] = g((\Gamma d)/(2\pi\,\text{rad}))^2 (x + Nd)^{-2}. \tag{11}$$

However, keep in mind that the actual apparatus is tuned by carefully timing the pendulum periods rather than by carefully measuring of the pendulum lengths.

Substituting Eq. (10) into Eq. (2) gives this final expression for $y[x,t]$, the continuous function underlying the pendulum patterns,

$$y[x,t] = A\cos[((2\pi\,\text{rad})(x + Nd)/(\Gamma d))t]. \tag{12}$$

Notice that $y[x,t]$ may be rewritten as

$$y[x,t] = A\cos[(2\pi\,\text{rad})(t/(\Gamma d))x + (2\pi\,\text{rad})(N/\Gamma)t], \tag{13}$$

the form suggested in Eq. (3). In this form it is apparent that the wave number k grows linearly with time according to

$$k[t] = (2\pi\,\text{rad}/(\Gamma d))t \tag{14}$$

and the angular frequency ω_0 to be applied at $x=0$ (i.e., at $n=0$) is

$$\omega_0 = (2\pi\,\text{rad})(N/\Gamma), \tag{15}$$

just as it should be according to Eq. (9).

Equation (13) may be visualized as follows. The $\omega_0\, t$ term drives the $x=0$ point (i.e., the bob of the longest pendulum) back and forth with period

$$T_0 = \Gamma/N. \tag{16}$$

At any non-negative time t and for positive values of x, the function $y[x,t]$ is momentarily sinusoidal in space, with wavelength

$$\lambda[t] = (2\pi\,\text{rad})/k[t] = \Gamma d/t. \tag{17}$$

The wavelength is infinite at $t=0$, resulting in a flat pattern, but thereafter λ shrinks as the reciprocal of the elapsed time.

IV. ALIASING

Equation (13) looks promising to describe the pendulum patterns from $t=0$ to $\Gamma/2$ [Figs. 2(a)–2(e)], up to the out-of-phase pattern. However beyond $t=\Gamma/2$, the wavelength in $y[x,t]$ continues to get smaller and smaller, whereas the pendulum patterns appear to reverse their evolution [Figs. 2(e)–2(i)] and get simpler and broader, until the pendula are all back in phase again when $t=\Gamma$.

Figure 3 shows the resolution of this apparent paradox. After $t=\Gamma/2$ there are actually more peaks and valleys in

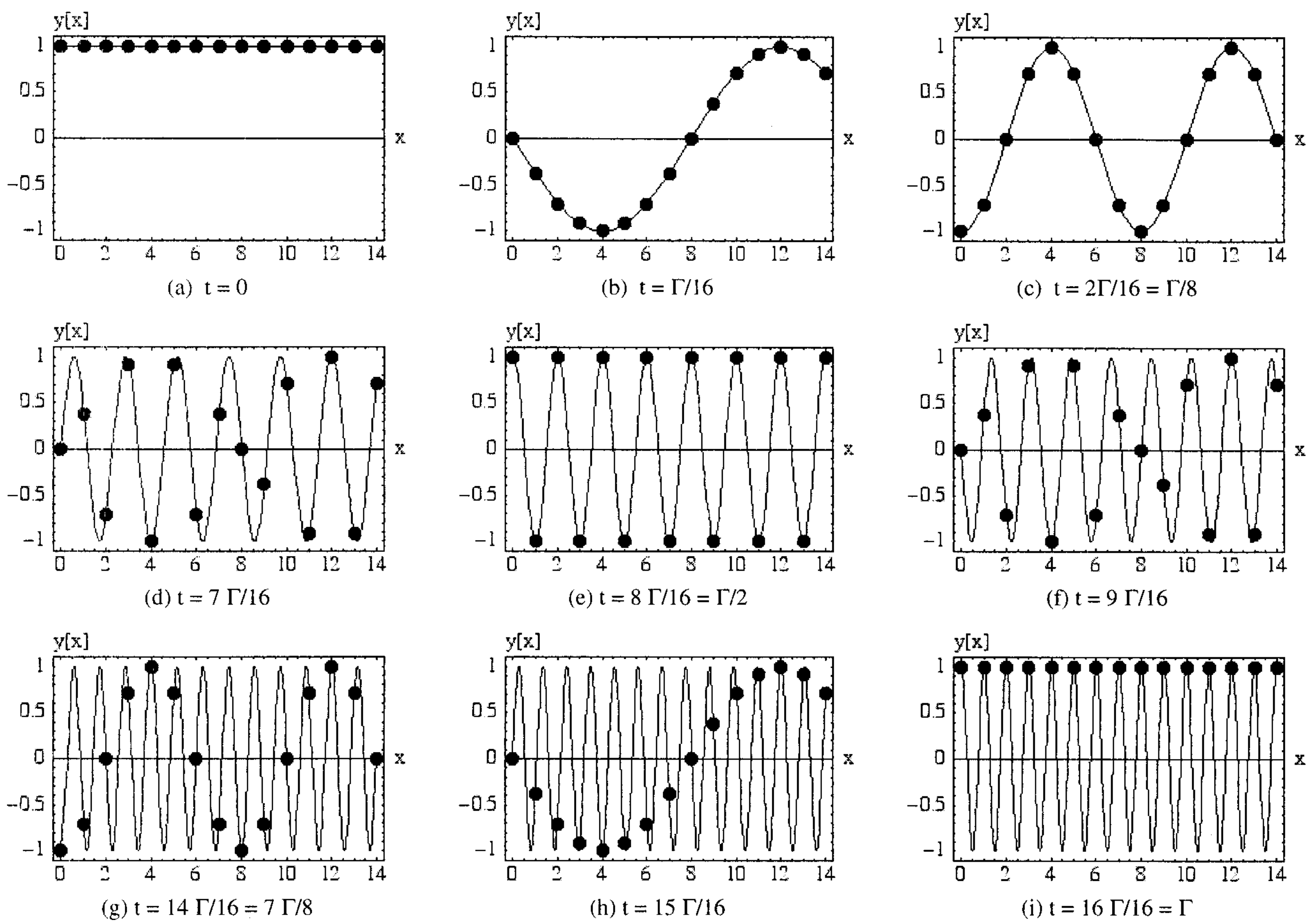


Fig. 3. The same pendulum patterns as in Fig. 2, fit by the continuous function $y[x,t]$ from Eq. (13). For times greater than $t=\Gamma/2$, aliasing results in relatively simple patterns in the pendula despite ever more complexity in the underlying continuous function $y[x,t]$.

$y[x,t]$ than there are pendula in the set, so the pendula cannot possibly display all the complexity of the underlying function. In fact, by the time $t=\Gamma$, the function $y[x,t]$ goes through one full wavelength between adjacent pendula [Fig. 3(i)]!

This is an example of ''aliasing,'' a term used to describe misleading patterns, such as those beyond $t=\Gamma/2$, that arise from the periodic sampling of periodic functions. A more familiar example of aliasing can sometimes be seen in movies or on TV when repetitive motion is viewed. Rotating objects like wagon wheels or helicopter blades can appear to be spinning too slowly, to be stopped, or even to be spinning backward. This optical illusion occurs because of the mismatch between the rate at which the motion cycles and the discrete frame rate of the camera used to film the motion.

Aliasing typically arises when a continuous signal is examined at some discrete sampling rate. A misleading pattern can emerge if the signal changes significantly on a time scale comparable to, or shorter than, the sampling time. The aliasing here is slightly different because there is a discrete sampling distance rather than a discrete sampling time. The pendulum bobs can be observed at all times, but they are located at discrete points along the x axis. The swinging pendula cannot possibly provide information about what the underlying continuous function is doing at points along the x axis between pendulum bobs, and aliasing can result.

Once this aliasing is recognized, an alternative derivation of Eq. (13) presents itself. One can guess the time dependence of λ by realizing that one extra wavelength of $y[x,t]$ must fit in distance d (i.e., must fit between adjacent pendula) for every repetition time Γ. Thus

$$\lambda[t=\Gamma]=d,\quad \lambda[t=2\Gamma]=d/2,\quad \lambda[t=3\Gamma]=d/3,\ldots, \tag{18}$$

which leads to the immediate conclusion that

$$\lambda[t]=d\Gamma/t, \tag{19}$$

as seen earlier in Eq. (17). From this result, $k[t]$ may be found immediately. Finally, ω_0 may be deduced using Eq. (9) with $n=0$, leading quite directly to Eq. (13) from the proposed form of $y[x,t]$ in Eq. (3).

V. CHECKING $y[x,t]$ ANALYTICALLY

Graphing is clearly helpful in visualizing the relationship between the pendulum patterns and the function $y[x,t]$, but all the salient features of the pendulum motion and the aliasing itself may also be extracted analytically from Eq. (13). For example, it is instructive to use Eq. (13) to answer each of the following questions.

Question 1. Does the function $y[x,t]$ at each discrete x_n (i.e., at the location of each pendulum) oscillate sinusoidally in time with the appropriate angular frequency ω_n? That is, does $y[x_n,t]$ equal $A\cos[\omega_n t]$?

Answer 1. Yes! One possible derivation is as follows:

$$y[x_n,t]=A\cos[(2\pi\,\text{rad})(t/(\Gamma d))x_n+(2\pi\,\text{rad})(N/\Gamma)t], \tag{20}$$

$$y[nd,t]=A\cos[(2\pi\,\text{rad})(t/(\Gamma d))nd+(2\pi\,\text{rad})(N/\Gamma)t], \tag{21}$$

$$y[nd,t]=A\cos[(2\pi\,\text{rad})((n+N)/\Gamma)t], \tag{22}$$

$$y[x_n,t]=A\cos[\omega_n t].\quad\text{QED} \tag{23}$$

Actually $y[x,t]$ is not the only continuous function that has this property, and hence it is not the only candidate that may be used to fit the pendula. As discussed in Question 2 below, $y[x,t+m\Gamma]$ also works, where m is any integer, as do other variants like $y[x,-t]$. However it seems most natural that the function used to fit the pendula be as simple as possible (i.e., flat) when $t=0$ and for time t to run forward, so we continue to use the $y[x,t]$ described in Eq. (13) instead of these unnecessarily complicated alternatives.

Question 2. Does the function $y[x,t+m\Gamma]$ at each discrete x_n (i.e., at the location of each pendulum) match the values of $y[x,t]$? Here m is any integer. Of course the function as a whole looks very different every time Γ has elapsed, so this is a check of the Γ periodicity in time of the pendulum patterns (i.e., of the cyclical nature of the aliasing).

Answer 2. Yes! Here is one way to argue this result:

$$y[x_n,t+m\Gamma]=A\cos[(2\pi\,\text{rad})((t+m\Gamma)/(\Gamma d))x_n + (2\pi\,\text{rad})(N/\Gamma)(t+m\Gamma)], \tag{24}$$

$$y[x_n,t+m\Gamma]=A\cos[(2\pi\,\text{rad})(t/(\Gamma d))x_n+(2\pi\,\text{rad}) \times(N/\Gamma)t+(2\pi\,\text{rad})m((x_n/d)+N)], \tag{25}$$

$$y[nd,t+m\Gamma]=A\cos[(2\pi\,\text{rad})(t/(\Gamma d))nd+(2\pi\,\text{rad}) \times(N/\Gamma)t+(2\pi\,\text{rad})m(n+N)], \tag{26}$$

$$y[x_n,t+m\Gamma]=A\cos[(2\pi\,\text{rad})(t/(\Gamma d))x_n + (2\pi\,\text{rad})(N/\Gamma)t]. \tag{27}$$

[*That last step works because* $m(n+N)$ *is an integer, and the cosine function is periodic with respect to adding* $(2\pi$ rad) *times any integer to its argument.*]

$$y[x_n,t+m\Gamma]=y[x_n,t].\quad\text{QED} \tag{28}$$

We think of this as the key property of the aliasing. Although $y[x,t]$ oscillates in less and less space as time goes by, the patterns exhibited by the pendula recur every time Γ has elapsed.

Question 3. Does the function $y[x,(\Gamma/2)+\epsilon]$ at each discrete x_n (i.e., at the location of each pendulum) match $y[x,(\Gamma/2)-\epsilon]$? This checks whether the (aliased) pendulum patterns are indeed symmetric in time with respect to the out-of-phase pattern at $t=\Gamma/2$.

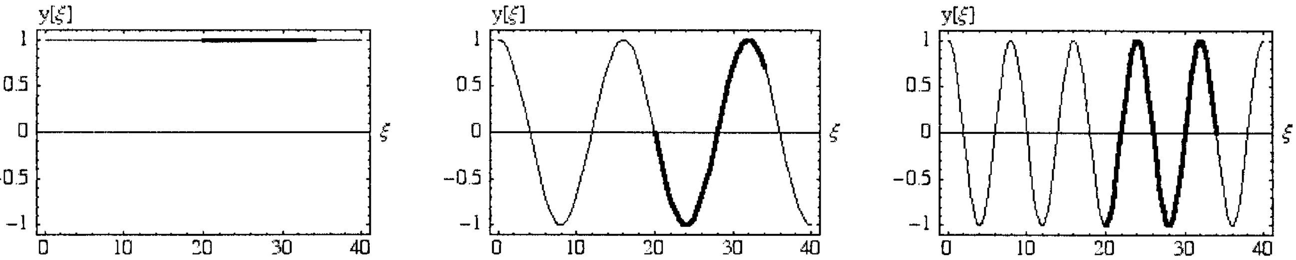
Answer 3. Yes! Consider the following derivation:

$$y[x_n,(\Gamma/2)+\epsilon]=A\cos[(2\pi\,\text{rad})(((\Gamma/2)+\epsilon)/(\Gamma d))x_n + (2\pi\,\text{rad})(N/\Gamma)((\Gamma/2)+\epsilon)], \tag{29}$$

$$y[x_n,(\Gamma/2)+\epsilon]=A\cos[(2\pi\,\text{rad})((x_n/(2d))+(N/2)) + (2\pi\,\text{rad})((\epsilon x_n/(\Gamma d))+(\epsilon N/\Gamma))], \tag{30}$$

$$y[nd,(\Gamma/2)+\epsilon]=A\cos[(2\pi\,\text{rad})((n+N)/2) + (2\pi\,\text{rad})\epsilon(n+N)/\Gamma], \tag{31}$$

$$y[x_n,(\Gamma/2)+\epsilon]=A\cos[(2\pi\,\text{rad})((n+N)/2) + (2\pi\,\text{rad})((n+N)/\Gamma)\epsilon], \tag{32}$$



(a) t = 0 (b) t = Γ/16 (c) t = 2Γ/16 = Γ/8

Fig. 4. A plot of $y[\xi,t]$ from Eq. (39), with $y[x,t]$ from Eq. (13) superimposed in bold, for the first three instants shown in Figs. 2 and 3. The values of x from the physical demonstration extend from $\xi = 20d$ to $\xi = 34d$ [i.e., from $\xi = Nd$ to $\xi = (N + (n_{max} - 1)d)$]. The relation between x and ξ is given in Eq. (37) and n_{max} is the number of pendula in the set.

Similarly,

$$y[x_n,(\Gamma/2)-\epsilon] = A\cos[(2\pi\,\text{rad})((n+N)/2) - (2\pi\,\text{rad})((n+N)/\Gamma)\epsilon]. \quad (33)$$

Now $(n+N)/2$ *is either an integer or a half integer, so*

$$\cos[(2\pi\,\text{rad})(n+N)/2] = \pm\pi. \quad (34)$$

That is to say, this argument corresponds to a peak or a valley in the cosine curve. In Eqs. (32) and (33), the cosine function is called with arguments that are equidistant above and below $(2\pi\,\text{rad})(n+N)/2$. *Since the cosine function is symmetric about both its peaks and its valleys, one may conclude that*

$$\cos[((2\pi\,\text{rad})((n+N)/2)) + \text{anything}] = \cos[((2\pi\,\text{rad})((n+N)/2)) - \text{that same thing}]. \quad (35)$$

Thus we have

$$y[x_n,(\Gamma/2)+\epsilon] = y[x_n,(\Gamma/2)-\epsilon]. \quad \text{QED} \quad (36)$$

This means that the pendulum patterns get more and more complicated until $t = \Gamma/2$, then the pendula go through exactly the same series of patterns backwards until they are all back in phase. This property is neat, and perhaps not unexpected, but we consider it of somewhat less importance than the cycling of the patterns discussed in Question 2.

VI. ANOTHER ORIGIN

The mathematical description of this physical system is somewhat simpler if the origin is shifted to the left by defining

$$\xi = x + Nd. \quad (37)$$

In terms of this new axis variable, the lengths of the pendula $l[\xi]$ are simply proportional to ξ^{-2} [see Eq. (11)] and the periods of the pendula become

$$T[\xi] = \Gamma d/\xi. \quad (38)$$

Thus Eq. (12) describing the underlying function can be written as

$$y[\xi,t] = A\cos[(2\pi\,\text{rad})(\xi/(\Gamma d))t]. \quad (39)$$

This shift of origin physically corresponds to adding longer and longer pendula to the left end of the set. During the cycling time, Γ, these pendula would go through $N-1$ cycles, $N-2$ cycles, and so on. Actually building these longer pendula quickly becomes impractical for a classroom demonstration if Γ is as large as 20 s. Even with a smaller Γ the $\xi = 0$ pendulum can never be constructed because it must be infinitely long, so as to go through 0 oscillations in time Γ!

Several graphs of the function $y[\xi,t]$ appear in Fig. 4. At every moment $y[\xi,t]$ is a sinusoidal function of ξ, with a fixed value of A at the $\xi = 0$ end. The wavelength of this sinusoid diminishes as $1/t$, in accordance with Eq. (17), so $y[\xi,t]$ looks something like a contracting accordion as time goes by. The function $y[x,t]$ describing the physical set of pendula is just a subset of $y[\xi,t]$ extending from $\xi = Nd$ to $\xi = (N + (n_{max} - 1))d$, where n_{max} is the number of pendula in the set. To illustrate how $y[x,t]$ is just a portion of the more extensive $y[\xi,t]$, the two functions are superimposed in Fig. 4, with $y[x,t]$ shown in bold.

VII. SUMMARY

The cyclic patterns exhibited by the pendula in this demonstration never fail to delight and intrigue audiences of all ages and backgrounds. Describing the continuous mathematics behind these patterns is a nifty exercise in mathematical modeling that is just one small step beyond the standard description of traveling waves in one dimension taught in introductory physics classes. The fact that the cyclic nature of the patterns arises from aliasing is also a valuable lesson, and serves to make the mathematics even more interesting.

[a]Electronic mail: flatenja@mrs.umn.edu

[1]Richard E. Berg, ''Pendulum waves: A demonstration of wave motion using pendula,'' Am. J. Phys. **59** (2), 186–187 (1991).

[2]Demo 08-25 ''Pendulum Waves,'' *The Video Encyclopedia of Physics Demonstrations* (The Education Group & Associates, Los Angeles, 1992).

[3]This web site is compatible with Netscape Communicator 4.7 and Internet Explorer 4.5. It is best viewed using a window between 800 and 1200 pixels wide.

[4]One may think of y as an angle instead of a distance, but in the small-angle approximation the perpendicular displacement is proportional to the angle.