QUANTUM GEOMETRY OF BOSONIC STRINGS

A.M. POLYAKOV

L.D. Landau Institute for Theoretical Physics, Moscow, USSR

We develop a formalism for computing sums over random surfaces which arise in all problems containing gauge invariance (like QCD, three-dimensional Ising model etc.). These sums are reduced to the exactly solvable quantum theory of the two-dimensional Liouville lagrangian. At $D = 26$ the string dynamics is that of harmonic oscillators as was predicted earlier by dual theorists, otherwise it is described by the nonlinear integrable theory.

There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned (and extremely useful) sums over random paths. The replacement is necessary, because today gauge invariance plays the central role in physics. Elementary excitations in gauge theories are formed by the flux lines (closed in the absence of charges) and the time development of these lines forms the world surfaces. All transition amplitude are given by the sums over all possible surfaces with fixed boundary. Now, what are the advantages and applications of that representation?

The general picture has been envisaged as follows [1]. We have, presumably, a theory of free strings which can move through each other without any interaction. These strings do not correspond to the general gauge theory. However, the interaction among the gauge strings is such that it does not destroy but only modifies conserved currents of the free strings. This picture is an analogue of what happens in 2d-integrable systems (like sine-Gordon) with a change of the word "particle" for the word "string".

The analogy goes even further in the case of the Z_2-gauge group (Ising model). In this case, as was stated in ref. [1] and developed in ref. [2], it was possible to introduce some sort of fermionic string which remains free, just as in the 2d-Ising model one finds free fermion representation.

All these considerations had one essential flaw: it was not known what was exactly meant by the "free string". It has been clear, that just as the amplitudes of free particles are defined as

$$G(x, x') = \sum_{(\text{paths})} \exp[-mL(P_{xx'})] , \qquad (1)$$

where $P_{xx'}$ is a path connecting points x and x' and L is the length of the path ..., one should define:

$$G(C) = \sum_{(S_C)} \exp[-m^2 A(S_C)] , \qquad (2)$$

here C is some loop, S_C is a surface bounded by the loop, $A(S_C)$ is the area of this surface.

Both formulas (1) and (2) are symbolic, but while in the case of (1) we know how to decipher and compute it, in the case of (2) such knowledge is not available. It is the purpose of the present work to overcome, at least partly, this drawback. This task is made even more tempting by the arguments given in ref. [2] which show that the fermionic analogue of (2) is directly connected with the physics of phase transitions. Another possible application would be multicolored QCD in which gauge strings also might become free, with the possible addition of fermionic degrees of freedom [3].

We start our analyses from the purely bosonic case,

and work in euclidean space. Let us describe our surface by the parametrization $x_\mu(\xi_1, \xi_2)$. First we need several facts concerning the classical geometry of surfaces. These facts are not new (some of them belong to the XIXth century) but it is good for our purpose to have their collection at hand (see also ref. [4]). The area spanned by the surface is given by

$$A = \int d^2\xi \, (\det \| h_{ab} \|)^{1/2} \, , \quad h_{ab} = \partial_a x_\mu \partial_b x_\mu \, . \quad (3)$$

The integral goes over a fixed region in the ξ-plane (say, a unit circle): $\xi^a = \xi^a(s)$. It is invariant under:

$$x_\mu(\xi) \to x_\mu(f(\xi)) \, . \quad (4)$$

The minimal area is given by

$$\delta A = \frac{1}{2} \int \sqrt{h} \, h^{ab} \delta h_{ab} = 0, \quad \delta h_{ab} = \partial_{(a} x_\mu \partial_{b)} \delta x_\mu \, ,$$

or, after integrating by parts:

$$\partial_a(\sqrt{h} \, h^{ab} \partial_b x_\mu) = 0 \, , \quad h_{ab} = \partial_a x_\mu \partial_b x_\mu. \quad (5)$$

The same equations can be obtained from another functional:

$$W = \frac{1}{2} \int \sqrt{g} \, g^{ab} \partial_a x_\mu \partial_b x_\mu \, d^2\xi \, , \quad (6)$$

which is supposed to be minimized both in $x_\mu(\xi)$ and $g_{ab}(\xi)$. Indeed, the first variation gives the Laplace equation

$$\partial_a(\sqrt{g} \, g^{ab} \partial_b x_\mu) = 0$$

and the g_{ab}-variation forces the energy–momentum tensor to be zero:

$$T_{ab} = \partial_a x_\mu \partial_b x_\mu - \tfrac{1}{2} g_{ab} g^{cd} \partial_c x_\mu \partial_d x_\mu = 0 \, . \quad (7)$$

From (7) follows that

$$g_{ab} = \partial_a x_\mu \partial_b x_\mu \, .$$

Another fact which we need is that it is possible to change coordinates $\xi \to f(\xi)$ in such a way that the metric becomes conformally euclidean. The function $f = f_1 + i f_2$ is defined by the equation

$$\partial_a f = i \epsilon_{aa} \sqrt{g} \, g^{a'b} \partial_b f \, . \quad (8)$$

The solution of (8) can be chosen in such a way that it maps a unit disc onto itself:

$$f(\xi(s)) = \xi(\alpha(s)) \quad (9)$$

(here $\alpha(S)$ is a reparameterisation of the boundary, uniquely connected with the original $g_{ab}(\xi)$). The transformed metric is of the form $\rho(\xi)\delta_{ab}$; $\rho(\xi) = |df/d\xi|^2$. If we minimize (6) with respect to $x_\mu(\xi)$ with the boundary condition

$$x_\mu(\xi(s)) = x_\mu(s) \, , \quad (10)$$

we get the relations

$$A_{min}[x(s)] = \min_{\{g_{ab}\}} \min_{\{x_\mu(\xi)\}} W[x(\xi), \{g_{ab}(\xi)\}]$$

$$= \min_{\{\alpha(S)\}} W_{min}[x_\mu(\alpha(S)), \delta_{ab}] \, . \quad (11)$$

The minimal area $A_{min}[x(s)]$ satisfies the equation

$$[\delta A_{min}/\delta x_\mu(s)]^2 = (dx_\mu/ds)^2 \, , \quad (12)$$

$$(dx_\mu/ds) \, \delta A_{min}/\delta x_\mu(s) = 0 \, .$$

Up to now we dealt with minimal "classical" surfaces. Let us proceed to the quantum theory. The most immediate problem is to define the proper measure for the summation over continuous surfaces. This measure must count all surfaces of a given area with the same weight. That means that if we have transformation Ω which maps a surface S_C (C is a boundary of it) onto another surface $\widetilde{S}_{\widetilde{C}}$ in such a way that $A(\widetilde{S}_{\widetilde{C}}) = A(S_C)$ we must have for any functional $\phi[S_C]$

$$\int d\mu(S) \, \phi(S_C) = \int d\mu(S) \, \phi(\widetilde{S}_{\widetilde{C}}) \, . \quad (13)$$

Condition (13) leads to the following expression for the measure (we are not giving the derivation here):

$$\int d\mu(S) \, \phi(S)$$

$$= \int [Dg_{ab}(\xi)] \exp\left(-\lambda \int \sqrt{g} \, d^2\xi\right)$$

$$\times \int Dx(\xi) \left[\exp\left(-\frac{1}{2} \int_D \sqrt{g} \, g^{ab} \partial_a x_\mu \partial_b x_\mu d^2\xi\right) \right]$$

$$\times \phi[x(\xi)] \, , \quad (14)$$

where λ is an arbitrary parameter, D is a unit disc in the ξ-plane, $[Dg_{ab}]$ is an integration measure over all possible metrics, the same as in general relativity, with some gauge condition applied (the gauge will be specified later). This expression may be regarded as a quantum counterpart of the action (6). The role of g_{ab} in the classical limit is that of a Lagrange multiplier,

ensuring us that $T_{ab} = 0$. We omitted in (14) the boundary terms associated with Euler characteristics. The integration is performed with the condition $x_\mu(\xi(s)) = x_\mu(s)$.

The most surprising feature of the measure (14) is that the functional integrals in it can be explicitly evaluated. The cosmological term in (14), which has been absent on the classical level, is necessary for renormalizability. The possibility of performing an x_μ-integration stems from the fact that by the coordinate transformation (8) one makes the metric conformally euclidean and the resulting integral depends on the conformal factor only through the conformal anomaly. In such a situation one can use a well-known trick [5] in order to find the $\rho(\xi)$-dependence. Namely, one has to use the relation

$$\int Dx(\xi) \exp\left(-\frac{1}{2}\int \sqrt{g}\, g^{ab}\, \partial_a x_\mu \partial_b x_\mu\right) = \exp(-F) ,$$

$$g^{ab}\delta F/\delta g^{ab} = g^{ab}\langle T_{ab}\rangle = (D/24\pi)\,[R(\xi) + \text{const}]$$

$$(15)$$

(here R is the scalar curvature, and the second equation (15) is the well known trace anomaly relation). Inserting (15) in the gauge $g_{ab}(\xi) = \rho\delta_{ab}$; $R = \rho^{-1}\partial^2 \log\rho$ one obtains

$$F = -\frac{D}{48\pi} \int d^2\xi \left[\frac{1}{2}(\partial_a \log\rho)^2 + \mu^2\rho\right]$$

and in arbitrary coordinates:

$$F = -\frac{D}{48\pi} \int d^2\xi\, d^2\xi'\, g^{1/2}(\xi)g^{1/2}(\xi')R(\xi)R(\xi')$$

$$\times K(\xi,\xi') + \text{const} \int \sqrt{g}\, d^2\xi \qquad (16)$$

(here K is a Green's function for the laplacian:

$$\partial_a(\sqrt{g}\, g^{ab}\, \partial_b)K(\xi,\xi') = \delta(\xi - \xi') ,$$

μ^2 is a quadratically divergent renormalization of the cosmological constant). The next step must be the integration over g_{ab}. In order to do this we have to specify a gauge and account for the Faddeev–Popov determinant. We would like to use the conformal gauge in which our expressions simplify considerably. In order to find the measure of integration we use the following decomposition of the metric variation:

$$\delta g_{ab}(\xi) = \delta\varphi(\xi)g_{ab}(\xi) + \nabla_a \epsilon_b + \nabla_b \epsilon_a \qquad (17)$$

and substitute it into the expression for the norm in the functional space of possible metrics:

$$\|\delta g_{ab}\|^2$$

$$= \int d^2\xi\, [g(\xi)]^{1/2}(g^{aa'}g^{bb'} + Cg^{ab}g^{a'b'})\delta g_{ab}\delta g_{a'b'} ,$$

$$(18)$$

where C is an arbitrary constant which will drop out of the final answer. The expression (18) is the only possible local covariant formula. Substituting (17) into (18) we get

$$\|\delta g_{ab}\|^2 = (1 + 2C) \int (\delta\varphi + \nabla_c \epsilon^c)^2\, d^2\xi\, \sqrt{g}$$

$$+ \int d^2\xi\, \sqrt{g}\, \phi_a^b \phi_b^a , \qquad (19)$$

$$\phi_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a - g_{ab}\nabla_c \epsilon^c .$$

From (19) we derive the expression for the integration measure in the space of all metrics:

$$d\mu(g_{ab}) = D\varphi(\xi)D\epsilon_a(\xi)\,(\det{}^{1/2} \mathcal{L}) , \qquad (20)$$

in which the operator $\mathcal{L}$ is obtained from the last term of (19) and given by

$$(\mathcal{L}\epsilon)_a = \nabla^b(\nabla_a \epsilon_b + \nabla_b \epsilon_a - g_{ab}\nabla_c \epsilon^c) . \qquad (21)$$

It resembles the ordinary vector laplacian but does not coincide with it. In two dimensions and in conformally euclidean metric $g_{ab} = \rho\delta_{ab}$ eigenvalues of $\mathcal{L}$ are determined from the equation

$$\rho^{-2}(\partial/\partial\bar{z})\,(\rho\partial\psi_n/\partial z) = -\lambda_n \psi_n \,, \; z = \xi_1 + i\xi_2 . \quad (22)$$

Since the operator at the left-hand side of (22) is a product of two conformally covariant operators. $\det \mathcal{L}$ is again determined by the conformal anomaly, and has the form

$$-\frac{1}{2}\log\det \mathcal{L} = A \int \left[\frac{1}{2}(\partial_\mu\varphi)^2 + \mu^2 e^\varphi\right] d^2\xi ,$$

$$\varphi = \log\rho . \qquad (23)$$

The constant A is most easily determined by matching (23) with perturbation theory for small φ and is found to be equal to

$$A = 13/24\pi . \qquad (24)$$

Combining (23) and (15) we obtain the partition function for the closed surfaces:

$$Z = \int D\varphi(\xi) \exp\left(-\frac{26-D}{48\pi} \int_\xi \left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \mu^2 e^\varphi\right]\right). \tag{25}$$

This expression shows very clearly the origin of the commonly known critical dimension 26 in the string theory: at this value of the dimension one could quantize the theory without bothering about the conformal anomaly, as has been done in dual models. However, for $D < 26$ in order to get proper quantization we must examine the quantum Liouville theory described by the lagrangian (25). This theory is two-dimensional, renormalizable, and completely integrable. All that means that it is exactly solvable, just as sine-Gordon theory, and thus it must be possible to evaluate explicitly the partition function of closed surfaces. This work is in progress now and in the present paper I shall only demonstrate how to express different physical quantities, like the spectrum, scattering amplitudes etc. through the correlation functions of quantum Liouville theory. The basic idea is to sum over surfaces which contain a given set of points $\{x_j\}$. The Fourier transform of such an amplitude has poles in p_j^2 (p_j is the momentum of point x_j) which define the mass spectrum. The residues of these poles can be identified with the scattering amplitude. So, we consider the average

$$A(p_1 ... p_N)$$

$$= \left\langle \prod_j \int d^2\xi_j \left[g(\xi_j)\right]^{1/2} \exp\left[ip_j x(\xi_j)\right] \right\rangle. \tag{26}$$

The average in (26) is understood in the sense of (14). All functional integrals being gaussian, they are easily evaluated with the result

$$A(p_1 ... p_N) = \int D\varphi(\xi) \exp\left(-\frac{26-D}{48\pi}\int d^2\xi\right.$$

$$\times \left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \mu^2 e^\varphi\right]\right)$$

$$\times \int \exp\left(\sum_j \varphi(\xi_j)\right)\prod_j d^2\xi_j$$

$$\times \exp\left(-\sum_i p_i p_j K(\xi_i, \xi_j, \varphi)\right). \tag{27}$$

The function $K(\xi, \xi', \varphi)$ is a Green's function for the laplacian in the metric $g_{ab} = e^\varphi \delta_{ab}$. If the points ξ and ξ' do not coincide it is just

$$K(\xi, \xi') = -(4\pi)^{-1}\log(\xi - \xi')^2. \tag{28}$$

However, when ξ is close to ξ' extra care is needed. We

have to recall about the cutoff built in our theory. The proper definition of K is given by

$$K(\xi, \xi') = \sum_n \left[\chi_n(\xi)\chi_n(\xi')/\lambda_n\right] \exp(-\epsilon\lambda_n), \tag{29}$$

where λ_n are eigenvalues, χ_n are eigenfunctions of the laplacian and ϵ is the proper time cut-off.

Using (29) one shows that

$$K(\xi, \xi^P) = -(4\pi)^{-1}\log(1/\epsilon) + (4\pi)^{-1}\varphi(\xi) \tag{30}$$

and in this way A functions are determined by the Liouville correlators. Note that at $D = 26$ only we obtain from (27) the standard dual model in the Koba–Nielsen form (see ref. [6] for a review). For physical D one has to solve the Liouville theory in order to find the scattering amplitudes.

A few words now about the quantization of the Liouville theory. The lagrangian possesses the symmetry

$$\varphi(z, \bar{z}) \rightarrow \varphi(w(z), \bar{w}(z)) + \log |dw/dz|^2, \tag{31}$$

which is all that remains from the general covariance after the specification of the conformal gauge. The theory must be quantized in such a way, that this invariance remains untouched. It is possible to prove that this is indeed possible and leads to a unique renormalization procedure.

So, our main conclusion is that the summation of random surfaces is reduced to the two-dimensional, exactly solvable theory, and that the old "dual" approach to the string is correct only at $D = 26$.

Extension of these results to the Fermi case and their physical applications are discussed in other papers [2,7].

References

[1] A.M. Polyakov, Phys. Lett. 82B (1979) 247.
[2] V.G. Dotsenko and A.M. Polyakov, to be published.
[3] A.A. Migdal, Nucl. Phys., to be published
[4] L. Brink and J. Schwarz, Nucl. Phys. B121 (1977) 285.
[5] A.S. Schwartz, Commun. Math. Phys. 64 (1979) 233.
[6] S. Mandelstam, Phys. Rep. 13C (1974) 261.
[7] A.M. Polyakov, Phys. Lett. 103B (1981) 211.