# THE MODERN LEGACIES OF THOMSON'S ATOMIC VORTEX THEORY IN CLASSICAL ELECTRODYNAMICS

SAMUEL J. LOMONACO, JR.

*Dedicated to the memory of Ralph H. Fox*

Abstract. The physicist Sir William Thomson (also known as Lord Kelvin) proposed in 1867 [56] [57] [58] that physical atoms were knotted vortex tubes in the then postulated all pervasive fluid called ether. The physicist Peter Guthrie Tait became so enamored with Thomson's theory that he undertook a study of the mathematical properties of knots, thus giving birth to the field of knot theory.

Although scientific evidence has since shown conclusively that physical atoms are by no means knotted vortices in the sense of Thomson, Thomson's theory has fragmented and relatively recently reemerged in many much more sophisticated forms in both classical and nonclassical physics. With the work of Jones, Witten, and others, knot theory has now begun to reassociate on a serious basis with its long lost ancestor, physics [28].

The first section of this paper gives a brief survey of the early Thomson atomic vortex theory as it developed within the James Clerk Maxwell milieu. The paper then focuses on the modern legacies of this theory in classical electrodynamics. In particular, the second and third sections of this paper look at some of the recent developments on knotted magnetic vortices and knotted electrostatic vortices, respectively.

In summary, this paper focuses on the study of the electromechanical behavior of knotted tubes of electrical charge and magnetic flux. Surprisingly, even within classical physics (more specifically, classical electrodynamics), there are many important unresolved questions about such objects. Many of these questions are relevant to such diverse fields as plasma physics, polymer physics, molecular biology, and, of course, knot theory.

*Date.* Published in AMS PSAPM, Vol. 51, (1996), pp. 145 - 166.

1991 *Mathematics Subject Classification.* Primary 57-02, 57M25, 78-02, 78A30, 76W05, 53C80; Secondary 01-02, 78-03, 45K05, 22E70.

*Key words and phrases.* Knots, Electrodynamics, Electromagnetism, Energy, Magnetic Knots, Electrostatic Knots, Minimum Energy Knots, Vortices, Gauge Theory.

Partially supported by the L-O-O-P Fund.

I would like to thank Mike Freedman and Keith Moffatt for their helpful comments.

## 1. **Introduction:** A TALE OF TWO CENTURIES BEGINS WITH KELVIN'S KNOTTED VORTICES

We begin our tale of two centuries in the late nineteenth century, a time when the atomic structure of matter was just beginning to unfold. Herman von Helmholtz had just published a paper [26] on what James Clerk Maxwell referred to as "water twists." In this paper, Helmholtz proved the surprising result that, within an incompressible, inviscid, constant density fluid, fluid vortices are actually permanent and indivisible.

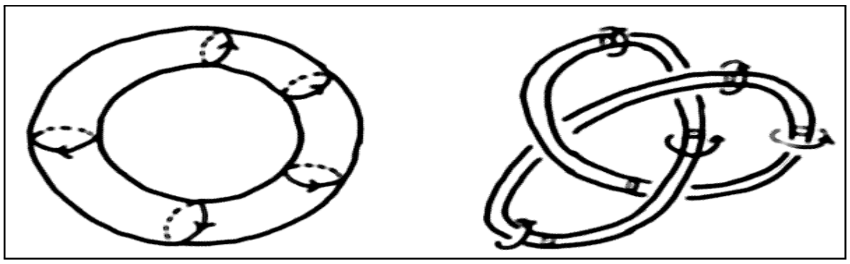


Figure 1. Fluid vortices, also called "water twists."

After reading Helmholtz's paper, the Edinburgh University physicist[1] Peter Guthrie Tait gave a series of lectures on Helmholtz's paper. To demonstrate Helmholtz's result in his lectures, he used an apparatus of his own design that produced vortex rings of smoke. His presentations illustrated quite vividly and dramatically that:

- The vortex rings behaved as independent solids.
- On collision with one another, the vortex rings rebounded as if they were quivering elastic solids, like rings of rubber.
- The smoke rings exhibited fascinating vibration modes about their circular form.
- On each attempt to cut the smoke rings with a knife, the smoke rings would simply wriggle around the knife. The rings were indivisible!

It just so happened that a professional colleague of Tait's, the physicist Sir William Thomson (also later known as Lord Kelvin) was in the audience of one of Tait's lecture-demonstrations. During Tait's presentation, Thomson was struck by the evident permanence and indivisibility of "water twists," as illustrated by Tait's smoke rings. It was at some time into this lecture that Thomson conceived of and created his atomic vortex theory, i.e., that atoms were nothing more than knotted and linked tubular vortices in the then postulated all pervasive fluid called ether. Starting in 1867, Thomson published a series of papers [56] [57] [58] that explained his theory.

[1] In the nineteenth century, physicists were called natural philosophers.

The atomic vortex theory appears to have been well received by the best minds of the nineteenth century. The physicist James Clerk Maxwell was much impressed with Thomson's atomic theory[3] [33]. Maxwell noted that, unlike the other atomic theories of the time, Thomson's theory was based on only a few assumptions. Because of its axiomatic simplicity, it was more likely to represent the physical world than the other existing theories of the time.

Peter Guthrie Tait became so enamored with the theory that he dedicated many years to constructing tables of knots (i.e. ,Tait's "periodical tables"), thus giving birth to the field of knot theory [53].

Tait is also known to have carried on a dialog with Maxwell. In a letter from Maxwell to Tait [33, page 106], Maxwell wrote,

GLENLAIR
DALBEATTIE,
Nov. 13, 1867.

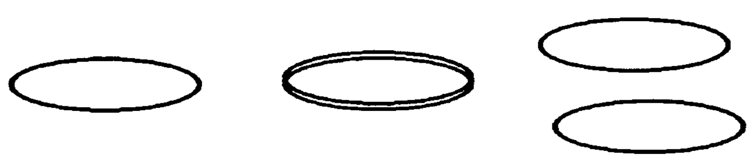
Dear Tait

If you have any spare copies of your translation of Helmholtz on "Water Twists" I should be obliged if you could send me one.

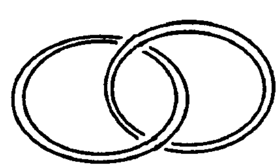
I set [sic] the Helmholtz dogma to the Senate House in '66, and got it very nearly done by some men, completely as to the calculation, nearly as to the interpretation.

Thomson has set himself to spin the chains of destiny out of a fluid plenum as M. Scott set an eminent person to spin ropes from the sea sand, and I saw you had put your calculus in it too. May you both prosper and disentangle your formulae in proportion as you entangle your worbles. But I fear the simplest indivisible whirl is either two embracing worbles or a worble embracing itself.

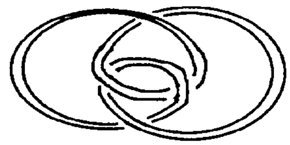
For a simple closed worble may be easily split and the parts separated



but two embracing worbles preserve each others solidarity thus



though each may split into many, every one of the one set must embrace every one of the other. So does a knotted one.



yours truly
J. Clerk Maxwell

Although scientific evidence has since shown conclusively that physical atoms are by no means vortices in the sense of Thomson, Thomson's theory has fragmented and relatively recently reemerged in many much more sophisticated forms in both classical and non classical physics.

In his 1991 Josiah Willard Gibbs Lecture, entitled "The Mysteries of Space," Sir Michael F. Atiyah pointed out four inherent properties of vortex atoms that explain the longevity of vortex theory [3] [4]. These four inherent characteristics are:

- **Stability:** Vortex atoms are stable, as are physical atoms.
- **Variety:** There is a great variety of knots, as there is a great variety of physical atoms.
- **Spectrum:** Vortex atoms have energy states and vibration modes, as do physical atoms.
- **Transmutation:** Knotted vortex atoms change their knot type if their energy is increased beyond a certain threshold, as do physical atoms change their atomic structure.

The recent resurgence of knot theory in physics can be thought of as simply an expression of a recurring major theme in physics, namely:

$$Physics = Geometry$$

Certainly the placement problem is a central part of geometry. A simplistic statement of the placement problem is given below:

**The Placement Problem.** When are two placements of the same space $X$ in a space $Y$ equivalent?

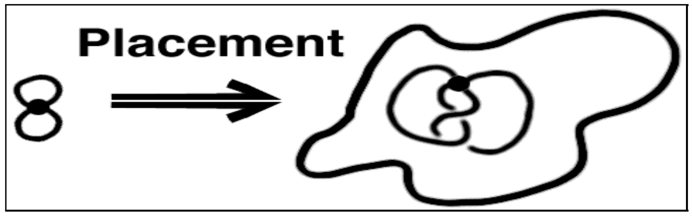



Figure 2. The Placement Problem

## 2. **Part 1:** KNOTTED MAGNETIC VORTICES IN THE TWENTIETH CENTURY

Our story resumes in the twentieth century with the works of Moffatt, Ricca, Berger, Arnold, Freedman, He, and others [38] [39] [40] [41] [42] [14] [49] [7] [8] [1] [15] [16] [61].

## 3. PRELIMINARIES: MAGNETIC FIELDS IN PERFECTLY CONDUCTING INCOMPRESSIBLE FLUIDS

Consider an incompressible constant density fluid in $\mathbb{R}^3$ with velocity field $\mathbf{v}(\mathbf{x}, t)$. If the fluid is contained within a closed surface, assume that the velocity field on that surface is tangent to the surface. Because of incompressibility, the velocity field is solenoidal, i.e.,

$$\nabla \cdot \mathbf{v} = 0$$

Let

$$\mathbf{x} \overset{g_t}{\longmapsto} \mathbf{X}(\mathbf{x}, t),$$

denote the fluid flow, where $\mathbf{X}(\mathbf{x}, t)$ denotes the position at time $t$ of a fluid particle that started at position $\mathbf{x}$ at time $t = 0$. Thus, $\mathbf{X}(\mathbf{x}, 0) = \mathbf{x}$. Because of incompressibility, $g_t$ is a parameterized family of volume preserving diffeomorphisms, i.e., $g_t \epsilon SDiff$. Thus, if $\mathcal{R}$ is a region bounded by a smooth surface $\mathcal{S} = \partial\mathcal{R}$, then, as the surface $g_t\mathcal{S}$ (and hence the region $g_t\mathcal{R}$) moves with the flow, the volume

$$\iiint_{g_t \partial\mathcal{R}} dvol$$

remains constant with time. The region $g_t\mathcal{R}$ may of course continuously change its shape. But its volume always remains the same.

Let $\mathbf{B}$ be a magnetic field[2] that is present in the fluid. Since $\mathbf{B}$ is solenoidal, i.e., $\nabla \cdot \mathbf{B} = 0$, there exists a vector potential $\mathbf{A}$ (uniquely determined up to the gradient of an arbitrary scalar function) such that

$$\mathbf{B} = \nabla \times \mathbf{A}$$

[2] With few exceptions, everything said within this paper for a magnetic field $\mathbf{B}$ is equally valid for an arbitrary solenoidal vector field.

Next, assume that the fluid is perfectly conducting. This assumption implies that the magnetic field is frozen in the fluid, and that its behavior is determined by the **frozen field equation**,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

By 'frozen,' we mean that the magnetic lines of force move with the fluid. Thus, if $\mathcal{S}$ is a surface bounded by a closed curve $\mathcal{C} = \partial \mathcal{S}$, then the magnetic flux $\Phi$ crossing the moving surface $g_t \mathcal{S}$, i.e.,

$$\Phi\left(g_t \mathcal{S}\right) = \iint\limits_{g_t \mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = \oint\limits_{g_t \mathcal{S}} A \cdot d\mathbf{x}$$

remains constant with time.

A surface $\mathcal{S}$ is said to be a **magnetic surface** if the magnetic field $\mathbf{B}$ on $\mathcal{S}$ is tangent to $\mathcal{S}$. Since $\mathbf{B}$ is frozen in the fluid, it follows that, if $\mathcal{S}$ is a magnetic surface at time $t = 0$, then the moving surface $g_t \mathcal{S}$ (i.e., the surface moving with the fluid) remains a magnetic surface for all time $t \geq 0$. Moreover, if $\mathcal{S}$ is also a closed surface, then as the parameterized family of magnetic surfaces $g_t \mathcal{S}$ dynamically changes its shape with time $t$, incompressibility implies that the volume $g_t \mathcal{R}$ which $g_t \mathcal{S}$ encloses

$$\iiint\limits_{g_t \mathcal{R}} d\,vol$$

remains constant. Of course, the magnetic flux leaving a closed magnetic surface is zero.

## 4. KNOTTED AND LINKED TUBES OF MAGNETIC FLUX

The following definition by the author captures much of the intent and content of the many constructions and examples found within Moffatt's and Berger's papers [39] [40] [41] [42] [7].

Let $\mathbb{T}$ denote the standard solid torus in $\mathbb{R}^3$ given by

$$\left( \quad (2 + \epsilon \cos\theta) \cos\varphi, \quad (2 + \epsilon \cos\theta) \sin\varphi, \quad \epsilon \sin\theta) \quad \right)$$

where $0 \leq \theta, \varphi < 2\pi$ and $0 \leq \epsilon < 1$. For relatively prime integers $p$ and $q$, let $\mathcal{F}_{p,q}$ denote the foliation of $\mathbb{T}$ by the curves $\gamma_{\epsilon,\theta}$ (where $0 \leq \epsilon \leq 1$ and $0 \leq \theta < 2\pi$) given by

$$\gamma_{\epsilon,\theta}(s) = \left( \quad (2 + \epsilon \cos(\theta + qs)) \cos(ps), \quad (2 + \epsilon \cos(\theta + qs)) sin(ps), \quad \epsilon \sin(\theta + qs)) \quad \right)$$

where $0 \leq s < 2\pi$.

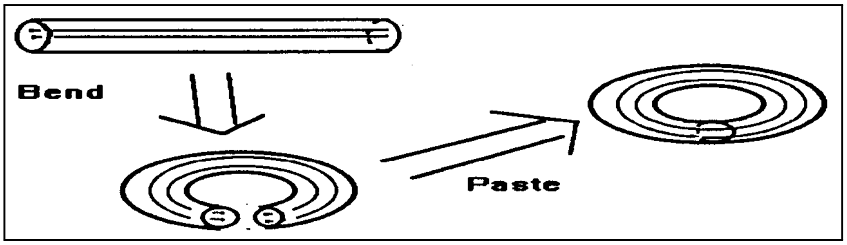



Figure 3. Construction of the foliation $\mathcal{F}_{1,0}$ of the standard torus $\mathbb{T}$.

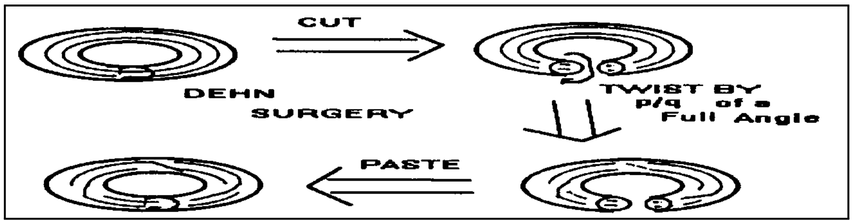



Figure 4. Construction of the foliation $\mathcal{F}_{p,q}$ from $\mathcal{F}_{1,0}$ by Dehn Surgery.

**Definition 4.1.** [3]*A* ***magnetic tubular link*** *(or more briefly, a* ***magnetic link****) is a smooth immersion into $\mathbb{R}^3$ of finitely many disjoint standard solid tori $\bigsqcup_{i=1}^{n} \mathbb{T}_i$*

$$L : \bigsqcup_{i=1}^{n} \mathbb{T}_i \to \mathbb{R}^3$$

*and a smooth magnetic field $\mathbf{B}$ on $\mathbb{R}^3$ such that*

(1) *$L$ is an imbedding when restricted to the interior of $\bigsqcup_{i=1}^{n} \mathbb{T}_i$*
(2) *The bounding surface of $\bigcup_i L(\mathbb{T}_i)$, i.e., $\bigcup_i L(\partial\mathbb{T}_i)$ is a magnetic surface*
(3) *For each component $L\mathbb{T}_i$, there exist relatively prime non-zero integers $p_i$ and $q_i$ such that $L$ maps the foliation $\mathcal{F}_{p_i,q_i}$ of $\mathbb{T}_i$ onto the integral curves of $\mathbf{B}$ in $L\mathbb{T}_i$.*

**Remark 4.1.** *In the above definition, we have allowed $L$, when restricted to the boundary, to be an immersion to deal with a phenomenon that occurs when the limit is taken under magnetic relaxation of a magnetic link.*

**Remark 4.2.** *Thus, for every fixed $i$ and $j$, the linking number between an arbitrary field line in $L\mathbb{T}_i$ and an arbitrary field line in $L\mathbb{T}_j$ is the same regardless of which integral curves are chosen from $L\mathbb{T}_i$ and $L\mathbb{T}_j$ respectively. This is true even when $i = j$.*

[3]This definition encompasses the examples of Moffatt and Berger constructed by Dehn surgery with a twist $2\pi h$ where $h$ is a rational number. To deal with twists for which $h$ is irrational, one should also allow foliations $\mathcal{F}_{p,q}$ where $p$ and $q$ are non-zero real numbers whose ratio $p/q$ is irrational. In this case the curves $\gamma_{\epsilon,\theta}(s)$, $-\infty<s<\infty$, are no longer closed. Arnold's asymptotic linking number [1] would then be required in what is to follow. This is considerably more complicated.

It follows that a magnetic link $\underset{i}{\cup} L\mathbb{T}_i$ remains a magnetic link under the action of the fluid flow. I.e., $\underset{i}{\cup} g_t L\mathbb{T}_i$ is a magnetic link for $t \geq 0$.

Keeping in mind that the magnetic field $\mathbf{B}$ is frozen in the fluid, our next objective is to find and study those properties of magnetic links that are invariant under the action of fluid flow. One obvious invariant is the **volume** $\mathcal{V}_i$ of each flux tube $g_t L\mathbb{T}_i$, i.e.,

$$\mathcal{V}_i = Vol(L\mathbb{T}_i) = Vol(g_t L\mathbb{T}_i) = \iiint_{g_t L\mathbb{T}_i} dvol$$

which remains unchanged because of incompressibility.

Another invariant of fluid flow is defined as follows:

**Definition 4.2.** *Let $L$ be a magnetic link. For each solid torus $\mathbb{T}_i$, choose a meridional disk $\mathcal{D}_i$. The **magnetic flux** $\Phi_i = \Phi(L\mathbb{T}_i)$ in the $i$-th component is the surface integral defined as*

$$\Phi_i = \Phi(L\mathbb{T}_i) = \iint_{L\mathcal{D}_i} \mathbf{B} \cdot \mathbf{U} \, d\,area$$

*where $\mathbf{U}$ denotes the normal to the surface $L\mathcal{D}_i$ pointing in the positive direction induced by the $\mathbf{B}$-field.*

It can be shown that $\Phi_i$ is independent of the chosen meridional disk. It also can be shown that each $\Phi_i$ is a fluid flow invariant, i.e.,

$$\Phi_i(g_t L\mathbb{T}_i) = \iint_{g_t L\mathcal{D}_i} \mathbf{B} \cdot \mathbf{U} \, d\,area$$

is independent of $t$.

One more fluid flow invariant that will play a central role in the energy minimization of magnetic links is given by the following definition.

**Definition 4.3.** *The **helicity**[4] [5] of a magnetic link $L$ is defined as:*

$$\mathcal{H}(L) = \iiint_{\underset{i}{\cup} L\mathbb{T}_i} \mathbf{A} \cdot \mathbf{B} \, dvol$$

[4]The term *helicity* was first introduced in a fluid context by Moffatt in [39]. The term *helicity* was previously used in particle physics for the scalar product of the momentum and spin of a particle.

[5]Please note that the *helicity* $\mathcal{H}(L)$ is the same as the *Chern-Simon action*. For $\mathcal{H}(L)$ can be expressed as

$$\mathcal{H}(L) = \int \mathbf{A} \wedge d\mathbf{A} = \int Trace\left(\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}\right)$$

where $\mathbf{A}$ now denotes the magnetic vector potential as a 1-form. For details, see [5].

It can be shown that $\mathcal{H}(L)$ is gauge invariant, and hence well defined.

**Theorem 4.1.** [41][1] [6] *The helicity is invariant under fluid flow, i.e.,*

$$\frac{d}{dt}\mathcal{H}(g_t L) = 0$$

The following theorem is believed by the author to summarize the many results found within the works of Moffatt, Ricca, and Berger relating the helicity of magnetic links to linking and to magnetic flux [38] [39] [40] [41] [42] [7] [49].

**Theorem 4.2.** *Let $L$ be a magnetic link. Then*

$$\mathcal{H}(L) = \sum_{1\le i\le n} \Phi_i^2 SL_{\mathcal{F}_i} + 2 \sum_{1\le i<j\le n} \Phi_i \Phi_j LK_{ij}$$

*where $SL_{\mathcal{F}_i}$ denotes the* ***self-linking number***[7] *of the axis curve of the tube $L\mathbb{T}_i$ with respect to the framing $\mathcal{F}_i$ induced by the integral curves of the magnetic field $\mathbf{B}$ within $L\mathbb{T}_i$, and $LK_{ij}$ denotes the* ***linking number*** *between any integral curve of the magnetic field $\mathbf{B}$ in $L\mathbb{T}_i$ with any integral curve of the magnetic field $\mathbf{B}$ in $L\mathbb{T}_j$.*

**Remark 4.3.** *Please note that $SL_{\mathcal{F}_i}$ is the same as the linking number between any two integral curves of the magnetic field $\mathbf{B}$ within the tube $L\mathbb{T}_i$.*

Thus, as noted by Moffatt [38] [39] [40] [41], Berger [7], Arnold [1], Freedman [15] [16], and others [49], the helicity does reflect the topology and the geometry of the magnetic lines of force within a magnetic link.

If for example $L$ has only one component, i.e., $L$ is a magnetic knot, then

$$\mathcal{H}(L) = \Phi^2 SL_{\mathcal{F}}(C)$$

where $SL_{\mathcal{F}}(C)$ is the self-linking number of the axis curve $\mathcal{C}$ of the knotted tube with respect to the framing $\mathcal{F}$ induced by the integral curves of the magnetic field $\mathbf{B}$ within the magnetic knot.[8] If for example the tube is knotted in the form of a trefoil and if the magnetic lines of force appear to be parallel to the axis curve when the trefoil is placed on a plane flat surface as shown Fig. 5,

[6] Arnold defines the helicity in a more abstract setting and shows that it is invariant under the group $SDiff$ of volume preserving diffeomorphisms.

[7] For a definition of self-linking number, see for example [22]

[8] This is the same as the linking number between any two distinct magnetic lines of force within the magnetic tube.

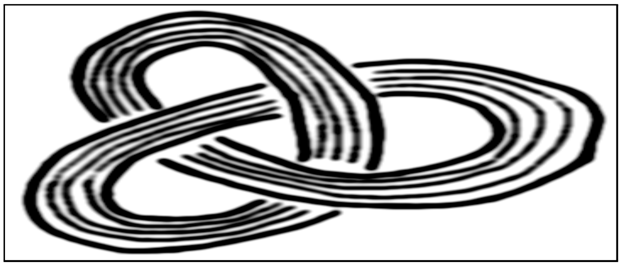


Figure 5. A magnetic trefoil with helicity $\mathcal{H} = \pm 3\Phi^2$.

then $SL = \pm 3$ and

$$\mathcal{H} = \pm 3\Phi^2 .$$

On the other hand, if for example the magnetic lines of force induce the trivial framing in each component, then

$$\mathcal{H}(L) = 2 \sum_{1 \le i < j \le n} \Phi_i \Phi_j LK_{ij}$$

Thus, if $L$ is a magnetic two component Hopf link with no twisting of the integral curves of the magnetic field within the components of $L$ as shown in Fig. 6,

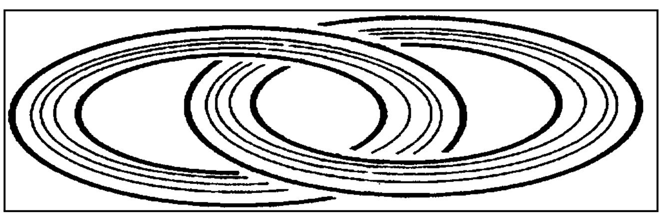


Figure 6. Magnetic Hopf link with trivial framing.

then

$$\mathcal{H}(L) = \pm 2\Phi_1 \Phi_2$$

because the self-linking number based on the **B**-field framing is zero for each component, and the linking number between the two components is $\pm 1$.

Finally, we close this section with the definition of the energy of a magnetic link.

**Definition 4.4.** *The* ***(magnetic) energy*** $\mathcal{E}_M(L)$ *of a magnetic link* $L$ *is defined by the classical formula*

$$\mathcal{E}_M(L) = \frac{1}{8\pi} \iiint\limits_{\underset{i}{\cup} L\mathbb{T}_i} |\mathbf{B}|^2 \ dvol \qquad\qquad (Gaussian \quad units).$$

Although the energy $\mathcal{E}_M$ is not flow invariant, it will play a central role when magnetic relaxation is discussed in the next section.

## 5. MAGNETIC RELAXATION AND MINIMUM ENERGY MAGNETIC LINKS

We now turn to the task of studying magnetic relaxation and minimum energy magnetic links.

Consider a magnetic link $L$ in a perfectly conducting, incompressible, viscous fluid. As a result of dissipative frictional fluid forces, the magnetic energy $\mathcal{E}_M(g_tL)$ of $g_tL$ will decrease with time $t$. In losing energy, the magnetic lines of force will contract. On the other hand, since this is a volume preserving process, the cross sections of the flux tubes of $g_tL$ will at the same time expand. These changes occur while the flux and helicity of $g_tL$ will remain the same.

This process can not continue indefinitely. For eventually the magnetic flux tubes of $g_tL$ must make contact with each other, as indicated in Fig. 7.

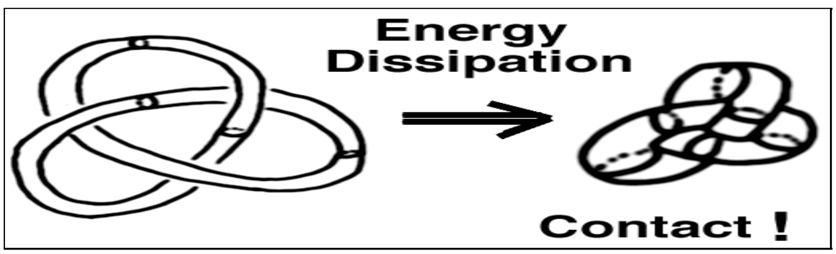



Figure 7. Magnetic relaxation of a magnetic trefoil.

In other words, the topology of the magnetic link $g_tL$, as expressed in knotting and linking, creates a barrier to the full dissipation of the magnetic link's energy. I.e., $\mathcal{E}_M(g_tL)$ has a positive lower bound that results from the topology of $g_tL$. Thus, the magnetic link will reach a non-trivial stable and invariant energy state, much as Kelvin conjectured his atomic vortices would.

Since the helicity $\mathcal{H}(g_tL)$ is both an invariant of fluid flow and an expression of the magnetic link $g_tL$'s topology, the following theorem is one mathematical expression of this topological bound.

**Theorem 5.1.** [40][41] *Let $L$ be a magnetic link. Then*

$$\mathcal{E}_M(L) \geq q_0 \left|\mathcal{H}(L)\right|$$

*where $q_0$ is a non-zero constant that is independent of the magnetic link.*

In [15] and [16], Freedman and He obtain more subtle and tighter topological bounds on the minimum energy of magnetic links. For example, for a magnetic knot $K$ they prove that

$$\mathcal{E}_M(K) \geq \frac{1}{4\pi^{\frac{5}{4}}} \frac{\Phi(K)^{\frac{3}{2}} ac(K)^{\frac{3}{4}}}{\mathcal{V}(K)^{\frac{1}{3}}} \geq \frac{1}{4\pi^{\frac{5}{4}}} \frac{\Phi(K)^{\frac{3}{2}} (2g(K)-1))}{\mathcal{V}(K)^{\frac{1}{3}}} \qquad (\textit{Gaussian units})$$

where $\mathcal{V}(K)$ denotes the volume of the magnetic knot $K$, where $\Phi(K)$ denotes the flux in $K$, where $ac(K)$ is the asymptotic crossing number, and where $g(K)$ is the genus of the knot $K$. Freedman and He conjecture that

$$ac(K) = c(K)$$

where $c(K)$ is the crossing number, i.e., the minimum number of crossings among all plane diagrams representing the knot $K$.

In [40] and [41], Moffatt suggests that the minimum energy spectrum of a magnetic knot can be used to construct new knot invariants.

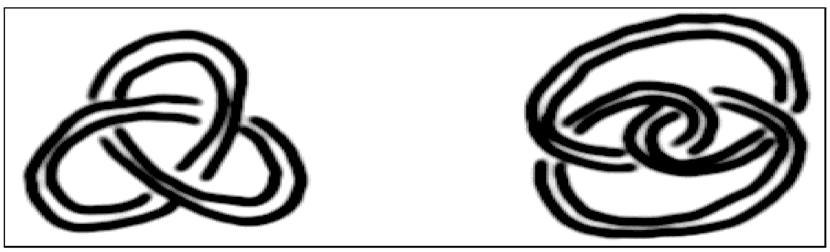


Figure 8. Two different reprsentatives of the trefoil knot type that could possibly magnetically relax to different minimal energy positions. Example taken from [40].

## 6. **Part 2:** KNOTTED ELECTROSTATIC VORTICES IN THE TWENTIETH CENTURY

A seemingly independent part of our tale of two centuries also resumes in the twentieth century with the works of Freedman, He, Birman, Lomonaco, Kusner, and others [18] [20] [9] [35] [12] [50] [51] [32] [45].

## 7. PRELIMINARIES: KNOTTED LOOPS AND TUBES OF ELECTRICAL CHARGE AND THE HONEY JAR PROBLEM

Consider a knotted loop $\alpha$ of string or wire that is perfectly flexible but non-extensible, hence of fixed length. Assume that $\alpha$ carries a fixed electrical charge $Q$. And let $\alpha$ be placed in a non-conducting, non-dielectric viscous fluid which we shall call "honey."

**The Honey Jar Problem (HJP)**[9]: *Find the minimal energy position or positions eventually assumed by the knotted loop $\alpha$ as a result of the repulsive Coulomb forces in combination with the dissipative frictional fluid forces.*

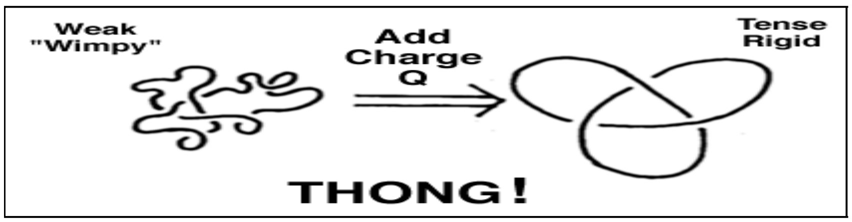



Figure 9. A possible example of an inextensible, flexible knot assuming a minimal energy position when a fixed charge $Q$ is added.

There are actually at least two honey jar problems, i.e.,

- The **conducting honey jar problem** where the charges are allowed to move freely on $\alpha$, i.e., $\alpha$ is assumed to be a perfect conductor. In this case, we refer to $\alpha$ as a **wire**.
- The **non-conducting honey jar problem** where the charges on $\alpha$ are immobile, i.e., $\alpha$ is assumed to be a perfect non-conductor. In this case, we refer to $\alpha$ as a **string**.

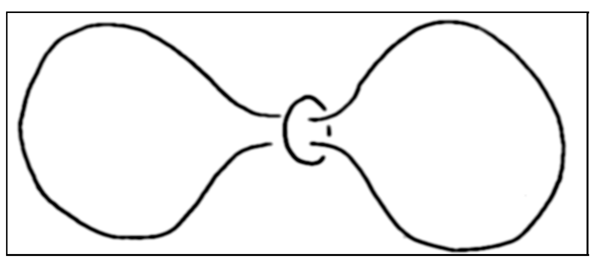


Figure 10. A possible example of a minimal energy electrostatic link.

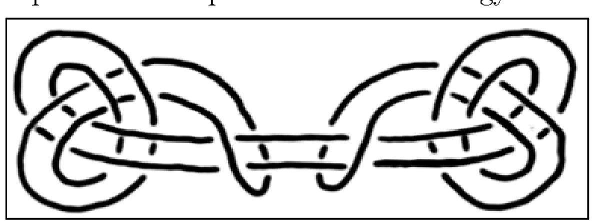


Figure 11. A possible example of a minimal energy electrostatic knot. Example taken from [18].

[9]The phrase 'honey jar' is taken from [20].

In [9] [35], Birman and Lomonaco discussed many of the aspects and difficulties that arise in this problem. In [9], it is pointed out that the loop $\alpha$ can be viewed in at least three ways, thus giving rise to three different honey jar problems for each of the above two cases, namely,

(1) The **HJP for curves** where the loop $\alpha$ is a knotted simple closed space curve in $\mathbb{R}^3$ carrying a charge given by a linear charge density function $\lambda$,
(2) The **HJP for hollow tubes** where the loop $\alpha$ is a knotted hollow tube (of constant cross-sectional radius $\epsilon$) in $\mathbb{R}^3$ carrying a surface charge given by a surface charge density function $\sigma$,
(3) The **HJP for solid tubes** where the loop $\alpha$ is a knotted solid tube (of constant cross-sectional radius $\epsilon$) in $\mathbb{R}^3$ carrying a charge given by a volume charge density function $\rho$.

In each of these cases the ultimate goal is to understand physical behavior. Success to this end can best be measured in terms of the following yardstick.

**The Asymptotic Behavior Objective (ABO) Yardstick.** *The objective is to understand minimum energy asymptotic behavior, i.e., the minimum energy behavior of a physical knotted charged string or wire as its cross section approaches zero.*

## 8. THE HONEY JAR PROBLEM FOR CURVES: MINIMUM ENERGY KNOTTED SPACE CURVES OF ELECTRICAL CHARGE

One of the major stumbling blocks encountered in the HJP for curves is a theorem in potential theory [29] that states that the potential energy of an electrically charged space curve is necessarily infinite. The potential energy of a simple closed curve $\mathbf{x}(s)$ parameterized by arclength $s$ and having a charge density function $\lambda(s)$ is given by the classical formula

$$\mathcal{E} = \frac{1}{2} \oint \oint \frac{\lambda(s)\lambda(s')dsds}{|\mathbf{x}(s) - \mathbf{x}(s')|}$$

which as stated is divergent. As Jackson points out in [27], the above integral diverges because of "self-energy" contributions of the electric field.

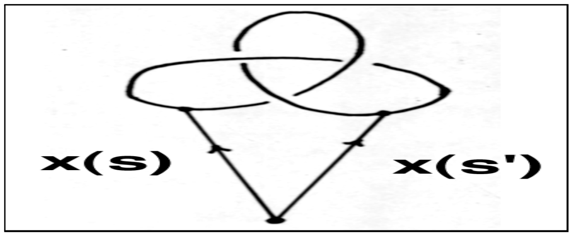



Figure 12. Two position vectors $x(s)$ and $x(s')$ pointing to two distinct points on a trefoil.

This singular behavior (pun intended) calls for great ingenuity for those in pursuit of the minimum energy positions of such curves! A number of renormalization techniques have been employed to navigate around this difficulty.

Fukuhara [21], for example, uses the discrete version of the above integral

$$\frac{1}{2}\sum_{1\le i<j\le n}\frac{q_iq_j}{|\mathbf{x}_i-\mathbf{x}_j|}\Delta_i s\Delta_j s$$

which is obviously finite. This approach yields physically meaningful results for a discrete set of charges. But are these results meaningful in the limit? I.e., in terms of the ABO yardstick, are these results physically meaningful as each $\Delta_i s$ is approaches zero?

Birman and Lomonaco in [9] and Lomonaco in [35] renormalize by replacing the Coulomb potential of classical electrodynamics $\frac{1}{r}$ with $\frac{1}{r+\epsilon}$, so that the energy $\mathcal{E}_\epsilon$ becomes

$$\mathcal{E}_\epsilon=\frac{1}{2}\oint\oint\frac{\lambda(s)\lambda(s')dsds}{|\mathbf{x}(s)-\mathbf{x}(s')|+\epsilon}$$

It is hoped, but not proven, that a result gained with this definition of energy will give meaningful results about asymptotic behavior, i.e., as $\epsilon$ tends to zero.

With this renormalization scheme for the conducting HJP for curves, it can be shown [9] [35] that a minimal $\epsilon$-energy knot with $\mathbf{x}(s)$ $C^2$ and $\lambda(s)$ $C^1$ satisfies the following **minimum energy equations**

$$\begin{cases}\nu_\epsilon\frac{d^2\mathbf{x}}{ds^2} & = & -\lambda_\epsilon\mathbf{E}_\epsilon\\ \\ \mu_\epsilon & = & \Phi_\epsilon\end{cases}$$

where the constant $\nu_\epsilon$ is the **tension** in the curve, where the constant $\mu_\epsilon$ is the constant **potential** of the conducting curve, where $\mathbf{E}_\epsilon=\mathbf{E}_\epsilon[\mathbf{x}]$ is the $\epsilon$-electric field functional, and where $\Phi_\epsilon=\Phi_\epsilon[\mathbf{x}]$ is the $\epsilon$-potential functional. It is not clear that the solutions $\mathbf{x}_\epsilon$ to the above $\epsilon$-equation approach a limit as $\epsilon$ tends to zero. And even if they do, is such a limit physically meaningful?

Although both $\nu_\epsilon$ and $\mu_\epsilon$ tend to infinity as $\epsilon$ tends to zero, the quantity

$$v_\epsilon+\frac{1}{2}\mu_\epsilon\lambda_\epsilon$$

can be shown to approach a limit. Thus, the limit of the following linear combination of the above $\epsilon$-equations

$$\left(v_\epsilon + \frac{1}{2}\mu_\epsilon\lambda_\epsilon\right)\ddot{\mathbf{x}} = -\lambda_\epsilon \mathbf{E}_\epsilon + \frac{1}{2}\lambda_\epsilon \Phi_\epsilon \ddot{\mathbf{x}}$$

exists.

In terms of the ABO yardstick, are the solutions of the limiting equation physically meaningful?

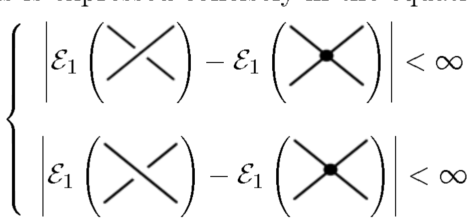
Freedman, He, and Wang in[18] [20] assume a constant charge density $\lambda$ of 1, and use a renormalization procedure similar to, but not the same as, one used by O'Hara [45] [46] [47] [48]. Their renormalized energy in [18] [20] for the $\frac{1}{r}$ potential of classical electrodynamics is:

$$\mathcal{E}_1[\mathbf{x}] = \oint\oint \left(\frac{1}{|\mathbf{x}(t) - \mathbf{x}(t')|} - \frac{1}{D\left(\mathbf{x}(t), \mathbf{x}(t')\right)}\right) \left|\frac{d\mathbf{x}(t)}{dt}\right| \left|\frac{d\mathbf{x}(t')}{dt'}\right| dt dt'$$

where $D(\mathbf{x}(t), \mathbf{x}(t'))$ denotes the distance between the points $\mathbf{x}(t)$ and $\mathbf{x}(t')$ on the curve $\mathbf{x}(t)$. It is proven that the minimizer of $\mathcal{E}_1$ over the class $\Gamma$ of all rectifiable, simple closed curves of length 1 is a planar, convex, simple, closed curve.[10] It is also conjectured that any minimizer of $\Gamma$ is a round circle of circumference 1.

But are the solutions physically meaningful? Does this approach measure up to the ABO yardstick?

In closing this section, it should be noted that there is no infinite energy barrier between knot types for the classical $\frac{1}{r}$-potential energy. I.e., the renormalized $\frac{1}{r}$-potential energy does not blow up as the loop $\alpha$ acquires double points. This is expressed concisely in the equations




The finiteness of this classical energy barrier fits neatly within Thomson's atomic vortex philosophy. In the last part of the first section of this paper, this is expressed in Atiyah's fourth listed reason for the longevity of Thomson's theory, namely, the *transmutation* characteristic. On the other hand, it does raise a fundamental question that has yet to be resolved. Are there any non-trivial minimal energy knots for the renormalized classical energy? Or in terms of the ABO yardstick, as the cross-section tends to zero, will every charged string or wire try to cross through itself as it seeks a minimal

[10] This does not imply that the minimizer of a representive of any specific knot type is always a planar, convex, simple, closed curve. Please refer to Figures 10 and 11 and to the discussion in the following last three paragraghs of this section.

energy position? Or will it in some cases seek a non-trivial minimal energy position? Figures 10 and 11 indicate possible examples of non-trivial minimal energy knots. Needless to say, it is important to know the answer to this question.

## 9. A DIGRESSION OUTSIDE THE REALM OF PHYSICS: MÖBIUS "ENERGY" AND OTHER FUNCTIONALS OF KNOTS

We now consider "energies" for knots and links that are based on non-physical potentials.[11] As a result, the ABO yardstick which is the original motivation for the original Honey Jar Problem (HJP) is completely abandoned. (Strictly speaking, this topic does not belong to the purview of this paper because the focus of this manuscript is classical electrodynamics, or at the very least, classical physics.)

The main motivation behind this shift to non-physical "energies" appears to stem mainly from the existence of a finite classical-energy barrier between knot types, as mentioned at the end of the last section. For the purpose of creating an infinite "energy" barrier between knot types, various non-physical "energy" functionals have been devised. For example, an "energy" based on a non-physical $\frac{1}{r^d}$-potential for $d \geq 2$ provides an infinite "energy" barrier between knot types.

Once a non-physical "energy" is chosen, the research objective shifts away from the Honey Jar Problem and from the ABO yardstick objective to the problem of finding the "best" shape, i.e., a shape of minimal "energy", for knots and links. More specifically, the new research objective is to find flows on isotopy classes of knots and links that terminate at a "best" shape for each given knot or link type, and then to use this "best" shape to develop new knot invariants.

One of the most prominent of these "energies" is the "energy" functional based on the non-physical $\frac{1}{r^2}$-potential [21] [45] [46] [47] [48] [32] [43] [12] [18] [20] [34] [13] [50] [51]. Freedman and He's version of this functional, which they call Möbius energy", [18] [20] is defined as

$$\mathcal{E}_2[\mathbf{x}] = \oint \oint \left( \frac{1}{\left| \mathbf{x}(t) - \mathbf{x}(t') \right|^2} - \frac{1}{D\left(\mathbf{x}(t), \mathbf{x}(t')\right)^2} \right) \left| \frac{d\mathbf{x}(t)}{dt} \right| \left| \frac{d\mathbf{x}(t')}{dt'} \right| dt dt'$$

where $D\left(\mathbf{x}(t), \mathbf{x}(t')\right)$ denotes the distance between the points $\mathbf{x}(t)$ and $\mathbf{x}(t')$ on the curve $\mathbf{x}(t)$. This functional is independent of the length of the curve $\mathbf{x}(t)$.

[11] Strictly speaking, these "energies" should be more correctly named knot functionals. But we bow begrudgingly to the overwhelming misleading misuse of this term in the mathematical literature.

In [18], Freedman and He prove the surprising result that, $\mathcal{E}_2[\mathbf{x}]$ is invariant under the action of the group $GM(\mathbb{R}^3 \cup \infty)$ of Möbius transformations.[12] More precisely, they prove that for every $T$ in $GM(\mathbb{R}^3 \cup \infty)$

$$\mathcal{E}_2[T\mathbf{x}] = \begin{cases} \mathcal{E}_2[\mathbf{x}] & \text{if } T \text{ takes no point } \mathbf{x} \text{ to } \infty \\ \\ \mathcal{E}_2[\mathbf{x}] - 4 & \text{if } T \text{ takes some point } \mathbf{x} \text{ to } \infty \end{cases}$$

They also show that the topological crossing number $c([\mathbf{x}])$ is bounded by the Möbius "energy," i.e.,

$$2\pi c([\mathbf{x}]) + 4 \leq \mathcal{E}_2[\mathbf{x}]$$

It immediately follows that the number of distinct knot types with representatives below any given Möbius "energy" threshold is bounded. They also prove that a minimal Möbius "energy" representative exists for every irreducible knot type. Moreover, they show that every locally extremal loop for the functional $\mathcal{E}_2$ is $C^{1,1}$.

In [32], Kim and Kusner construct the first explicit example of non-trivially knotted curves which are critical for $\mathcal{E}_2$.

## 10. THE HONEY JAR PROBLEMS FOR HOLLOW AND SOLID TUBES: MINIMUM ENERGY KNOTTED TUBES OF ELECTRICAL CHARGE

One of the chief difficulties with the HJP for curves is that infinitely thin charged strings and wires simply do not exist.

The HJP problems for hollow and solid tubes are more in keeping with the Kelvin approach. But most importantly, the classical electrostatic energy of charged hollow and solid knotted and linked tubes is a well defined, finite physical quantity [29]. No renormalization is needed.

The classical electrostatic energy of charged hollow tubular links is given by

$$\mathcal{E}_{Surface} = \frac{1}{2} \iint\limits_{\bigcup_i L\partial\mathbb{T}_i} \iint\limits_{\bigcup_i L\partial\mathbb{T}_i} \frac{\sigma\sigma'}{|\mathbf{y}-\mathbf{y}'|} da da'$$

where $\sigma$, and $\sigma'$ denote respectively the surface charge densities at points $P$ and $P'$ on the surface, and where $P$ and $P'$ are given respectively by the position vectors $\mathbf{y}$ and $\mathbf{y}'$. The energy $\mathcal{E}_{Solid}$ for a solid tube is defined in like manner.

In [35], Lomonaco studies both the conductor and non-conductor HJP for hollow and solid tubes. For the conductor HJP for hollow tubes, he proves that a minimal energy electrically charged hollow tubular link of constant

[12] For a definition of $GM(\mathbb{R}^3 \cup \infty)$ see [6].

circular cross section of radius $\epsilon$ must satisfy the following **minimum energy equations**

$$\begin{cases} \left(v_\epsilon + \mu_\epsilon \frac{Q}{L}\right) \frac{d^2\mathbf{x}}{ds^2} & = & -\oint \sigma_\epsilon \mathbf{E}_\epsilon \omega d\theta \\ \\ \mu_\epsilon & = & \Phi_\epsilon \end{cases}$$

where $\mathbf{x}(s)$ is the center axis curve of the tube parameterized by arclength $s$, where the constant $\nu_\epsilon$ is the **tension** in the center curve $\mathbf{x}$, where the constant $\mu_\epsilon$ is the **electrical potential** of the conducting surface, where $\sigma_\epsilon$ is the surface charge density, where $\mathbf{E}_\epsilon[\mathbf{x}]$ is the electric field functional on the surface, where $\Phi_\epsilon[\mathbf{x}]$ is the electrical potential functional on the surface, and where $\omega d\theta ds$ is the surface area 2-form.

In [35], results are also obtained for the non-conducting HJP for hollow tubes. Moreover, in [35], results for conducting and non-conducting HJP for solid tubes are given.

These approaches are more likely to measure up to the ABO yardstick. But do they?

## 11. **Conclusion:** CONCLUSIONS FOR THIS CENTURY AND OPEN QUESTIONS FOR THE NEXT MILLENNIUM

In our tale of two centuries, we have traced the impact of Sir William Thomson's (Lord Kelvin's) work on his atomic vortex theory in the nineteenth century to current research in the twentieth century on magnetic and electrostatic knots and links. Clearly, there are many open questions to be answered, especially in regard to dynamic behavior of knotted loops of electrical charge and current, that most certainly will keep researchers occupied much into the next millennium.

## References


[1] Arnold, V.I., **The asymptotic Hopf invariant and its applications**, Sel. Math. Sov., Vol. 5, No. 4 (1986), pp327-345. (English translation)

[2] Arnold, V.I., **"Mathematical Methods of Classical Mechanics,"** Springer-Verlag, New York (1978).

[3] Atiyah, Sir Michael F., **"The Mysteries of Space,"** The 1991 Josiah Willard Gibbs Lecture, San Francisco, California, January 1991, AMS Selected Lectures in Mathematics (video tape).

[4] Atiyah, Sir Michael F., **"The Geometry and Physics of Knots,"** Cambridge University Press 1990.

[5] Baez, John, and Javier P. Muniain, **"Gauge Fields, Knots and Gravity,"** World Scientific, River Edge, New Jersey (1994).

[6] Beardon, Alan F., **"The Geometry of Discrete Groups,"** Springer-Verlag, New York (1983).

[7] Berger, Mitchell A., and George B. Field, **The topological properties of magnetic helicity**, J. Fluid Mech. (1984), vol. 147, pp 133-148.

[8] Berger, M.A., **Energy-crossing number relations for braided magnetic fields**, Physical Rev. Letters, Vol. 70, No. 6, 8 February 1993, pp 705 - 708.

[9] Birman Joan and S. J. Lomonaco, Jr., **Electrostatic knots,** talk presented at a special session on Low Dimensional Topology at the joint AMS/CMS meeting held in Vancouver, British Columbia, Canada, June 2, 1983.

[10] Brakke, Kenneth A., **"The Motion of a Surface by its Mean Curvature,"** Princeton University Press, Princeton, New Jersey, 1978.

[11] Brakke, Kenneth A., **Surface evolver manual,** Minnesota Geometry Center, Research Report GCG55, July 1993.

[12] Bryson, Steve, Michael H. Freedman, Zheng-Xu He, and Zhenghan Wang, **Möbius invariance of knot energy**, A.M.S. Bull. Vol. 28, No. 1, January 1993, pp. 99 - 103.

[13] Buck, G. and J. Simon, **Knots as dynamical systems**, Topology and its Applications 51 (1993) 229 -246.

[14] Chui, A.Y.K. and H.K. Moffatt, **Minimum energy magnetic fields with toroidal topology**, in "Topological Aspects of the Dynamics of Fluids and Plasmas," ed. H. K. Moffatt, G.M. Xaslavsky, P. Comte, and M. Tabor, Kluwer Academic Publishers (1992), pp 195-218.

[15] Freedman, Michael H., **A note on topology and magnetic energy in incompressible perfectly conducting fluids**, J. Fluid Mech. (1988), vol. 194, pp 549 - 551.

[16] Freedman, Michael H. and Zheng-Xu He, Divergence free fields: **Energy and asymptotic crossing number**, Ann. Math.,134,(1991), 189-229.

[17] Freedman, Michael H. and Zheng-Xu He, **Links of tori and the energy of incompressible flows**, Topology, vol. 30, No. 2 (1991), pp 283 - 287.

[18] Freedman, Michael H., Zheng-Hu He, and Zhenghan Wang, **Möbius Energy of Knots and Unknots**, Ann. Math.,139 (1994),I-50.

[19] Freedman, M.H. and F. Luo, **Equivariant isotopy of unknots to round circles**. Preprint submitted to Topology and Its Applications.

[20] Freedman, Michael H. and Zheng-Xu He, **Research announcement on the "energy" of knots**, in "Topological Aspects of the Dynamics of Fluids and Plasmas," ed. H.K. Moffatt, G.M. Xaslavsky, P. Comte, and M. Tabor, Kluwer Academic Publishers (1992), pp 219 - 222.

[21] Fukuhara, Shinji, **Energy of a knot**, pp 443-451, in "A Fete of Topology: Papers Dedicated to Itiro Tamura," edited by Matsumoto,Y.T., and S. Morita, Academic Press, New York (1988).

[22] Fuller, F. Brock, **The writhing number of a space curve**, Proc. Nat'l Acad. Sci. USA, Vol. 68, No. 4, 1971, pp 815 - 819.

[23] Gauss, Karl Friedrich, **Gauss 1833 Werke. Königlichen Gesellschaft der Wissenschaften zu Göttingen**, 1877, p 605 -

[24] Goldstein, Herbert, **"Classical Mechanics,"** (second edition) Addison-Wesley, Reading, Massachusetts (1980).

[25] Green, George and John T. Lloyd (editors), **"Kelvin Papers: Index to the manuscript collection of William Thomson, Baron Kelvin in Glasgow University Library,"** Glasgow University Library (1977).

[26] Helmholtz, Herman von, **Über Integrale, der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen**, Jour. Reine Angewandte Mathematik, 55 (1858), pp 25 - 55.

[27] Jackson, John David, **"Classical Electrodynamics,"** (second edition) John Wiley & Sons, New York (1975).

[28] Kauffman, Louis H., **"Knots and Physics"** (second Edition), World Scientific, New Jersey (1993).

[29] Kellogg, Oliver Dimon, **"Foundations of Potential Theory,"** Dover, New York (1954).

[30] Kelvin, Lord (also known as Sir William Thomson), **"Baltimore Lectures on Molecular Dynamics and the Theory of Light,"** C.J. Clay and Sons, Cambridge University Press Warehouse, Ave Maria Lane, London (1904).
[31] Kelvin, Lord (also known as Sir William Thomson), and Peter Guthrie Tait, **"Treatise on Natural Philosophy,"** Volumes I & 2, Cambridge University Press (1923).
[32] Kim, Denise and Rob Kusner, **Torus knots extremizing the Möbius energy**, Experimental Math, 2, (1993), pp I - 9.
[33] Knott, Cargill Gilston, **"Life and Scientific Work of Peter Guthrie Tait,"** Cambridge University Press (1911).
[34] Kusner, Robert B., and John M. Sullivan, **Möbius energies for knots and links, surfaces and manifolds**, Minnesota Geometry Center, Research Report GCG64, University of Minnesota, Minneapolis, MN.
[35] Lomonaco, S.J., Jr., **Minimum energy knotted tubes of electrical charge**, in preparation.
[36] Maxwell, James Clerk, **On physical lines of force**, Parts I-11, March, April, May 1861, Parts III-IV, Jan., Feb., 1862.
[37] Moffatt, H.K., **"Magnetic Field Generation in Electrically Conducting Fluids,"** Cambridge University Press, New York, New York (1978).
[38] Moffatt, H.K., **The degree of knottedness of tangled vortex lines**, J. Fluid Mech., vol. 35, part 1 (1969) pp. 117 -129.
[39] Moffatt, H.K., **Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1. Fundamentals**, J. Fluid Mech. (1985), vol. 159, pp. 359 -378.
[40] Moffatt, H.K., **The energy spectrum of knots**, Nature, vol. 347, September 1990, pp 367 -369.
[41] Moffatt, H.K., **Relaxation under topological constraints**, in "Topological Aspects of the Dynamics of Fluids and Plasmas," ed. H.K. Moffatt, G.M. Xaslavsky, P. Comte, and M. Tabor, Kluwer Academic Publishers (1992), pp 3-28.
[42] Moffatt, H.K. and Renzlo L. Ricca, **Helicity and the Calugareanu invariant**, Proc. R. Soc. London A (1992) 439, pp 411-429.
[43] Nakauchi, Nobumitsu, **A remark on O'Hara's energy of knots**, Proc. AMS, vol. 118, No. 1, May 1993, pp 293 - 296.
[44] Niven, W.D. (editor), **"The Scientific Papers of James Clerk Maxwell,"** Dover, New York, 1966.
[45] O'Hara, Jun, **Energy of a knot**, Topology, vol. 30, No.2, pp241 -247, 1991.
[46] O'Hara, Jun, **Energy functional of knots**, "Topology Hawaii," K.H. Dovernan ed., World Scientific.
[47] O'Hara, Jun, **Energy functionals of knots**, Topology and its Applications.Ohara4
[48] O'Hara, Jun, **Energy functionals of knots II**, preprint
[49] Ricca, R. L., and H.K. Moffatt, **The helicity of a knotted vortex filament**, in "Topological Aspects of the Dynamics of Fluids and Plasmas," ed. H.K. Moffatt, G.M. Xaslavsky, P. Comte, and M. Tabor, Kluwer Academic Publishers (1992), pp 225 -236.
[50] Simon, J., **Energy functions of knots**, preprint.
[51] Simon, Jonathan K., **Energy functions of polygonal knots**, J. Knot Theory and Its Ramifications, vol. 3, no. 3 (1994) pp 299-320.
[52] Stewart, Ian, **Finding the energy to solve a knotty problem**, Science, 6 March, 1993, p. 18.
[53] Tait, P.G., **"On Knots 1, 11, III,"** Scientific Papers, Vol. I, Cambridge University Press, London (1898), pp273-347.
[54] Tait, P.G., **Scientific Papers**, Vol. 11, Cambridge University Press (1900).

[55] Thomson, J.J., **"Notes on Recent Researches in Electricity and Magnetism,"** Clarendon Press, Oxford (1893).

[56] Thomson, W.T. (Lord Kelvin), **On vortex atoms**, Philosophical Magazine, 34 July 1867, p15-24. Mathematical and Physical Papers, Vol.4, Cambridge (1910).

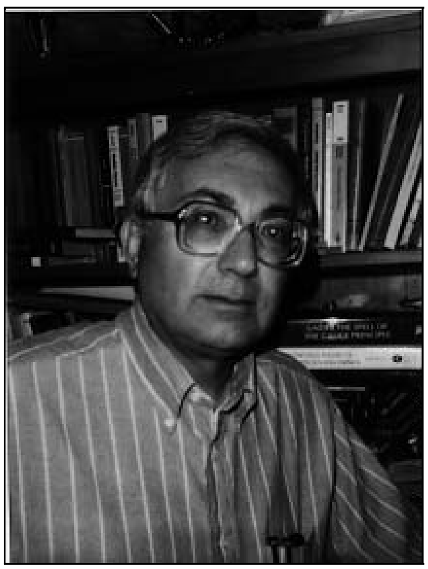
[57] Thomson, Sir William (Lord Kelvin), **On vortex motion**, Trans. Royal Soc. Edinburgh, 25 (1868/69), pp 217-260.

[58] Thomson, Sir William (Lord Kelvin), **Vortex statics**, Proceedings of the Royal Society of Edinburgh, Vol. IX , No. 94 (1875), pp 59 - 78.

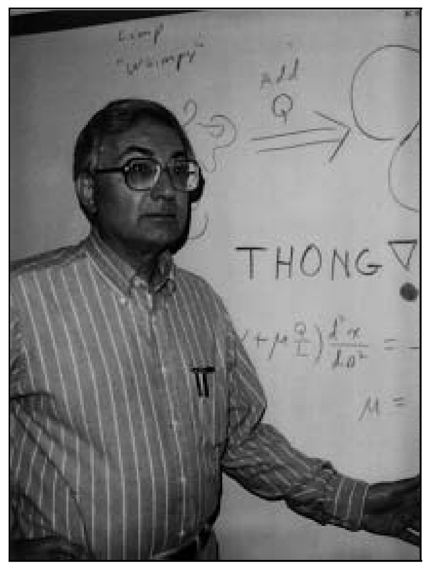
[59] Thomson, Sir William (Lord Kelvin), **"Reprints of Papers on Electrostatics and Magnetism,"** (second edition) Macmillan & Co., London (1884).

[60] Wilson, David B. (editor), **"Catalogue of the Manuscript Collections of Sir George Stokes and Sir William Thomson, Baron Kelvin of Largs in the Cambridge University Library,"** Cambridge University Library (1976).

[61] Woltjer, L., **A theorem on force-free magnetic fields**, Proc. Nat'l Acad. Sci. U.S.A., **44** (1958), pp 489 - 491.



The author writing




The author lecturing

University of Maryland Baltimore County, Baltimore, Maryland 21228
*E-mail address*: Lomonaco@UMBC.Edu