

PHI, THE GOLDEN RATIO (to 4599 Decimal Places), AND FIBONACCI NUMBERS

Murray Berg
Oakland, Calif.

The golden ratio can be obtained by the division of a line segment into two parts such that the ratio of the total line to the larger segment is the same as the larger segment to the smaller segment. This ratio was given the name, phi, by the U.S. mathematician, Mark Barr, over 50 years ago [1].

If B were the larger segment and A the smaller segment, then

$$(1) \quad \frac{A + B}{B} = \frac{B}{A}$$

Letting A = unity, we have

$$\frac{1 + B}{B} = B$$

and the positive root, B is the number, phi:

$$B = \frac{1 + \sqrt{5}}{2}$$

If we let B equal unity, then the result would be one less than phi. Phi is the only number whose reciprocal is obtained by subtracting one from itself.

Phi is an irrational number, and its determination depends on the calculation of $\sqrt{5}$. It can also be calculated from the Fibonacci series [1] and [2].

The Fibonacci numbers are an infinite series, where each element in the series is the sum of the preceding two elements, and the first two are equal to 1.

$$\text{Thus} \quad F_1 = F_2 = 1$$

$$\text{and for } i > 2 \quad F_i = F_{i-1} + F_{i-2},$$

where F_i = the i th number in the Fibonacci series.

A program was written for the IBM 1401 which produced Fibonacci numbers up to 4000 digits in length. A pair of consecutive numbers were printed out at each 100 digit level.

F_{476} is the first 100 digit Fibonacci number. For each succeeding 100 digit level, the F number increased by 478 and 479 alternately, i. e., F_{954} is the first Fibonacci number with 200 digits, F_{1433} has 300, F_{1911} has 400, up to F_{19137} with 4000 digits. Table 1 lists the Fibonacci numbers F_{11003} and F_{11004} .

It is evident from equation (1) that phi is the ratio between two consecutive members of the Fibonacci series. Theoretically, we can obtain the ratio to any number of decimal places by performing one division with Fibonacci numbers that are far out in the series. However, this must have practical limits.

A separate program performed such a division for F_{11003} and F_{11004} , both 2300 digits in length. The division instruction took 21 minutes. This was repeated with the numerator and denominator reversed. The ratios coincided to 4598 decimal places.

Empirical observation of members early in the series show that the true ratio lies somewhere between the ratios obtained by the two divisions. In fact, it is approximately one fourth the distance between the first two non-coinciding digits of the two divisions. These two digits for the first division were 42 and for the second division, 10. The true first digit of the two would be 3. This is the 4599th digit of phi. Table 2 lists the fractional digits of phi.

REFERENCES

1. Martin Gardner, Mathematical Games, Scientific American, Vol. 201, No. 2, Aug. 1959, pp. 128-134.
2. N. Vorob'ev, Fibonacci Numbers, Translated from the Russian by Halina Moss, Blaisdell Publishing Co., 1961.

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TABLE I

F11003

1385226108	1389849146	3318369579	2494356958	4648374158	2397008279	8631194478	0123270814	9606278507	1751399903
2651224797	6473891243	0891846980	3039604879	0046956080	4905717207	6604928573	5218136522	1731643799	6629392564
7173511856	2146914169	6708707974	6699151188	2442466984	5710547919	6199746044	7472341072	4003782106	0553357908
7574996945	6924869639	6917120736	5268922184	1896192623	7606833193	0271352823	3209213511	6654782175	9454851487
2496093268	9753082994	9214939556	6355848406	4062988440	6134620903	7912031476	6964248790	8726924515	7016612608
0051225211	5428814745	3016761256	7294777233	1289541117	0885655450	2312356840	1372262062	2298301027	7269258591
2012890852	4806673466	9663354501	1142199481	3650636786	5229287829	8848188137	8172683771	0267970914	5115676896
2723799650	8596935856	5423569649	1078336886	5232864678	6569734815	6337166059	1819627031	1521792082	0236762110
5572088355	6215509544	0202887462	8129578616	9581239454	0372956238	2134273674	4384466075	5128521668	0497974568
7660404686	1519119027	4619727906	1507794036	8906861517	2622180521	1023706129	5731454302	7936180979	1895837832
2764124938	6088403251	4049695388	4991667711	8842003238	3436710506	9902097057	1865017517	9126821677	1994293318
2740869994	0851091246	9478056732	4448753439	5432510823	1301627655	8708508729	2151315800	9109439700	1412268114
8090725106	3521862288	6871088487	7752208419	8640236523	8079861118	4773187478	6521109077	2668319024	3505313833
5635374951	5618951083	0648797240	0490175133	8680223685	3297035838	1198854328	8383937835	9669940528	7162356283
5632860204	6956657639	5257855778	2371306204	3737166938	2038668754	7674835212	3478098396	4289148123	8919252426
4129306207	6694621619	2343575456	2042187238	4602706582	1109668539	8459488464	3113389750	2861398637	8833783107
0887842148	5647170984	3084772401	9888001330	8095527704	2088481463	9095526048	6913994736	0451421700	6160580357
4139509331	5149971174	6116928633	0190593393	4778407869	5760226150	4212756331	3772061206	7071231484	0297335718
8165137824	6242619136	9332914897	6535760412	9776981220	2863991636	1855595685	2283544770	1435962831	8080741054
3977633239	4708095587	4875953506	4381158531	0839843746	7700728139	7778864638	1922731227	0284538002	6509070757
6975227979	1208062350	8359916939	5154135119	9519556612	8740630214	0954042220	3240718052	7657932922	9545448966
0339811947	9365812919	9616056923	2347309682	3963617027	26334743224	7003593412	0651665497	7760111150	99822857292
5460935494	6299557873	1660491087	1345701086	7269788440	18711245899	2914722272	8848008668	9496842560	1752488377

TABLE 1 (Continued)

F11004	2241342925	0726149417	8768445221	7640750624	9224305335	3277323564	4619416117	2235392377	3056632665	5745110745
	2509731507	7828619535	7415711132	5130674465	4508593892	5043085035	6940122848	8191759950	7301724135	8171825183
	0928585522	0324203872	2540352820	1125292205	7271034243	4836044721	0363252053	1266365415	0010467211	5995273570
	0519690070	8154592872	8421330888	7876866356	2200570692	4022736279	1533037075	6075911934	2980240833	5229508679
	4835803161	5626140930	3642851474	0824316080	8489859296	8152543605	7710729043	4702322192	0394506576	7556471880
	1852815052	8910088959	0065297343	9636830863	1040299947	0813959319	2799580310	1107921801	9909982891	8337347082
	0729628005	3193120486	2310564502	2262602234	2833899932	2885933136	1942355534	9314355005	4676741907	4830542906
	9507437916	1485588617	1102677540	5378988772	6054590766	7167333068	9901230463	3515170420	2771791077	4707282953
	9807088368	5168935663	7998120065	4173341196	6919087513	2306681796	1109461965	4626972652	4844032105	7089563389
	2558962811	8372532811	5690534216	0287770603	6864679470	1130003741	4686266305	5064980434	5989905718	8508082738
	9223333007	6393902074	7344803185	0112496308	7426219214	3034580269	9566190926	5696653468	2397757935	0645559192
	8527203580	1513269923	9354011433	8718644225	3582205650	1943318605	8535551141	2072772788	9582224424	1176699658
	4223655959	0237488568	5519687985	3969117752	3387338406	7085151201	3165688111	6727373403	7025157771	5171781686
	3583422096	2143997905	2819057484	3271191933	0627247913	5974124740	7459818257	2713876742	4295849216	4540420708
	5664623931	7181718163	5574390609	3286721969	0540640885	3679865367	6770799571	2239359060	4474264222	5386481817
	1129799983	1819808009	7828847818	6912050400	6977241945	5576981463	4364542393	3025531789	7895313665	2905867204
	8280069287	1985281490	6219182278	2098843186	3500754266	8667392361	4481010338	5608921945	4604373723	2083981697
	0743459356	3245433618	0826392315	5182193383	9849278009	0521701221	6729698803	2212727621	4583381448	8919561483
	2472352138	0294211905	8572350989	5990695907	6354573336	3605187057	7977696355	4843478023	8250297146	4130520029
	7263043600	5077368545	7178988583	5679066442	2174890839	9413215948	1841222396	8058820056	7544797041	7480859537
	1468275379	5884519860	4036505692	7219638895	4773751421	4006846575	5214682501	8676764452	2065126375	9990118794
	3738874031	9405122694	2526076260	6900250347	6468505567	2906807306	8337622732	0580472090	7584467309	6730487626
	8214395238	0405570837	5413169662	4660432562	1471942062	3375517429	9699405764	8550102519	4559914918	54362210128

PLACES), AND FIBONACCI NUMBERS

TABLE 2

2045868343	6563811772	0309179805	7628621354	4862270526	0462818902	4497072072	0418939113	7484754088	0753868917
5212663386	2223536931	7931800607	6672635443	3389086595	9395829056	3832266131	9928290267	8806752087	6692550171
1696207032	2210432162	6954862629	6313614438	1497587012	2034080588	7954454749	2461856953	6486444924	1044320771
3449470495	6584678850	9874339442	2125448770	6647809158	8460749988	7124007652	1705751797	8834166256	2494075890
6970400028	1210427621	7711177780	5315317141	0117046665	9914669798	7317613560	0670874807	1013179523	6894275219
4843530567	8300228785	6997829778	3478458782	2891109762	5003026961	5617002504	6433824377	6486102838	3126833037
2429267526	3116533924	7316711121	1588186385	1331620384	0052221657	9128667529	4654906811	3171599343	2359734949
8509040947	6213222981	0172610705	9611645629	9098162905	5520852479	0352406020	1727997471	7534277759	2778625619
4320827505	1312181562	8551222480	9394712341	4517022373	5805772786	1600868838	2952304592	6478780178	8992199027
0776903895	3219681986	1514378031	4997411069	2608867429	6226757560	5231727775	2035361393	6210767389	3764556060
6059216589	4667595519	0040055590	8950229530	9423124823	5521221241	5444006470	3405657347	9766397239	4949946584
4948730231	5417676893	7521030687	3788034417	0093954409	6279558986	7872320951	2426893557	3097045095	9568440175
5519881921	8020640529	0551893494	7592600734	8522821010	8819464454	4222318891	3192946896	2200230144	3770269923
0078030852	6118075451	9288770502	1096842493	6271359251	8760777884	6658361502	3891349333	3122310533	9232136243
1926372891	0670503399	2822652635	5620902979	8642472759	7725655086	1548754357	4826471814	1451270006	0238901620
7773224499	4353088999	0950168032	8112194320	4819643876	7586331479	8571911397	8153978074	7615077221	1750826945
8639320456	5209896985	5567814106	9683728840	5874610337	8105444390	9436835835	8138113116	8993855576	9754841491
4453415091	2954070050	1947754861	6307542264	1729394680	3673198058	6183391832	8599130396	0720144559	5044977921
2076124785	6459161608	3705949878	6006970189	4098864007	6443617093	3417270919	1433650137	1576601148	0381430626
2380514321	1734815100	5590134461	0118007905	0638142152	7093085880	9287570345	0507808145	4588199063	3612982798
1411745339	2731208092	8972792221	3298064294	6878242748	7401745055	4067787570	8323731097	5915117762	9784432847
4790817651	8097787268	4161176325	0386121129	1436834376	7023503711	1633072586	98883258710	3363222381	0980901211
0198991768	4149175123	3134015273	3843837234	5009347860	4979294599	1582201258	1045982309	2552872124	1370436149

• 61803398 8749894848

TABLE 2 (Continued)

1020547185 5496118087 6426576511 0605458814 7560443178 4798584539 7312863016 2544876114 8520217064 4041116607
 6695059775 7832570395 1108782308 2710647893 9021115691 0392768384 5386333321 5658296597 7310343603 2322545743
 6372041244 0640888267 3758433953 6795931232 2134373209 9574988946 9956564736 0072959998 3912881031 9742631251
 7971414320 1231127955 1894778172 6914158911 7799195648 1255800184 5506563295 2859859100 0908621802 9775637892
 5999164994 6428193022 2935523466 74759322695 1654214021 0913630181 94722270789 0122087287 3617073486 4999815625
 5472811373 4798716569 5274890081 4438405327 4837813782 4669174442 2963491470 8157007352 5457070897 7267546934
 3822619546 8615331209 5335792380 1460927351 0210119190 2183606750 9730895752 8957746814 2295433943 8549315533
 9630380729 1691758461 0146099505 5064803679 3041472365 7203986007 3550760902 3173125016 1320484358 3648177048
 4818109916 0244252327 1672190189 3345963786 0878752870 1739359303 0133590112 3710239171 2659047026 3494028307
 6687674363 8651327106 2803231740 6931733448 2343564531 8505813531 0854973335 0759966778 71244490583 6367541328
 9086240632 4563953572 1252426117 0278028656 0432349428 3730172557 4405837278 2679960317 3936401328 7627701243
 6798311446 4369476705 3127249241 0471670013 8247831286 5650649343 4180390041 0178053395 0587724586 6557552293
 9158239708 4177298337 2823115256 9260929959 4224000056 0626678674 3579239724 5408481765 1973436265 2689448885
 5272027477 8747335983 5367277614 0759171205 1326934483 7529916499 8093602461 7844267572 7767900191 9190703805
 2204612324 8239132610 4327191684 5123060236 27899354543 2461769975 7536890417 6365025478 5138246314 6583363833
 7602357789 9267298863 2161858395 9036399818 3845827644 9124598093 7043055559 6137973432 6134830494 9496868108
 9535696348 2817812886 2536460842 0339465381 9441945714 2666823718 3949183237 0908574850 2665680398 9744066210
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 8468831137 1852991921 6319052015 6863122282 0715599876 4684235520 5928537175 7807656050 3677313097 5191223973
 8872246825 8057159744 5740484298 7807352215 98422667662 5780770620 1943040054 2550158312 5030175340 9411719101
 9298903844 7250332988 0245014367 9684416947 9595453045 9103138116 2187045679 9786636617 4605957000 3445970113
 5251813460 0656553520 3478881174 1499412748 2641521355 6776394039 0710387088 1823380680 48