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MODEL THEORY – EXERCISE 11

To be submitted on Wednesday 29.06.2011 by 14:00 in the mailbox.

Definition.

(1) The *Prüfer p-group* $\mathbb{Z}_{p^\infty}$, for a prime number p is the multiplicative group of all p^nth roots of unity in $\mathbb{C}^\times$ for all $n < \omega$.
(2) The *fundamental theorem for finitely generated abelian groups* states that every finitely generated abelian group G is isomorphic to a direct sum of primary cyclic groups and infinite cyclic groups. A primary cyclic group is one whose order is a power of a prime.
(3) An abelian group G is divisible if for every $n < \omega$ and $x \in G$ there is some $y \in G$ such that $ny = x$.
(4) An abelian group G, equipped with a linear order $<$ is an *ordered abelian group* iff it satisfies $\forall xyu\,(x < y \Rightarrow x + u < y + u)$.
(5) An abelian group G is said to be *orderable* iff there exists some $<$ such that $(G, <)$ is an ordered abelian group.
(6) A group is called *locally finite* if every finitely generated subgroup is finite.
(7) A class of structures K is called *hereditary* if whenever $M \in K$ and $N \subseteq M$ (a substructure) $N \in K$.

Question 1.
Let p be a prime number and n a positive natural number.

(1) Prove that the group G of all p^n roots of unity in $\mathbb{C}^\times$ is cyclic of order p^n.
Solution: (there are many possible, and also one can generalize this to all finite subgroups. here is one suggestion) Consider the polynomial $f(X) = X^{p^n} - 1$. First of all this polynomial is separable (it splits into distinct linear factors – no double roots): if it had, then $(f, f') \neq 1$, but $f' = nX^{n-1}$, so X is the only prime factor of f' but it isn't a factor of f. So the set of solutions is of size p^n – the order of G is p^n. In particular it is finite. Let m be the maximal order of an element from G. Then m is of the form p^k for $k \leq n$. This means that for all $x \in G$, $x^m = 1$, but there are at most m such elements, so $m = p^n$.
(2) Conclude that the Prüfer p-group is a union $\bigcup_{i<\omega} G_i$ of finite cyclic groups of order p^i such that $G_i \leq G_{i+1}$.
Solution: Let G_i be the group of p^i-roots of unity (note that $G_0 = \{1\}$).
(3) Prove that the Prüfer p-group is a divisible abelian group.
Solution: Let G be the p-Prüfer group. It's enough to show that for every prime q, G is q-divisible. Let $x \in G$. Suppose $q = p$. Let ζ be a generator of G_{i+1} where $x \in G_i$ for some i. Let $\varepsilon = \zeta^p$, so ε is of order p^i and a generator of G_i. And so $x = \varepsilon^k$. Let $y = \zeta^k$. Then $y^p = \zeta^{pk} = \varepsilon^k = x$. If $q \neq p$, then there are $a, b \in \mathbb{Z}$ such that $ap + bq = 1$. Suppose $x \in G_i$. Then $x = x^{ap+bq} = x^{ap} \cdot x^{bq} = x^{bq} = \left(x^b\right)^q$.
(4) Conclude that if G is an abelian group, then G can be embedded in a divisible abelian group.

Work in the signature $L = \{\cdot\}$. Consider the theory $D(G)$ (the diagram of G) and the theory T of divisible abelian groups (T consists of the axioms saying that the universe is an abelian group and it is n-divisible for all n). We should show that $D(G) \cup T$ is consistent (in the language $L(G)$). By compactness, it's enough to show that every finite part of it is consistent, i.e. that $D(G_0) \cup T$ is consistent for all finitely generated $G_0 \leq G$. By the fundamental theorem, G_0 can be written as

$$\mathbb{Z}^n \oplus \mathbb{Z}/p_1^{n_1}\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_k^{n_k}\mathbb{Z}$$

where p_i are prime numbers. So $\mathbb{Z}^n \leq \mathbb{Q}^n$ which is divisible, and $\mathbb{Z}/p_i^{n_i}\mathbb{Z} \leq \mathbb{Z}_{p_i^\infty}$ which is also divisible, so G_0 is a subgroup of

$$\mathbb{Q}^n \oplus \mathbb{Z}_{p_1^\infty} \oplus \ldots \oplus \mathbb{Z}_{p_k^\infty}$$

which is divisible.

Question 2.

Suppose G is an abelian group. Show that G is orderable iff G is torsion free.

Solution: If G is orderable, say by $<$, then if $x \neq 0$, then $nx \neq 0$ for all $n < \omega$ because for instance if $x > 0$ then $0 < x = x + 0 < x + x$ and if $x < 0$ then $-x > 0$. If G is torsion free consider the theory $T \cup D(G)$ where T is the theory of ordered abelian groups in the signature $\{<, +\}$. If this theory is consistent, then G is embeddable in an ordered abelian group, but then since the axiom relating the order and the group structure is universal, this axiom remains true in G with the reduction of that ordering to G. By compactness, it's enough to show that $D(G_0) \cup T$ is consistent where G_0 is a finitely generated subgroup of G. By the fundamental theorem, G_0 is isomorphic to $\mathbb{Z}^n$ for some n (here we use the fact that G is torsion free). Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \mathbb{R}$ be n-linearly independent (over $\mathbb{Q}$) elements from $\mathbb{R}$ (which exist because as a vector space over $\mathbb{Q}$, $\mathbb{R}$ has dimension $2^{\aleph_0}$). Then G is isomorphic to the subgroup of $\mathbb{R}$ generated by this set, and in particular it is orderable.

Question 3.

(1) Suppose I and J are 2 linear orders and that J is infinite. Show that there is embedding of I into a model of $Th(J)$.
Solution: by compactness, we must show that $D(I) \cup Th(J)$ is consistent, so it is enough to embed any finite subset I_0 of I into J. But that's obvious since J is infinite.
(2) In particular show conclude that every linear order can be embedded into a dense linear order.
Solution: let $J = (\mathbb{Q}, <)$ and use the fact that denseness is elementary.

Question 4.

Show that the class of locally finite groups is hereditary, but not elementary.

Solution: If it were elementary, then let T be a set of axioms in $\{\cdot\}$ such that $G \models T$ iff G is locally finite. Add a constant c to the language. Let Σ be $T \cup \left\{c^k \neq 1 \,|k \in \omega\right\}$. Then Σ is consistent, because we can find finite groups (so also locally finite) with elements of bigger and bigger orders. But obviously in every locally finite group the order of each element is finite.