Near-horizon geometries of supersymmetric branes

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D=11 Supergravity

Eleven-dimensional supergravity consists of the following fields: [Nahm, 77; Cremmer et al., 78]

- a Lorentzian metric g;
- a closed 4-form F; and
- a gravitino Ψ.

Supersymmetric vacua $(g, F, \Psi = 0)$ are solutions of the equations of motion for which the supersymmetry variation $\delta_\varepsilon \Psi = 0$, (as an equation on ε) has solutions.

Elementary brane solutions preserving $\frac{1}{2}$ of the supersymmetry:

- *electric* membrane [Duff+Stelle, 91]
- *magnetic* fivebrane [Güven, 92]

Supermembranes

The elementary membrane solution has the following form: [Duff+Stelle, 91]

$$ds^2 = H^{-\frac{2}{3}} ds^2(\mathbb{E}^{2,1}) + H^{\frac{1}{3}} ds^2(\mathbb{E}^8)$$
$$F = \pm \,\mathrm{dvol}(\mathbb{E}^{2,1}) \wedge dH^{-1},$$

where

- $ds^2(\mathbb{E}^{2,1})$ and dvol$(\mathbb{E}^{2,1})$ are the metric and volume form, respectively, of 3-dimensional Minkowski spacetime $\mathbb{E}^{2,1}$;

- $ds^2(\mathbb{E}^8)$ is the metric of 8-dimensional Euclidean space $\mathbb{E}^8$; and

- H is a harmonic function on $\mathbb{E}^8$. For example, we can take

  $$H(r) = 1 + \frac{\alpha}{r^6},$$

  corresponding to one or more coincident membranes at $r = 0$.

Some remarks:

- More general H are possible, corresponding to parallel membranes localised at the singularities of H. In fact, we can take $H(\boldsymbol{x})$ to be arbitrary harmonic function on $\mathbb{E}^8$ with suitable asymptotic behaviour: $H(\boldsymbol{x}) \to 1$ as $|\boldsymbol{x}| \to \infty$, say.

- There exist solutions where H is invariant under some subgroup of isometries. In this case, the interpretation is often less clear: e.g., delocalised membranes,...

  [Gauntlett et al., 97]

- Although the membrane solution with H given above preserves only $\frac{1}{2}$ of the supersymmetry, it interpolates between two maximally supersymmetric solutions: $\mathbb{E}^{10,1}$ for $r \to \infty$ and $\mathrm{AdS}_4 \times S^7$ for $r \to 0$.

  [Gibbons+Townsend, 93; Duff et al., 94]

Near-horizon geometry

Notice that

$$ds^2(\mathbb{E}^8) = dr^2 + r^2 ds^2(S^7),$$

where $ds^2(S^7)$ is the metric on the unit 7-sphere $S^7 \subset \mathbb{E}^8$.

In the limit $r \to 0$,

$$\lim_{r \to 0} H(r) \sim \frac{\alpha}{r^6}.$$

Therefore in this limit,

$$ds^2 = \alpha^{-\frac{2}{3}} r^4 ds^2(\mathbb{E}^{2,1}) + \alpha^{\frac{1}{3}} r^{-2} dr^2 + \alpha^{\frac{1}{3}} ds^2(S^7).$$

The last term is the metric on a round 7-sphere of radius $R = \alpha^{1/6}$. The first two terms combine to produce the metric on 4-dimensional *anti de Sitter* spacetime with "radius" $R_{\text{adS}} = \frac{1}{2}\alpha^{1/6}$:

$$ds^2_{\text{adS}} = R^2_{\text{adS}} \left[\frac{du^2}{u^2} + \left(\frac{u}{R_{\text{adS}}} \right)^2 \frac{ds^2(\mathbb{E}^{2,1})}{R^2_{\text{adS}}} \right],$$

where $u = \frac{r^2}{4R_{\text{adS}}}$.

Other branes

Similar considerations apply to other branes.

e.g., 3-brane in $D = 10$ [Horowitz+Strominger, 91]

$$ds^2 = H^{-\frac{1}{2}} ds^2(\mathbb{E}^{3,1}) + H^{\frac{1}{2}} ds^2(\mathbb{E}^6),$$

where

$$H(r) = 1 + \frac{\beta}{r^4},$$

whose near-horizon geometry is given by

$$\mathrm{AdS}_5(\beta^{1/4}) \times S^5(\beta^{1/4}).$$

Generally there are supersymmetric p-branes in D dimensions with near-horizon geometry

$$\mathrm{AdS}_{p+2} \times S^{D-p-2}.$$

These solutions are all maximally supersymmetric. Sacrificing some (but not all!) of the supersymmetry, one can obtain p-branes with more interesting near horizon geometries.

[Gibbons+Townsend, 93; Duff et al., 95; Castellani et al., 98]

Solutions exist whose near-horizon geometries are of the form

$$\mathrm{AdS}_{p+2} \times M^{D-p-2},$$

where M is compact Einstein with positive cosmological constant $\Lambda = D - p - 3$, just as for the standard sphere $(D-p-2)$-sphere.

The transverse space to the p-brane will be the (deleted) *metric cone* $C(M)$ of M.

Topologically, $C(M) \cong \mathbb{R}^+ \times M$ with metric

$$ds^2_{\text{cone}} = dr^2 + r^2 ds^2(M).$$

If $M = S^{D-p-2}$, then $C(M) = \mathbb{E}^{D-p-1} \setminus \{0\}$. In this case, the metric extends smoothly to the apex of the cone.

More generally,
M is Einstein with $\Lambda = \dim M - 1$
$\Rightarrow C(M)$ is *Ricci-flat* and the metric has a conical singularity.

Supersymmetry $\Rightarrow$

$$M \text{ admits } \textit{real Killing spinors}$$
$$\Updownarrow$$
$$C(M) \text{ admits } \textit{parallel spinors}$$

[Bär, 93]

Simply-connected spin manifolds admitting parallel spinors are classified by their *holonomy group*. [Wang, 89]

Fact: $C(M)$ Ricci-flat $\Rightarrow$ not locally symmetric.

Fact: $C(M)$ is either flat or has irreducible holonomy group. [Gallot, 79]

Therefore, we need only consider irreducible holonomy groups of manifolds which are not locally symmetric.
In other words, those in Berger's table.

Holonomy and parallel spinors

Let (X^D, g) be a simply-connected irreducible spin riemannian manifold which is not locally symmetric and ∇ the riemannian connection.

The *holonomy group* Hol(∇) of ∇ is a compact Lie subgroup of SO(D). These were classified by Berger, with simplifications due to Simons, Alekseevskiĭ.

Of those, the ones which admit parallel spinors are given by the following table, which also lists the number N (or (N_L,N_R) in even D) of linearly independent parallel spinors.

[Besse, 87; Wang, 89]

D	Hol(∇)	Geometry	N
$4k+2$	SU($2k+1$)	Calabi–Yau	(1,1)
$4k$	SU($2k$)	Calabi–Yau	(2,0)
$4k$	Sp(k)	hyperkähler	($k+1$,0)
7	G_2	exceptional	1
8	Spin(7)	exceptional	(1,0)

The *holonomy principle* guarantees the existence of certain parallel tensors on $C(M)$, whenever the holonomy group reduces.

For the geometries in the Table, we find

- **Calabi–Yau n-fold**
  Orthogonal complex structure I and complex holomorphic volume n-form Λ.

- **Hyperkähler**
  Quaternionic structure I, J and K.

- G_2 **holonomy**
  3-form Φ and 4-form $\tilde{\Phi} \equiv \star\Phi$.

- Spin(7) **holonomy**
  Self-dual 4-form Ω.

The parallel tensors on $C(M)$, together with the *Euler vector* $\xi = r\partial_r$, induce interesting geometric structures on M.
(We identify M and $\{1\} \times M \subset C(M)$.)

Example: $C(M)$ has Spin(7) holonomy, Ω the Cayley 4-form. Then define a 3-form ϕ on M by

$$\phi \equiv \imath(\xi) \cdot \Omega \quad \text{so that} \quad \Omega = dr \wedge \phi + \star\phi.$$

$\nabla\Omega = 0$ on $C(M)$ implies $\nabla\phi = \star\phi$ on M.
$\Rightarrow M$ has *weak G_2 holonomy*.

Example: $C(M)$ has G_2 holonomy, Φ the associative 3-form. The 2-form $\omega \equiv \imath(\xi) \cdot \Phi$ defines an almost complex structure J on M by

$$\langle X, JY \rangle = \omega(X, Y).$$

$\nabla\Phi = 0$ on $C(M)$ implies $\nabla_X J(X) = 0$ but $\nabla_X J \neq 0$ on M. $\Rightarrow M$ is *nearly Kähler*.

Similarly one can recognise the geometric structures for the hyperkähler and Calabi–Yau cases.

Every parallel complex structure I on $C(M)$ gives rise to a *Sasaki* structure on M:

- a unit norm Killing vector $X = I\,\xi$;
- a dual 1-form $\theta = \langle X, - \rangle$;
- a (1,1) tensor $T = -\nabla X$ satisfying

$$\nabla_V T(W) = \langle V, W \rangle X - \theta(W)\, V\,.$$

(In fact, $C(M)$ is Kähler $\Leftrightarrow$ M is Sasaki.)

It follows that

$C(M)$ is Calabi–Yau $\Leftrightarrow$ M is *Sasaki–Einstein*
$C(M)$ is Hyperkähler $\Leftrightarrow$ M is *3-Sasaki*

[Bär, 93]

New supersymmetric horizons

In summary, if the transverse space of a supersymmetric p-brane in D dimensions is a metric cone $C(M)$, then the near-horizon geometry is $\mathrm{AdS}_{p+2} \times M^d$, where $d \equiv D - p - 2$.

The fraction ν of the supersymmetry which is preserved <u>relative to the round sphere</u> will depend on the number of Killing spinors. We also list the fraction $\bar{\nu}$ corresponding to the opposite orientation for M.

d	Geometry of M	$(\nu,\ \bar{\nu})$
7	weak G_2 holonomy	$(\frac{1}{8}, 0)$
	Sasaki–Einstein	$(\frac{1}{4}, 0)$
	3-Sasaki	$(\frac{3}{8}, 0)$
6	nearly Kähler	$(\frac{1}{8}, \frac{1}{8})$
5	Sasaki–Einstein	$(\frac{1}{4}, \frac{1}{4})$

Notice that this is particularly rich in dimension 7, when the transverse space to the brane has dimension 8.

Some examples

Consider the generalised membrane solution where $\mathbb{E}^8$ is replaced by a metric cone $C(M)$. Its near-horizon geometry is of the form

$$\mathrm{AdS}_{3+1} \times M^7.$$

Supersymmetric M include:

- 3-**Sasaki** ($\frac{3}{16}$ supersymmetry)
  $\mathrm{SU}(3)/\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_1)$
  N_{010} [Castellani+Romans, 84]
  Infinite toric family [Bielawski, 97; Boyer et al, 98]

- **Sasaki–Einstein** ($\frac{1}{8}$ supersymmetry)
  M_{pqr} [Castellani et al., 84]
  Circle bundles over $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2 \times \mathbb{CP}^1$, $\mathbb{CP}^3$, $\mathrm{SU}(3)/\mathbb{T}^2$, $\mathrm{Gr}(2|5)$ [Boyer+Galicki]

- **weak G_2 holonomy** ($\frac{1}{16}$ supersymmetry)
  Any squashed 3-Sasaki manifold (e.g., S^7)
  $\mathrm{SO}(5)/\mathrm{SO}(3)$, N_{pqr} [Castellani+Romans, 84]
  N_{kl} [Aloff–Wallach, 75]

In $D = 10$ a membrane has a 7-dimensional transverse space which can be chosen to be a cone over a nearly Kähler manifold.

In the 3-brane solution in $D = 10$, we can also substitute the cone over the round S^5 for the cone over any Sasaki–Einstein manifold. Examples include circle bundles over $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Notice that for 5 transverse dimensions ($d = 4$), there are no examples except the sphere. Therefore there does not exist a generalised M5-brane in this context.

General remarks

Homogeneous examples have all been known for some time. [Castellani et al., 98]
There are infinitely many homotopy types of weak G_2 holonomy manifolds, and even examples of *exotic differentiable structures*.
[Kreck+Stolz, 88]

Non-homogeneous examples are also plentiful, although their supergravity spectrum is much harder to compute. For example, all possible rational homotopy types ($b_i = 0$ except for $b_2 = b_5$) of 3-Sasaki manifolds appear.

There exists a 3-Sasaki quotient construction which, via the cone construction, corresponds to the hyperkähler quotient construction. [Boyer et al., 94]
All known examples are toric quotients of spheres whose cones are toric hyperkähler manifolds. Some of these toric hyperkähler are dual to intersecting branes.
[Gauntlett et al., 97]

Outlook

The relation between supersymmetry and geometry is still going strong. Many questions are open:

Questions:

- What can one say about the spectra of the dual CFTs in the non-homogeneous examples?

- Does duality relate the near-horizon geometries in an interesting way? Can some of these geometries be dual to near-horizon geometries of intersecting branes?

- Is there a more direct relationship between the 3-Sasaki quotient and supersymmetry?

- Can one substitute S^7 for an exotic S^7 and remain with a supergravity vacuum solution? Is it supersymmetric?