THEORETICAL SURVEY

Theories of Superconductivity

1. **Phenomenological** equations: the London equations and the Landau-Ginzburg equations

In 1950, a psuedo wave function ψ *for the SC state,* $n_s = |\psi|^2$

2. **Quantum** theory of superconductivity was given by Bardeen, Cooper, and Schrieffer (BCS).

--Microscopic theory, 1957

--1959, Gorkov derives a macroscopic form of BCS theory near T_c, and **the order parameter is proportional to the gap function Δ_g**

3. Subsequent work of Josephson and Anderson discovered the importance of the **phase** of the superconducting wave function.

Josephson effect:

- as the first case of theory leading experiment in SC !!

(1) *Thermodynamics of the Superconducting Transition*

1. The transition between the normal and superconducting state is thermo-dynamically reversible.

2. The critical field H_c is a quantitative measure of the free energy difference between the superconducting and normal states at constant temperature.

3. The stabilization free energy of the superconducting state with respect to the normal state can be determined by *calorimetric,* or *magnetic* measurements.

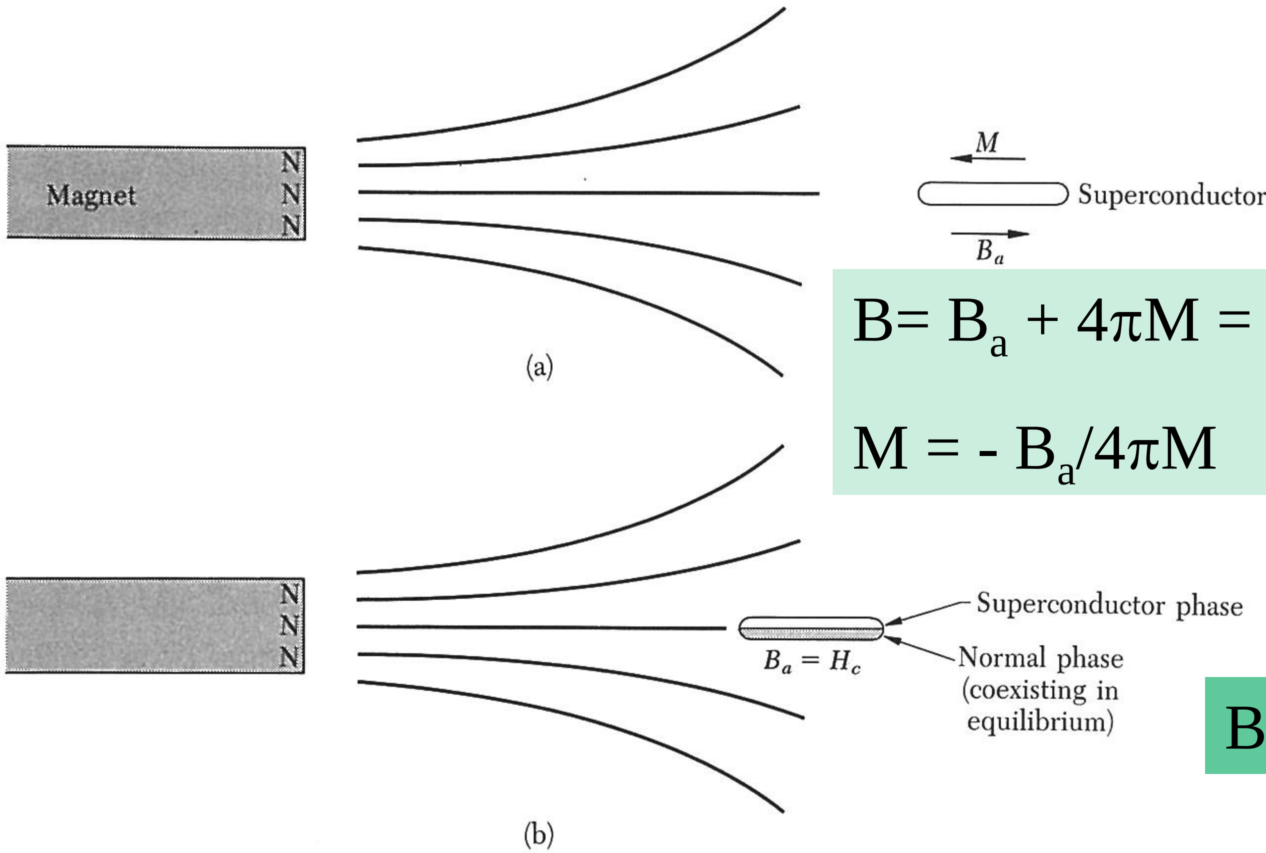
a. In the calorimetric method: From the difference of the heat capacities we can compute the free energy difference, which is the stabilization free energy of the superconducting state.

b. In the magnetic method: The stabilization free energy is found from the value of the applied magnetic field, that will destroy the superconducting state at constant temperature.

H_c: *Thermodynamic critical field*

Consider the work done (Fig. 11) on a superconductor, when it is brought reversibly at constant temperature from a position at infinity (where the applied field is zero) to a position **r** in the field of a permanent magnet:

$$W = -\int_0^{B_a} \mathbf{M} \cdot d\mathbf{B}_a \,, \qquad (3)$$



$B = B_a + 4\pi M = 0$, inside SC

$M = - B_a/4\pi M$

$B_a = H_c$

Figure 11 (a) A superconductor in which the Meissner effect is complete has $B = 0$, as if the magnetization were $M = -B_a/4\pi$, in CGS units. (b) When the applied field reaches the value B_{ac}, the normal state can coexist in equilibrium with the superconducting state. In coexistence the free energy densities are equal: $F_N(T, B_{ac}) = F_S(T, B_{ac})$.

The thermodynamic identity for the process is

$$dF = -\mathbf{M} \cdot d\boldsymbol{B}_a \,, \qquad (4)$$

For a superconductor with **M** related to $\boldsymbol{B}_a$ by (1)

M= (-1/4π)B_a

$$dF_S = \frac{1}{4\pi} \boldsymbol{B}_a \; d\boldsymbol{B}_a \quad ; \qquad (5)$$

The increase in the free energy density of the superconductor is

$$\boxed{F_S(\boldsymbol{B}_a) - F_S(0) = \boldsymbol{B}_a^2 / 8\pi} \; ; \qquad (6)$$

Now consider a normal nonmagnetic metal. Then M = 0 the energy of the normal metal is independent of field. At the critical field we have

$$F_N(\boldsymbol{B}_{ac}) = F_N(0) \;. \qquad (7)$$

At the critical value $\boldsymbol{B}_{ac}$ of the applied magnetic field the energies are equal in the normal and superconducting states:

$$F_N(\boldsymbol{B}_{ac}) = F_S(\boldsymbol{B}_{ac}) = F_S(0) + \boldsymbol{B}_a^2 / 8\pi \,. \qquad (8)$$

$$\Delta F \equiv F_N(0) - F_S(0) = B_{ac}^2 / 8\pi \,, \qquad (9)$$

Where ΔF is the stabilization free energy density of the superconducting state.

At a finite temperature the normal and superconducting phases are in equilibrium, when the magnetic field is such that their free energies **F = U - TS** are equal.

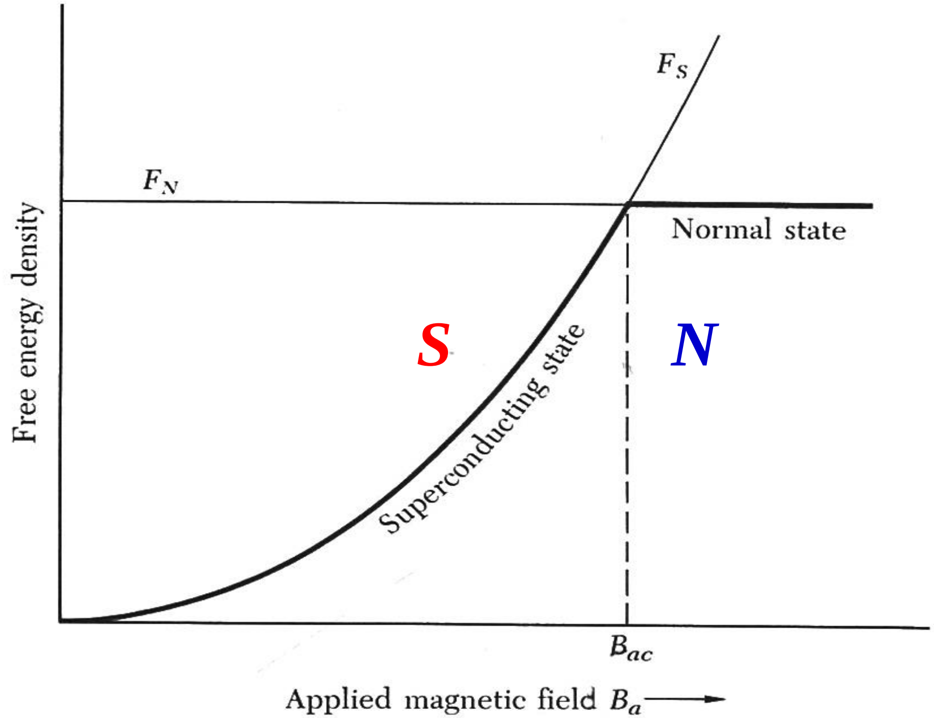
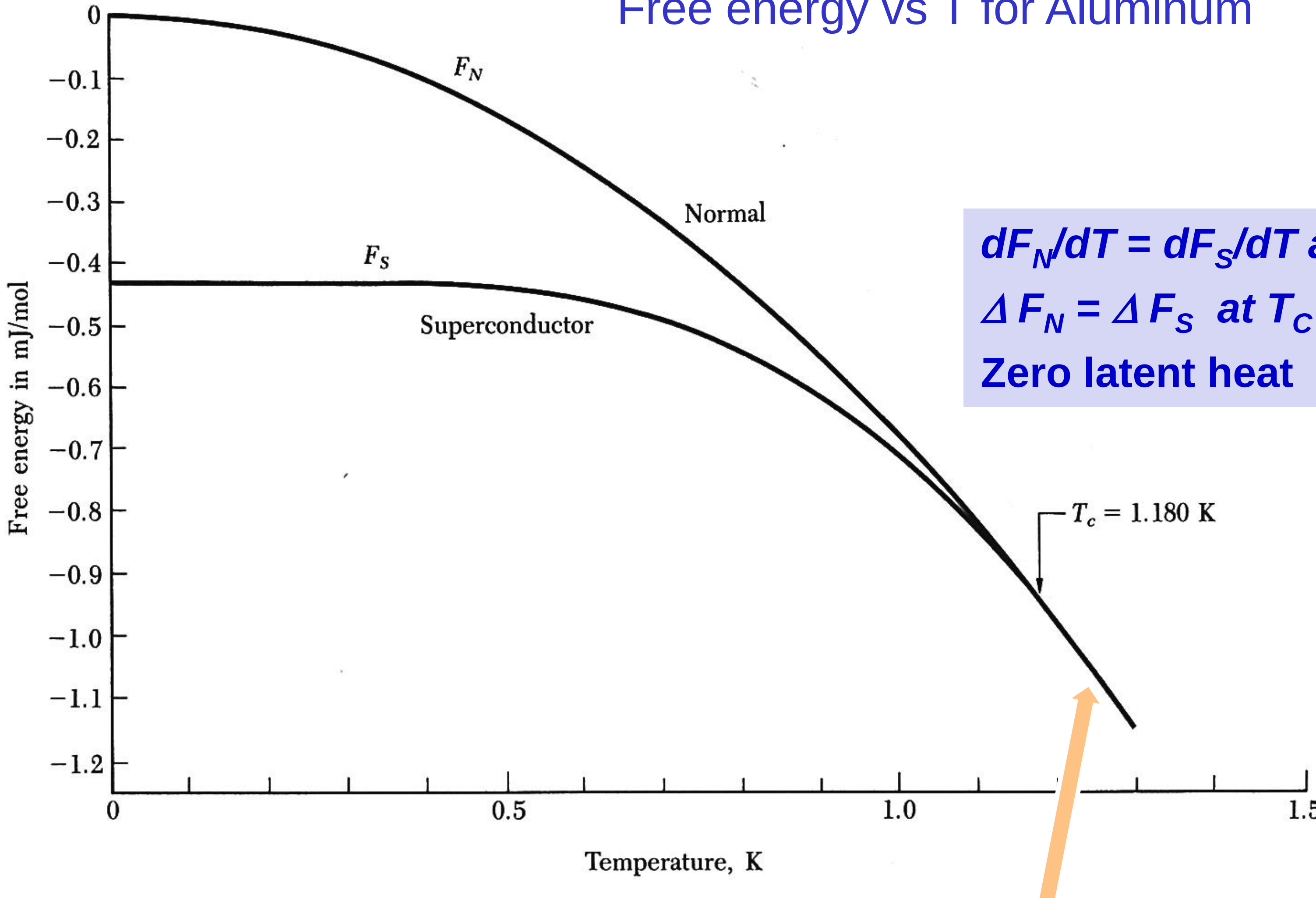


Figure 12 The free energy density F_N of a nonmagnetic normal metal is approximately independent of the intensity of the applied magnetic field B_a. At a temperature $T < T_c$ the metal is a superconductor in zero magnetic field, so that $F_S(T, 0)$ is lower than $F_N(T, 0)$. An applied magnetic field increases F_s by $B_a^2/8\pi$, in CGS units, so that $F_S(T, B_a) = F_S(T, 0) + B_a^2/8\pi$. If B_a is larger than the critical field B_{ac} the free energy density is lower in the normal state than in the superconducting state, and now the normal state is the stable state. The origin of the vertical scale in the drawing is at $F_S(T, 0)$. The figure equally applies to U_S and U_N at $T = 0$.

Free energy vs T for Aluminum



$dF_N/dT = dF_S/dT$ at T_C

$\Delta F_N = \Delta F_S$ at T_C

Zero latent heat

So that the phase transition is second order.
(There is no latent heat of transition at T_c).

(2) London Equation

Electrical conduction in the normal state of a metal is described by Ohm's law. $J = \sigma E$

We postulate that in the superconducting state the current density is directly proportional to the vector potential **A** of the local magnetic field **B**,

$$\mathbf{j} = -\frac{c}{4\pi\lambda_L^2}\mathbf{A} \quad ;$$

The concept of "Local Field"

⇨ *London Equation*

Since B = curl A

$$\text{curl}\ \mathbf{j} = -\frac{c}{4\pi\lambda_L^2}\mathbf{B} \quad ;$$

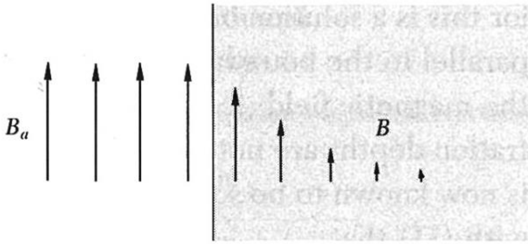
$$\text{curl}\ \mathbf{B} = (4\pi/c)\ \mathbf{j}$$

from *Maxwell Equation*

$$\text{curl curl}\ \mathbf{B} = -\nabla^2\mathbf{B} = (4\pi/c)\ \text{curl}\ \mathbf{j}$$

$$\nabla^2\mathbf{B} = \mathbf{B}/\lambda_L^2$$

$$\boldsymbol{B}(x) = \boldsymbol{B}(0)\exp(-x/\lambda_L)\ ,$$



$$\boldsymbol{B}(x) = \boldsymbol{B}(0) \exp(-x / \lambda_L) ,$$

Figure 13 Penetration of an applied magnetic field into a semi-infinite superconductor. The penetration depth λ is defined as the distance in which the field decreases by the factor e^{-1}. Typically, $\lambda \approx 500$ Å in a pure superconductor.

Table 5 Calculated intrinsic coherence length and London penetration depth, at absolute zero

Metal	Intrinsic Pippard coherence length ξ_0, in 10^{-6} cm	London penetration depth λ_L, in 10^{-6} cm	λ_L/ξ_0
Sn	23.	3.4	0.16
Al	160.	1.6	0.010
Pb	8.3	3.7	0.45
Cd	76.	11.0	0.14
Nb	3.8	3.9	1.02

Type I SC (Sn, Al, Pb, Cd)

Type II SC (Nb)

After R. Meservey and B. B. Schwartz.

See slide #24

$\boldsymbol{\lambda_L = (mc^2/4\pi nq^2)^{1/2}}$; London Penetration Depth

An applied magnetic field B_a will penetrate into a thin film fairly uniformly, if the thickness is much less than λ_L; thus in a thin film the Meissner effect is not complete. In a thin film, the induced field is much less than B_a.

(3) *Coherence Length*

1. **Coherence length** is a measure of the distance within which the SC electron concentration cannot change drastically in a spatially varying magnetic field.
2. The **coherence length** is a measure of the range over which we should average **A** to obtain **j.**
3. It is also a measure of **the minimum spatial extent** of a transition layer between normal and SC.

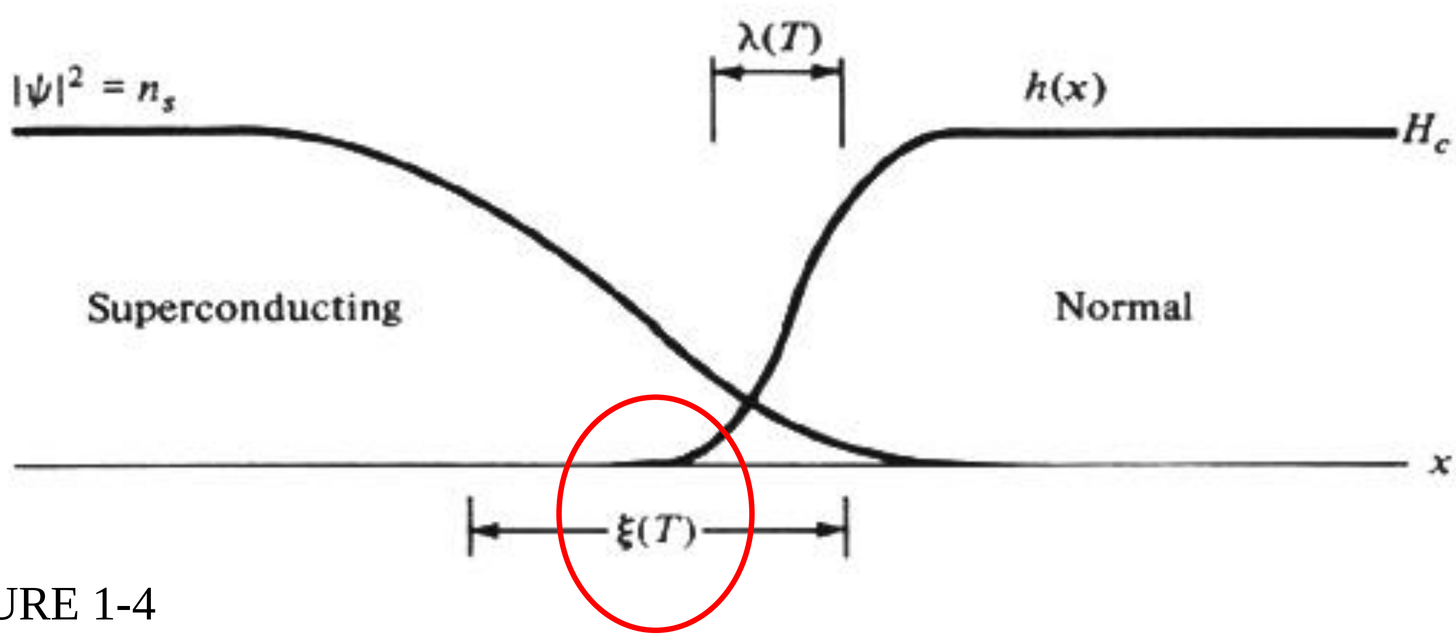


FIGURE 1-4
Interface between superconducting and normal domains in the intermediate state.

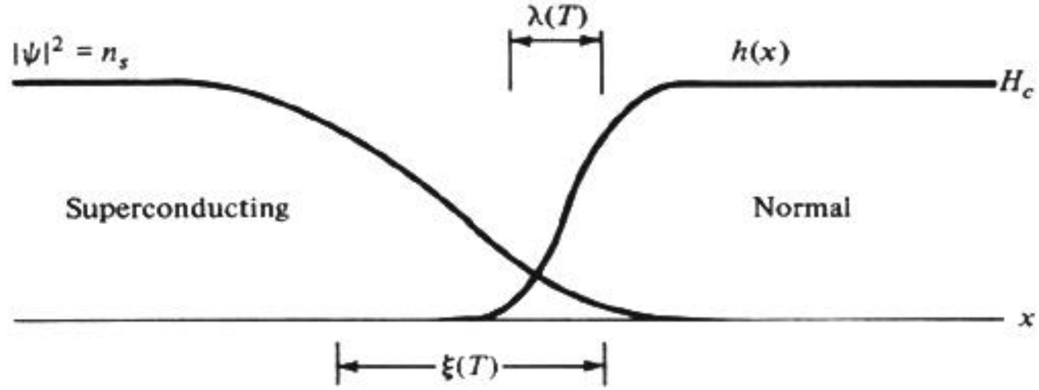


FIGURE 1-4
Interface between superconducting and normal domains in the intermediate state.

$$\kappa = \frac{\lambda_{\text{eff}}(T)}{\xi(T)} = \frac{2\sqrt{2}\,\pi H_c(T)\lambda_{\text{eff}}^2(T)}{\Phi_0}$$

Ginsburg Landau Parameters
Tinkham, eq. (4-27)

κ << 1
Type I

κ >> 1
Type II

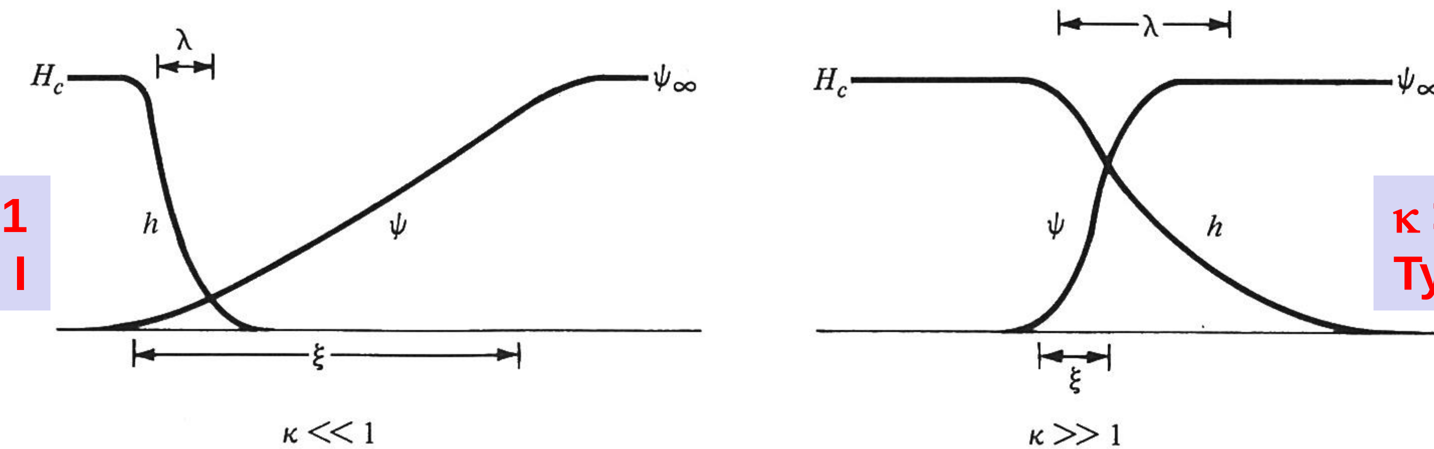


FIGURE 4-2
Schematic diagram of variation of h and ψ in a domain wall. The case $\kappa \ll 1$ refers to a type I superconductor (positive wall energy); the case $\kappa \gg 1$ refers to a type II superconductor (negative wall energy).

Any spatial variation in the state of an electronic system requires extra kinetic energy. It is reasonable to restrict the spatial variation of ***j (r)*** in such a way that the extra energy is less than the stabilization energy of the SC state.

$$\varphi(x) = 2^{-1/2}\left(e^{i(k+q)x} + e^{ikx}\right)$$

Whereas $\psi^*\psi$ is modulated with the wavevector ***q***

$$\varphi^*\varphi = \tfrac{1}{2}\left(e^{-i(k+q)x} + e^{-ikx}\right)\left(e^{i(k+q)x} + e^{ikx}\right)$$
$$= \tfrac{1}{2}\left(2 + e^{iqx} + e^{-iqx}\right) = 1 + \cos qx \ . \qquad (15b)$$

$$\int dx\, \varphi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\varphi = \frac{1}{2}\left(\frac{\hbar^2}{2m}\right)\left[(k+q)^2 + k^2\right] \cong \frac{\hbar^2}{2m}k^2 + \frac{\hbar^2}{2m}kq$$

E_g

The increase of the energy required to modulate is $\hbar^2 kq/2m$.

If this increase exceeds the energy gap E_g, superconductivity will destroy.

We define an **intrinsic coherence length** ξ_o related to the critical modulation by $\xi_o = 1/q_o$ at $k = k_F$.

$$\xi_0 = \hbar^2 k_F/2mE_g = \hbar v_F/2E_g \ ,$$

From the BCS theory,
for a pure SC, the exact form

$$\xi_0 = 2\hbar v_F/\pi E_g \ .$$

Another derivation

$\Delta t \bullet \Delta E \sim \hbar$

$\Delta x/v_F \bullet E_g \sim \hbar$

$\xi_o / v_F \bullet E_g \sim \hbar$

$\xi_o \sim \hbar v_F / E_g$

In impure materials and in alloys the coherence length ξ is shorted than ξ_o. The coherence length and the actual penetration depth λ depends on the mean free path l of the electrons measured in the normal state; the relationships are indicted in Fig. 14. When the superconductor is very impure, with a very small l.

then $\xi \approx (\xi_o \ell)^{1/2}$

$$\lambda \approx \lambda_L (\xi_o / \ell)^{1/2}$$

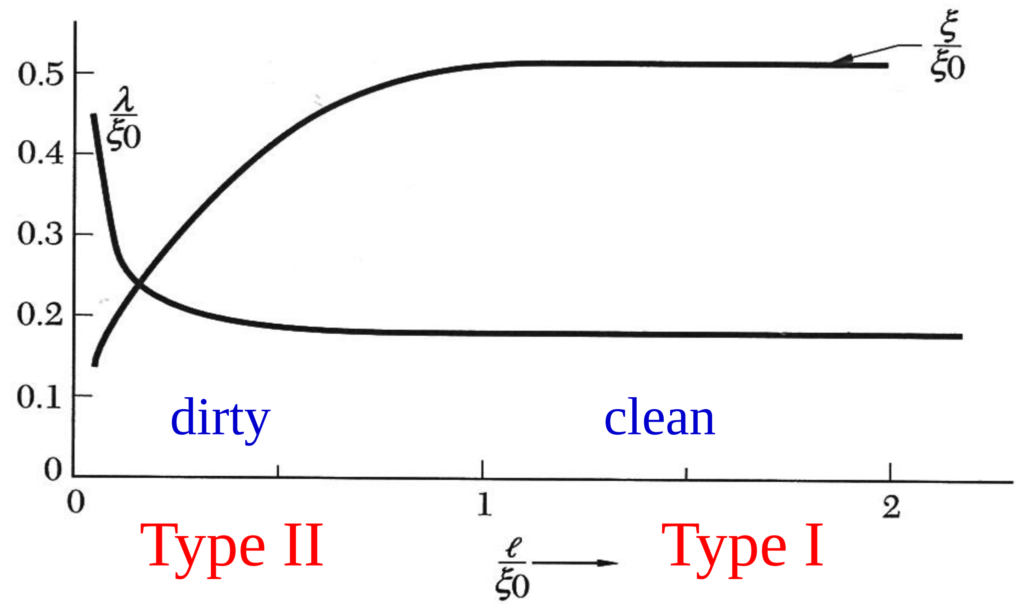
at very small mean free path ℓ in impure SC

so that $\lambda/\xi \approx \lambda_L / \ell$.

This is the "dirty superconductor" limit.

The ratio λ/ξ is denoted by κ .

Figure 14 Penetration depth λ and the coherence length ξ as functions of the mean free path ℓ of the conduction electrons in the normal state. All lengths are in units of ξ_0, the intrinsic coherence length. The curves are sketched for $\xi_0 = 10\lambda_L$. For short mean free paths the coherence length becomes shorter and the penetration depth becomes longer. The increase in the ratio $\kappa\lambda/\xi$ favors type II superconductivity.



(4) *BCS Theory of Superconductivity*

1. The Cooper Pair :

The "BCS wave function" is composed of particle pairs $\boldsymbol{k}\uparrow$ and $-\boldsymbol{k}\downarrow$, when treated by the BCS theory, gives the familiar electronic superconductivity observed in metals, and exhibits the energy gaps of Table 3. This pairing is known as ***s*-wave pairing** ($l = 0$) .

Postulated by Cooper in 1956

- A weak attraction can bind pairs of electrons into a bound state
- The Fermi sea of electrons is unstable against the formation at least one bound pair, regardless how weak the interaction is, so long it is attractive.
- The lowest energy state to have the **total zero momentum**, so that two electrons must have **equal** and **opposite momenta**.
- Introduce $\mathbf{V}_{kk'} = -\mathbf{V}$ for all $\boldsymbol{k}$ out to a cut-off energy $\hbar\omega_c$ away from E_f , and $V_{kk'} = 0$ for $\boldsymbol{k}$ beyond $\hbar\omega_c$.

$$E \sim 2E_F - 2\hbar\omega_c\, e^{-2/N(0)V} \qquad \Delta = 2E_F - E = 2\hbar\omega_c\, e^{-2/N(0)V} > 0$$

- The contribution to the energy of the attractive potential outweights the excess kinetic energy, leading to a binding energy regardless how small **V** is.

Origin of the Attractive Interaction:

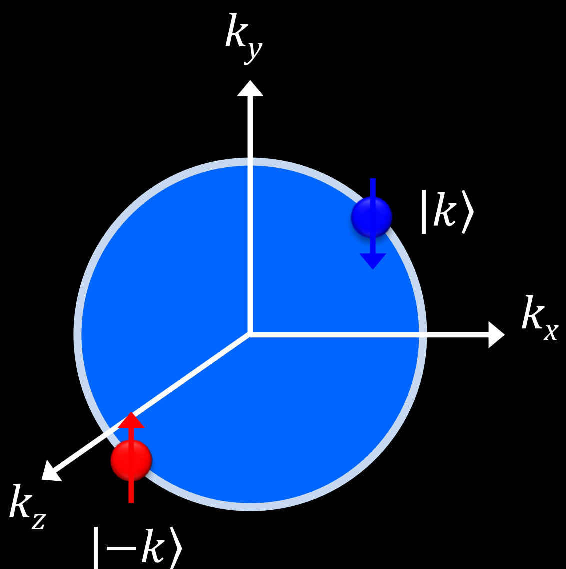
2. The electron-lattice-electron interaction leads to an ***energy gap*** of the observed magnitude. The indirect interaction proceeds when one electron interacts with the lattice and deforms it; a second electron sees the deformed lattice and adjust itself to take advantage of the deformation to lower its energy. Thus the second electron interacts with the first electron via the **lattice deformation.**

e.g. the mattress theory

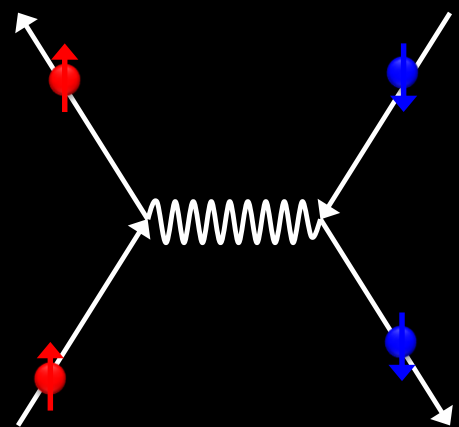
- In 1950 **Frohlich** first suggested ***the electron phonon interaction*:** The physical idea is that the first electron polarizes the medium by attractive positive ions; these excessive positive ions, in turn, attract the second electron, giving an ***effective attractive*** interaction between the electrons.
- If this attractive interaction is strong enough to override the repulsive screened Coulomb interaction, it gives rise to a ***net attractive*** interaction, and the **superconductivity** results.
- The cut-off frequency $\hbar\omega_c$ of the Cooper pair's attraction is expected to be of the order of the Debye frequency, $\hbar\omega_D$, as a measure of the ***stiffness*** of the lattice.

Superconducting Ground State

Normal state

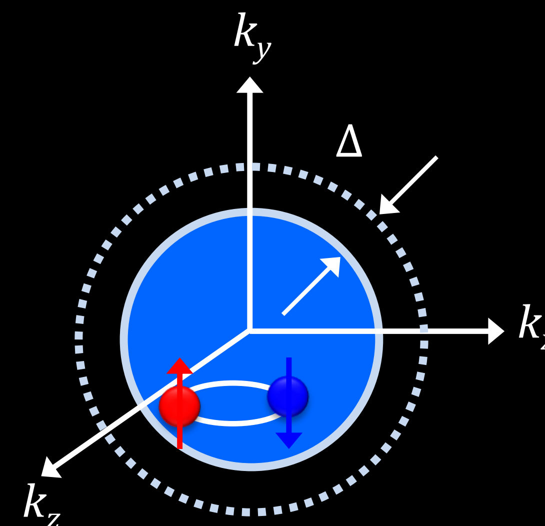


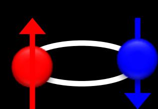
Cooper Pairs



Exchange boson:
Lattice Vibration Mode

Superconducting ground state

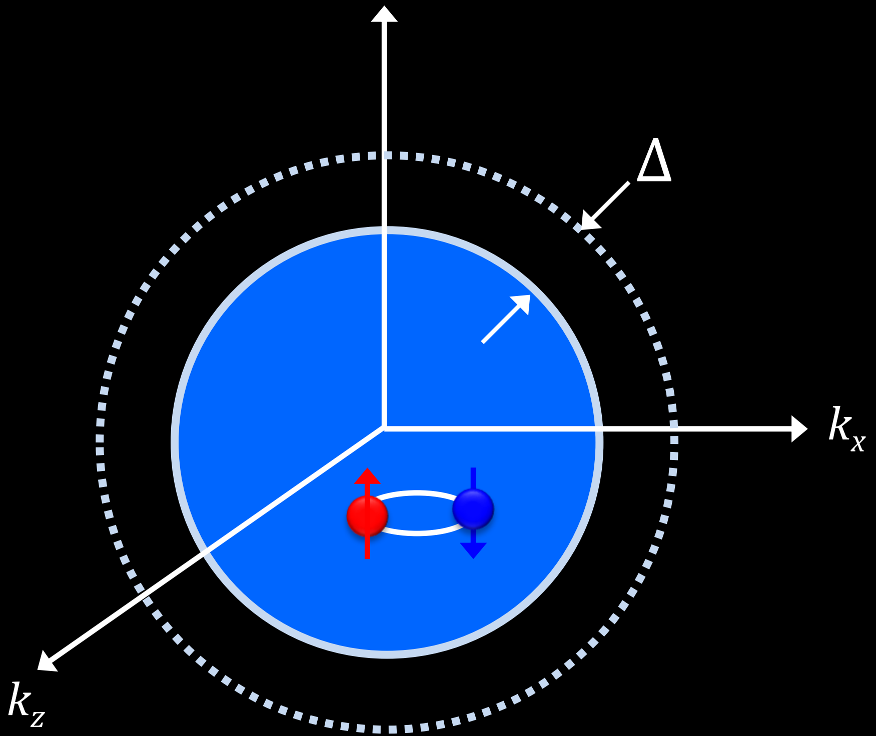




- Spin singlet
- L=0; S=0
- Binding energy: Δ

Superconducting Ground States

SC Ground State



$$\Psi_{BCS} = \prod_k \left(u_k + v_k c^*_{k\uparrow} c^*_{-k\downarrow}\right) | 0 >$$

u_k and v_k : coherence factor

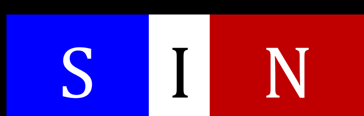
BCS, Phys Rev 108, 1175 (1957)

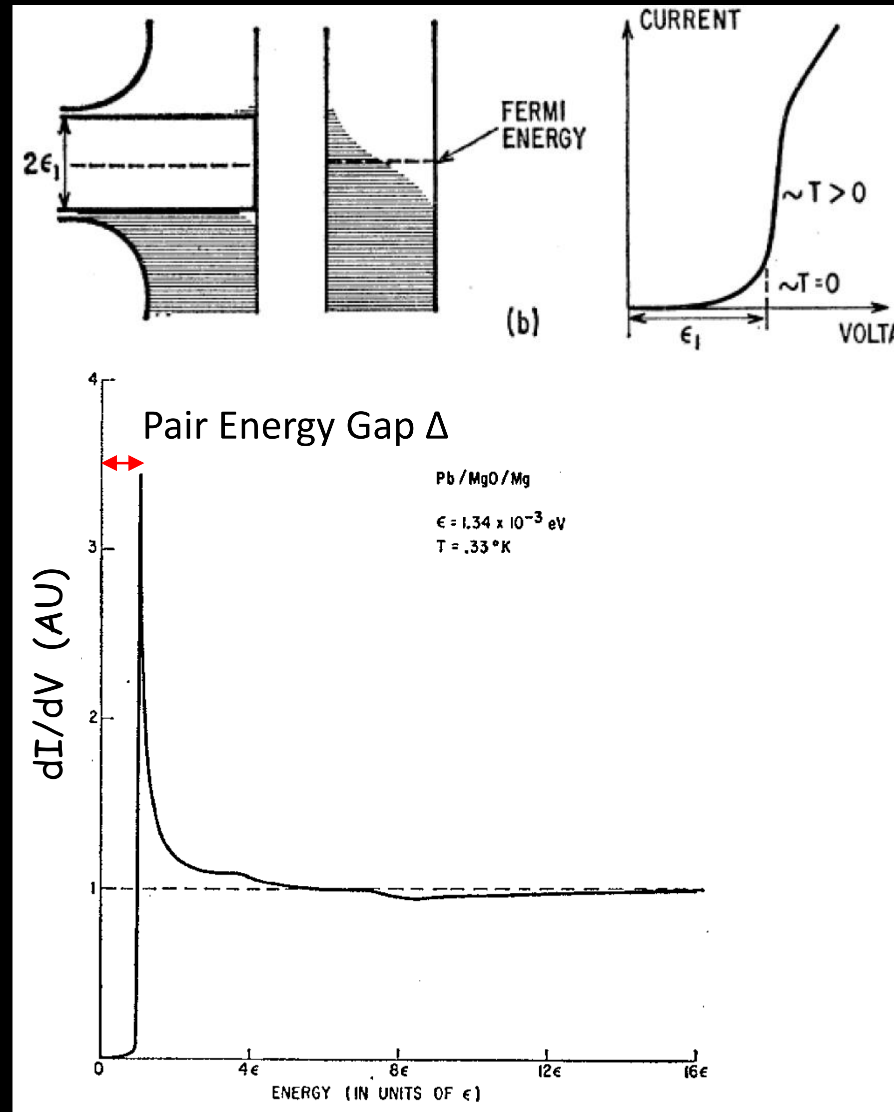
Superconducting Energy Gap in 1960

Ivar Giaever

Nobel Prize in 1973

Tunneling junction





I. Giaever, Phys. Rev. Lett. 5, 147 (1960)
I. Giaever, Phys. Rev. 126, 941 (1962)

(5) *BCS Ground State*

1. The BCS theory shows that, with an appropriate attractive interaction between electrons, the new ground state is superconducting, and is separated by a finite energy E_g from its lowest excited state.
2. With the attractive potential energy of the BCS state, the total energy of the BCS state will be lower with respect to the Fermi state.
3. The central feature of the BCS state is that one–particle orbitals are occupied in pairs: if an orbital with the wavevector ***k*** and **spin up** is occupied, then the orbital with the wavevector ***–k*** and **spin down** is also occupied.
4. **Cooper pairs**: they have a spin zero, and have many attributes of **bosons**.

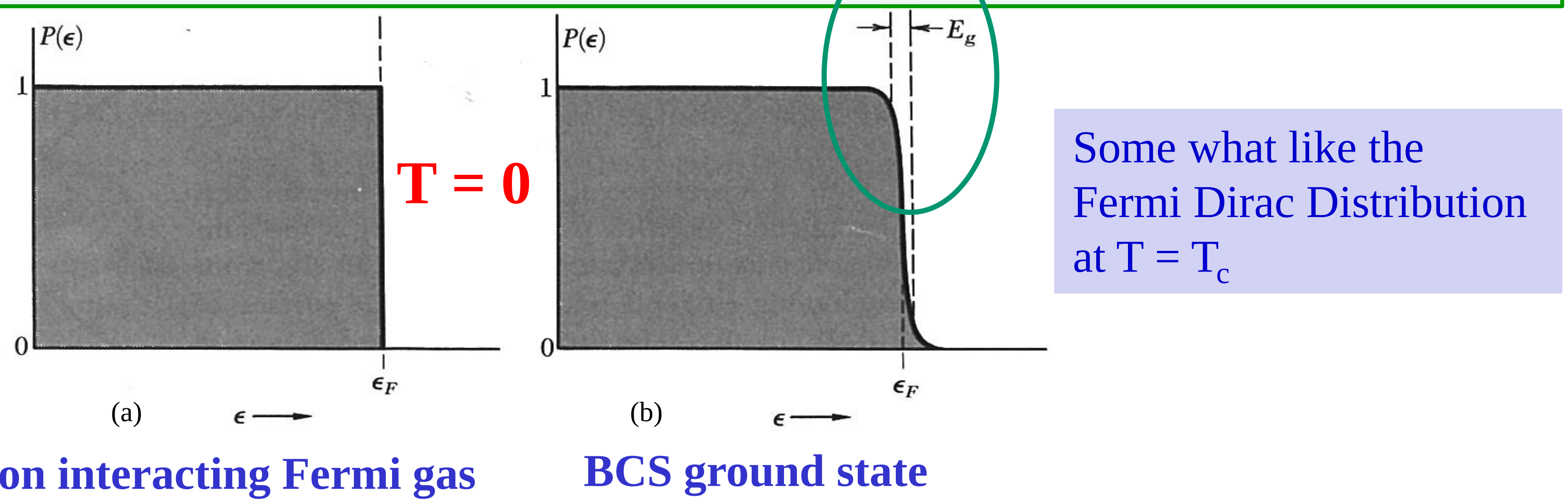


Figure 15 (a) Probability P that an orbital of kinetic energy ϵ is occupied in the ground state of the noninteracting Fermi gas; (b) the BCS ground state differs from the Fermi state in a region of width of the order of the energy gap E_g. Both curves are for absolute zero.

The BCS Ground State

Tinkham, Chapter 2

Singlet wave function, a vacuum state with no particles present

$$|\psi_0\rangle = \sum_{k>k_F} g_{\mathbf{k}} c^*_{\mathbf{k}\uparrow} c^*_{-\mathbf{k}\downarrow} |F\rangle \qquad (2\text{-}11)$$

Creation operator C_k^*
Annihilation operator C_k

where $|F\rangle$ = Fermi sea filled up to k_F

Using a **Hartree self consistent field**, or **a mean field theory**

the BCS Ground state wave function

$$|\psi_G\rangle = \prod_{\mathbf{k}=\mathbf{k}_1,\ldots,\mathbf{k}_M} (u_{\mathbf{k}} + v_{\mathbf{k}} c^*_{\mathbf{k}\uparrow} c^*_{-\mathbf{k}\downarrow}) |\phi_0\rangle \qquad (2\text{-}14)$$

where $u_k^2 + v_k^2 = 1$, *and* $u_k = e^{i\phi} v_k$

The pairing Hamiltonian

$$\mathscr{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} n_{\mathbf{k}\sigma} + \sum_{\mathbf{kl}} V_{\mathbf{kl}} c^*_{\mathbf{k}\uparrow} c^*_{-\mathbf{k}\downarrow} c_{-\mathbf{l}\downarrow} c_{\mathbf{l}\uparrow} \qquad (2\text{-}20)$$

The gap equation

$$\Delta_{\mathbf{k}} = -\frac{1}{2}\sum_{\mathbf{l}} \frac{\Delta_{\mathbf{l}}}{E_{\mathbf{l}}} V_{\mathbf{kl}} = -\frac{1}{2}\sum_{\mathbf{l}} \frac{\Delta_{\mathbf{l}}}{(\Delta_{\mathbf{l}}^2 + \xi_{\mathbf{l}}^2)^{1/2}} V_{\mathbf{kl}} \qquad (2\text{-}30)$$

$E_k = (\Delta_k^2 + \xi_k^2)^{1/2}$ *Quasi-particle excitation energy*

$$V_{\mathbf{kl}} = \begin{cases} -V & \text{if} \quad |\xi_{\mathbf{k}}| \quad \text{and} \quad |\xi_{\mathbf{l}}| \leq \hbar\omega_c \\ 0 & \text{otherwise} \end{cases} \tag{2-31}$$

$$\Delta_{\mathbf{k}} = \begin{cases} \Delta & \text{for} \quad |\xi_{\mathbf{k}}| < \hbar\omega_c \\ 0 & \text{for} \quad |\xi_{\mathbf{k}}| > \hbar\omega_c \end{cases} \tag{2-32}$$

in weak coupling limit

$$\Delta = \frac{\hbar\omega_c}{\sinh\,[1/N(0)V]} \approx 2\hbar\omega_c e^{-1/N(0)V} \tag{2-34}$$

The BCS Pairing occupation number

$$v_{\mathbf{k}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) = \frac{1}{2}\left[1 - \frac{\xi_{\mathbf{k}}}{(\Delta^2 + \xi_{\mathbf{k}}^2)^{1/2}}\right] \tag{2-35}$$

$$u_{\mathbf{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) = 1 - v_{\mathbf{k}}^2$$

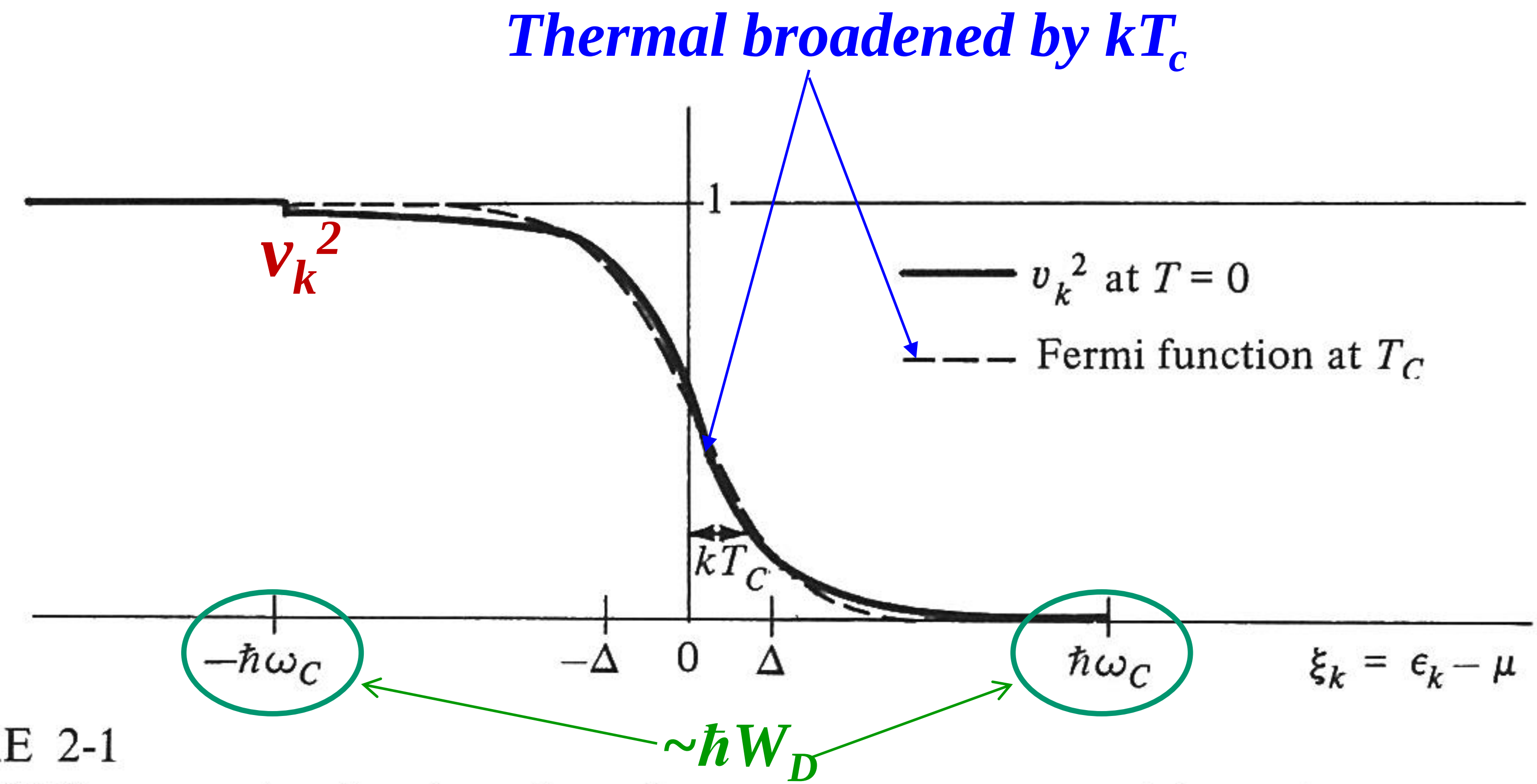
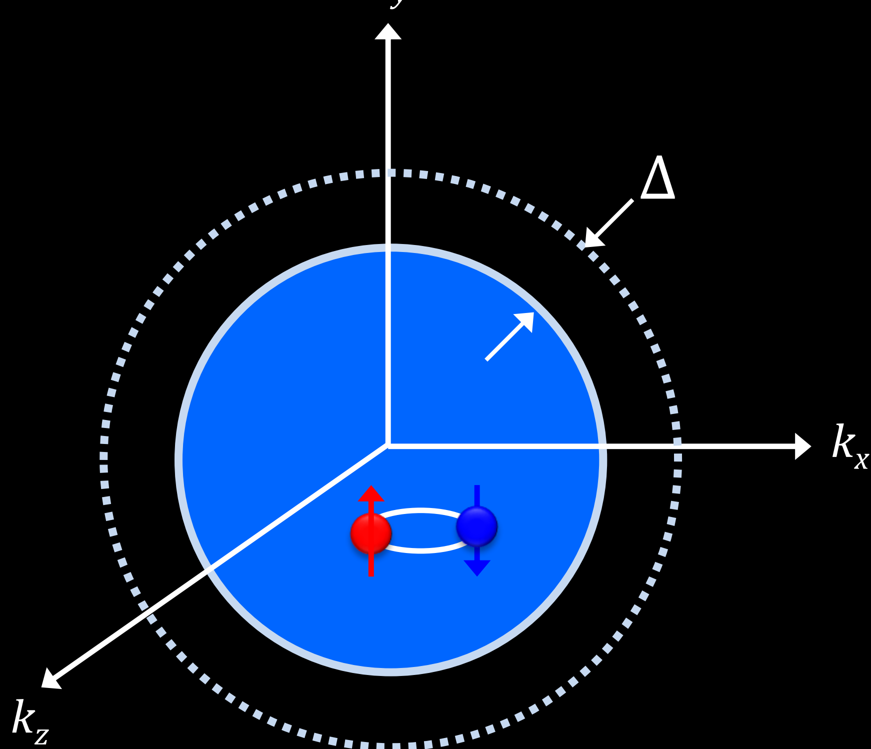


FIGURE 2-1
Plot of BCS occupation fraction v_k^2 vs. electron energy measured from the chemical potential (Fermi energy). To make the cutoffs at $\pm\hbar\omega_c$ visible, the plot has been made for a strong-coupling superconductor with $N(0)V = 0.43$. For comparison, the Fermi function for the normal state at T_c is also shown on the same scale, using the BCS relation $\Delta(0) = 1.76kT_c$.

$$\omega_C \sim \omega_D >> \Delta = 1.76\ kT_c$$

Superconducting Excited States

SC Ground State

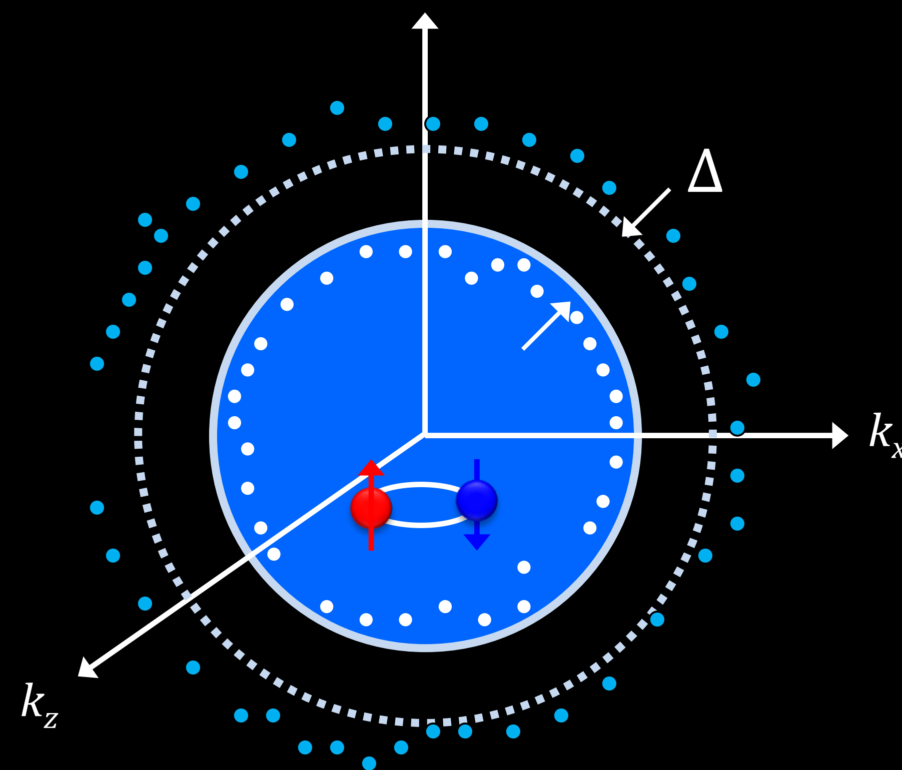


$$\Psi_{BCS} = \prod_k \left(u_k + v_k c^*_{k\uparrow} c^*_{-k\downarrow} \right) | 0 >$$

u_k and v_k : coherence factor

BCS, Phys Rev 108, 1175 (1957)

SC Excited States



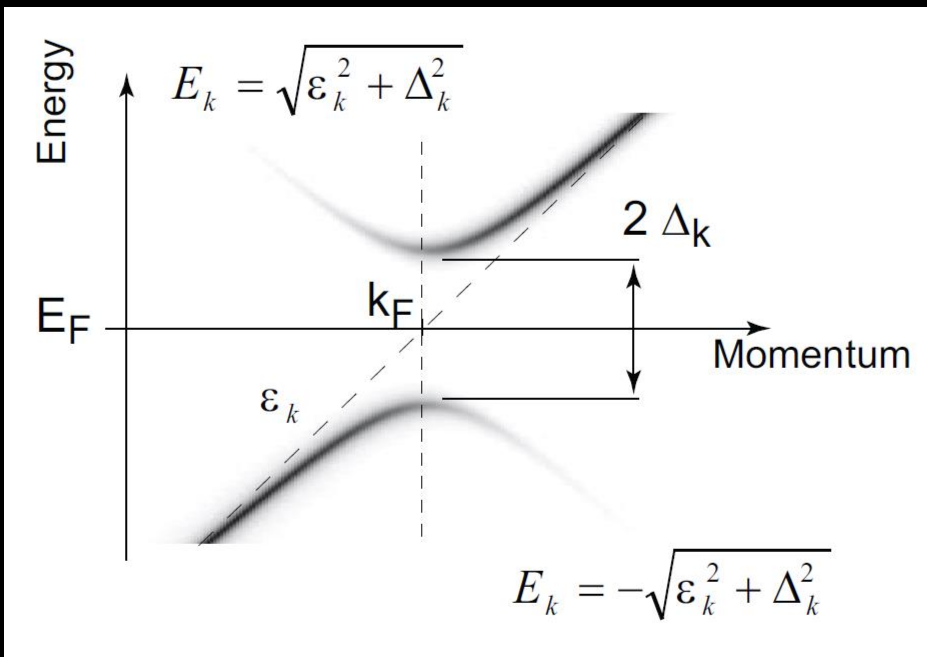
Bogoliubov quasiparticle

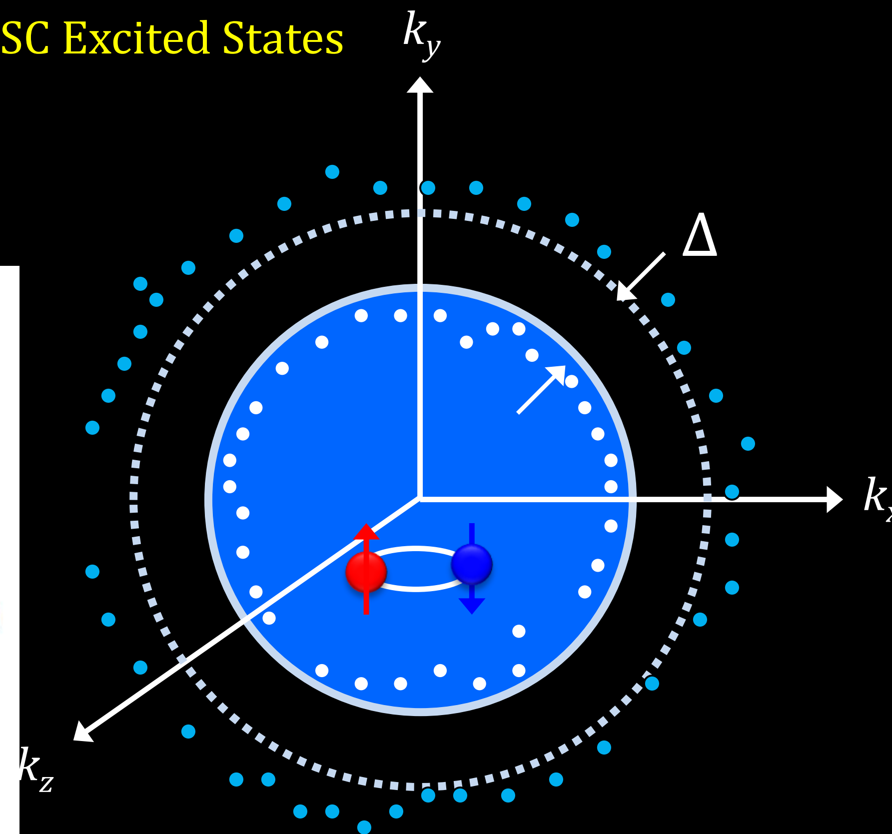
$$\gamma^*_{k\uparrow} = u_k c_{k\uparrow} + v_k c^*_{-k\downarrow}$$

Bogoliubov, Nuovo Cimento 7, 794 (1958)

Superconducting Excited States

$$E_{\pm}(\vec{k}) = \pm\sqrt{\varepsilon(\vec{k})^2 + \Delta(\vec{k})^2}$$





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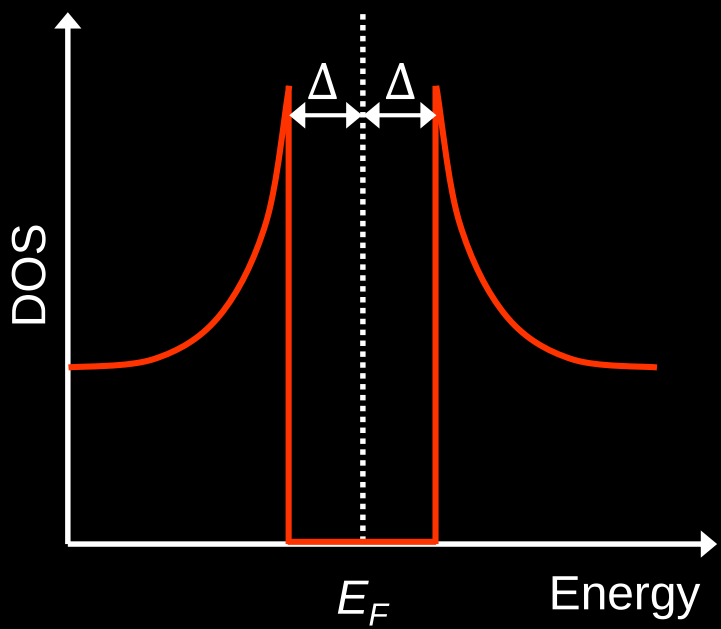
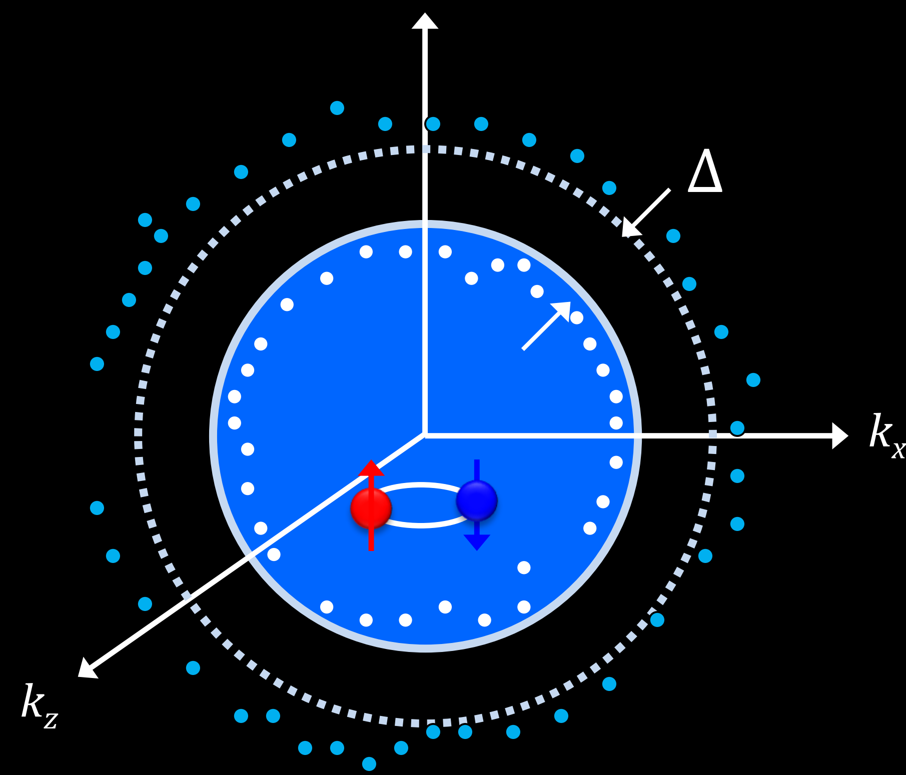
Superconducting Excited States

Superconducting energy gap=2Δ
(T=0)



SC Excited States



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Superconducting Gap

Pair wave function : $\Psi_{kss'} = \langle \Psi_{BCS} | c_{-ks'} c_{ks} | \Psi_{BCS} \rangle = g(k)\,\chi_{ss'}$

Spin part : $\chi_{ss'}$ $(\uparrow\downarrow - \downarrow\uparrow)$ $S = 0$

$(\uparrow\uparrow, \uparrow\downarrow + \downarrow\uparrow, \downarrow\downarrow)$ $S = 1$

Orbital part : $g(k)$

$$\psi(\mathbf{r}) \propto \sum_{\mathbf{k}} \frac{\Delta(\mathbf{k})}{\sqrt{\epsilon(\mathbf{k})^2 + \Delta(\mathbf{k})^2}} \exp(-i\mathbf{k}\mathbf{r})$$

Spin	Orbital
anti-symmetric (S = 0)	symmetric (s, d, …)
symmetric (S = 1)	anti-symmetric (p, f, …)

$l = 0$: s wave (conventional SC)

$l = 1$: p wave (superfluid ^{3}He)

$l = 2$: d wave (cuprate SC)

If $l > 0$, $\psi(0) = 0$

→ { repulsive interaction; $\Delta(k)$ must change its sign }

Gap Equation

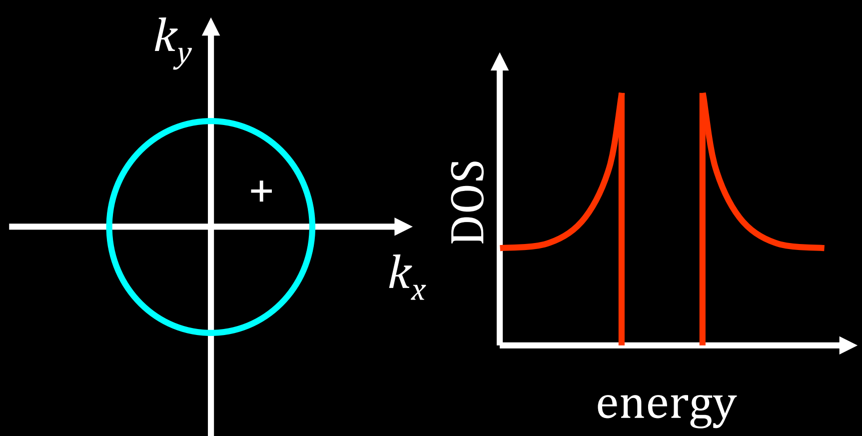
$$\Delta(\mathbf{k}) = -\frac{1}{2}\sum_{\mathbf{q}} V(\mathbf{q}) \frac{\Delta(\mathbf{k})}{\sqrt{\epsilon(\mathbf{k}+\mathbf{q})^2 + \Delta(\mathbf{k}+\mathbf{q})^2}} \tanh \frac{\sqrt{\epsilon(\mathbf{k}+\mathbf{q})^2 + \Delta(\mathbf{k}+\mathbf{q})^2}}{2k_B T}$$

Pairing interaction

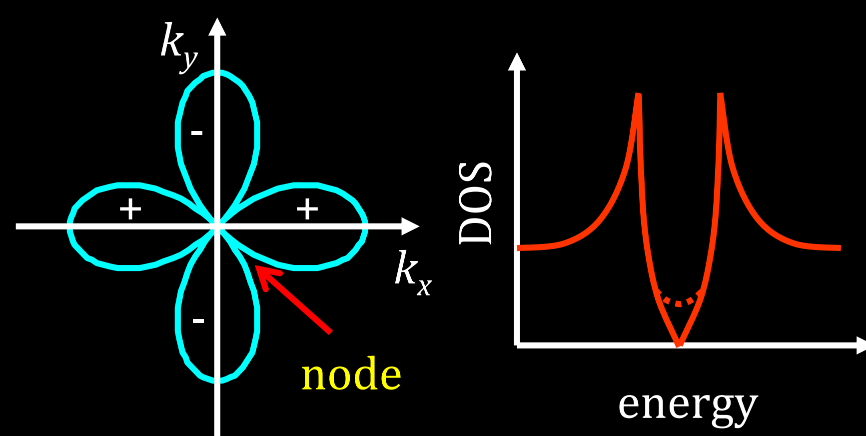
In conventional BCS, $V(\mathbf{q}) = -|V| < 0$: Δ is always positive.

If $V(\mathbf{q} = \mathbf{Q}) > 0$ plays a role, $\Delta(\mathbf{k})$ and $\Delta(\mathbf{k}+\mathbf{Q})$ have a different sign.

s wave



d wave



BCS theory:

$$\Delta \;=\; 2\,\hbar\omega_c\, e^{-2/N(0)V}$$
$$kT_c = 1.14\,\hbar\omega_c\, e^{-2/N(0)V}$$

$\Delta(0)/kT_c = 2/1.14 = 1.76$, weak el-ph coupling

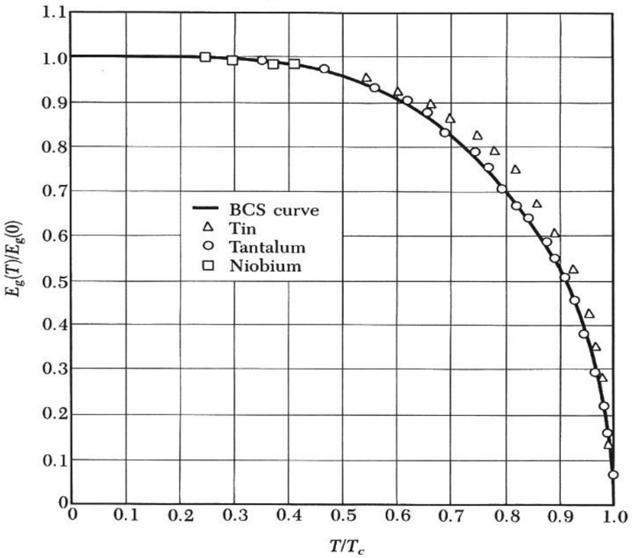
If $\Delta(0)/kT_c > 2$, strong el-ph coupling

$\beta = 1/kT$

$$\frac{1}{V} = \frac{1}{2}\sum_k \frac{\tanh(\beta E_k/2)}{E_k}$$

(2-50)

Determines the temperature dependence of $\Delta(T)$



$\Delta(T)/\Delta(0) \sim 1.74\,(1 - T/T_c)^{1/2}$

at $T \sim T_c$

In the mean field theory,

Δ_k is the order parameter !

2-6.1 Determination of T_c

The critical temperature T_c is the temperature at which $\Delta(T) \to 0$. In this case, $E_\mathbf{k} \to |\xi_\mathbf{k}|$, and the excitation spectrum becomes the same as in the normal state. Thus, T_c is found by replacing $E_\mathbf{k}$ with $|\xi_\mathbf{k}|$ in (2-50) and solving. After changing the sum to an integral, taking advantage of the symmetry of $|\xi_\mathbf{k}|$ about the Fermi level, and changing to a dimensionless variable of integration, this condition becomes

$$\frac{1}{N(0)V} = \int_0^{\beta_c \hbar\omega_c/2} \frac{\tanh x}{x}\, dx$$

This integral can be evaluated and yields $\ln(A\beta_c \hbar\omega_c)$, where $A = 2\gamma/\pi \approx 1.13$, γ here being Euler's constant. Consequently,

$$kT_c = \beta_c^{-1} = 1.13\hbar\omega_c e^{-1/N(0)V}. \tag{2-51}$$

Comparing this with (2-34), we see that

$$\frac{\Delta(0)}{kT_c} = \frac{2}{1.13} = 1.764 \tag{2-52}$$

so that the gap at $T = 0$ is indeed comparable in energy to kT_c. The numerical factor 1.76 has been tested in many experiments and found to be reasonable. That is, experimental values of 2Δ for different materials and different directions in k space generally fall in the range 3.0 to $4.5kT_c$, with most clustered near the BCS value of $3.5kT_c$.

2-6.2 Temperature Dependence of the Gap

Given (2-50), or its integral equivalent

$$\frac{1}{N(0)V} = \int_0^{\hbar\omega_c} \frac{\tanh \frac{1}{2}\beta(\xi^2 + \Delta^2)^{1/2}}{(\xi^2 + \Delta^2)^{1/2}} d\xi \tag{2-53}$$

$\Delta(T)$ can be computed numerically. For weak-coupling superconductors, in which $\hbar\omega_c/kT_c \gg 1$, $\Delta(T)/\Delta(0)$ is a universal function of T/T_c which decreases monotonically from one at $T = 0$ to zero at T_c, as shown in Fig. 2-2. Near $T = 0$, the temperature variation is exponentially slow, since $e^{-\Delta/kT} \approx 0$, so that the hyperbolic tangent is very nearly unity and insensitive to T. Physically speaking, Δ is nearly constant until a significant number of quasi-particles are thermally excited. On the other hand, near T_c, $\Delta(T)$ drops to zero with a vertical tangent, approximately as

$$\frac{\Delta(T)}{\Delta(0)} \approx 1.74\left(1 - \frac{T}{T_c}\right)^{1/2} \qquad T \approx T_c \tag{2-54}$$

The variation of the order parameter Δ with the square root of $(T_c - T)$ is characteristic of all mean-field theories. For example, $M(T)$ has the same dependence in the molecular-field theory of ferromagnetism.

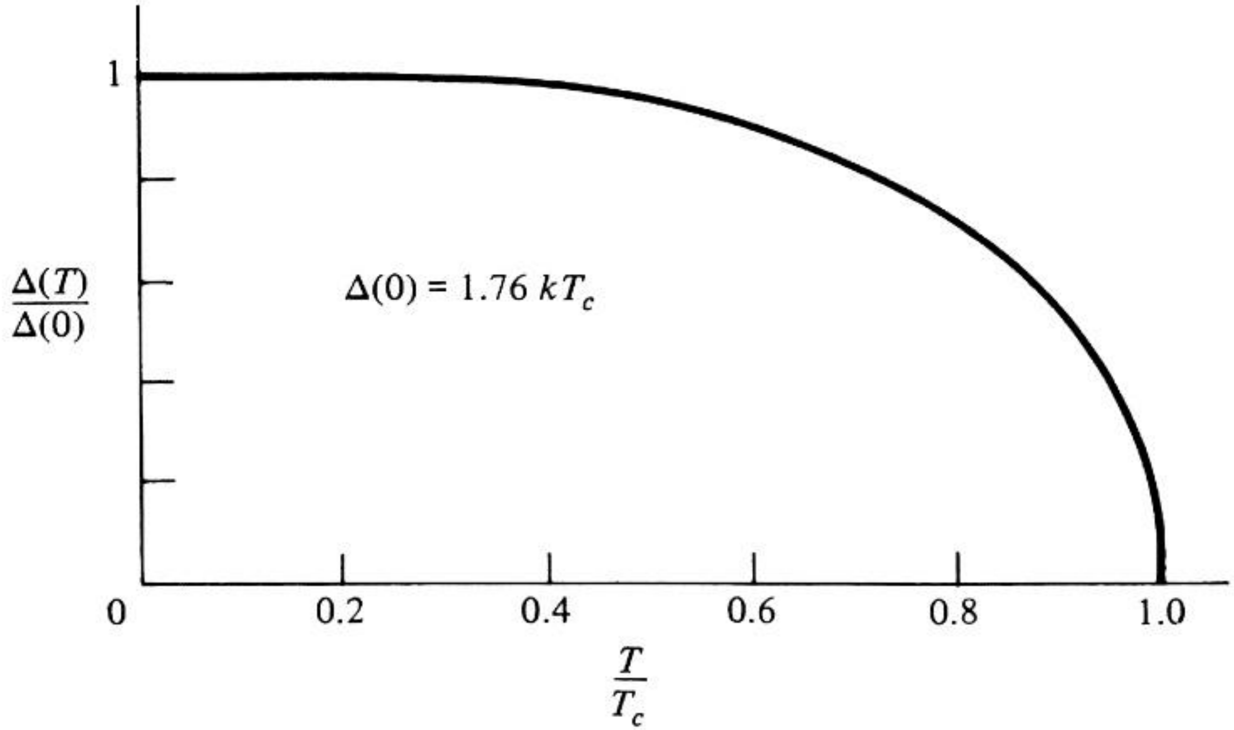


FIGURE 2-2
Temperature dependence of the energy gap in the BCS theory. Strictly speaking, this universal curve holds only in the weak-coupling limit, but it is a good approximation in most cases.

Low temperature Superconductors

-- Mediated by *electron phonon coupling*

-- *In strong electron phonon coupling, modified by Elishberg et al*

W. McMillian's formula for T_c

$$T_c = \frac{\Theta_D}{1.45} \exp\left\{ - \left[\frac{(1+\lambda_{ep})}{\lambda_{ep} - \mu^*(1+0.62\lambda_{ep})} \right] \right\}$$

λ : electron phonon coupling constant

μ^* : Coulomb repulsion of electrons

$\lambda \propto N(0) < I^2 > / \omega^2$

Are electrons or phonons more important to give rise to high T_c ?

Important E&M properties from the BCS theory

(1) We first show that a charged boson gas obeys the London equation. Let $\psi(\mathbf{r})$ be the particle probability amplitude. We suppose that the pair concentration $n = \psi^*\psi$ = constant.

$$\psi = n^{1/2}\, e^{i\theta(\mathbf{r})} \quad ; \qquad \psi^* = n^{1/2}\, e^{-i\theta(\mathbf{r})} \tag{19}$$

The phase $\theta(\mathbf{r})$ is important

$$\mathbf{v} = \frac{1}{m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right) = \frac{1}{m}\left(-i\hbar\nabla - \frac{q}{c}\mathbf{A}\right)$$

The particle flux is given by, and from eq. (19)

$$\psi^*\mathbf{v}\psi = \frac{n}{m}\left(\hbar\nabla\theta - \frac{q}{c}\mathbf{A}\right) \tag{20}$$

that the electric current density is $\mathbf{j} = q\psi^*\mathbf{v}\psi = \dfrac{nq}{m}\left(\hbar\nabla\theta - \dfrac{q}{c}\mathbf{A}\right)$ (21)

London equation: $\operatorname{curl} \mathbf{j} = -\dfrac{nq^2}{mc}\mathbf{B}$

$$\mathbf{j} = -\frac{c}{4\pi\lambda_L^2}\mathbf{A} \; ; \tag{22}$$

London penetration depth $\boldsymbol{\lambda_L = (mC^2/4\pi nq^2)^{1/2}}$

(2) Quantization of the magnetic flux through a ring is a dramatic consequence of Eq. (21). Let us take a closed path C through the interior of the superconducting material, well away from the surface (Fig. 16).

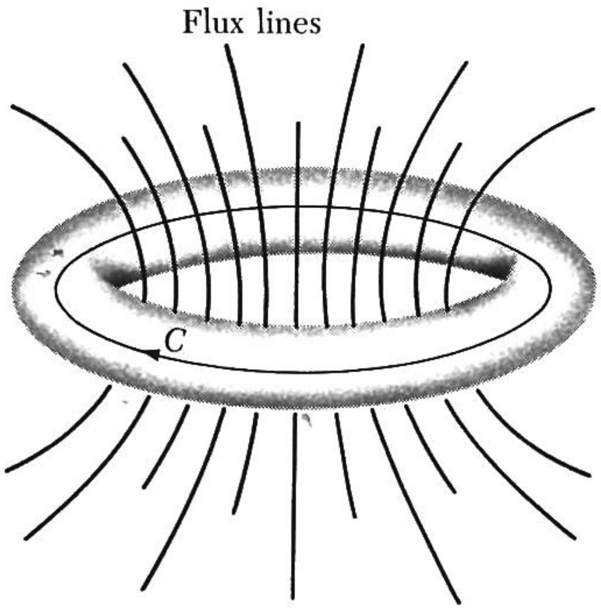


Figure 16 Path of integration C through the interior of a superconducting ring. The flux through the ring is the sum of the flux Φ_{ext} from external sources and the flux Φ_{sc} from the superconducting currents which flow in the surface of the ring; $\Phi = \Phi_{ext} + \Phi_{sc}$. The flux Φ is quantized. There is normally no quantization condition on the flux from external sources, so that Φ_{sc} must adjust itself appropriately in order that Φ assume a quantized value.

B and **j** are zero in the interior. from the Meissner effect

$$\hbar c \nabla\theta = q\mathbf{A} \ . \qquad \text{From Eq. 20, 21} \tag{23}$$

We form

$$\oint_C \nabla\theta \cdot d\boldsymbol{l} = \theta_2 - \theta_1 \tag{23'}$$

for the change of phase on going once around the ring.

The probability amplitude ψ is measurable in the classical approximation, so that ψ must be single-valued and

$$\theta_2 - \theta_1 = 2\pi s \quad , \tag{24}$$

S is an integer

where s is an integer. By the Stokes theorem,

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_C (\mathrm{curl}\ \mathbf{A}) \cdot d\boldsymbol{\sigma} = \int_C \mathbf{B} \cdot d\boldsymbol{\sigma} = \Phi \quad , \tag{25}$$

$d\boldsymbol{\sigma}$ is an element of area on a surface bounded by the curve C, and Φ is the magnetic flux through C.

$$\Phi = (2\pi\hbar c/q)s \quad . \tag{26}$$

S is an integer

Thus the flux through the ring is quantized in integral multiples of $2\pi\hbar c/q$.

Flux Quantization: The evidence of pairing of electrons !

By experiment $q = -2e$

$\Phi = \Phi_o \, \mathbf{s}$

S is an integer

$$\Phi_0 = 2\pi\hbar c/2e \cong 2.0678 \times 10^{-7} \text{ gauss cm}^2 = \pi\hbar c/e$$

This unit of flux is called a **fluxoid** or **fluxon**. (27)

$\Phi = \Phi_{ext} + \Phi_{sc}$ The total flux Φ is quantized. (28)

There is normally no quantization condition on the flux from external sources, so that Φ_{sc} must adjust itself appropriately in order that Φ assume a quantized value.

Flux Quantization Theory in 1950

* We note that in order for Ψ to be a single-valued function, as required by quantum mechanics, it is necessary that the moduli of χ fulfill a kind of quantum condition:

$$\langle\chi\rangle = \oint \bar{p}_s \cdot ds = Kh$$

where K must be an integer. This means that there exists a universal unit for the fluxoid:

$$\Phi_1 = hc/e \simeq 4 \cdot 10^{-7}\ \text{gauss} \cdot \text{cm}^2$$

~ 2 larger

Fritz London

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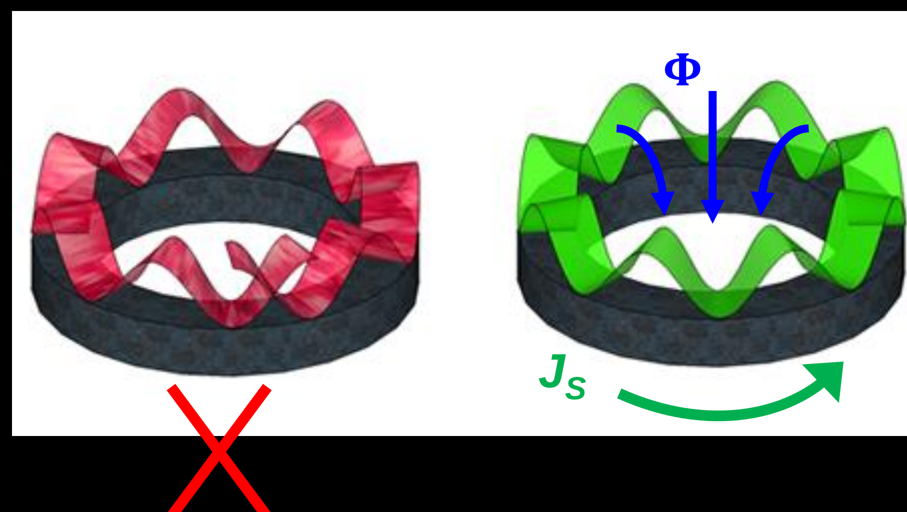
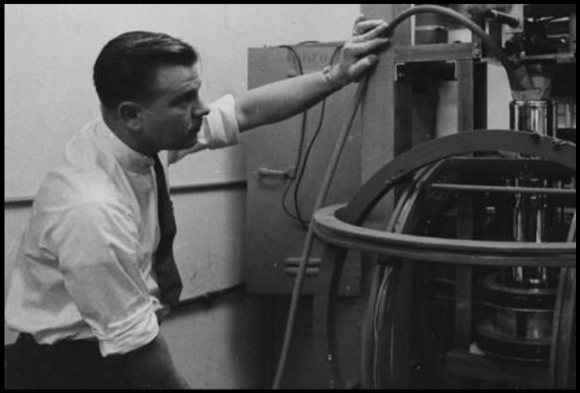
Superconducting ring



Superfluids, Macroscopic Theory of Superconductivity, Structure of Matter Vol. 1 (Wiley, New York, 1950)

Flux Quantization Experiments in 1961

Bascom Deaver William Fairbank



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$|\Phi| = n\, {}^{hc}/_{2e} = n\Phi_0,$

where $\Phi_0 = 2.0 \times 10^{-15} Tesla - m^2$

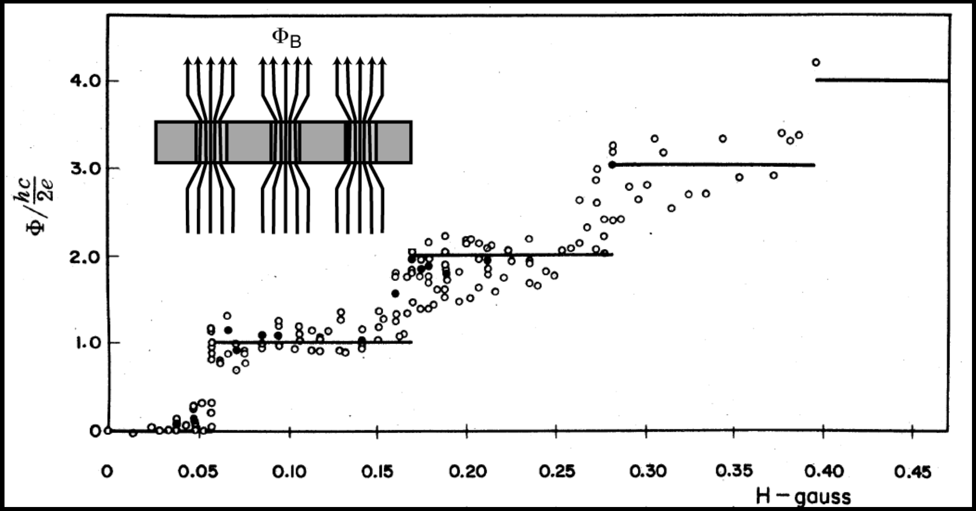
Each vortex carries one flux quanta

SC carriers are 2e !

Confirmation of Cooper pairs !

Robert Doll Martin Näbauer

©Walther-Meißner-Institute



B. D. Deaver and W. M. Fairbank, PRL **7**, 43 (1961)
R. Doll and M. Näbauer, PRL **7**, 51 (1961)

(3) *Duration of Persistent Currents*

A fluxoid cannot leak out of the ring and thereby reduce the persistent current unless by a thermal fluctuation a minimum volume of the superconducting ring is momentarily in the normal state.

The probability per unit time that a fluxoid will leak out is the product

$$P = (\text{attempt frequency})(\text{activation barrier factor}) \ . \qquad (28)$$

The activation barrier factor is $\exp(-\Delta F/k_B T)$, where the free energy of the barrier is

$$\Delta F \approx (\text{minimum volume})(\text{excess free energy density of normal state}) \ .$$

$$\Delta F \approx R\xi^2 H_c^2/8\pi \ . \qquad (29)$$

$$\exp(-\Delta F/k_B T) \approx \exp(-10^8) \approx 10^{-(4.34\times 10^7)} \qquad (29')$$

The characteristic frequency with which the minimum volume can attempt to change its state must be of order of $E_g/\hbar$. If $E_g = 10^{-15}$ erg, the attempt frequency is $\approx 10^{-15}/10^{-27} \approx 10^{12}\ \mathrm{s}^{-1}$. The leakage probability (28) becomes

$$P \approx 10^{12} 10^{-4.34\times 10^7}\ \mathrm{s}^{-1} \approx 10^{-4.34\times 10^7}\ \mathrm{s}^{-1} \ .$$

The reciprocal of this is a measure of the time required for a fluxoid to leak out, $T = 1/P = 10^{4.34\times 10^7}$ s.

The age of the universe is only 10^{18} s, so that a fluxoid will never leak out in the age of the universe, under our assumed conditions. Accordingly, the current is maintained.

(4) *Type II Superconductors*

1. A good type I superconductor excludes a magnetic field until superconductivity is destroyed suddenly, and then the field penetrates completely.

2. (a) A good type II superconductor excludes the field completely up to a field H_{c1} .

   (b) Above H_{c1} the field is partially excluded, but the specimen remains electrically super conducting.

   (c) At a much higher field, H_{c2} , the flux penetrates completely and superconductivity vanishes.

   (d) An outer **surface** layer of the specimen may remain superconducting up to a still higher field H_{c3} .

3. An important difference in a type I and a type II superconductor is in the mean free path of the conduction electrons in the normal state.

$$\textbf{type I, with } k = \lambda/\xi < 1$$

$$\textbf{type II, with } k = \lambda/\xi > 1$$

1. A superconductor is type I if the surface energy is always positive as the magnetic field is increased, for $H < H_c$
2. It is type II SC, if the surface energy becomes negative, as the magnetic field is increased. for $H_{c1} < H < H_{c2}$

The free energy of a bulk superconductor is increased when the magnetic field is expelled. However, a parallel field can penetrate a very thin film nearly uniformly (Fig.17), only a part of the flux is expelled, and the energy of the superconducting film will increase only slowly as the external magnetic field is increased.

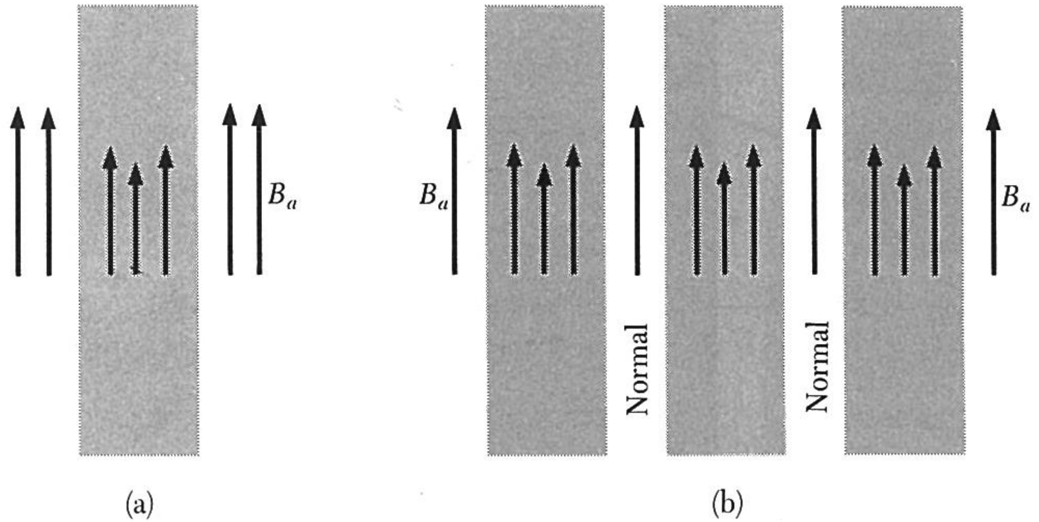


Figure 17 (a) Magnetic field penetration into a thin film of thickness equal to the penetration depth λ. The arrows indicate the intensity of the magnetic field. (b) Magnetic field penetration in a homogeneous bulk structure in the mixed or vortex state, with alternate layers in normal and superconducting states. The superconducting layers are thin in comparison with λ. The laminar structure is shown for convenience; the actual structure consists of rods of the normal state surrounded by the superconducting state. (The N regions in the vortex state are not exactly normal, but are described by low values of the stabilization energy density.)

Vortex State

In such a mixed state, called the vortex state, the external magnetic field will penetrate the thin normal regions uniformly, and the field will also penetrate somewhat into the surrounding superconducting material

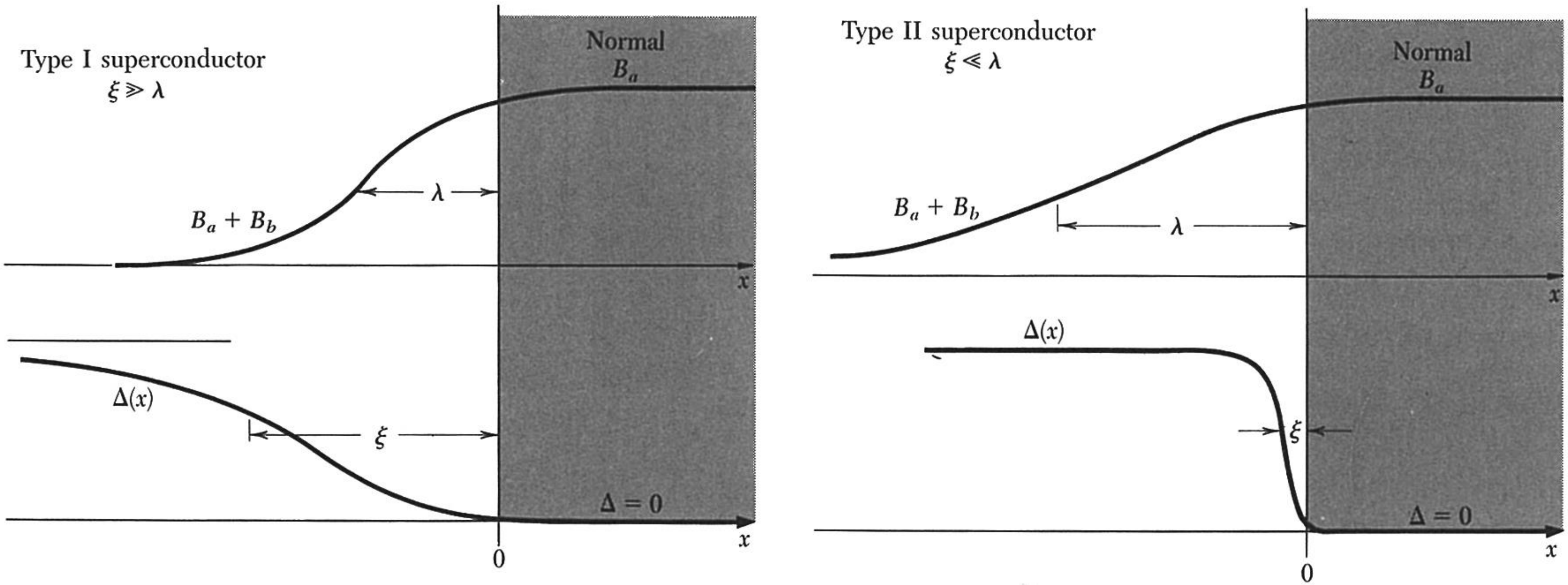
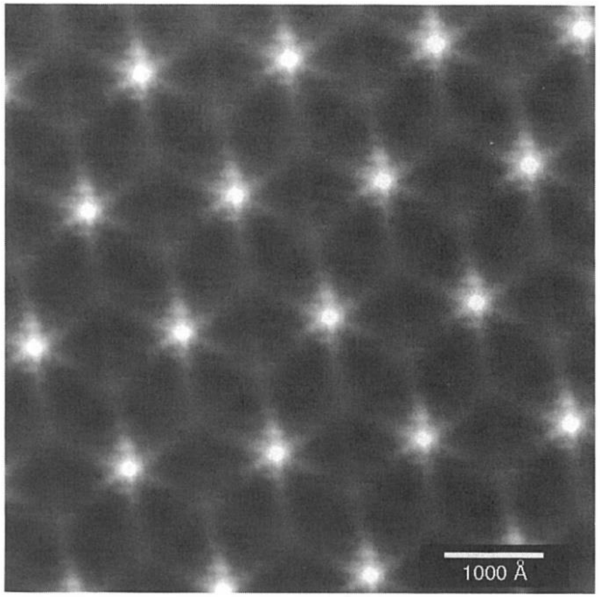


Figure 18 Variation of the magnetic field and energy gap parameter $\Delta(x)$ at the interface of superconducting and normal regions, for type I and type II superconductors. The energy gap parameter is a measure of the stabilization energy density of the superconducting state.

The term **vortex** state describes the circulation of superconducting currents in vortices throughout the bulk specimen.

Flux lattice of $NbSe_2$ at 0.2K



Abrikosov triangular lattice, as imaged by LT-STM, H. Hess et al

Figure 19 Flux lattice in $NbSe_2$ at 1,000 gauss at 0.2K, as viewed with a scanning tunneling microscope. The photo shows the density of states at the Fermi level, as in Figure 23. The vortex cores have a high density of states and are shaded white; the superconducting regions are dark, with no states at the Fermi level. The amplitude and spatial extent of these states is determined by a potential well formed by $\Delta(x)$ as in Figure 18 for a Type II superconductor. The potential well confines the core state wavefunctions in the image here. The star shape is a finer feature, a result special to $NbSe_2$ of the sixfold disturbance of the charge density at the Fermi surface. Photo courtesy of H. F. Hess, AT&T Bell Laboratories.

The vortex is stable when the penetration of the applied field into the superconducting material causes the surface energy become negative. A type II superconductor is characterized by a vortex state stable over a certain range of magnetic field strength; namely, between H_{c1} and H_{c2} .

Observation of Hexagonally Correlated Flux Quanta In $YBa_2Cu_3O_7$

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L. F. Schneemeyer, and J. V. Waszczak
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(Received 26 August 1987)

The high-resolution Bitter pattern technique has been used to reveal the magnetic structure of single-crystal samples of high-T_c superconductor $YBa_2Cu_3O_7$ at 4.2 K. Typical patterns consist of hexagonally correlated, singly quantized vortices of flux $hc/2e$. That is, the structures are comparable to those that would be observed in conventional type-II superconductors under similar conditions.

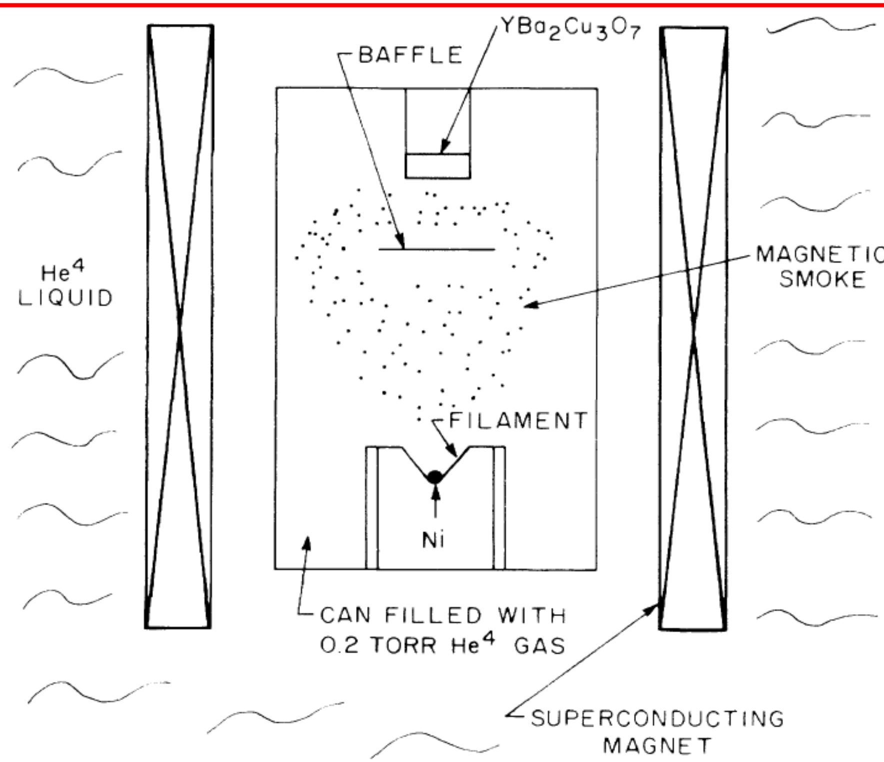

FIG. 1. Sketch of the decoration apparatus.

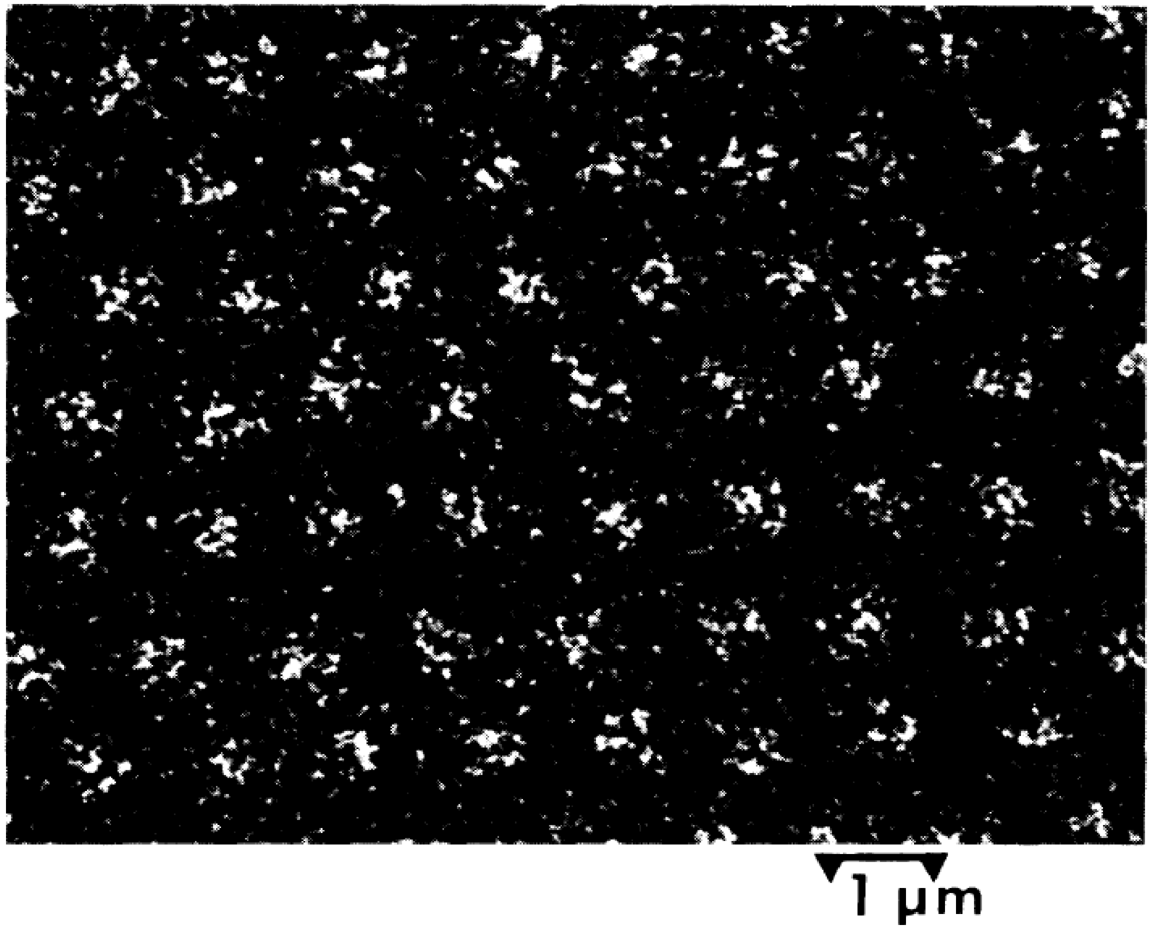

FIG. 2. Flux spots in a $YBa_2Cu_3O_7$ sample decorated after cooling in a field of 13 G.

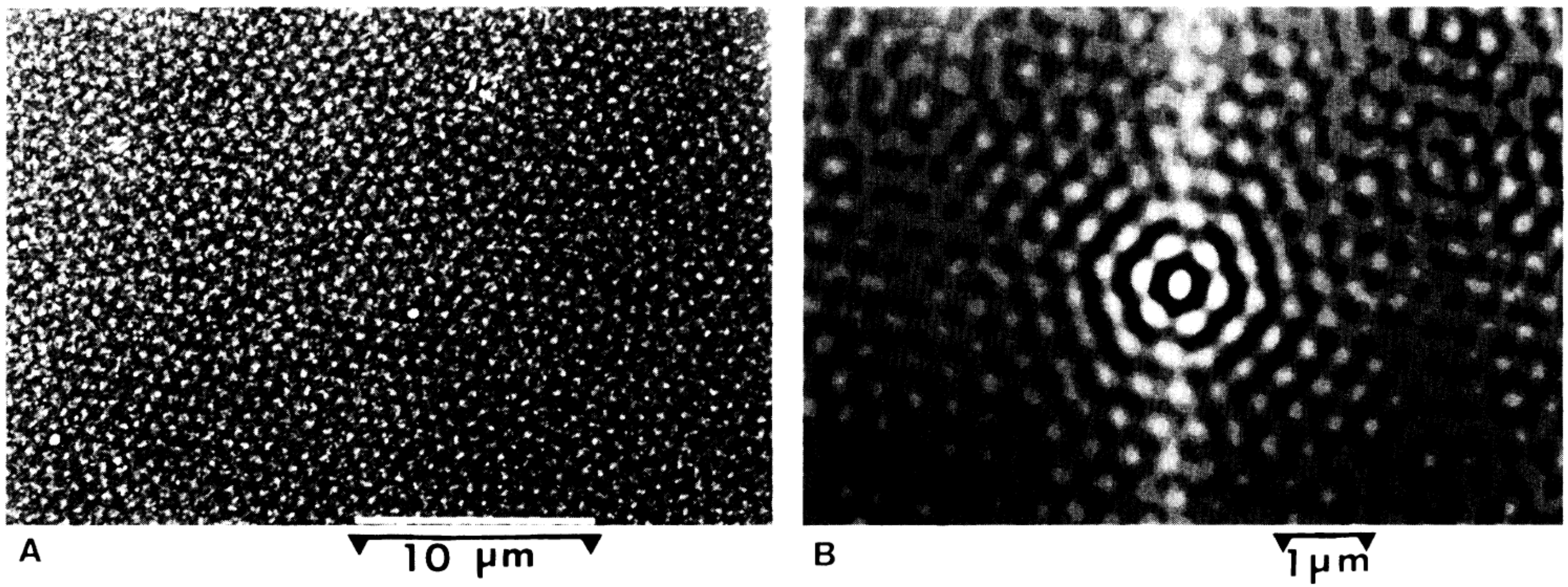

FIG. 3. (a) Typical area of a sample cooled in a 52-G field. (b) Central portion of the autocorrelation function of the pattern in (a).

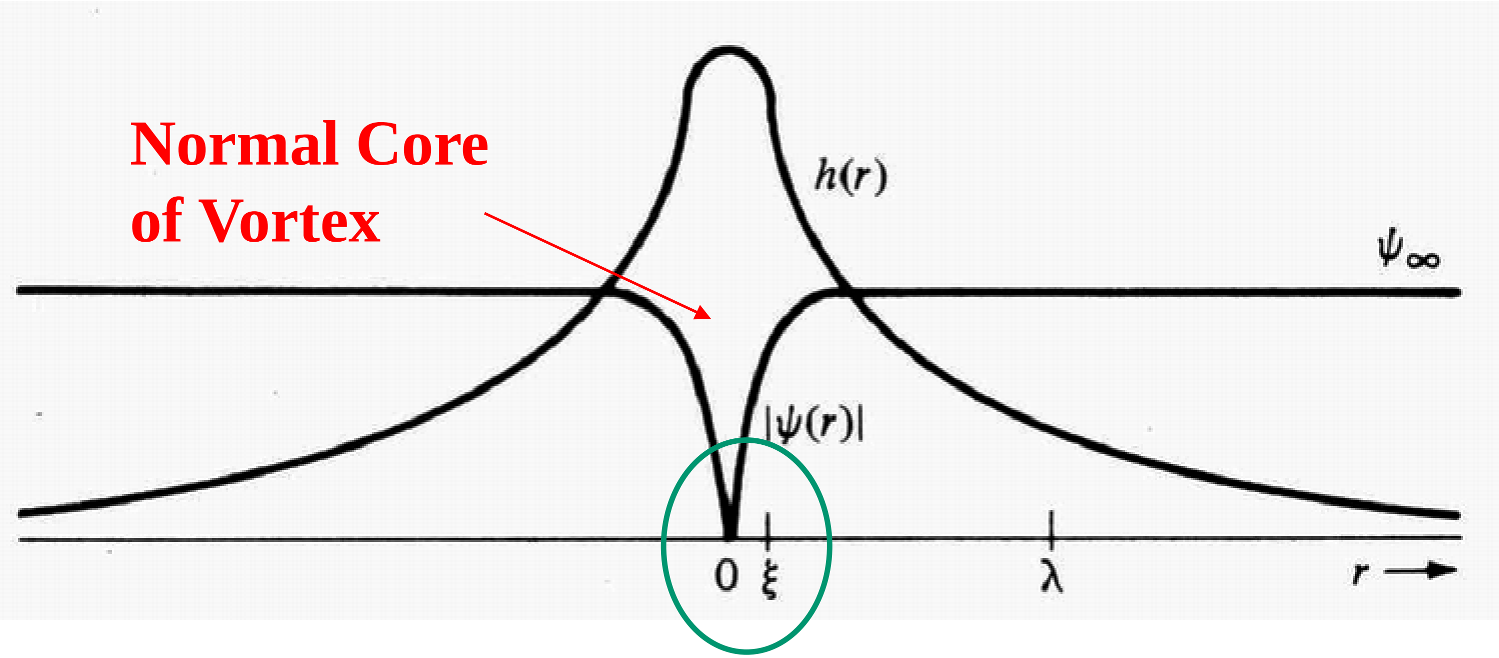

FIGURE 5-1
Structure of an isolated Abrikosov vortex in a material with $\kappa \approx 8$. The maximum value of $h(r)$ is approximately $2H_{c1}$.

Estimation of H_{c1} and H_{c2}

The field will extend out from the normal core a distance $\boldsymbol{\lambda}$ into the superconducting environment. The flux thus associated with a single (first) core is $\pi\lambda^2 H_{c1}$, and this must be equal to the flux quantum Φ_0.

$$\boldsymbol{H_{c1} \approx \Phi_0/\pi\lambda^2} \qquad (30)$$

This is the field for nucleation of a single fluxoid.

The external field penetrates the specimen almost uniformly, with small ripples on the scale of the fluxoid lattice. Each (last) core is responsible for carrying a flux of the order of $\pi\xi^2 H_{c2}$,

$$\boldsymbol{H_{c2} \approx \Phi_0/\pi\xi^2} \qquad (31)$$

The larger the ratio λ/ξ , the larger is the ratio of H_{c2} to H_{c1}.

The estimate H_{c1} in terms of H_c , we consider the stability of the vortex state at absolute zero in the impure limit $\xi < \lambda$; here $\kappa > 1$ are the coherence length is short in comparison with the penetration depth.

We estimate in the vortex state the stabilization energy of a fluxoid core viewed as a normal metal cylinder which carries an average magnetic field B_a. The radius is of the order of the coherence length, as the thickness of the boundary between **N** and **S** phases.

$$f_{core} \approx \frac{1}{8\pi} H_c^2 \times \pi\xi^2 \;, \tag{32}$$

But there is also a decrease in magnetic energy because of the penetration of the applied field B_a into the superconducting material around

$$f_{mag} \approx -\frac{1}{8\pi} B_a^2 \times \pi\lambda^2 \;. \tag{33}$$

$$f = f_{core} + f_{mag} \approx \tfrac{1}{8}(H_c^2\xi^2 - B_a^2\lambda^2) \;. \tag{34}$$

The threshold field for a stable fluxoid is at $f = 0$, or, with H_{c1} written for B_a,

$$H_{c1}/H_c \approx \xi/\lambda \;. \sim 1/\kappa \tag{35}$$

The threshold field divides the region of positive surface energy from the region of negative surface energy.

for H < H_{c1}, f > 0 ; for H > H_{c1}, f < 0

(30) + (35) $\pi\xi\lambda H_c \approx \Phi_0$ (36)

(30) + (31) $(H_{c1}H_{c2})^{1/2} \approx H_c$ (37a)

(31) + (37a) $H_{c2} \approx (\lambda/\xi)H_c = \kappa H_c$ (37b)