Surface Defects and the BPS Spectrum of 4d N=2 Theories

Solvay Conference, May 19, 2011

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Wall-crossing in Coupled 2d-4d Systems: 1103.2598

Framed BPS States: 1006.0146

Wall-crossing, Hitchin Systems, and the WKB Approximation: 0907.3987

Four-dimensional wall-crossing via three-dimensional Field theory: 0807.4723

A Motivating Question

Given an arbitrary four-dimensional field theory with N=2 supersymmetry, is there an algorithm for computing its BPS spectrum?

Who cares?

A good litmus test to see how well we understand these theories…


CUNARD
``Getting there is half the fun!”

Goal For Today

We describe techniques which should lead to such an algorithm for

``A_k theories of class S"

Some isolated examples of BPS spectra are known:

1. Bilal & Ferrari: SU(2) $N_f = 0,1,2,3$
2. Ferrari: SU(2), N = 2*, SU(2) $N_f=4$
3. GMN: A_1 theories of class S

Outline

1. Review of some N=2,d=4 theory

2. Theories of Class S
   a. 6d (2,0) and cod 2 defects
   b. Compactified on C
   c. Related Hitchin systems
   d. BPS States and finite WKB networks

3. Line defects and framed BPS States

4. Surface defects
   a. UV and IR
   b. Canonical surface defects in class S
   c. 2d4d BPS + Framed BPS degeneracies

5. 2d/4d WCF

6. Algorithm for theories of class S

7. Overview of results on hyperkahler geometry

1

N=2, d=4 Field Theory

Coulomb branch: $\mathcal{B}$,

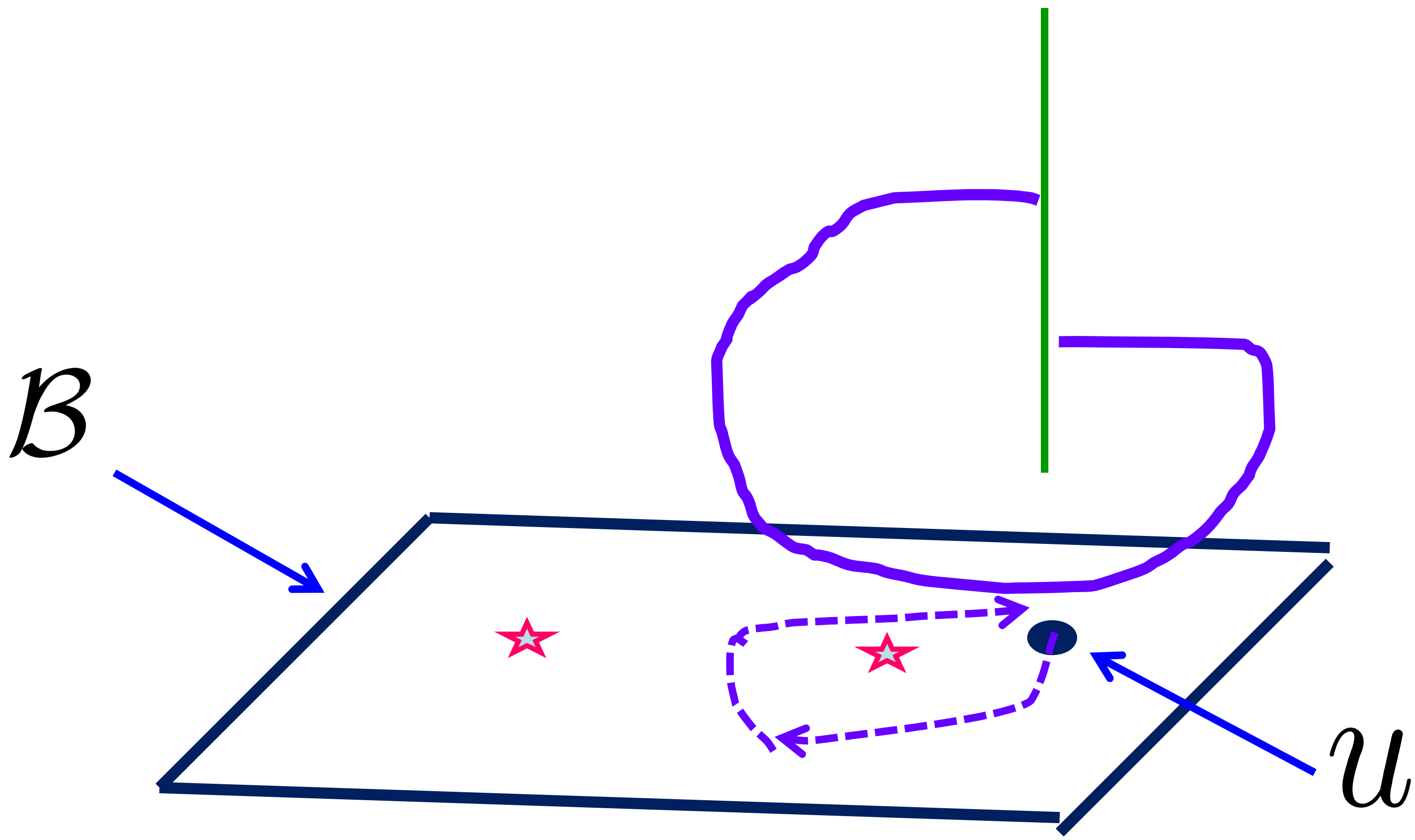
generic point $u \in \mathcal{B}$

Local system of charges, with integral antisymmetric form:

$$0 \to \Gamma_f \to \Gamma \to \Gamma_g \to 0,$$

Γ_f: Charges of unbroken flavor symmetries

Γ_g: Symplectic lattice of (elec,mag) charges of IR abelian gauge theory


$\mathcal{B}$
u

Self-dual IR abelian gauge field

$$V = \Gamma_g \otimes \mathbb{R} \qquad \mathbb{F} \in \Omega^2(M_4) \otimes V$$

$$d\mathbb{F} = 0 \qquad \mathbb{F} = *\mathbb{F}$$

$$\mathbb{F} = d\mathbb{A} = e_I F^I + e^I G_I$$

$$S = \int \mathrm{Im}\tau_{IJ} F^I * F^J + \mathrm{Re}\tau_{IJ} F^I F^J$$

Central charge: $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$

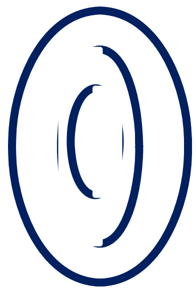
$$\mathcal{H} = \oplus_{\gamma\in\Gamma}\mathcal{H}_\gamma \qquad E \geq |Z_\gamma|$$

$$\mathcal{H}_\gamma^{\mathrm{BPS}} = \{\psi : E\psi = |Z_\gamma|\psi\}$$

$$\Omega(\gamma; u; y) := \mathrm{Tr}_{h_\gamma}(-1)^{2J_3}(-y)^{2J_3+2I_3}$$

$$\Omega(\gamma; u) := \mathrm{Tr}_{h_\gamma}(-1)^{2J_3}$$

Seiberg-Witten Moduli Space



$$\subset \quad \mathcal{M} = \Gamma_g^* \otimes \mathbb{R}/(2\pi\mathbb{Z})$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{B} \end{array}$$

Hyperkahler target space of 3d sigma model from compactification on $\mathbb{R}^3 \times S^1$

Seiberg & Witten

2

Theories of Class S

Consider nonabelian (2,0) theory $T[\mathfrak{g}]$ for ``gauge algebra'' $\mathfrak{g}$

The theory has half-BPS codimension two defects D(m)

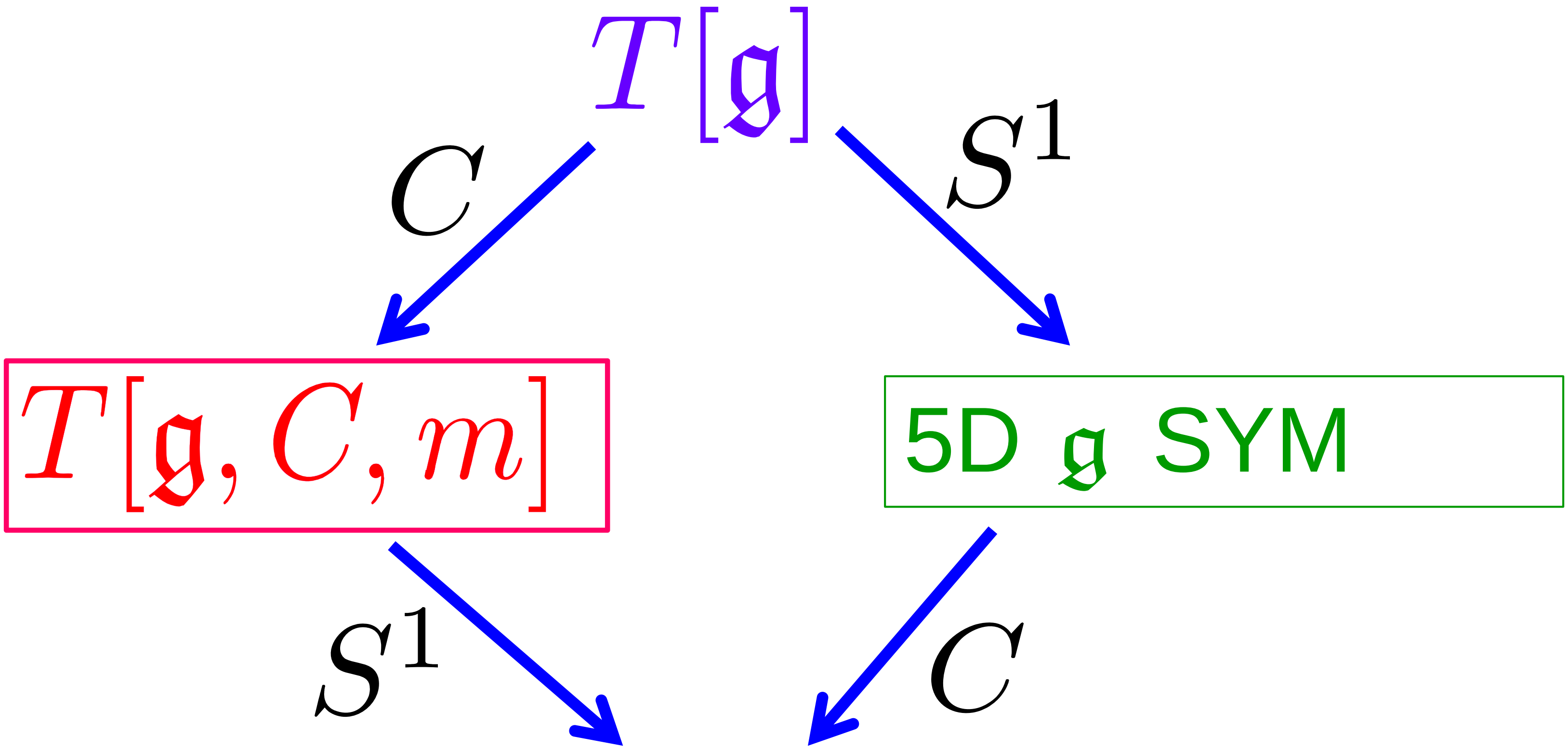
Compactify on a Riemann surface C with $D(m_a)$ inserted at punctures z_a

$$so(5)_R \to so(3)_R \oplus \underbrace{so(2)_R}_{\text{Twist to preserve d=4,N=2}}$$

Witten, 1997
GMN, 2009
Gaiotto, 2009

$$T[\mathfrak{g}, C, m]$$

Seiberg-Witten = Hitchin



σ-Model: $\mathbb{R}^{1,2} \rightarrow \mathcal{M}$

$$F + R^2[\varphi, \bar{\varphi}] = 0$$
$$\bar{\partial}_A \varphi = 0$$

Digression: ***Puncture Zoo***

Regular singular points:

$$\varphi \sim \frac{dz}{z - z_a}\mathfrak{r} + \mathrm{reg}$$

$\mathfrak{r}$:

$\mathrm{Diag}\{m_1, m_2, \dots, m_k\}$ ``Full puncture''

$\mathrm{Diag}\{m, \dots, m, -(k-1)m\}$ ``Simple puncture''

Irregular singular points:

$$\varphi \sim \frac{dz}{(z - z_a)^{\ell}}\mathfrak{r} + \mathrm{reg}$$

Seiberg-Witten Curve

$$\Sigma : \det(\lambda - \varphi(z, \bar{z})) = 0 \subset T^*C$$

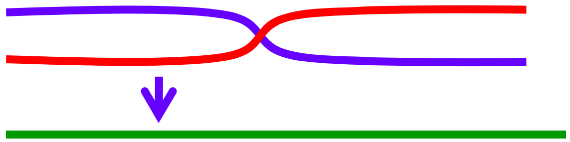
$\lambda = pdq$ $\quad\quad$ $\lambda|_\Sigma$ $\quad\quad$ SW differential

For $\mathfrak{g}$ = su(k) $\quad\quad$ $\pi : \Sigma \to C$

is a k-fold branch cover

$$\lambda^k + \lambda^{k-2}\phi_2(z) + \cdots + \phi_k(z) = 0$$

Local System of Charges



$$\ker \pi_* : Jac(\Sigma) \to Jac(C)$$

determines $\Gamma \subset H_1(\Sigma; \mathbb{Z})$

A local system over a torsor for spaces of holomorphic differentials…

BPS States: Geometrical Picture

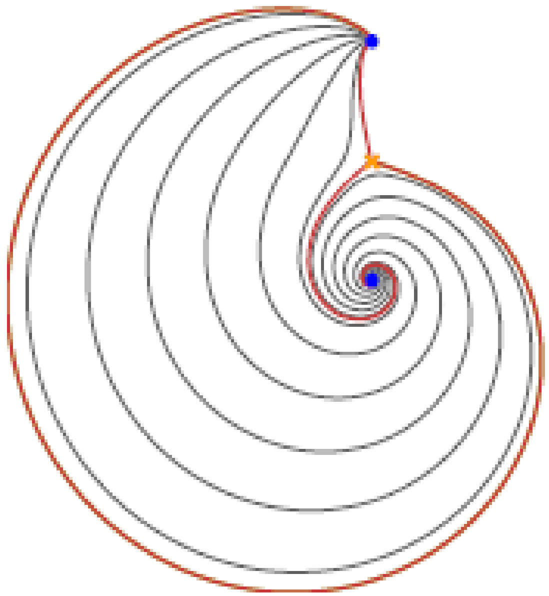
BPS states come from open M2 branes stretching between sheets i and j. Here i,j, =1,…, k. This leads to a nice geometrical picture with string networks:

Klemm, Lerche, Mayr, Vafa, Warner; Mikhailov; Mikhailov, Nekrasov, Sethi,

Def: A WKB path on C is an integral path

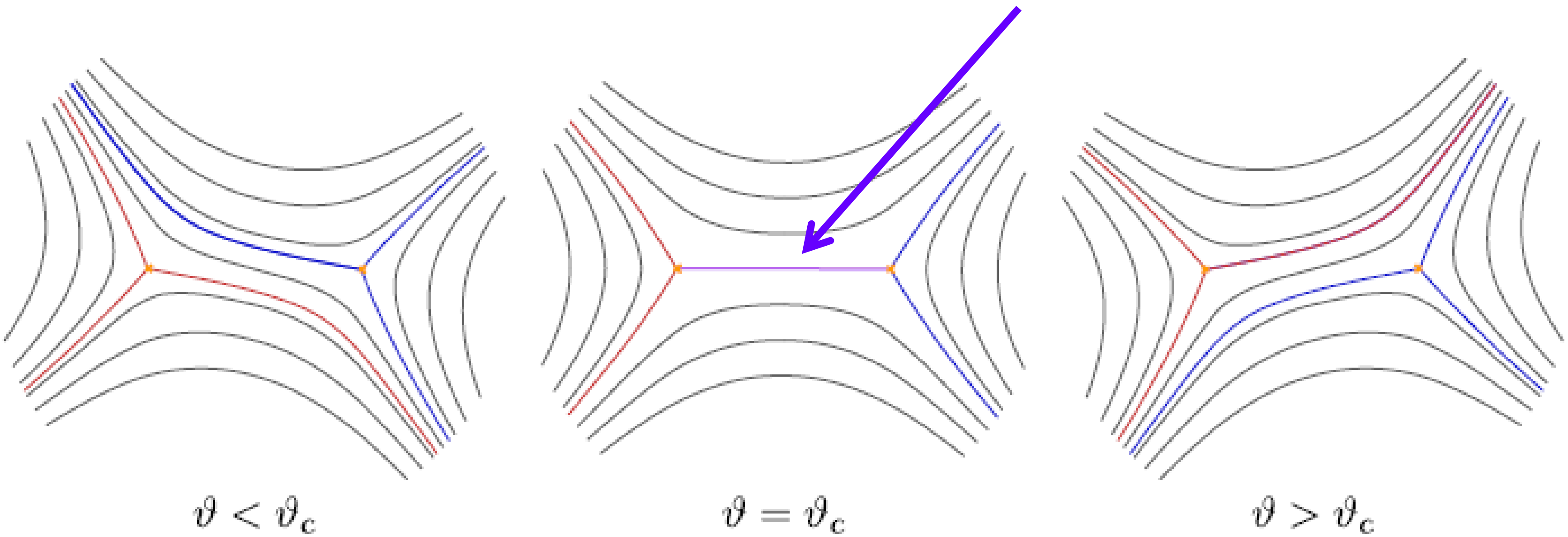
$$\langle \lambda_i - \lambda_j , \partial_t \rangle = e^{i\vartheta}$$

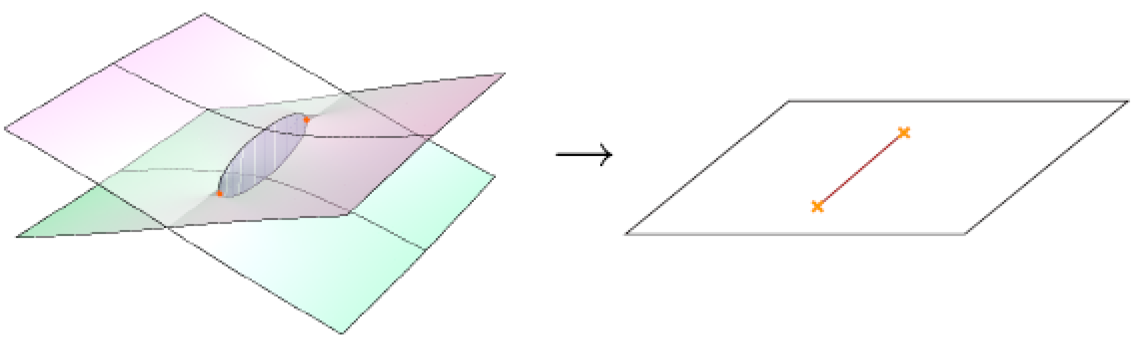
Generic WKB paths have both ends on singular points z_a



Finite WKB Networks - A

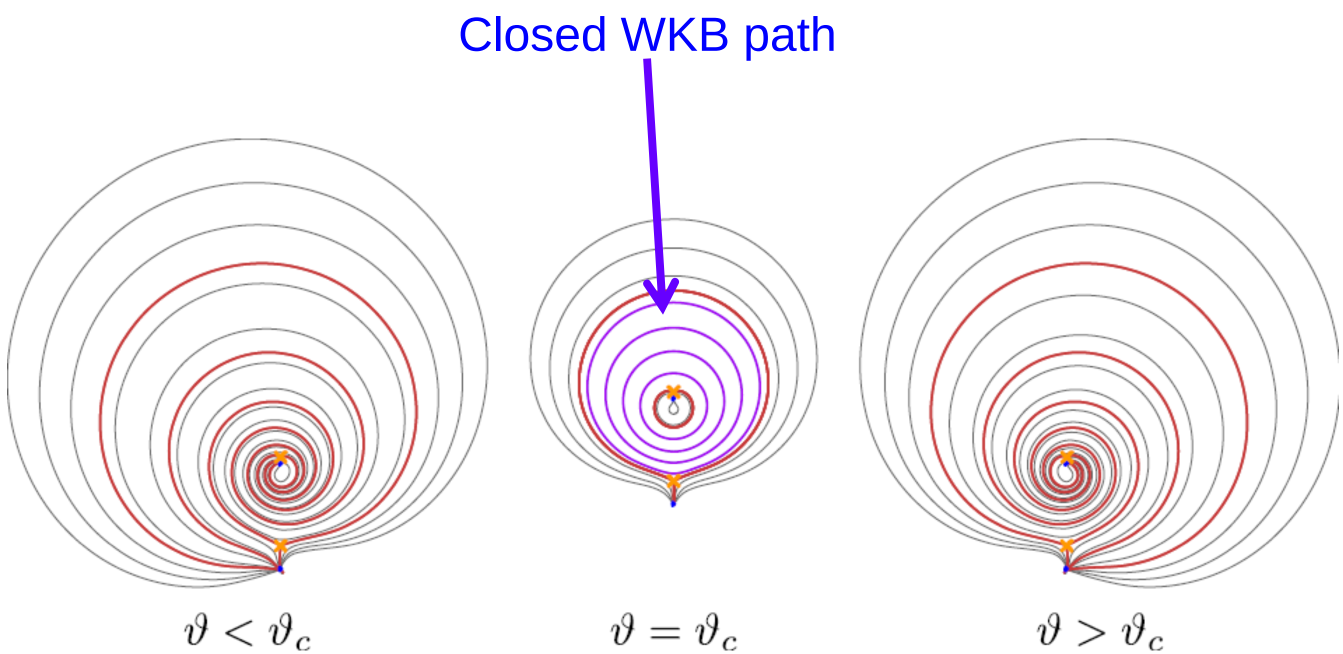
But at critical values of $\vartheta=\vartheta_*$ ``finite WKB networks appear'':

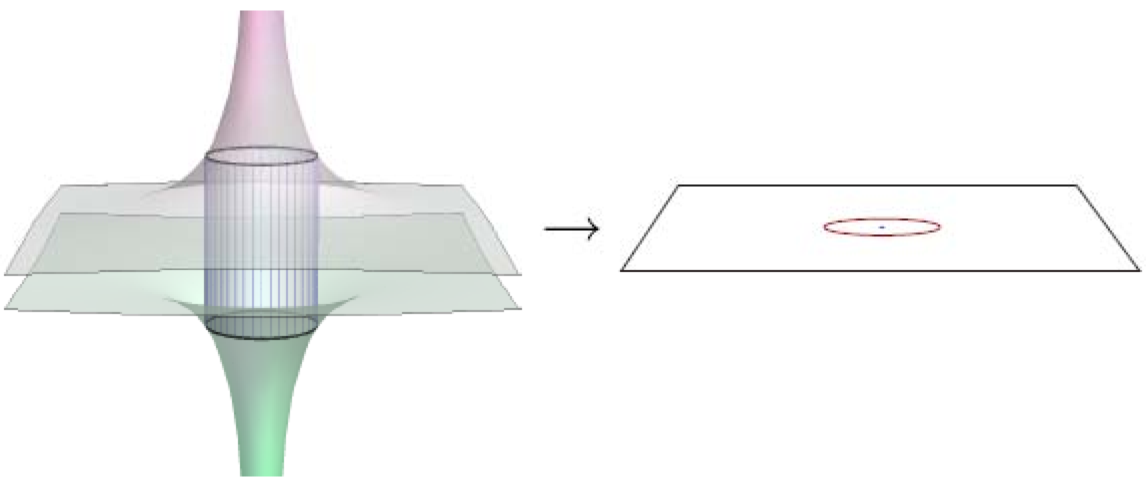




Hypermultiplet

Finite WKB Networks - B

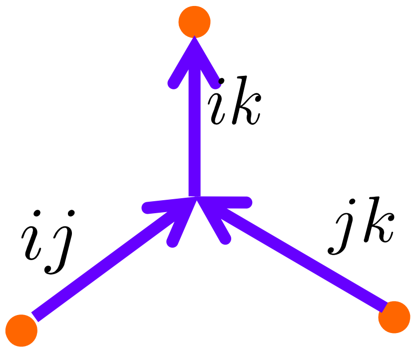




Vectormultiplet

Finite WKB Networks - C

At higher rank, we get string networks at critical values of ϑ:



A ``finite WKB network'' is a union of WKB paths with endpoints on branchpoints or such junctions.

These networks lift to closed cycles γ in Σ and represent BPS states with

$$Z_\gamma = \oint_\gamma \lambda = e^{i\vartheta_*} |Z_\gamma|$$

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Line Defects & Framed BPS States

A ***line defect*** (say along $\mathbb{R}_t$ x {0 }) is ***of type ζ*** if it preserves the susys:

$$Q_{\alpha}^{A} + \zeta \sigma^{0}_{\alpha \dot\beta} \bar{Q}^{\dot\beta A}$$

Example: $$L_{\zeta} = \exp \int_{\mathbb{R}_t \times \vec{0}} \left(\frac{\varphi}{2\zeta} + A + \frac{\zeta}{2} \overline{\varphi} \right)$$

$$\mathcal{H}_L = \oplus_{\gamma \in \Gamma} \mathcal{H}_{L,\gamma}$$

$$E \geq -\mathrm{Re}(Z_{\gamma}/\zeta)$$

Framed BPS States saturate this bound, and have framed protected spin character:

$$\overline{\underline{\Omega}} := \mathrm{Tr}_{\mathcal{H}^{bps}_{L,\gamma}} (-1)^{2J_3} (-y)^{2J_3+2I_3}$$

$$\overline{\underline{\Omega}}(L, \gamma; y; \zeta; u)$$

Piecewise constant in ζ and u, but has wall-crossing across ``BPS walls'' (for $\Omega(\gamma) \neq 0$):

$$W_\gamma := \{(u,\zeta) : Z_\gamma(u)/\zeta \in \mathbb{R}_-\}$$

Particle of charge γ binds to the line defect:

→ Similar to Denef's halo picture

Wall-Crossing for $\overline{\underline{\Omega}}$

$$F(L) = \sum_{\gamma} \overline{\underline{\Omega}}(L,\gamma;y) X_{\gamma}$$

$$X_{\gamma_1} X_{\gamma_2} = y^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}$$

Across $W(\gamma_h)$ Denef's halo picture leads to:

$$F^{+}(L) = \Phi(X_{\gamma_h}) F^{-}(L) \Phi(X_{\gamma_h})^{-1}$$

$\Phi(X_{\gamma_h})$ constructed from $\Omega(\gamma_h; y)$

Wall-Crossing for Ω

Consistency of wall crossing of framed BPS states implies the Kontsevich-Soibelman ``motivic WCF'' for

$$\Omega(\gamma; y)$$

This strategy also works well in supergravity to prove the KSWCF for BPS states of Type II/Calabi-Yau
(but is only physically justified for y=-1)

Andriyash, Denef, Jafferis, Moore

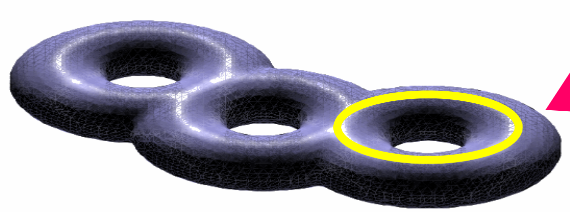
Line defects in T[$\mathfrak{g}$,C,m]

6D theory T[$\mathfrak{g}$] has supersymmetric surface defects S($\mathcal{R}$, σ)

For T[$\mathfrak{g}$,C,m] consider

$$\sigma = \mathbb{R} \times \{\vec{0}\} \times \wp$$

C



$$L_\zeta(\mathcal{R}, \wp)$$

Line defect in 4d *labeled* by isotopy class of a ***closed*** path $\wp$ and $\mathcal{R}$

k=2: Drukker, Morrison, Okuda

Complex Flat Connections

(A,ϕ) solve Hitchin equations iff

$$\mathcal{A} = R\zeta^{-1}\varphi + A + R\zeta\overline{\varphi}$$

is a complex flat connection on C $\forall \zeta \in \mathbb{C}^*$

On $\mathbb{R}^3$ x S^1 line defects become local operators in the 3d sigma model:

$$\langle L_\zeta(\mathcal{R}, \wp) \rangle = \mathrm{Tr}_{\mathcal{R}} Hol(\mathcal{A}, \wp)$$

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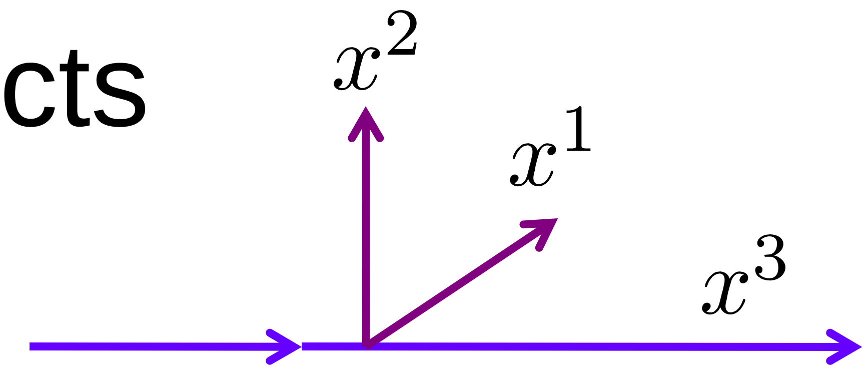
Surface defects

$\mathbb{S}$ at $x^1 = x^2 = 0$



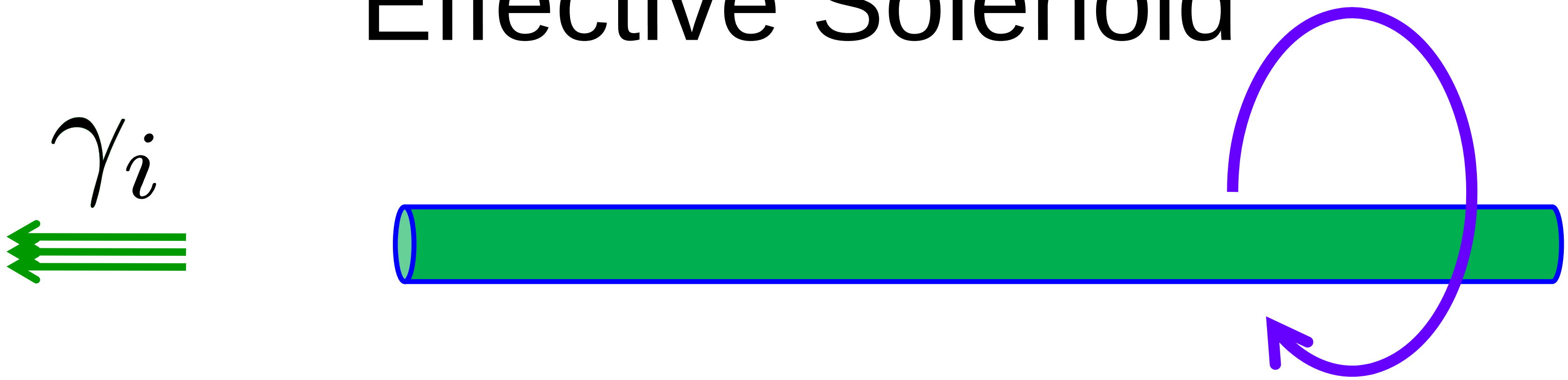
Preserves d=2 (2,2) supersymmetry subalgebra

Twisted chiral multiplet : $\Upsilon = \varphi + \cdots$

Finite set of vacua $i \in \mathcal{V}$

$$S_{IR} = \int d^4x d^4\theta \ \mathcal{F}(a) + \int d^2x d^2\theta \ \mathcal{W}^{eff}(\Upsilon)$$

Effective Solenoid



$$\oint \mathbb{A} = \gamma_i \in V = \Gamma_g \otimes \mathbb{R}$$

$$\gamma_i = \eta_I e^I + \alpha^I e_I$$

$$\eta + \tau \cdot \alpha = \frac{\partial W^{eff}}{\partial a}$$

$\Rightarrow$ α,η ARE NOT QUANTIZED

Torsor of Effective Superpotentials

A choice of superpotential =

a choice of gauge =

a choice of flux γ_i

$$Z_{\gamma_i} := \mathcal{W}^{eff}$$

$$Z_{\gamma_i+\gamma} := \mathcal{W}^{eff} + Z_\gamma$$

Extends the central charge to a Γ - torsor Γ_i

Canonical Surface Defect in T[$\mathfrak{g}$,C,m]

For z $\in$ C we have a ***canonical surface defect*** $\mathbb{S}_z$

It can be obtained from an M2-brane ending at $x^1=x^2=0$ in $\mathbb{R}^4$ and z in C

In the IR the different vacua for this M2-brane are the different sheets in the fiber of the SW curve over z.

Therefore the chiral ring of the 2d theory should be the same as the equation for the SW curve!

Alday, Gaiotto, Gukov, Tachikawa, Verlinde; Gaiotto

$$\lambda^k + \lambda^{k-2}\phi_2(z) + \cdots + \phi_k(z) = 0$$

Example of SU(2) SW theory

$$\lambda^2 = \left(\frac{1}{z} + \frac{2u}{z^2} + \frac{\Lambda^2}{z^3}\right)(dz)^2$$

$$\lambda = x dz \qquad z = e^t$$

$$x^2 = e^t + 2u + \frac{\Lambda^2}{e^t}$$

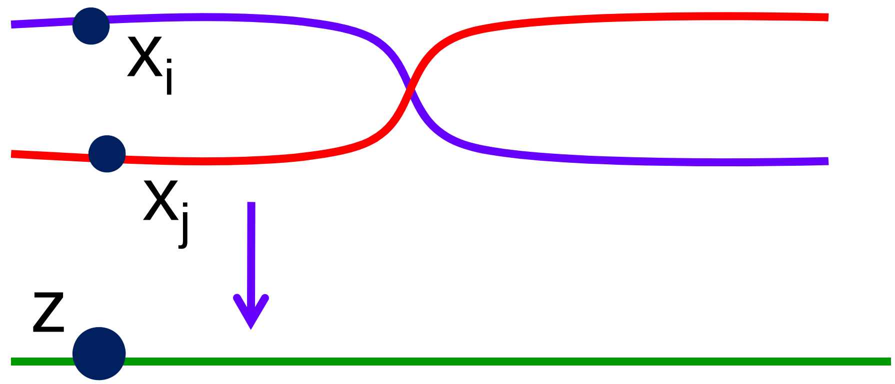
Chiral ring of the $\mathbb{C}P^1$ sigma model.

Twisted mass

2d-4d instanton effects

Gaiotto

Superpotential for $\mathbb{S}_z$ in T[$\mathfrak{g}$,C,m]



Σ

C

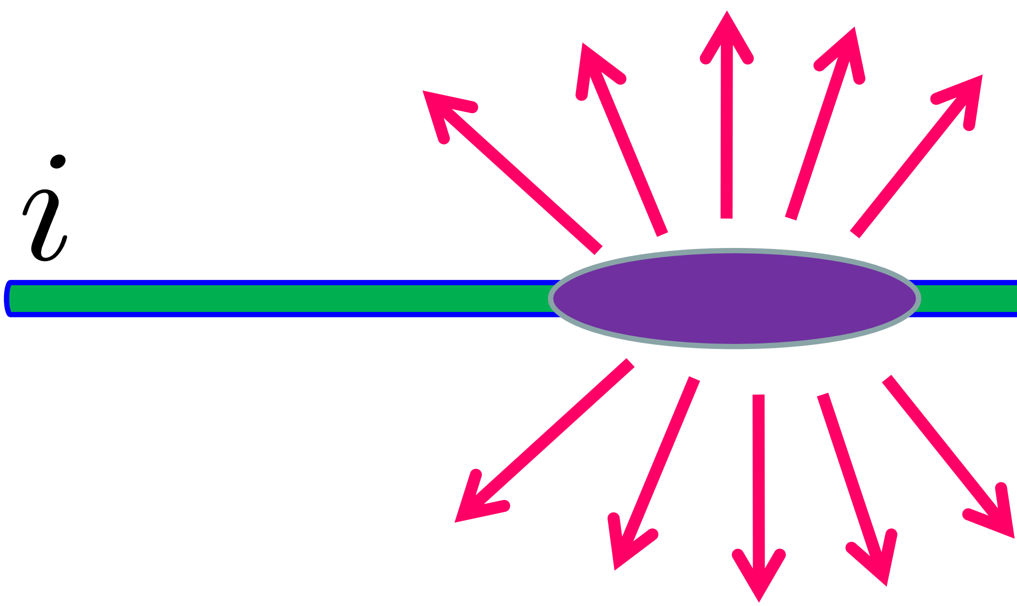
$$Z_{\gamma_i} - Z_{\gamma_j} = \int_{\gamma_{ij}} \lambda$$

γ_{ij} Homology of an ***open*** path ***on Σ*** joining x_i to x_j in the fiber over $\mathbb{S}_z$

$$\gamma_{ij} \in \Gamma_{ij} \subset H_1(\Sigma, \{x_i, x_j\}; \mathbb{Z})$$

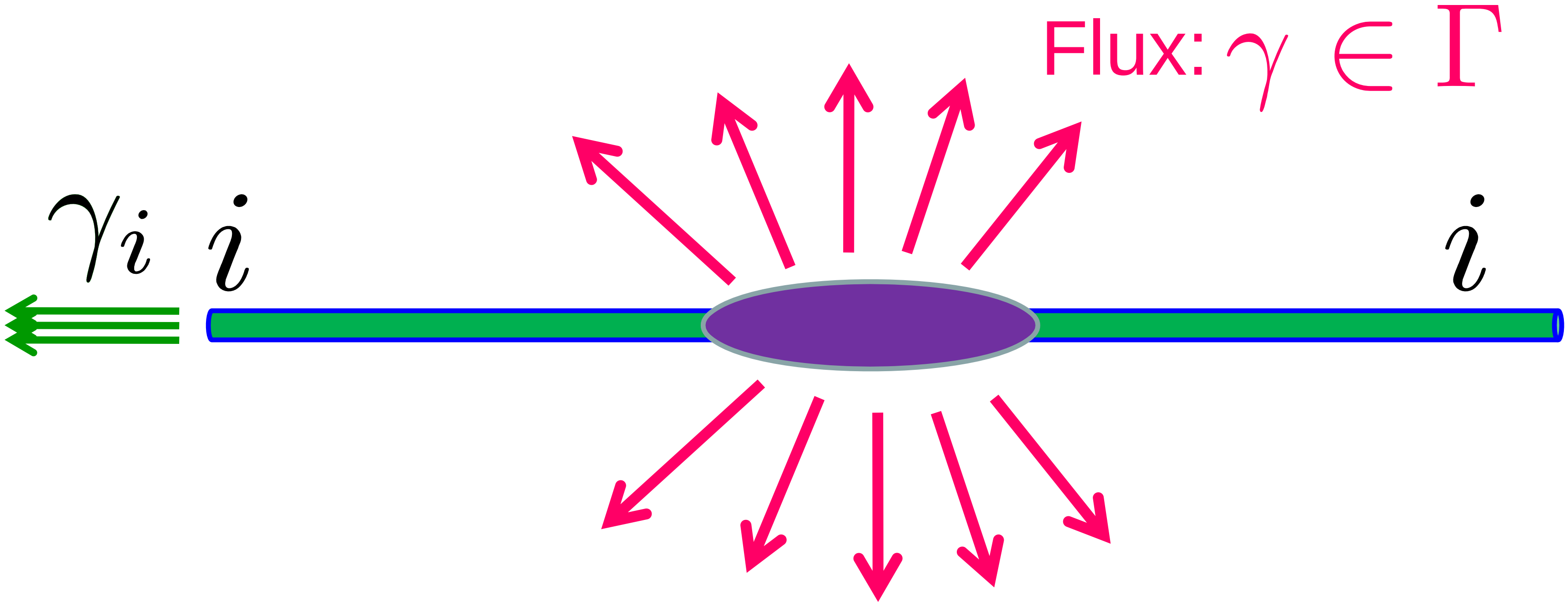
New BPS Degeneracies: μ

$\mu(\gamma_{ij})$ 2D soliton degeneracies.



For $\mathbb{S}_z$ in T[su(k),C,m], μ is a signed sum of open finite BPS networks ending at z

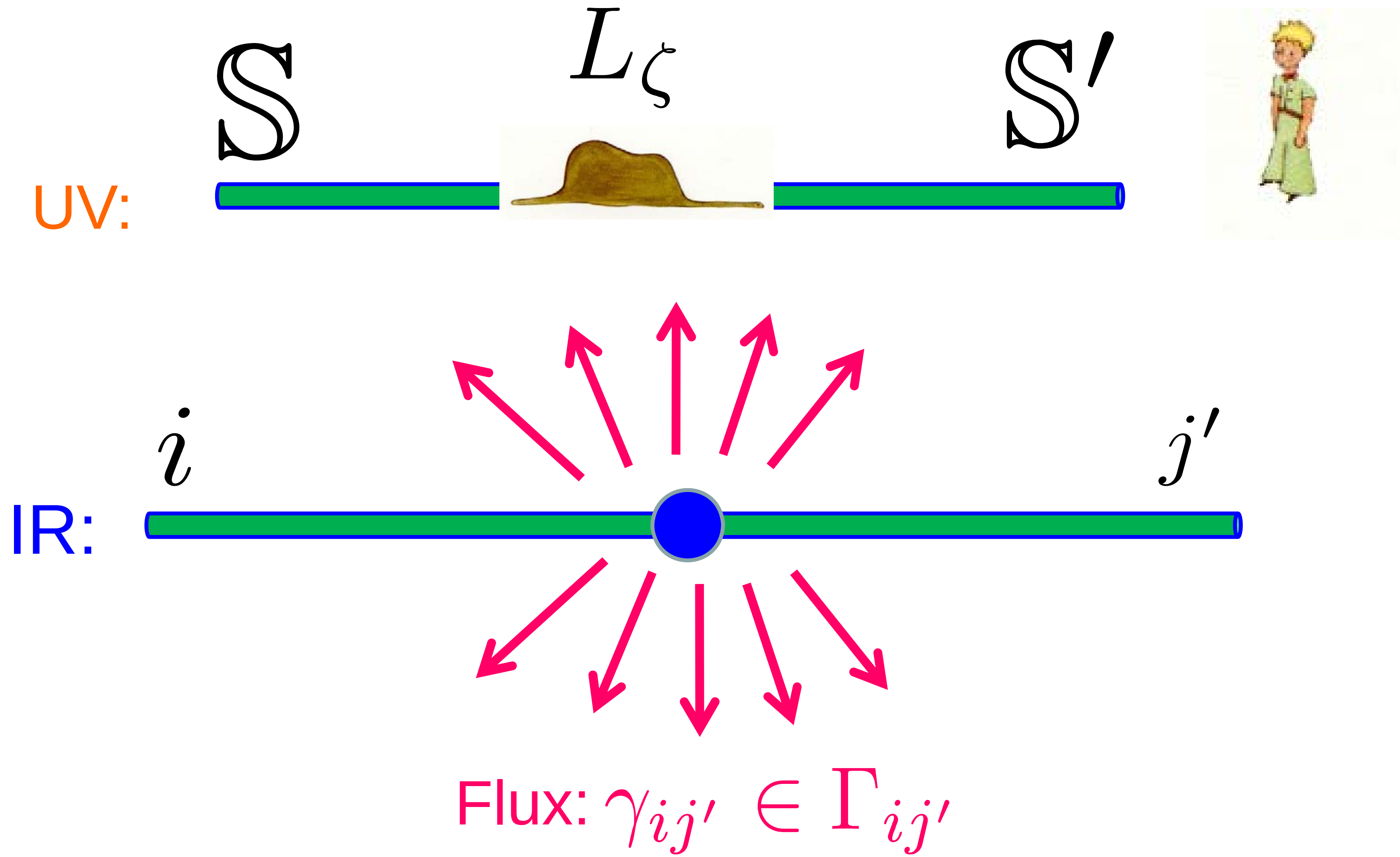
New BPS Degeneracies: ω



Degeneracy: $\omega(\gamma;\gamma_i)$

$$\omega(\gamma;\gamma_i+\gamma') = \omega(\gamma;\gamma_i) + \Omega(\gamma)\langle\gamma,\gamma'\rangle$$

Supersymmetric Interfaces - A



Supersymmetric Interfaces -B

$$\mathcal{H}_{\mathbb{S}L\mathbb{S}'} = \oplus_{\gamma_{ij'} \in \Gamma_{ij'}} \mathcal{H}_{\mathbb{S}L\mathbb{S}', \gamma_{ij'}}$$

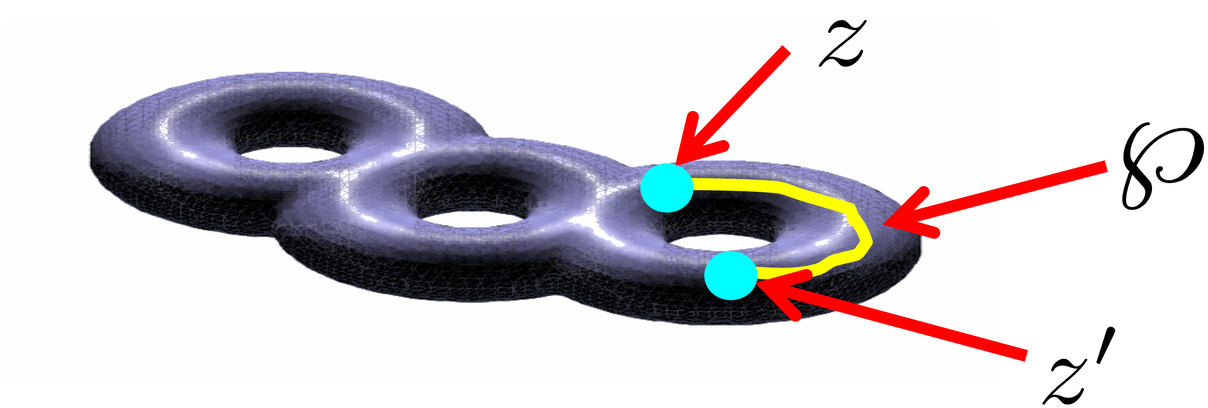
Our interfaces preserve two susy's of type ζ and hence we can define framed BPS states and form:

$$F(L) = \sum_{\Gamma_{ij'}} \overline{\underline{\Omega}}(L, \gamma_{ij'}) X_{\gamma_{ij'}}$$

$$X_{\gamma_{ij'}} X_{\gamma_{k''l'''}} = \begin{cases} \pm X_{\gamma_{ij'} + \gamma_{j'l'''}} & j' = k'' \\ 0 & \text{else} \end{cases}$$

Susy interfaces for T[$\mathfrak{g}$,C,m] - A

Interfaces between $\mathbb{S}_z$ and $\mathbb{S}_{z'}$ are labeled by open paths $\wp$ on C



So: framed BPS states are graded by open paths $\gamma_{ij'}$ on Σ with endpoints over z and z'

$$\Gamma_{ij'} \subset H_1(\Sigma, \{x_i, x_{j'}\}; \mathbb{Z})$$

Susy interfaces for T[$\mathfrak{g}$,C,m] - B

Wrapping the interface on a circle in $\mathbb{R}^3 \times S^1$ compactification:

$$\langle L_\zeta(\mathcal{R}, \wp) \rangle = \rho_{\mathcal{R}} Hol(\mathcal{A}, \wp)$$

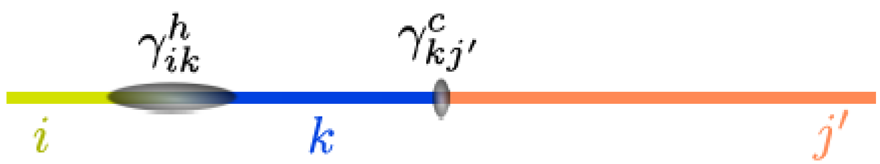
$$\mathcal{A} = R\zeta^{-1}\varphi + A + R\zeta\overline{\varphi}$$

Framed BPS Wall-Crossing

Across BPS W_γ walls the framed BPS degeneracies undergo wall-crossing.

Now there are ***also*** 2d halos which form across walls

$$W_{\gamma_{ik}} := \{(u,\zeta) : Z_{\gamma_{ik}}(u)/\zeta \in \mathbb{R}_-\}$$



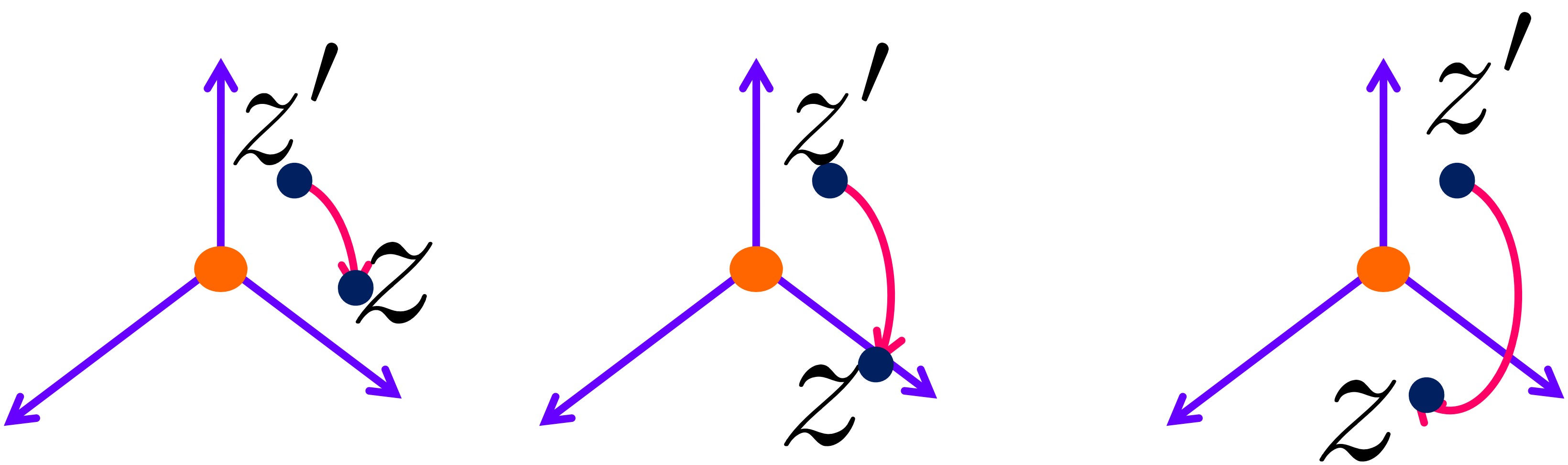
$$F(L) \to \mathcal{S} F(L) \mathcal{S}^{-1} \qquad \mathcal{S} = 1 + \mu(\gamma_{ik}) X_{\gamma_{ik}}$$

As before, consistency of the wall-crossing for the framed BPS degeneracies implies a general wall-crossing formula for unframed degeneracies μ and ω.

Framed Wall-Crossing for T[$\mathfrak{g}$,C,m]

The separating WKB paths of phase ζ on C are the BPS walls for

$$\overline{\Omega}(L_{\zeta,\mathcal{P}(z,z')}, \gamma_{ij'})$$



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Formal Statement of 2d/4d WCF

1. Four pieces of data

2. Three definitions

3. Statement of the WCF

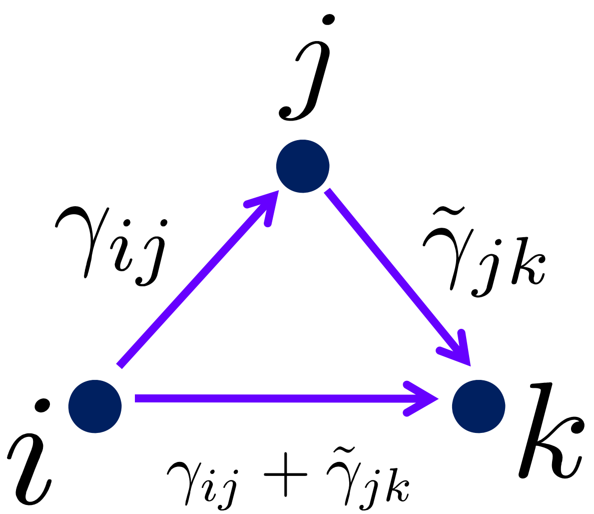
4. Relation to general KSWCF

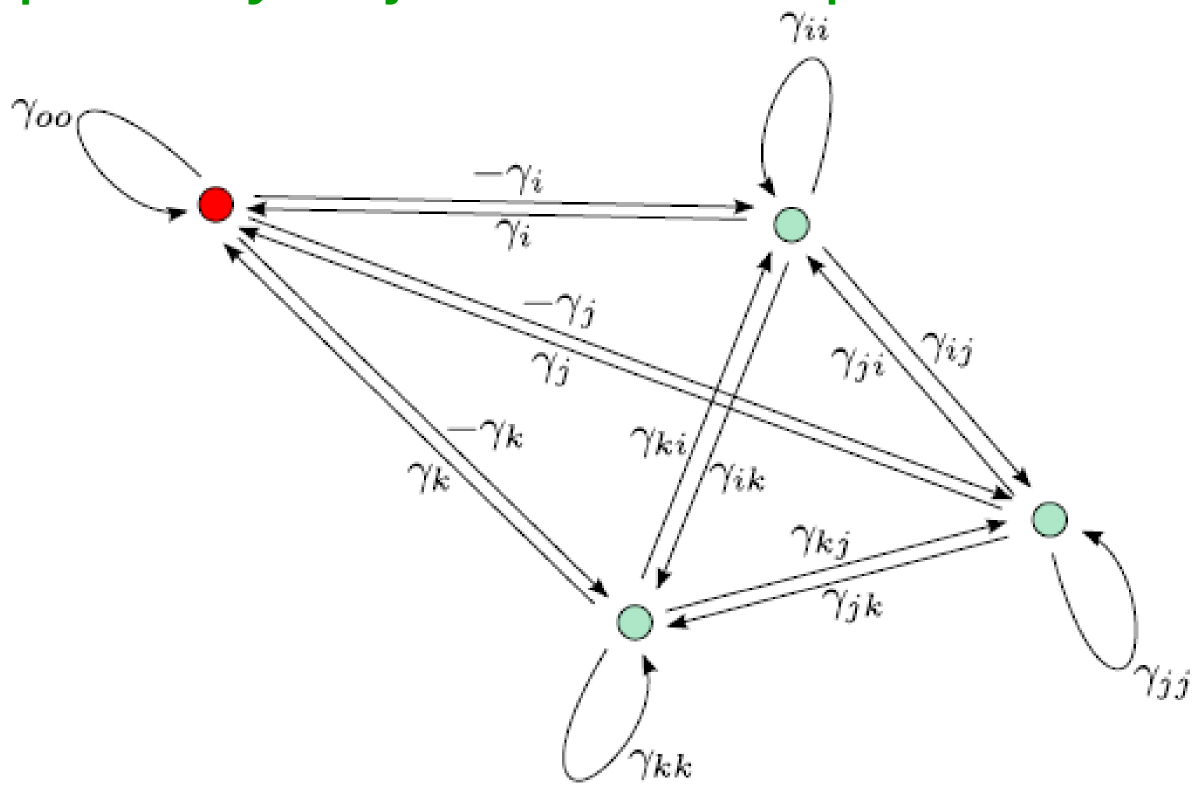
5. Four basic examples

2d-4d WCF: Data

A. Groupoid of vacua, $\mathbb{V}$:   Objects = vacua of $\mathbb{S}$:
i = 1,…, k & one distinguished object 0.

Morphism spaces are torsors for Γ, and the automorphism group of any object is isomorphic to Γ:





2d-4d WCF: Data

B. Central charge Z $\in$ Hom($\mathbb{V}$, $\mathbb{C}$) :

$$Z(a+b) = Z(a) + Z(b)$$

Here a, b are morphisms γ γ_i γ_{ij} ; valid when the composition of morphisms a and b, denoted a+b, is defined.

C. BPS Data: $\mu(\gamma_{ij}) \in \mathbb{Z}$ & $\omega(\gamma, a) \in \mathbb{Z}$

$$\omega(\gamma; a+\gamma') = \omega(\gamma; a) + \Omega(\gamma)\langle\gamma, \gamma'\rangle$$

D. Twisting function: $\sigma(a,b) \in \mathbb{Z}_2$ when a+b is defined

2d-4d WCF: 3 Definitions

A. A ***BPS ray*** is a ray in the complex plane:

$$\ell_\gamma = Z(\gamma)\mathbb{R}_- \qquad \text{IF} \qquad \omega(\gamma, \cdot) \neq 0$$

$$\ell_{\gamma_{ij}} = Z(\gamma_{ij})\mathbb{R}_- \qquad \text{IF} \qquad \mu(\gamma_{ij}) \neq 0$$

B. The ***twisted groupoid algebra*** $\mathbb{C}[\mathbb{V}]$:

$$X_a X_b = \begin{cases} \sigma(a,b) X_{a+b} & a+b \quad \text{composable} \\ 0 & \text{else} \end{cases}$$

2d-4d WCF: 3 Definitions

C. Two automorphisms of $\mathbb{C}[\mathbb{V}]$:

CV-like: $\mathcal{S}^{\mu}_{\gamma_{ij}}$:

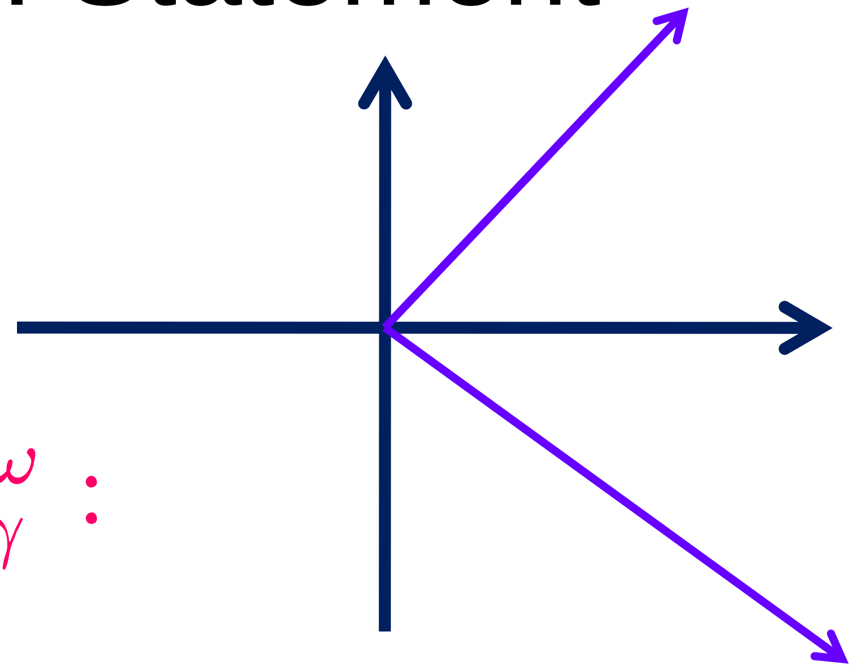
$$X_a \to (1 - \mu(\gamma_{ij}) X_{\gamma_{ij}}) X_a (1 + \mu(\gamma_{ij}) X_{\gamma_{ij}})$$

KS-like: $\mathcal{K}^{\omega}_{\gamma}$

$$X_a \to (1 - X_\gamma)^{-\omega(\gamma;a)} X_a$$

2d-4d WCF: Statement

Fix a convex sector: $\sphericalangle$



$$A(\sphericalangle) =: \prod S^{\mu}_{\gamma_{ij}} K^{\omega}_{\gamma} :$$

The product is over the BPS rays in the sector, ordered by the phase of Z

WCF:

$A(\sphericalangle)$ is constant as a function of Z, so long as no BPS line enters or leaves the sector

2d-4d WCF: Relation to KSWCF

Kontsevich & Soibelman stated a general WCF attached to any graded Lie algebra $\mathfrak{g}$ with suitable stability data.

The 2d-4d WCF (with y= -1) is a special case for the following Lie algebra

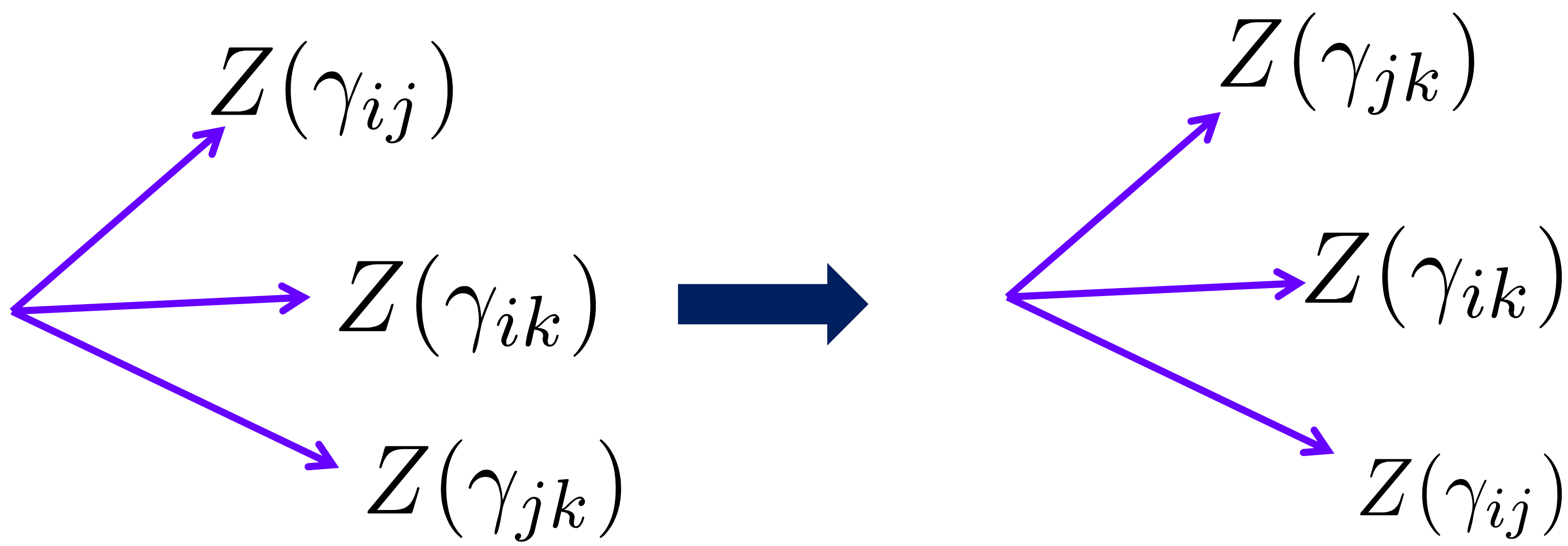
$\mathcal{A}$ Twisted algebra of functions on the Poisson torus $T = \Gamma^* \otimes \mathbb{C}^*$

Generated by $\mathcal{Y}_\gamma$ $\mathcal{Y}_\gamma \mathcal{Y}_{\tilde{\gamma}} = \sigma(\gamma, \tilde{\gamma}) \mathcal{Y}_{\gamma+\tilde{\gamma}}$

$$\mathfrak{g} = M_k(\mathcal{A}) \oplus \mathrm{SymplVect}(T)$$

Four ``types'' of 2d-4d WCF-A

A. Two 2d – central charges sweep past each other:

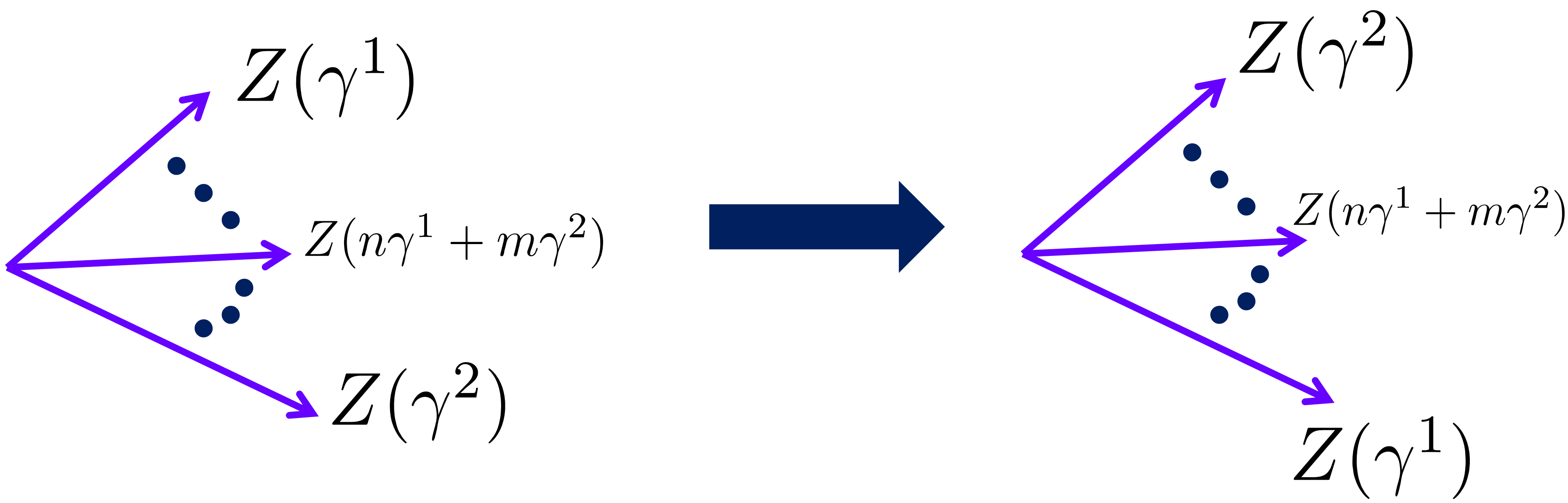


$$\mathcal{S}^{\mu}_{\gamma_{ij}} \mathcal{S}^{\mu}_{\gamma_{il}} \mathcal{S}^{\mu}_{\gamma_{jl}} = \mathcal{S}^{\mu'}_{\gamma_{jl}} \mathcal{S}^{\mu'}_{\gamma_{il}} \mathcal{S}^{\mu'}_{\gamma_{ij}}$$

Cecotti-Vafa

Four ``types'' of 2d-4d WCF - B

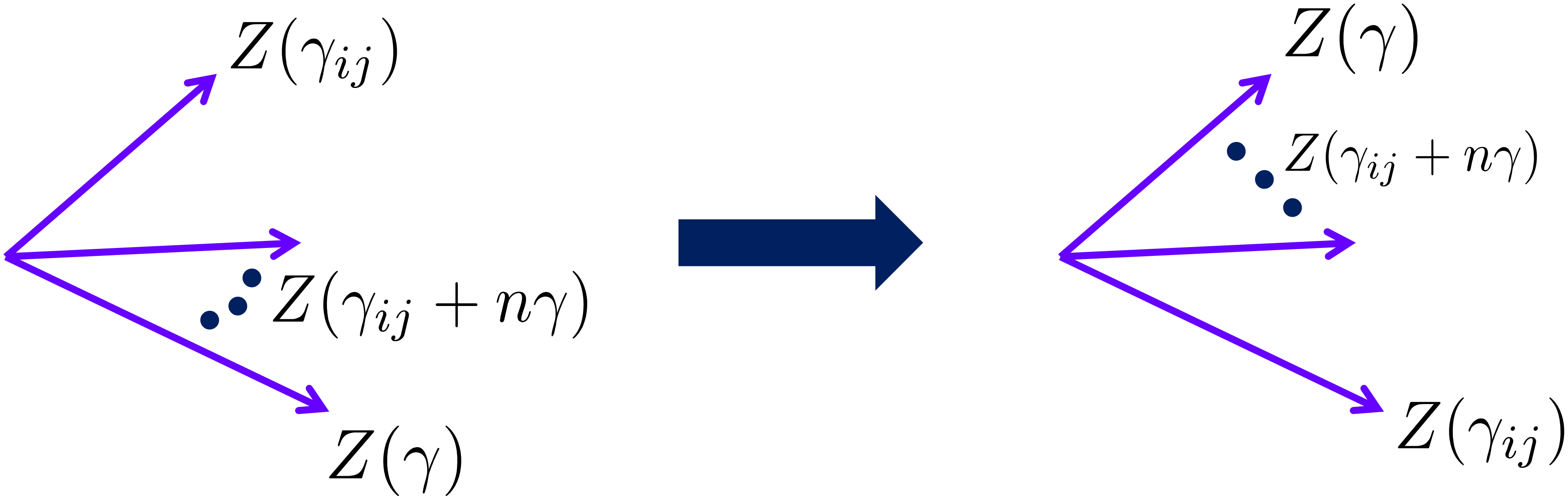
B. Two 4d – central charges sweep past each other:



$$\prod_{\frac{n}{m}\nearrow} \mathcal{K}^{\omega}_{n\gamma^1+m\gamma^2} = \prod_{\frac{n}{m}\searrow} \mathcal{K}^{\omega'}_{n\gamma^1+m\gamma^2}$$

Four ``types'' of 2d-4d WCF - C

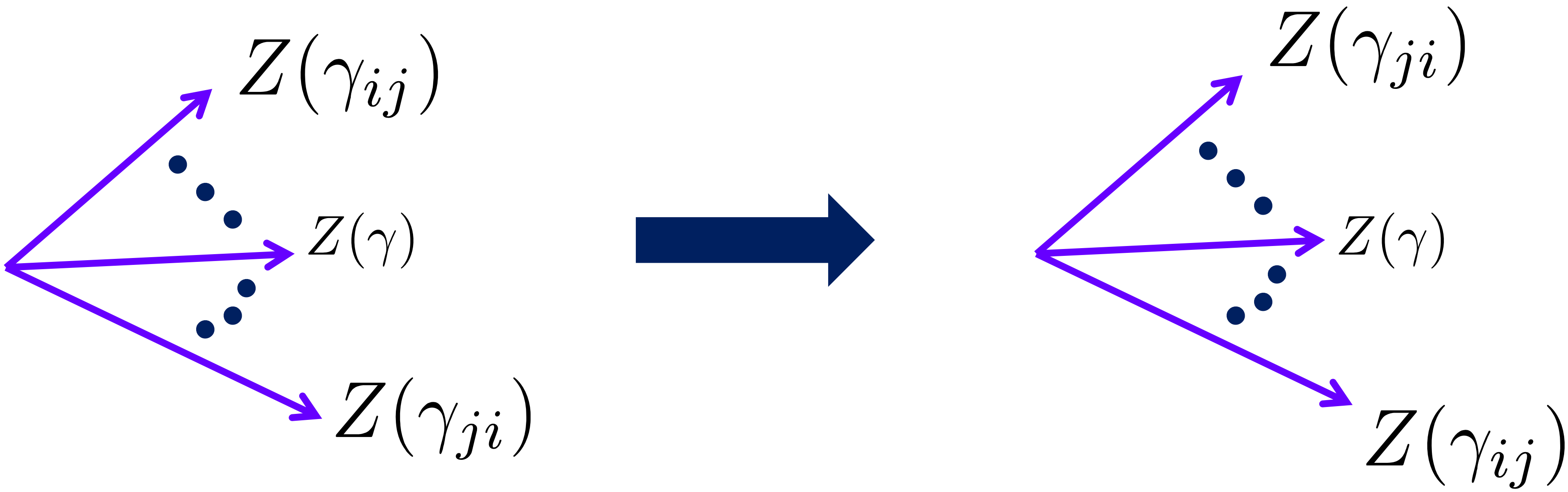
C. A 2d and 4d central charge sweep past each other:



$$\mathcal{K}_{\gamma}^{\omega} \prod_{n \searrow} \mathcal{S}_{\gamma_{ij}+n\gamma}^{\mu} = \prod_{n \nearrow} \mathcal{S}_{\gamma_{ij}+n\gamma}^{\mu'} \mathcal{K}_{\gamma}^{\omega'}$$

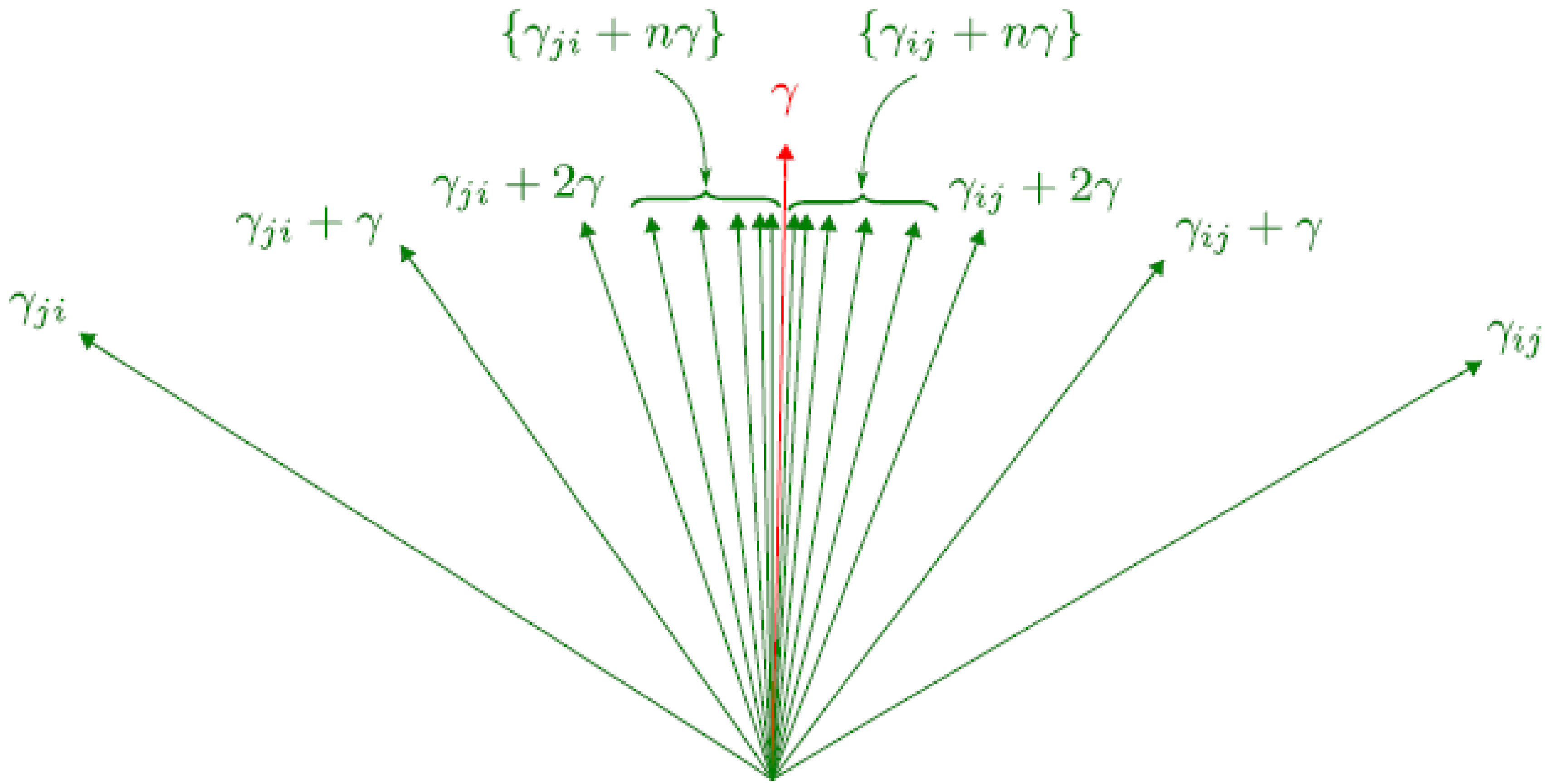
Four ``types'' of 2d-4d WCF - D

D. Two 2d central charges sweep through a 4d charge:



$$\prod_{n\nearrow} \mathcal{S}^{\mu}_{\gamma_{ij}+n\gamma} \prod_{m=1}^{\infty} \mathcal{K}^{\omega}_{m\gamma} \prod_{n\searrow} \mathcal{S}^{\mu}_{\gamma_{ji}+n\gamma} =$$

$$\prod_{n\nearrow} \mathcal{S}^{\mu'}_{\gamma_{ji}+n\gamma} \prod_{m=1}^{\infty} \mathcal{K}^{\omega'}_{m\gamma} \prod_{n\searrow} \mathcal{S}^{\mu'}_{\gamma_{ij}+n\gamma}$$

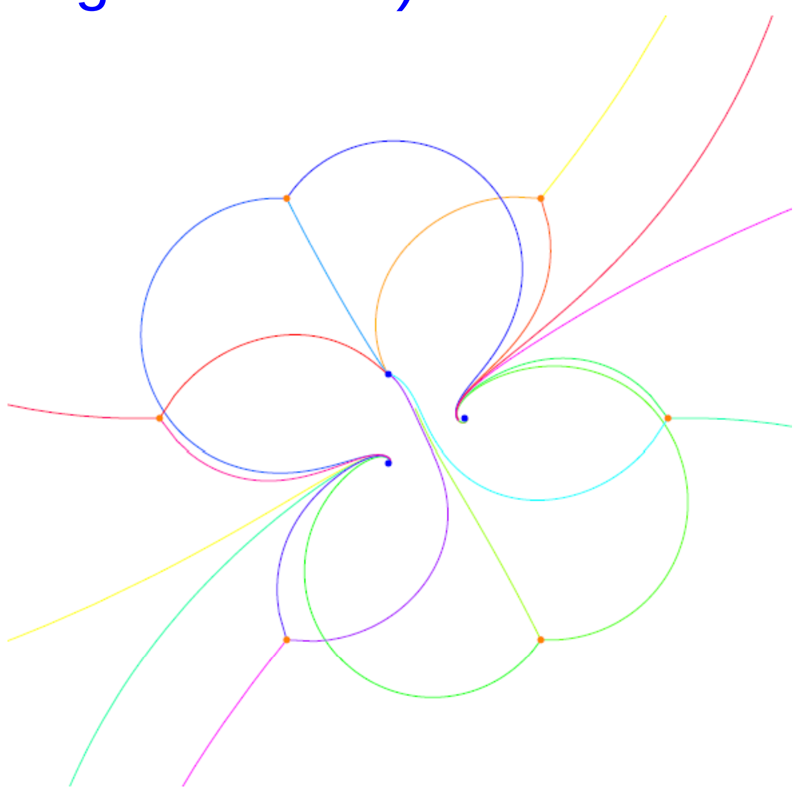


$\{\gamma_{ji} + n\gamma\}$
$\{\gamma_{ij} + n\gamma\}$
γ
$\gamma_{ji} + 2\gamma$
$\gamma_{ij} + 2\gamma$
$\gamma_{ji} + \gamma$
$\gamma_{ij} + \gamma$
γ_{ji}
γ_{ij}

6

The Algorithm

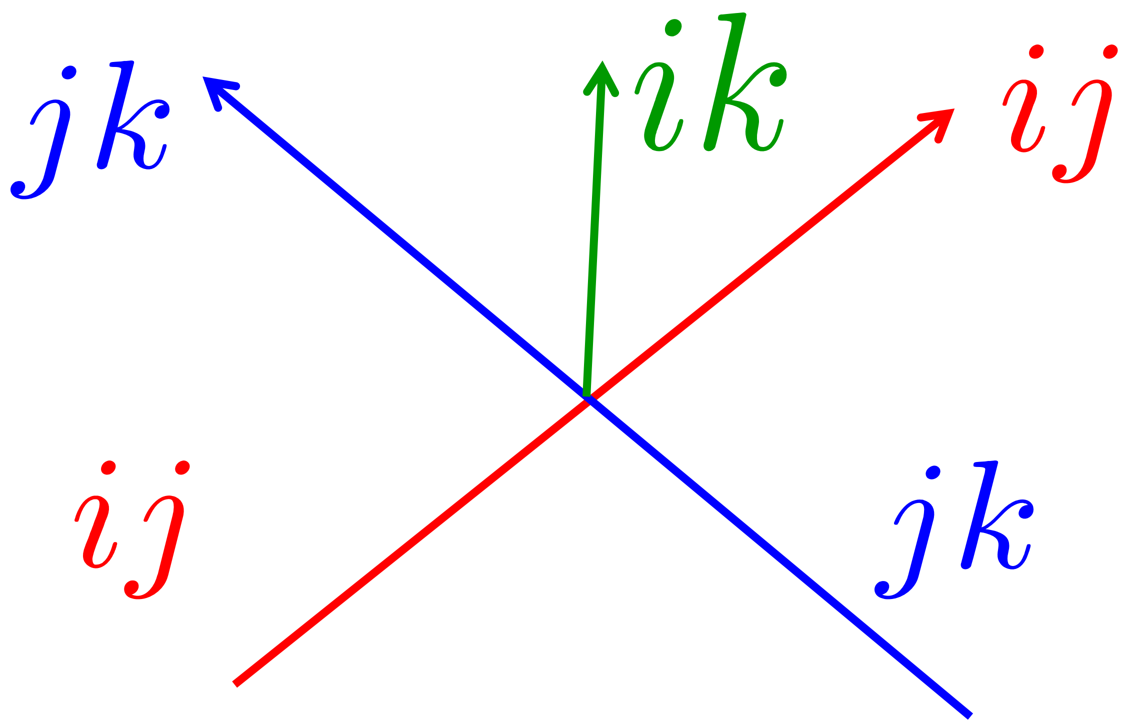
A Fix a phase ϑ . On the UV curve C draw the separating WKB paths of phase ϑ : These begin at the branch points but end at the singular points (for generic ϑ) :



Massive Nemeschansky-Minahan E_6 theory, realized as a trinion theory a la Gaiotto.

B Label the walls with the appropriate S^{μ} factors – these are easily deduced from wall-crossing.

Now, when a *ij*-line intersects a *jk*-line, new lines are created. This is just the CV wall-crossing formula SSS = SSS.



C: Iterate this process.

Conjecture: It will terminate after a finite number of steps (given a canonical structure near punctures).

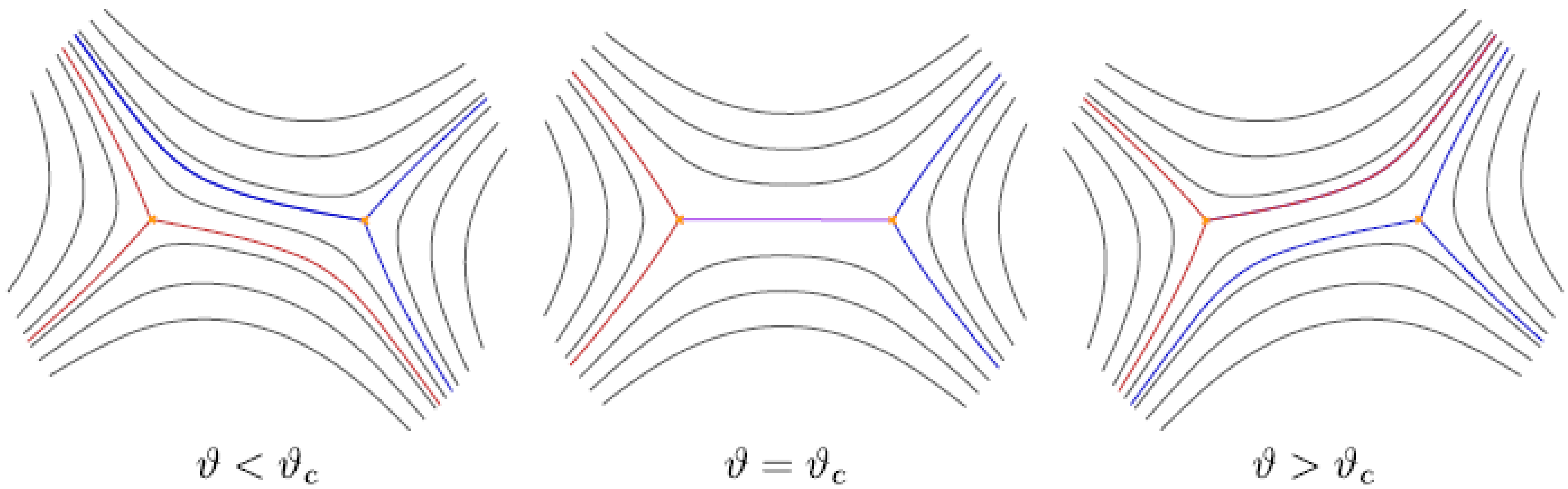
Call the resulting structure a ``minimal S-wall network'' (MSWN)

D: Now vary the phase ϑ.

This determines the entire 2d spectrum

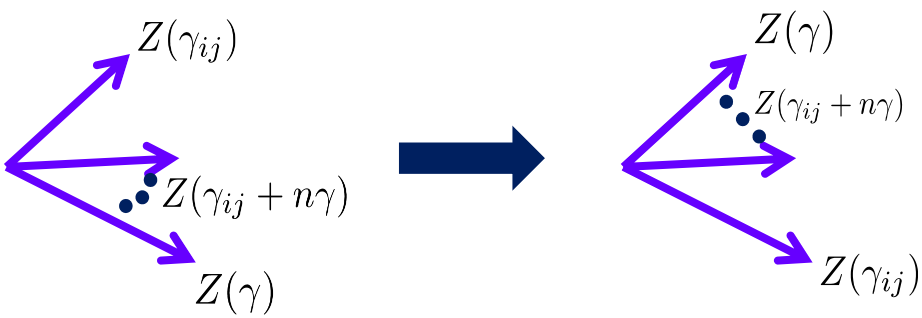
$$\mu(\gamma_{ij}) \quad \text{for all} \quad \mathbb{S}_z, \, i, \, j$$

The MSWN will change isotopy class precisely when an S-wall sweeps past a K-wall in the ζ- plane. Equivalently, when an (ij) S-wall collides with an (ij) branch point:



E: Finally, use the 2d/4d WCF to determine the 4d BPS spectrum:

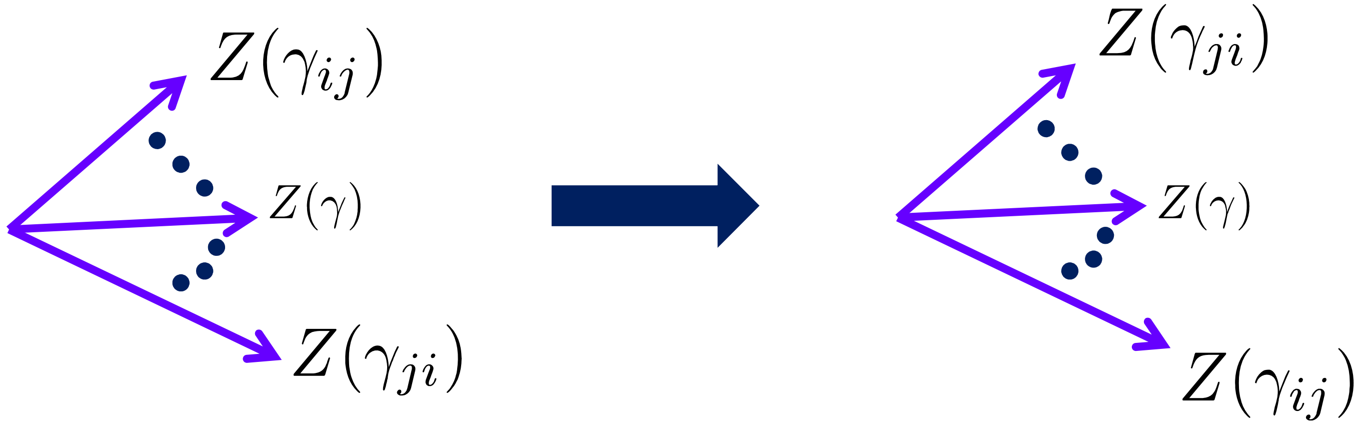
$$Z_{\gamma_{ij}} \parallel Z_{\gamma}$$



$$(1-X_{\gamma})^{\omega(\gamma,\gamma_{ij})} = \Sigma'_{ij}/\Sigma_{ij}$$

$$\Sigma_{ij} := \sum_{n=0}^{\infty} \mu(\gamma_{ij}+n\gamma) X_{\gamma}^{n}$$

$$Z_{\gamma_{ij}} \parallel Z_{\gamma} \parallel Z_{\tilde{\gamma}_{ji}}$$



$$\Pi_{ij} = \frac{\Sigma'_{ij}}{\Sigma_{ij}} - \sigma(\gamma_{ij}, \gamma_{ji}) \Sigma_{ij} \Sigma_{ji} X_{\gamma}$$

$$\Pi_{ij} = \prod_{n=1}^{\infty} (1 - X_{\gamma}^{n})^{\omega(n\gamma, \gamma_{ij})}$$

Concluding slogan for this talk

The 2D spectrum

controls

the 4D spectrum.

Spectrum Generator?

Can we work with just one ζ ?

Perhaps yes, using the notion of a ``spectrum generator'' and ``omnipop''

This worked very well for T[su(2),C,m] to give an algorithm for computing the BPS spectrum of these theories.

Stay tuned….

Hyperkahler Summary - A

1. Hyperkahler geometry: A system of holomorphic Darboux coordinates for SW moduli spaces can be constructed from a TBA-like integral equation, given Ω.

2. From these coordinates we can construct the HK metric on $\mathcal{M}$.

3. $\langle L_{\zeta,\wp} \rangle = \sum_\gamma \overline{\underline{\Omega}}(L,\gamma)\mathcal{Y}_\gamma$

Hyperkahler Summary - B

4. For T[su(2),C,m], $\mathcal{Y}_\gamma$ turn out to be closely related to Fock-Goncharov coordinates

5. We are currently exploring how the coordinates for T[su(k),C,m] are related to the ``higher Teichmuller theory" of Fock & Goncharov

Hyperkahler Summary - C

6. For T[su(2),C,m] the analogous functions: $\mathcal{Y}_{\gamma_{ij'}}$ associated to

$$\langle L_{\zeta,\mathcal{P}(z,z')} \rangle$$

are sections of the universal bundle over $\mathcal{M}$, and allow us moreover to construct hyper-holomorphic connections on this bundle.

7. Explicit solutions to Hitchin systems (a generalization of the inverse scattering method)


That's all Folks!