Using Generating Functions to Solve Linear Inhomogeneous Recurrence Equations

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Abstract: This paper looks at the approach of using generating functions to solve linear inhomogeneous recurrence equations with constant coefficients. It will be shown that the generating functions for these recurrence equations are rational functions. By decomposing a generating function into partial fractions, one can derive explicit formula as well as asymptotic estimates for its coefficients.

Key–Words: Linear recurrence equations, generating functions, partial fractions decomposition.

1 Introduction

Solving recurrence equations (REs) is an important technique in the analysis of algorithms, for example in [**?**, 6]. A lot of REs occurring in the analysis of algorithms are linear inhomogeneous recurrence equations with constant coefficients which we shall abbreviate as LIREs. It's the aim of our paper to study how to solve these REs using the approach of generating functions (GFs). Our main result, Theorem 4, shows that the GFs for those REs are rational functions. By applying the method of partial fractions decomposition (PFD) in Theorem 5 to derive exact formulas and asymptotic estimates for the coefficients of a GF, we can solve these REs exactly or approximately to a high degree of accuracy. The organization of our paper is then as follow. We start off with a discussion of some preliminary materials, namely recurrence equations, generating functions, and locations of zeros, in Section 2. Section 3 contains the main result, Theorem 4. We then illustrate our approach with the exact and approximate solutions of several REs in Section 4. We end our paper with a discussion of possible works in the future in Section 5.

2 Background

This section will cover some materials regarding REs and GFs. Our discussion of GFs will cover the formal power series aspect, which enable us to manipulate the GFs just like power series, and the analytic aspect, which enable us to find asymptotic estimate for the coefficients of the GFs.

2.1 Recurrence Equation (RE)

Let $\{a_n\}$ be a sequence of (real) numbers. A RE for $\{a_n\}$ of the form

$$c_0(n)a_n + c_1(n)a_{n-1} + \cdots + c_k(n)a_{n-k} = r(n) \tag{2.1}$$

or, equivalently

$$\sum_{i=0}^{k} c_i(n)a_{n-i} = r(n)$$

is called a linear RE. Eq. (2.1) is homogeneous if $r(n) \equiv 0$ and inhomogeneous otherwise. If $c_j(n)$ in Eq. (2.1) are constants then Eq. (2.1) is a linear recurrence equation with constant coefficients. In this paper, we deal exclusively with LIREs such that the $r(n)$ for these RES are of the form

$$r(n) = \sum_{j=1}^{m} (b_i)^n P_i(n) \tag{2.2}$$

2.2 Generating Function (GF)

Let $\{a_n\}_{n\geq 0} = a_0, a_1, a_2, \ldots$ be a sequence of real numbers. The *ordinary generating function* (OGF) for $\{a_n\}$ is the power series $a(z)$ defined as

$$a(z) = \sum_{n\geq 0} a_n z^n \tag{2.3}$$

In this paper, we will only consider OGFs. Therefore, in the scope of this paper, a generating function (GF) is to be understood as meaning an OGF. We will write $a(z) \leftrightarrows \{a_n\}$ to denote that $a(z)$ is the generating function for the sequence $\{a_n\}$. If $a(z) \leftrightarrows \{a_n\}$, then $[z^n]a(z) = a_n$ where $[z^n]$ is the coefficient extractor operator. If we regard GFs as *formal power series* over the ring $\mathbb{K}[[z]]$ where $\mathbb{K}$ is usually the complex field $\mathbb{C}$, then we can formally manipulate the GFs. For example, let $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$ be sequences of numbers and $a(z)$, $b(z)$ be their generating functions, respectively. A short list of valid operations on $a(z)$ and $b(z)$ [12] is

$$\{ca_n\} \leftrightarrows ca(z) \tag{2.4}$$

$$\{a_n + b_n\} \leftrightarrows a(z) + b(z) \tag{2.5}$$

$$\{\sum_{k=0}^{n} a_k b_{n-k}\} \leftrightarrows a(z)b(z) \tag{2.6}$$

$$\{a_{n-1}\} \leftrightarrows za(z) \tag{2.7}$$

$$\{a_{n+k}\} \leftrightarrows \frac{1}{z^k}\Big(a(z) - \sum_{i=0}^{k-1} a_i z^i\Big) \tag{2.8}$$

$$\{P(n)a_n\} \leftrightarrows P(zD)a(z) \tag{2.9}$$

$$\{a_n/n\} \leftrightarrows \int \frac{a(z)}{z}\mathrm{d}z \tag{2.10}$$

where D is the differentiation operation, $Df = \frac{\mathrm{d}f}{\mathrm{d}z}$, and $P(n)$ is an arbitrary polynomial.

The formal power series aspect of GFs allow us to manipulate them while disregarding their convergences. However, if the power series do converges and that they represent valid functions, then we are in a position to find additional analytic information, such as the asymptotic behaviour, regarding the coefficients of the series. Assume for the following discussion that $f(z) = \sum f_n z^n$ is a power series over the ring $\mathbb{C}[[z]]$.

Theorem 1 ([12]): *There exists a $r \in \mathbb{R}$, $0 \leq r \leq \infty$, called the* radius of convergence *of f such that the series $\sum f_n z^n$ converges, i.e. $f(z)$ is analytic, if $|z| < r$ and diverges if $|z| > r$. r is expressed in terms of $\{f_n\}_0^\infty$ by*

$$\frac{1}{r} = \limsup_{n\to\infty} |f_n|^{1/n} \tag{2.11}$$

If r is the radius of convergence of $f(z)$ then $f(z)$ must have a singularity point, a point where $f(z)$ fails to converge, on the circle $\{|z| = r\}$. ♢

From Eq. (2.11), we can deduce that if α is the nearest singularity point from the origin, then $r = |\alpha|$ and for any $\epsilon > 0$ with n sufficiently large

$$(1-\epsilon)\,|\alpha|^{-n} \leq f_n \leq (1+\epsilon)\,|\alpha|^{-n}$$

which is the basis of most methods for find asymptotic estimates for the coefficients of a generating function [10, 12, 4, 8, 5].

2.3 Locations of Zeros

We will show in Theorem 4 that the GF for a LIRE is a rational function. Since the singularity points of a rational function are always poles located at the zeros of its denominator, by locating the zeros of the denominator we can derive asymptotic estimates for the coefficients of the GF. The following two results from complex analysis will aid us in the location of singularities for the rational GFs discussed in this paper. For proofs, please consult [9].

Theorem 2 (Pringsheim's Theorem [9]): *If $f(z)$ is representable at the origin by a series expansion that has non-negative coefficients and radius of convergence r, then the point $z = r$ is a singularity of $f(z)$.*♢

Pringsheim's Theorem is important since practically all GFs that arise in combinatorics and analysis of algorithms have non-negative coefficients.

Theorem 3 (Rouché's Theorem [9]): *Let $f(z)$ and $g(z)$ be analytic functions in a region containing in its interior the closed simple curve γ. Assume that $|f(z)| < |g(z)|$ on the curve γ. Then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside the interior domain delimited by γ.* ♢

3 Statement of the Main Results

Theorem 4: *Let $\{a_n\}_{n\geq 0}$ be a sequence of numbers satisfying a RE of the form*

$$\sum_{i=0}^{k} c_i a_{n-i} = \sum_{j=1}^{m} (b_j)^n P_j(n) \quad n \geq k \tag{3.1}$$

with $\{c_i\}_{i=0}^k$ being real numbers and $\{b_j\}_{j=1}^m$ being positive real numbers as well as $\{P_j(n)\}_{j=1}^m$ being polynomials. Then the GF $A(z)$ for $\{a_n\}_{n\geq 0}$ is a rational function. ♢

PROOF: Upon multiplying the left hand side (LHS) of Eq. (3.1) by z^n and summing from $n = k$ up to ∞, we have

$$\begin{aligned} S_L &= \sum_{n=k}^{\infty}\left[\sum_{i=0}^{k} c_i a_{n-i}\right] z^n \\ &= c_0 \sum_{n=k}^{\infty} a_n z^n + c_1 z \sum_{n=k-1}^{\infty} a_n z^n + \ldots + c_k z^k \sum_{n=0}^{\infty} a_n z^n \\ &= c_0[A(z) - A_{k-1}(z)] + c_1 z[A(z) - A_{k-2}(z)] \\ &\qquad + \ldots + c_k z^k A(z) \end{aligned} \tag{3.2}$$

where $A(z)$ is the GF for $\{a_n\}_{n\geq 0}$ and $A_k(z)$ is $A(z)$ truncated after the k^{th} term, i.e.

$$A_k(z) = \sum_{i=0}^{k-1} a_i z^i \tag{3.3}$$

The function $(1 - bz)^{-1}$ with $b > 0$ is the generating function for the sequence $\{b^n\}_{n\geq 0}$. If we define the operator Θ as $\Theta(f) = z\frac{\mathrm{d}f}{\mathrm{d}z}$, then by Eq. (2.9) we have

$$\{n^k b^n\} \leftrightarrows \Theta^k((1 - bz)^{-1}) \tag{3.4}$$

The GF $\Theta^k((1-bz)^{-1})$ in Eq. (3.4) is a rational function. Therefore, by Eq. (2.5), the GF for the sequence $\{P(n)b^n\}_{n\geq 0}$ where $P(n)$ is a polynomial will be a sum of rational functions, i.e. $\{P(n)b^n\}$ admits a rational GF. It's then therefore obvious that the sequence given by the RHS of Eq. (3.1) also admits a rational GF, i.e.

$$\sum_{n=k}^{\infty}\sum_{j=1}^{m} (b_j)^n P_j(n) z^n = \frac{\alpha(z)}{\beta(z)} \tag{3.5}$$

where $\alpha(z)$ and $\beta(z)$ are polynomials. From Eqs. (3.1), (3.2) and (3.5) we can conclude that $A(z)$ is rational. ∎

Theorem 5 ([5]): *Let $f(z)$ be a rational function that's also analytic at 0 and has poles at $\alpha_1, \alpha_2, \ldots \alpha_m$ of order $s_1, s_2, \ldots s_m$, respectively. Then there exists m polynomials $\{S_j(n)\}_{j=1}^{m}$ such that the coefficients f_n of $f(z)$ are given by*

$$f_n \equiv [z^n]f(z) = \sum_{j=1}^{m} S_j(n)\alpha_j^{-n} \tag{3.6}$$

Furthermore, the degree of each $S_j(n)$ is $s_j - 1$. ◇

PROOF: Since $f(z)$ is rational, there exists a partial fraction decomposition (PFD) for $f(z)$ as

$$f(z) = Q(z) + \sum_{j=1}^{m}\sum_{k=1}^{s_j} \frac{c_{\alpha_j,k}}{(z-\alpha_j)^k} \tag{3.7}$$

where $Q(z)$ is a polynomial and $c_{\alpha_j,k}$ is given by

$$c_{\alpha_j,k} = \frac{1}{(s_j-k)!}\lim_{z\to\alpha_j}\frac{\mathrm{d}^{s_j-k}}{\mathrm{d}z^{s_j-k}}\left[(z-\alpha_j)^{s_j} f(z)\right] \tag{3.8}$$

Coefficients extraction of the right hand side of Eq. (3.7) is facilitated by Newton's binomial expansion

$$[z^n]\frac{1}{(z-\alpha)^r} = \frac{(-1)^r}{\alpha^r}\binom{n+r-1}{r-1}\alpha^{-n}$$

The binomial coefficient is a polynomial of degree $r-1$ in n, and so by collecting the terms associated with a given α we arrive at Theorem 5.
If the $\{\alpha_j\}_1^m$ are such that there exists a α_j with $|\alpha_j| < |\alpha_k|$ for all other k, then for n sufficiently large we have,

$$f_n = \alpha_j^{-n} S_j(n) + o(\alpha_j^{-n}) \tag{3.9}$$

which gives us a quick asymptotic estimate for the coefficients f_n of $f(z)$. ∎

For a lot of LIREs theirs GFs are not easily determined. For example, the REs appearing in Example 3 and 4 have quite complicated GFs. Therefore, the exact values for the terms of some REs are difficult to obtain. However, if those REs satisfy some easily checked constraints, then we can obtain simple asymptotic estimates for their terms. The asymptotic estimates obtained will then be of the form

$$a_n = c_k n^k \alpha^n + o(n^k \alpha^n)$$

for sufficiently large n.

Theorem 6: *Let $\{a_n\}$ be a sequence satisfying a RE of the form*

$$\sum_{i=0}^{k} c_i a_{n-i} = \sum_{j=1}^{m} (b_j)^n P_j(n)$$

and $\mathscr{R}$ be the set of all roots of the characteristic equation

$$C(z) = 0 \quad \textit{where } C(z) = \sum_{i=0}^{k} c_i z^i$$

Define the constant α as the smallest modulus of all $z \in \mathscr{R}$, i.e.

$$\alpha = \min_{z\in\mathscr{R}} |z|$$

If $\alpha^{-1} < \beta^{-1} = \max\{b_i\}$, then the following estimate holds

$$a_n \sim \frac{\beta^{-n} n^d}{C(\beta)} \tag{3.10}$$

where d is the degree of the polynomial $P(n)$ corresponding to β. ◇

PROOF: Let $A(z)$ be the GF for $\{a_n\}$. From the proof of Theorem 4 we know that $\mathscr{R} \cup \{b_j^{-1}\}_{j=1}^m$ is the set of poles of $A(z)$. Since $\beta^{-1} = \max\{b_i\} > \alpha^{-1}$, β is the unique dominant singularity of $A(z)$. Therefore by Eq. (3.9), an asymptotic estimate for the a_n is

$$a_n \sim [z^n]\frac{c}{(z-\beta)^s} \tag{3.11}$$

where s is the order of the pole $z = \beta$ and c is given by Eq. (3.8) as

$$c = \lim_{z\to\beta}(z-\beta)^s A(z)$$

Again, from the construction of $A(z)$ as described in the proof of Theorem 4, we have

$$\lim_{z\to\beta}(z-\beta)^s A(z) = \lim_{z\to\beta}\frac{(z-\beta)^s \Theta^{s-1}((1-\beta^{-1}z)^{-1})}{C(z)}$$

since in all the other terms in the PFD of $A(z)$, either $z = \beta$ isn't a pole or is a pole of a lower order than s. We have

$$\Theta^{s-1}((1-\beta^{-1}z)^{-1}) = \frac{\beta^{-s+1}(s-1)!z^{s-1}}{(1-\beta^{-1}z)^s} + \dots \tag{3.12}$$

where the $\dots$ in Eq. (3.12) refers to terms of the form $c(1-\beta^{-1}z)^{-s'}$ with $s' < s$. Therefore,

$$\begin{aligned} c &= \lim_{z\to\beta}(z-\beta)^s A(z) \\ &= \lim_{z\to\beta}\frac{(z-\beta)^s \Theta^{s-1}((1-\beta^{-1}z)^{-1})}{C(z)} \\ &= \lim_{z\to\beta}\frac{(z-\beta)^s}{C(z)}\left[\frac{\beta^{-s+1}(s-1)!z^{s-1}}{(1-\beta^{-1}z)^s C(z)} + \dots\right] \\ &= \lim_{z\to\beta}\frac{(z-\beta)^s\beta^{-s+1}(s-1)!z^{s-1}}{(1-\beta^{-1}z)^s C(z)} \\ &= \frac{(-1)^s\beta^s(s-1)!}{C(\beta)} \end{aligned} \tag{3.13}$$

From Eq. (3.11), we then have

$$\begin{aligned} a_n &\sim [z^n]\frac{c}{(z-\beta)^s} \\ &\sim \frac{(-1)^s\beta^s(s-1)!}{C(\beta)}(-1)^s\beta^{-s}\binom{n+s-1}{s-1}\beta^{-n} \\ &\sim \frac{n^{s-1}\beta^{-n}}{C(\beta)} \end{aligned}$$

which is Eq. (3.10) upon replacing $d = s - 1$. ∎

The above proof shows why we must constraint that $\alpha^{-1} < \beta^{-1}$. If $\alpha^{-1} \geq \beta^{-1}$, then there is no simple equation to compute the required constant c like that of Eq. (3.13) since the value of c will then depends on all terms in the PFD of the GF.

4 Some Examples

Example 1: Consider the following Fibonacci-like recurrence equation

$$a_n = \begin{cases} n & n = 0, 1 \\ a_{n-1} + a_{n-2} + n^2 3^n & n \geq 2 \end{cases} \tag{4.1}$$

We first find the generating function for the sequence $\{n^2 3^n\}_{n\geq 2}$

$$\begin{aligned} S(z) &= \sum_{n=2}^{\infty} n^2 3^n z^n \\ &= \underbrace{\sum_{n=0}^{\infty} n^2 3^n z^n}_{S_1(z)} - 3z \end{aligned}$$

By Eq. (3.4), we have

$$S_1(z) = \Theta^2((1-3z)^{-1}) \tag{4.2}$$

Therefore,

$$\begin{aligned} S(z) &= \Theta^2((1-3z)^{-1}) - 3z \\ &= \frac{18z^2}{(1-3z)^3} + \frac{3z}{(1-3z)^2} - 3z \end{aligned}$$

If $A(z)$ is the generating function for the sequence $\{a_n\}_{n\geq 0}$, then Eq. (4.1) gives us

$$(1-z-z^2)A(z) - z = S(z) \tag{4.3}$$

From Eq. (4.3) and by a PFD, we have

$$\begin{aligned} A(z) = &-\frac{146z+153}{25(1-z-z^2)} \\ &+ \frac{288}{25(1-3z)} - \frac{9}{(1-3z)^2} + \frac{18}{5(1-3z)^3} \end{aligned}$$

and so

$$\begin{aligned} a_n &= [z^n]A(z) \\ &= [z^n]\left\{-\frac{146z+153}{25(1-z-z^2)}\right\} \\ &+ [z^n]\left\{\frac{288}{25(1-3z)} - \frac{9}{(1-3z)^2} + \frac{18}{5(1-3z)^3}\right\} \\ &= -\frac{146F_n + 153F_{n+1}}{25} + \frac{(45n^2-90n+153)3^n}{25} \end{aligned} \tag{4.4}$$

where F_n is the n^{th} Fibonacci number. △

Example 2: Let's now find an asymptotic estimate for the sequence $\{a_n\}_{n\geq 0}$ in Example 1. Out of all the poles of $A(z)$, the one with smallest modulus is the pole at $z = 1/3$ of order three. Therefore without carrying out a partial fractions decomposition, we know that the partial fractions decomposition of $A(z)$ contains a term of the form $c_0(z-1/3)^{-3}$ where

$$c_0 = A(z)(z-1/3)^3\Big|_{z=1/3} = -\frac{2}{15}$$

An asymptotic estimate for $\{a_n\}_{n\geq 0}$ is then

$$a_n \sim [z^n]\frac{-2}{15(z-1/3)^3} \sim \frac{9n^2 3^n}{5}$$

which is the highest order term of Eq. (4.4). A more accurate estimate can be found by finding the polynomial $S(n)$ associated with 3^n. Since $S(n)$ is known to be of degree two, we have

$$S(n) = \frac{-27c_0}{2}(n+1)(n+2) + 9c_1(n+1) - 3c_2$$

where

$$c_1 = \frac{1}{1!}\lim_{z\to 1/3}\frac{\mathrm{d}}{\mathrm{d}z}\left[A(z)(z-1/3)^3\right] = -1$$

$$c_2 = \frac{1}{2!}\lim_{z\to 1/3}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[A(z)(z-1/3)^3\right] = -\frac{96}{25}$$

from which we gather that

$$f_n \sim \frac{(45n^2 - 90n + 153)3^n}{25}$$

△

Example 3: Let $\{f_n\}_{n\geq 0}$ be a sequence satisfying the following complicated looking RE

$$f_n = \begin{cases} n & n \leq 4 \\ f_{n-2} + f_{n-3} + f_{n-5} + n^2 2^n + n^3 3^n & n \geq 5 \end{cases}$$

We now attempt to find an asymptotic estimate for the sequence $\{f_n\}_{n\geq 0}$. If $f(z)$ is the generating function for the sequence $\{f_n\}_{n\geq 0}$, then it's easy to see that the poles of $f(z)$ lie in the set $\{1/3, 1/2\}$ or the set $\{z : 1 - z^2 - z^3 - z^5 = 0\}$. From Rouche's theorem we know that all the roots of $1 - z^2 - z^3 - z^5 = 0$ lies outside the circle $\{|z| = 1/3\}$. Therefore, $f(z)$ has a unique dominant singularity at $z = 1/3$. Therefore, the partial fractions decomposition of $f(z)$ contains a term of the form $c(z-1/3)^{-4}$ for some constant c and that $[z^n]c(z-1/3)^{-4}$ is the desired asymptotic estimate. By Eq. (3.8), the value of c is given as

$$c = \lim_{z\to 1/7} f(z)(z-1/3)^4$$

$f(z)$, however, is not easily determined. But from the construction of $f(z)$ as was described in the proof of Theorem 4, we can easily deduce that

$$\lim_{z\to 1/3} f(z)(z-1/3)^4 = \lim_{z\to 1/3}\widehat{f}(z)(z-1/3)^4$$

where

$$\widehat{f}(z) = \frac{\Theta^3((1-3z)^{-1})}{1-z^2-z^3-z^5}$$

and so

$$c = \lim_{z\to 1/3}\frac{\Theta^3((1-3z)^{-1})(z-1/3)^4}{1-z^2-z^3-z^5} = \frac{9}{103}$$

The asymptotic estimate for f_n is then

$$\begin{aligned} f_n &\sim \frac{9}{103}[z^n](z-1/3)^{-4} \\ &\sim \frac{9}{103}(-3)^4\binom{n+3}{3}3^n \\ &\sim \frac{243n^3 3^n}{206} = \widehat{f_n} \end{aligned} \tag{4.5}$$

The relative error of $\widehat{f_n}$ in comparison to f_n is less than 4.13×10^{-3} for $n \geq 300$. △

Example 4: Our last example will be that of a RE whose PFD will be very complicated. Let $\{a_n\}$ be a sequence satisfying the following RE

$$a_n = \begin{cases} n & n \leq 2 \\ a_{n-1} + a_{n-2} + a_{n-3} + n^7 4^n - n^4 & n \geq 3 \end{cases}$$

It's quite unlikely that we will be able to obtain exact values for the sequence $\{a_n\}$. However, Eq. (3.10) with $\beta = 1/4$ gives

$$\begin{aligned} a_n &\sim \frac{4^n n^7}{1-(1/4)^1-(1/4)^2-(1/4)^3} \\ &\sim \frac{64}{43}n^7 4^n \end{aligned}$$

The above estimate is accurate for $n \geq 500$. △

5 Conclusions

We have seen in this paper how GFs can help us in solving linear inhomogeneous REs with constant coefficients. It will be interesting to study how GFs can be applied to linear inhomogeneous REs where the coefficients are polynomials since the GFs for these type of REs will satisfy inhomogeneous differential equations whose closed form solutions don't always exist. The set of rational functions is a subset of the set of algebraic functions. The problem of determining the type of REs whose GFs are algebraic functions is therefore a possibly highly interesting problem.

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