On Massive Neutron Cores

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It has been suggested that, when the pressure within stellar matter becomes high enough, a new phase consisting of neutrons will be formed. In this paper we study the gravitational equilibrium of masses of neutrons, using the equation of state for a cold Fermi gas, and general relativity. For masses under $\frac{1}{3}\odot$ only one equilibrium solution exists, which is approximately described by the nonrelativistic Fermi equation of state and Newtonian gravitational theory. For masses $\frac{1}{3}\odot < m < \frac{3}{4}\odot$ two solutions exist, one stable and quasi-Newtonian, one more condensed, and unstable. For masses greater than $\frac{3}{4}\odot$ there are no static equilibrium solutions. These results are qualitatively confirmed by comparison with suitably chosen special cases of the analytic solutions recently discovered by Tolman. A discussion of the probable effect of deviations from the Fermi equation of state suggests that actual stellar matter after the exhaustion of thermonuclear sources of energy will, if massive enough, contract indefinitely, although more and more slowly, never reaching true equilibrium.

I. INTRODUCTION

FOR the application of the methods commonly used in attacking the problem of stellar structure[1] the distribution of energy sources and their dependence on the physical conditions within the star must be known. Since at the time of Eddington's original studies not much was known about the physical processes responsible for the generation of energy within a star, various mathematically convenient assumptions were made in regard to the energy sources, and these led to different star models (e.g. the Eddington model, the point source model, etc.). It was found that with a given equation of state for the stellar material many important properties of the solutions (such as the mass-luminosity law) were quite insensitive to the choice of assumptions about the distribution of energy sources, but were common to a wide range of models.

In 1932 Landau[2] proposed that instead of making arbitrary assumptions about energy sources chosen merely for mathematical convenience, one should attack the problem by first investigating the physical nature of the equilibrium of a given mass of material in which no energy is generated, and from which there is no radiation, presumably in the hope that such an investigation would afford some insight into the more general situation where the generation of energy is taken into account. Although such a model gives a good description of a white dwarf star in which most of the material is supposed to be in a degenerate state with a zero point energy high compared to thermal energies of even 10^7 degrees, and such that the pressure is determined essentially by the density only and not by the temperature, still it would fail completely to describe a normal main sequence star, in which on the basis of the Eddington model the stellar material is nondegenerate, and the existence of energy sources and of the consequent temperature and pressure gradients plays an important part in determining the equilibrium conditions. The stability of a model in which the energy sources have to be taken into account is known to depend also on the temperature sensitivity of the energy sources and on the presence or absence of a time-lag in their response to temperature changes. However, if the view which seems plausible at present is adopted that the principal sources of stellar energy, at least in main sequence stars, are thermonuclear reactions, then the limiting case considered by Landau again becomes of interest in the discussion of what will eventually happen to a normal main sequence star after all the elements available for thermonuclear reactions are used up. Landau showed that for a model consisting of a cold degenerate Fermi gas there exist no stable equilibrium configurations for

[1] A. Eddington, *The Internal Constitution of the Stars* (Cambridge University Press, 1926); B. Strömgren, Ergebn. Exakt. Naturwiss. **16**, 465 (1937); Short summary in G. Gamow, Phys. Rev. **53**, 595 (1938).
[2] L. Landau, Physik. Zeits. Sowjetunion **1**, 285 (1932).

masses greater than a certain critical mass, all larger masses tending to collapse. For a mixture of electrons and nuclei in which on the average there are two protonic masses per electron Landau found the critical mass to be roughly $1.5\odot$, and in general the critical mass is inversely proportional to the square of the mass per particle obtained by spreading out the total mass over only those particles which essentially determine the pressure of the Fermi gas.

The possibility has been suggested[3] that in sufficiently massive stars after all the thermonuclear sources of energy, at least for the central material of the star, have been exhausted a condensed neutron core would be formed. The minimum mass for which such a core would be stable has been estimated by Oppenheimer and Serber,[4] who on taking into account some effects of nuclear forces give approximately $0.1\odot$ as a reasonable minimum mass. The gradual growth of such a core with the accompanying liberation of gravitational energy is suggested by Landau as a possible source of stellar energy.

In this connection it seems of interest to ask whether this model of the final state of a star can be right for arbitrarily heavy stars, i.e., to investigate whether there is an upper limit to the possible size of such a neutron core. Landau's original result for a cold relativistically degenerate Fermi gas quoted above gives in the case of a neutron gas an upper limit of about $6\odot$ beyond which the core would not be stable but would tend to collapse. Two objections might be raised against this result. One is that it was obtained on the basis of Newtonian gravitational theory while for such high masses and densities general relativistic effects must be considered. The other one is that the Fermi gas was assumed to be relativistically degenerate throughout the whole core, while it might be expected that on the one hand, because of the large mass of the neutron, the nonrelativistically degenerate equation of state might be more appropriate over the greater part of the core, and on the other hand the gravitational effect of the kinetic energy of the neutrons could not be neglected. The present investigation seeks to establish what differences are introduced into the result if general relativistic gravitational theory is used instead of Newtonian, and if a more exact equation of state is used. A discussion of the general relativistic treatment of the equilibrium of spherically symmetric distributions of matter is first given, and then the special ideal case of a cold neutron gas is treated. A discussion of the results, and comparison with some results of Professor R. C. Tolman reported in an accompanying paper are given in the concluding sections.

II. Relativistic Treatment of Equilibrium

It is known[5] that the most general static line element exhibiting spherical symmetry may be expressed in the form

$$ds^2 = -e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + e^{\nu}dt^2, \qquad \lambda = \lambda(r), \quad \nu = \nu(r). \tag{1}$$

If the matter supports no transverse stresses and has no mass motion, then its energy momentum tensor is given by[6]

$$T_1{}^1 = T_2{}^2 = T_3{}^3 = -p, \quad T_4{}^4 = \rho \tag{2}$$

where p and ρ are respectively the pressure and the macroscopic energy density measured in proper coordinates. With these expressions for the line element and for the energy momentum tensor, and with the cosmological constant Λ taken equal to zero, Einstein's field equations reduce to:[7]

$$8\pi p = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{3}$$

$$8\pi \rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{4}$$

$$\frac{dp}{dr} = -\frac{(p+\rho)}{2}\nu', \tag{5}$$

where primes denote differentiation with respect to r. These three equations together with the equation of state of the material $\rho = \rho(p)$ de-

[3] G. Gamow, *Atomic Nuclei and Nuclear Transformations* (Oxford, 1936), second edition, p. 234. L. Landau, Nature **141**, 333 (1938) and others.

[4] J. R. Oppenheimer and R. Serber, Phys. Rev. **54**, 540 (1938).

[5] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), pp. 239–241.

[6] R. C. Tolman, reference 5, p. 243.

[7] R. C. Tolman, reference 5, p. 244.

termine the mechanical equilibrium of the matter distribution as well as the dependence of the $g_{\mu\nu}$'s on r.

The boundary of the matter distribution is the value of $r=r_b$ for which $p=0$, and such that for $r<r_b$, $p>0$. For $r<r_b$ the solution depends on the equation of state of the material connecting p and ρ. For many equations of state a sharp boundary exists with a finite value of r_b.

In empty space surrounding the spherically symmetric distribution of matter $p=\rho=0$, and Schwarzschild's exterior solution is obtained:[8]

$$e^{-\lambda(r)}=1+A/r, \quad e^{\nu(r)}=B(1+A/r). \tag{6}$$

The constants A and B are fixed by the requirement that at great distances away from the matter distribution the $g_{\mu\nu}$'s must go over into their weak-field form, i.e., $B=1, A=-2m$ where m is the total Newtonian mass of the matter as calculated by a distant observer.[9]

Inside the boundary Eqs. (3), (4) and (5) may be rewritten as follows. Using the equation of state $\rho=\rho(p)$ Eq. (5) may be immediately integrated.

$$\nu(r)=\nu(r_b)-\int_0^{p(r)}\frac{2dp}{p+\rho(p)},$$

$$e^{\nu(r)}=e^{\nu(r_b)}\exp\left[-\int_0^{p(r)}\frac{2dp}{p+\rho(p)}\right].$$

The constant $e^{\nu(r_b)}$ is determined by making e^ν continuous across the boundary.

$$e^{\nu(r)}=\left(1-\frac{2m}{r_b}\right)\exp\left[-\int_0^{p(r)}\frac{2dp}{p+\rho(p)}\right]. \tag{7}$$

Thus e^ν is known as a function of r if p is known as a function of r. Further in Eq. (4) introduce a new variable

$$u(r)=\tfrac{1}{2}r(1-e^{-\lambda}) \quad \text{or} \quad e^{-\lambda}=1-2u/r. \tag{8}$$

Then Eq. (4) becomes:

$$du/dr=4\pi\rho(p)r^2. \tag{9}$$

In Eq. (3) replace $e^{-\lambda}$ by its value from (8) and ν' by its value from (5). It becomes:

$$\frac{dp}{dr}=-\frac{p+\rho(p)}{r(r-2u)}[4\pi pr^3+u]. \tag{10}$$

[8] R. C. Tolman, reference 5, p. 203.
[9] R. C. Tolman, reference 5, pp. 203 and 207.

Equations (9) and (10) form a system of two first-order equations in u and p. Starting with some initial values $u=u_0$ $p=p_0$ at $r=0$, the two equations are integrated simultaneously to the value $r=r_b$ where $p=0$, i.e., until the boundary of the matter distribution is reached. The value of $u=u_b$ at $r=r_b$ determines the value of $e^{\lambda(r_b)}$ at the boundary, and this is joined continuously across the boundary to the exterior solution, making

$$u_b=\frac{r_b}{2}[1-e^{-\lambda(r_b)}]=\frac{r_b}{2}\left[1-\left(1-\frac{2m}{r_b}\right)\right]=m.$$

Thus the mass of this spherical distribution of matter as measured by a distant observer is given by the value u_b of u at $r=r_b$.

The following restrictions must be made on the choice of p_0 and u_0, the initial values of p and u at $r=0$:

(a) In accordance with its physical meaning as pressure, $p_0\geqq 0$.

(b) From Eq. (8) it is seen that for all finite values of $e^{-\lambda}$, $u_0=0$. Since $g_{11}=-e^\lambda$ must never be positive, $u_0\leqq 0$ for infinite values of $e^{-\lambda}$ at the origin. However, it may be shown that of all the finite values of p_0 at the origin $p_0=0$ is the only one compatible with a negative value of u_0, and that for equations of state of the type occurring in this problem even this possibility is excluded, so that u_0 must vanish.[10]

(c) A special investigation for any particular equation of state must be made to see whether solutions exist in which $0\geqq u_0\geqq -\infty$ and $p\rightarrow\infty$ as $r\rightarrow 0$.

III. Particular Equations of State

The above arguments show that Eqs. (9) and (10) together with a given equation of state completely determine the distribution of matter.

[10] This can be seen from the following argument. Having chosen some particular value of p_0 one may usually represent the equation of state in that pressure range by $\rho=Kp^s$ with some appropriate value of s. Using this equation of state and taking the approximate form of Eq. (10) near the origin for the case $u_0<0$, and finite p_0, one obtains:

$$\frac{dp}{dr}=\frac{p+\rho(p)}{2r}=\frac{p+Kp^s}{2r}.$$

Integration of this equation shows that for $s<1$ $p_0\geqq 0$ can not be satisfied, and for $s\geqq 1$ only the value $p_0=0$ is possible. For the equations of state used in this problem always $s<1$ holds. It may also be noted that the above equation together with Eq. (7) show that $e^{\nu(r)}\rightarrow\infty$ as $r\rightarrow 0$.

The assumption $\rho=\text{const.}$, $u_0=0$ makes it possible to integrate Eqs. (9) and (10) explicitly and leads to Schwarzschild's interior solution.[11] Other matter distributions corresponding to other equations of state are given by Professor Tolman in an accompanying paper.

If the matter is taken to consist of particles of rest mass μ_0 obeying Fermi statistics, and their thermal energy[12] and all forces between them are neglected, then it may be shown that a parametric form for the equation of state is:[13]

$$\rho=K(\sinh t-t), \qquad (11)$$

$$p=\tfrac{1}{3}K(\sinh t-8\sinh \tfrac{1}{2}t+3t), \qquad (12)$$

where $K=\pi\mu_0{}^4c^5/4h^3$ (13)

and
$$t=4\log\left(\frac{\hat{p}}{\mu_0 c}+\left[1+\left(\frac{\hat{p}}{\mu_0 c}\right)^2\right]^{\frac{1}{2}}\right), \qquad (14)$$

where $\hat{p}$ is the maximum momentum in the Fermi distribution and is related to the proper particle density N/V by

$$\frac{N}{V}=\frac{8\pi}{3h^3}\hat{p}^3. \qquad (15)$$

Substituting the above expressions for p and ρ into Eqs. (9) and (10) one gets:

$$\frac{du}{dr}=4\pi r^2K(\sinh t-t), \qquad (16)$$

$$\frac{dt}{dr}=-\frac{4}{r(r-2u)}\frac{\sinh t-2\sinh\frac{1}{2}t}{\cosh t-4\cosh\frac{1}{2}t+3}\times[(4/3)\pi Kr^3(\sinh t-8\sinh\tfrac{1}{2}t+3t)+u]. \qquad (17)$$

These equations are to be integrated from the values $u=0$, $t=t_0$ at $r=0$ to $r=r_b$ where $t_b=0$ (which makes $p=0$), and $u=u_b$.

A note must be made of the units employed in these equations. Eqs. (3), (4) and (5) from which (16) and (17) are derived are stated in relativistic units,[14] i.e., such that $c=1$, $G=1$ (c is the velocity of light, G is the gravitational constant). This determines the unit of time and the unit of mass in terms of a still arbitrary unit of length. The unit of length is now fixed by the requirement that $K=1/4\pi$. Eqs. (16) and (17) now become:

$$\frac{du}{dr}=r^2(\sinh t-t), \qquad (18)$$

$$\frac{dt}{dr}=-\frac{4}{r(r-2u)}\frac{\sinh t-2\sinh\frac{1}{2}t}{\cosh t-4\cosh\frac{1}{2}t+3}\times[\tfrac{1}{3}r^3(\sinh t+8\sinh\tfrac{1}{2}t+3t)+u]. \qquad (19)$$

The unit of length has been fixed to be

$$a=\frac{1}{\pi}\left(\frac{h}{\mu_0 c}\right)^{\frac{3}{2}}\frac{c}{(\mu_0 G)^{\frac{1}{2}}},$$

while the unit of mass is

$$b=\frac{c^2}{G}a=\frac{1}{\pi}\left(\frac{h}{\mu_0 c}\right)^{\frac{3}{2}}\frac{c^3}{(\mu_0 G^3)^{\frac{1}{2}}}.$$

For a neutron gas $a=1.36\times10^6$ cm, $b=1.83\times10^{34}$ g. The general character of the solution is seen to be independent of the mass of the neutron which determines only the scale of the result.

No way was found to carry out the integration analytically, so Eqs. (18) and (19) were integrated numerically for several finite values of t_0. For all these cases u_0 was taken to be equal to zero, since the equation of state near the origin for finite t_0 behaves like $\rho(p)=Kp^s$, $s<1$. The first four entries in Table I were thus obtained.

For $t_0\rightarrow\infty$ Eqs. (18) and (19) may be replaced by their asymptotic expressions:

$$du/dr=\tfrac{1}{2}r^2e^t, \qquad (20)$$

$$\frac{dt}{dr}=-\frac{4}{r(r-2u)}\left[\frac{r^3}{6}e^t+u\right]. \qquad (21)$$

[11] R. C. Tolman, reference 5, pp. 246–247.

[12] The condition for thermal equilibrium in a static gravitational field is given by Tolman (reference 5, p. 318) as $T_0(g_{44})^{\frac{1}{2}}=\text{const.}$ where T_0 is the proper temperature. The equilibrium state of a matter distribution which no longer radiates appreciably corresponds to a low surface temperature T_0. If g_{44} is everywhere finite, then T_0 will be small throughout the matter distribution. For those singular solutions in which g_{44} vanishes at the origin it is conceivable that the central temperature may be high. However, on the one hand from Eq. (7) it is seen that the vanishing of g_{44} at the origin corresponds to infinite central pressure, and in this limit the equation of state given below reduces to $\rho=3p$ so that temperature introduces no radically new effects, and on the other hand zero values of g_{44} indicate the slowing down of all physical processes near the origin and thus may correspond to nonstatic solutions describing states which have not yet attained equilibrium, and which are not discussed in this paper.

[13] Cf. S. Chandrasekhar, Monthly Notices of R.A.S. **95**, 222 (1935), but introduce *energy* density in place of his *mass* density.

[14] R. C. Tolman, reference 5, pp. 201–202.

An exact solution[15] of these equations is:

$$e^{t}=3/7r^{2}, \qquad u=3r/14, \tag{22}$$

which corresponds to $t_0=\infty$, $u_0=0$. A careful examination of Eqs. (20) and (21) shows that there are no other solutions corresponding to $t_0=\infty$, $0\geqq u_0\geqq -\infty$. The exact solution (22) of the approximate equations (20) and (21) was taken out to that value of r where $t=6$ (the approximation in the form of Eq. (20) and (21) it quite good for $t\geqq 6$), and then the integration of the exact equations (18) and (19) was carried out numerically to $r=r_b$ where $t=0$. This gave the last entry in Table I.

It is of interest to ask whether perhaps a finite gravitational mass might correspond to an infinite number of particles, and an infinite gravitational binding energy. It may be seen that this is not the case by the following argument. Although the proper particle density becomes infinite when the central pressure becomes infinite, still it remains integrable, so that the total number of particles always remains finite. The element of proper volume of a spherical shell is $4\pi e^{\lambda/2}r^2dr$. As the solution of the approximate equations shows in the neighborhood of the origin:

$$e^{\lambda/2}=\left(1-\frac{2u}{r}\right)^{-\frac{1}{2}}=\left(1-\frac{3}{7}\right)^{-\frac{1}{2}}=\left(\frac{7}{4}\right)^{\frac{1}{2}},$$

$$N/V\propto \hat{p}^3\propto e^{3t/4}\propto 1/r^{\frac{3}{2}} \text{ for large } t \text{ and } \hat{p};$$

$$\therefore \quad N\propto\int_0^r \frac{r^2}{r^{\frac{3}{2}}}dr\propto r^{\frac{3}{2}} \text{ near the origin.}$$

A fortiori the number of particles is finite for nonsingular solutions.

For very small values of t the equation of state (11), (12) reduces to $p=K\rho^{5/3}$ and $\hat{p}\propto t$. Using this equation of state and Newtonian gravitational theory (which is expected to give a good result for small masses and densities), one finds that $\hat{p}\propto m^{\frac{2}{3}}$, or that $m\propto t^{\frac{3}{2}}$. Fig. 1 gives a schematic plot of the dependence of m on t_0 for the case that the elementary particles are neutrons. The mass m is plotted in units of sun's mass (2×10^{33} g) against $\tan^{-1} t_0$. The curve near the origin is dotted since, as has been already pointed out, a neutron core with a mass less than about $0.1\odot$ will disintegrate into nuclei and electrons.

The striking feature of the curve is that the mass increases with increasing t_0 until a maximum is reached at about $t_0=3$, after which the curve drops until a value roughly $\frac{1}{3}\odot$ is reached for $t_0=\infty$. In other words no static solutions at all exist for $m>\frac{3}{4}\odot$, two solutions exist for all m in $\frac{3}{4}\odot>m>\frac{1}{3}\odot$, and one solution exists for all $m<\frac{1}{3}\odot$.

Some insight into this situation may be gained from the following considerations. In the nonrelativistic polytrope solutions of Emden[16] the equation of state was assumed to be $p=K\rho^{\gamma}=K\rho^{1+1/n}$. Solutions which at first sight seem to be quite satisfactory (i.e., giving a finite mass within a finite radius) were found for values of $n<5$ or $\gamma>6/5$. But Landau[2] pointed out that although these solutions in every case give an equilibrium configuration, they do not in every case give *stable* equilibrium. Thus, unless $\gamma\geqq 4/3$ the equilibrium configuration is unstable. This may be seen from the following rough calculation. The gravitational part of the free energy of the system is negative and proportional to $\rho^{\frac{1}{3}}$ where

[15] This solution is a limiting form of the solutions V, VI given by Tolman in the accompanying paper.

[16] Emden, *Gaskugeln* (1907), or cf. *Handbuch der Astrophys.* Vol. 3, p. 186.

TABLE I. *Mass, radius and neutron density for various values of* t_0.

	Mass		Radius			
t_0	In Units of Eqs. (18), (19)	In Units of $\odot$ for Neutrons	In Units of Eqs. (18), (19)	In Kilometers, for Neutrons	$\left(\frac{\hat{p}}{\mu_0 c}\right)_{r=0}$	$\left(\frac{N}{V}\right)_{r=0}$ Neutrons cm³
1	0.033	0.30	1.55	21.1	0.25	0.062×10^{39}
2	0.066	0.60	0.98	13.3	0.52	0.56×10^{39}
3	0.078	0.71	0.70	9.5	0.82	2.2×10^{39}
4	0.070	0.64	0.50	6.8	1.17	6.4×10^{39}
∞	0.037	0.34	0.23	3.1	∞	∞

ρ is an appropriate average density (Newtonian gravitational theory is used). The part of the free energy caused by compression is proportional to $\int p dv$, and hence to $\rho^{\gamma-1}$ ($\gamma \neq 1$). Thus

$$F = -a\rho^{\frac{1}{3}} + b\rho^{\gamma-1}.$$

Polytrope solutions exist for both $\gamma = 5/3 (> 4/3)$, i.e., for $n = 3/2$ and for $\gamma = 5/4 (< 4/3$, but $> 6/5)$, i.e., for $n = 4$, but as may be seen from the schematic plot of the free energy curves in Fig. 2, the former corresponds to stable equilibrium and the latter to unstable equilibrium.

In the present relativistic calculations the results for small masses and small central densities and pressures (small values of t_0), as was already mentioned above, may be expected to agree quite closely with nonrelativistic calculations with the equation of state $p = K\rho^{5/3}$. Since ρ is a monotonic function of t, the curves of free energy against t_0 for fixed total number of particles, and thus for a fixed M_0 (gravitational mass at zero density; the gravitational mass will vary somewhat along a curve of constant particle number, as the density increases), will for small masses have the same general character as the curves of free energy against some average density in the nonrelativistic case (cf. the curve for $\gamma = 5/3$ in Fig. 2). Then as the number of particles is increased the character of the free energy curves must change in order to admit the possibility of a second equilibrium position. Since the free energy must be a continuous function of t_0, and since we know from nonrelativistic calculations that for small masses (and low densities) we have a position of stable equilibrium (a minimum in the free energy curve) we can conclude that the second equilibrium position corresponds either to a maximum or to an inflection point in the free energy curve (and certainly not to a minimum). Fig. 3 gives a schematic plot of free energy against t_0 for different values of M_0 which would explain the existence of one equilibrium position for small masses, two for intermediate masses, and none for large masses. The masses marked on the curves are the actual gravitational masses corresponding to the equilibrium points of the critical free energy curves separating the solutions into the three types mentioned above.

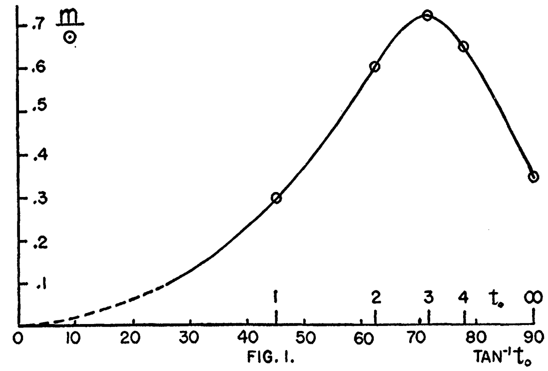

FIG. 1. Dependence of m on t_0 for neutrons.

IV. Discussion—Relation to Tolman's Solutions

Before we study the physical implications of these results, we may try to show how their qualitative features may be obtained from the analytic solutions recently discovered by Professor Tolman.[17] This will also help us to understand the probable effect of alterations in the equation of state of the neutron gas at high densities.

On the one hand Tolman's solution IV, discussed in §6 of his paper, enables us to understand the existence of a limiting mass for static solutions and to give an estimate of its magnitude; on the other hand Tolman's solution VI, discussed in §8 (and less directly solution V), has for $n = \frac{1}{2}$, very much the character of our singular solution for $t_0 \to \infty$, and, with appropriately chosen constants, gives a mass of the same order of magnitude as we have found.

Tolman's solution IV is nonsingular, and corresponds to the quadratic equation of state (6.5) of his paper:

$$8\frac{(p_c - p)^2}{p_c + \rho_c} - 5(p_c - p) + \rho_c - \rho = 0 \quad \text{(Tolman, 6.5)},$$

where ρ_c and p_c are the central density and pressure. From Eqs. (6.4), (6.6) and (6.9) of Tolman's paper the mass corresponding to this solution is given in terms of p_c and ρ_c by

$$m = 4\left(\frac{p_c}{\rho_c + 3p_c}\right)^{\frac{3}{2}}\left[\frac{8\pi}{3}(\rho_c - 3p_c)\right]^{-\frac{1}{2}}. \qquad (23)$$

[17] We are very much indebted to Professor Tolman for letting us see these results before publication, and for helpful discussions of them.

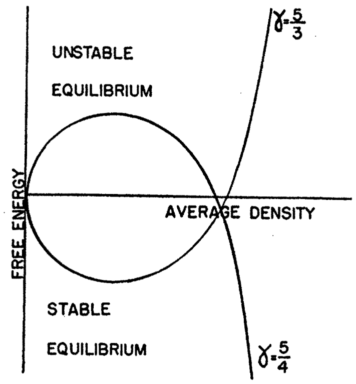

FIG. 2. **Free** energy as a function of average density.

If ρ_c and p_c are now themselves connected by the Fermi equation of state (11), (12), then $p_c \propto \rho_c^{5/3}$ as $\rho_c \rightarrow 0$, and $\rho_c - 3p_c \sim 0(\rho_c^{\frac{1}{2}})$ as $\rho_c \rightarrow \infty$, and m is seen to have a maximum value. For values of ρ_c corresponding to this maximum the equation of state (Tolman 6.5) does not differ qualitatively from the Fermi equation of state (11), (12), as may be seen by comparing for the two solutions the values of $d \ln p / d \ln \rho$; and the maximum mass in fact turns out from (23) to be $\sim 0.4 \odot$, agreeing in order of magnitude with our value of $\sim 0.7 \odot$.

Tolman's solution V, with $n = \frac{1}{2}$, $R \rightarrow \infty$, and his solution VI, with $n = \frac{1}{2}$, $B/A \rightarrow 0$, are just our solution (22) corresponding to the equation of state $p = \frac{1}{3}\rho$, a unique unstable singular solution. For solution V, with $n = \frac{1}{2}$ and finite R, the pressure differs from $\frac{1}{3}\rho$ by terms of the order of $\rho^{-1/6}$; however, for VI with $n = \frac{1}{2}$ and finite B/A, for large ρ, $\rho - 3p = \text{const.}\ \rho^{\frac{1}{2}}$, which is just the behavior of a highly compressed Fermi gas. Using for the mass of this solution

$$m = A/42B \qquad \text{(Tolman, 8.9)}$$

and adjusting the ratio B/A to make the equation of state of VI, i.e.,

$$p = \frac{1}{3}\rho \frac{1 - 9(B/A)(3/56\pi)^{\frac{1}{2}}\rho^{-\frac{1}{2}}}{1 - (B/A)(3/56\pi)^{\frac{1}{2}}\rho^{-\frac{1}{2}}} \qquad \text{(Tolman, 8.5)}$$

agree to terms of order $\rho^{\frac{1}{2}}$ with (11), (12), we get $B/A = (7/3)^{\frac{1}{2}}$, and $m \sim (1/7) \odot$, to compare with the value of $\frac{1}{3} \odot$ which the Fermi equation gives.

These necessarily somewhat rough comparisons may thus serve to give an idea of the analytic character of the solutions corresponding to maximum mass and to maximum (infinite) central density which we obtained above.

V. Discussion—Application to Stellar Matter

We have seen that for a cold neutron core there are no static solutions, and thus no equilibrium, for core masses greater than $m \sim 0.7 \odot$. The corresponding maximum mass M_0 before collapse is some ten percent greater than this. Since neutron cores can hardly be stable (with respect to formation of electrons and nuclei) for masses less than $\sim 0.1 \odot$, and since, even after thermonuclear sources of energy are exhausted, they will not tend to form by collapse of ordinary matter for masses under $1.5 \odot$ (Landau's limit), it seems unlikely that static neutron cores can play any great part in stellar evolution;[18] and the question of what happens, after energy sources are exhausted, to stars of mass greater than $1.5 \odot$ still remains unanswered. It should be observed that for the critical solution with $m \sim 0.7 \odot$ the potentials $g_{\mu\nu}$ are nowhere singular, and that in particular such a core does not tend to "protect itself" from the addition of further matter by the vanishing of g_{44} at the boundary. There would then seem to be only two answers possible to the question of the "final" behavior of very massive stars: either the equation of

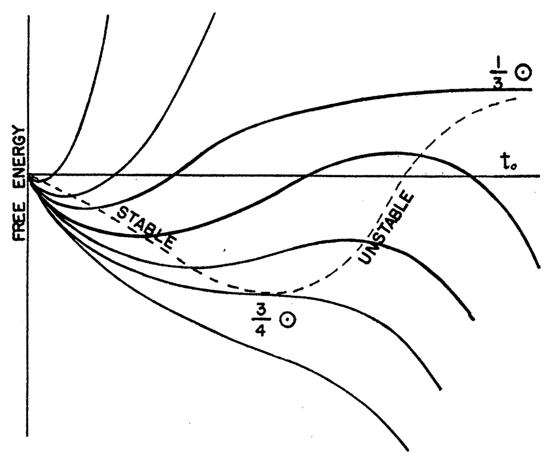

FIG. 3. Schematic plot of free energy as a function of t_0.

[18] The mass of the shell of ordinary (but dense) matter surrounding the core must be small for cores much more massive than the lightest core stable with respect to disintegration into electrons and nuclei.

state we have used so far fails to describe the behavior of highly condensed matter that the conclusions reached above are qualitatively misleading, or the star will continue to contract indefinitely, never reaching equilibrium. Both alternatives require serious consideration.

The central density in the "critical" core is even higher than nuclear density, so that our extrapolation of the Fermi equation of state can hardly rest on a very sure basis. Under these conditions the disintegration of neutrons, either into protons and electrons, or into mesotrons, will be energetically unfavorable and will not occur. And the relatively weak attractive forces which are known to act between neutrons will facilitate, and not prevent, the collapse of the core. If, however, under extreme compression, phenomena occurred which have the effect of repulsive forces, i.e., of raising the pressure for a given density above the value given by the Fermi equation of state, this could tend to prevent the collapse.

Such repulsive forces, even if they exist, will hardly make possible static solutions for arbitrarily large amounts of matter. For at low densities they cannot appreciably affect the equation of state, so that the dimensions of the core will necessarily be finite, and so will be the gravitational mass m of the core

$$m=\tfrac{1}{2}r_b(1-e^{-\lambda_b})\quad \text{(Tolman, 5.5).}$$

Nor can the mass M_0 before collapse be infinite. For this to be true we should have to have a singular solution. But the effect of repulsive forces can for high density at most be to make $3p$ even more nearly equal to ρ than for the Fermi equation of state; and for $\rho=3p$, as has been remarked above, and as is also suggested by Tolman's solutions V and VI, the *only* singular static solution is (22), for which the total particle number is finite.

We may obtain an extreme limit on the increase in the limiting mass which strong repulsive forces at high densities could give, by the following simple argument. For $\rho<10^{15}$ g/cm^3 these forces can hardly be important. Let us assume that for $\rho\geqslant 10^{15}$, they have the extreme effect of making $p=\frac{1}{3}\rho$. Then the mass of a sphere for which this equation of state holds down to $\rho=10^{15}$, and for which p falls rapidly as $\rho\rightarrow 0$, is given by our solution (22), and is of the order of $\odot$. It seems likely that our limit of $\sim 0.7\odot$ is near the truth.

This argument is based on the requirement that even for arbitrarily high densities, $\rho-3p$ shall not be negative; and this is in turn closely related to the positive definite character of the (proper) energy density of neutrons and of the fields of force (apart from gravitation) associated with them. It seems probable that if p could be very much greater than $\frac{1}{3}\rho$, static solutions of arbitrarily large mass could be found.[19]

From this discussion it appears probable that for an understanding of the long time behavior of actual heavy stars a consideration of nonstatic solutions must be essential. Among all (spherical) nonstatic solutions one would hope to find some for which the rate of contraction, and in general the time variation, become slower and slower, so that these solutions might be regarded, not as equilibrium solutions, but as quasi-static. Some reason for this we may see in the following argument: for large enough mass the core will collapse; near the center the density and pressure will grow, and $g_{44}=e^{\nu}$ will be small (cf. Eq. (7)); and as e^{ν} grows smaller, all processes will, as seen by an outside observer, slow down in the central region. Formally one sees this, in the occurrence, in Einstein's equations, of products of the form

$$e^{-\nu}\frac{d^2\lambda}{dt^2},\quad e^{-\nu}\left(\frac{d\lambda}{dt}\right)^2,\quad e^{-\nu}\frac{d\lambda}{dt}\frac{d\nu}{dt}.$$

For high enough central densities it is no longer justified to neglect even a very slow time variation; and the singular solutions which presumably represent very massive neutron cores cannot be obtained unless this is taken into account. These solutions are now being investigated.

[19] Thus for $\rho=$const. there is a class of singular static solutions, for which $p\sim k/r^3$, and which would seem to lead, for $K\rightarrow\infty$, to infinite masses, and which one of us (G.M.V.) hopes to discuss in detail elsewhere.