Multiparticle Quantum Mechanics Obeying Fractional Statistics

Yong-Shi Wu
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 2 April 1984)

We obtain the rule governing many-body wave functions for particles obeying fractional statistics in two (space) dimensions. It generalizes and continuously interpolates the usual symmetrization and antisymmetrization. Quantum mechanics of more than two particles is discussed and some new features are found.

PACS numbers: 03.65.Ca, 03.65.Ge, 05.30.−d

In two (space) dimensions, there are allowed to be particles of fractional angular momentum or spin.[1,2] If there is a generalized spin-statistics connection, such particles are expected to have unusual (fractional) statistics which continuously interpolates between the normal bosons and fermions. (An example for such interpolation is known in one dimension.[3]) The intriguing problem of how it works is interesting both from the viewpoint of theoretical principles and from the prospect of physical applications. A possible relevance of fractional statistics to the quantized Hall effect has been recently suggested.[4]

Two simple models have been proposed for particles obeying fractional statistics by Wilczek[1,5] Yang and Yang,[3] and Wilczek and Zee.[6] Two-particle quantum mechanics was analyzed in detail. A low-density expansion of the partition function interpolating the standard statistics was obtained. As pointed out in these papers, Feynman's path-integral formulation is a good starting point. However, the formalism in terms of wave functions may be practically more convenient. An immediate problem is the general rule governing the many-body wave functions, namely how to generalize the usual rule to obtain a continuous interpolation between symmetrization and antisymmetrization. In this note I answer this question by deriving the desired rule in the two models mentioned above. As an application, I discuss the quantum mechanics of three particles, not yet touched in the literature. Some new features are found which are not present in the two-particle case.

Anyons revisited.—Following Wilczek,[5] I denote composites formed from charged particles and magnetic flux tubes as anyons, since their spin

$$\Delta = q\Phi/2\pi = \theta/2\pi \tag{1}$$

can take any real values. Here q is the charge and Φ the flux. That[5] interchange of two anyons leads to a phase $e^{i\theta}$ is an indication of the fractional statistics. We here consider quantum mechanics for more than two anyons.

The Hamiltonian for a charged particle orbiting around a flux tube can be written as

$$H_0 = \frac{1}{2m_q}\left[-i\frac{\partial}{\partial \vec{r}_q} - q\vec{A}(\vec{r}_q - \vec{r}_f)\right]^2 + \frac{1}{2m_f}\left[-i\frac{\partial}{\partial \vec{r}_f} + q\vec{A}(\vec{r}_q - \vec{r}_f)\right]^2. \tag{2}$$

Here we consider the limit in which the size of the flux tube can be neglected. $\vec{r}_q$ and $\vec{r}_f$ are two-dimensional vectors. Let us assume that the flux tube has a finite effective mass m_f in two dimensions. The form (2) has the advantage that the effect of the interaction is confined to the wave function in the relative coordinate. In a regular gauge the vector potential is

$$q\vec{A}(\vec{r}_q - \vec{r}_f) = -q\vec{A}(\vec{r}_f - \vec{r}_q) = (\theta/2\pi)[\vec{n}\times(\vec{r}_q - \vec{r}_f)]/|\vec{r}_q - \vec{r}_f|^2 \tag{3}$$

(with $\vec{n}$ being the unit vector normal to the two-dimensional plane), and the wave function is single-valued everywhere.

Now we proceed to consider n identical anyons and neglect the electrostatic forces between them (i.e., consider the limit $q \to 0$ with $\theta = q\Phi$ fixed). The charged particle in each anyon feels the vector potential of the flux tube in the other. Using the Hamiltonian (1) and applying a procedure similar to that in Goldhaber[7] for the charge-monopole composites, one finds that the anyon-anyon potential is equivalent to that of a charge interacting with twice the flux in one flux tube; namely

$$H = \sum_{i=1}^{n}\frac{1}{2m_a}\left[-i\frac{\partial}{\partial \vec{r}_i} - 2q\sum_{j\neq i}\vec{A}(\vec{r}_i - \vec{r}_j)\right]^2. \tag{4}$$

Let us adopt Eq. (3) for $\vec{A}(\vec{r}_i-\vec{r}_j)$ in the regular gauge in which the wave function ψ is single-valued as in the one anyon case. To eliminate the long-range vector potential between anyons, we make the gauge transformation

$$\psi'(\vec{r}_1,\ldots,\vec{r}_n)=\prod_{i<j}\exp\left\{i\frac{\theta}{\pi}\phi_{ij}\right\}\psi(\vec{r}_1,\ldots,\vec{r}_n), \qquad (5)$$

where ϕ_{ij} is the azimuthal angle of the relative vector $\vec{r}_i-\vec{r}_j$. Now the new wave function ψ' satisfies the free Schrödinger equation with no vector potential.

At first sight the multivaluedness of the new wave function ψ' seems to be very discomforting. One can manage to avoid it by imposing appropriate boundary conditions for ψ' on certain cuts in the two-dimensional plane[5] or formulating quantum mechanics on sections on fiber bundles.[8] However, these two methods are very hard to put into practice for more than two anyons. Actually, nothing is wrong with the multivaluedness of the wave function (5). The modulus squared, $|\psi'|^2$, is single-valued, and the multivalued phase factors are just right to keep track of the Aharonov-Bohm effect.[9] In my opinion once one understands the need for extending the notion of a wave function (i.e., not requiring it to be necessarily 2π periodic in ϕ_{ij}), there is no difficulty in accepting and directly using the multivalued wave function (5) as everybody does with the double-valued spinors in three dimensions.[10]

By use of the complex coordinates $z_i=x_i+iy_i$ and $z_i^*=x_i-iy_i$ instead of $\vec{r}_i=(x_i,y_i)$, the wave function (5) can be put into a more elegant form[11]:

$$\psi'(z_i,z_i^*)=\prod_{i<j}(z_i-z_j)^{\theta/\pi}f(z_i,z_i^*), \qquad (6)$$

with $f(z_i,z_i^*)=(r_{ij})^{-\theta/\pi}\psi(z_i,z_i^*)$ single-valued. f is totally symmetric (antisymmetric) in the pairs (z_i,z_i^*), if all the fields describing the flux tube and charged particle are bosonic (if the charged particle is fermionic). The equation (6) is the desired rule for many-body wave functions obeying θ statistics.

Solitons in point approximation.—The solitons in the $(2+1)$-dimensinal O(3) nonlinear sigma model, with a topological action, also provide a model for particles with fractional spin and statistics.[3,6] When widely separated solitons are approximately treated as point particles, the topological term (with the parameter θ) leads to an additional term

$$S'=\int dt\,L', \quad L'=(\theta/\pi)(d/dt)\textstyle\sum_{i<j}\phi_{ij}, \qquad (7)$$

to the ordinary action $S_0=\int dt\,\frac{1}{2}m\sum_i\dot{\vec{r}}_i^{\,2}$. While this term does not affect the equation of motion, it determines the statistics of the particles via path integral.

When one goes from path integral to wave functions, the term (7) also leads to the rule (6) for many-body wave functions associated with usual Hamiltonian containing no peculiar interactions. In fact, the change of ϕ_{ij} can be always written as

$$\phi_{ij}(t)|_{t'}^{t''}=2\pi n_{ij}+\phi_{ij}''-\phi_{ij}', \qquad (8)$$

with $0\le\phi_{ij}''-\phi_{ij}'<2\pi$. Thus, the propagator in the n-particle configuration space is a sum of "partial amplitudes," each corresponding to a distinct class of paths having the same winding numbers $\{n_{ij}\}$:

$$K(\vec{r}_i'',t'';\vec{r}_i',t')=\exp[i\sum_{i<j}(\phi_{ij}''-\phi_{ij}')]\sum_{n_{ij}}\exp(i2\theta\sum_{i<j}n_{ij})\int_{\vec{r}_i'}^{\vec{r}_i''}[\mathscr{D}\vec{r}_i(t)]_{n_{ij}}\exp(iS_0). \qquad (9)$$

As usual, a single-valued wave function $\psi(\vec{r}_i,t)$ can be introduced such that

$$\psi(\vec{r}_i'',t'')=\int d\vec{r}_i'\,k(\vec{r}_i'',t'';\vec{r}_i',t')\psi(\vec{r}_i',t'). \qquad (10)$$

We can eliminate the sum in Eq. (9) by introducing a new wave function

$$\tilde{\psi}(\vec{r}_i,t)=\exp\left\{i\frac{\theta}{\pi}\sum_{i<j}\phi_{ij}\right\}\psi(\vec{r}_i,t). \qquad (11)$$

Then, corresponding to Eq. (10), now we have

$$\tilde{\psi}(\vec{r}_i'',t'') = \int d\vec{r}_i' \tilde{K}(\vec{r}_i'',t'';\vec{r}_i',t')\tilde{\psi}(\vec{r}_i',t'), \quad (12)$$

$$\tilde{K}(\vec{r}_i'',t'';\vec{r}_i',t') = \int_{\vec{r}_i'}^{\vec{r}_i''} [\mathscr{D}\vec{r}_i(t)]\exp(iS_0). \quad (13)$$

Note that the wave function $\tilde{\psi}(\vec{r}_i,t)$ is single-valued on the universal covering space (or Reimann surface) of the n-particle configuration space. The integration over $\vec{r}_i'$ in Eq. (12) is taken on this covering space. By use of the complex coordinates, it is easy to recover Eq. (6) from Eq. (11).

Another way to derive the same result is the following. The Hamiltonian corresponding to L_0+L' is

$$H=\frac{m}{2}\sum_i \dot{\vec{r}}_i^{\,2}, \quad m\dot{\vec{r}}_i=\vec{p}_i-\frac{\theta}{\pi}\sum_{i<j}\frac{\vec{r}_i-\vec{r}_j}{|\vec{r}_i-\vec{r}_j|^2}. \quad (14)$$

Here $\vec{p}_i$ is the canonical momentum conjugate to $\vec{r}_i$. It is easy to see that H is the same as given by Eq. (4) together with Eq. (3). We can repeat the same procedure in the last section to arrive at Eq. (6). However, the argument given from Eq. (8) to Eq. (14) has the advantage that it elucidates the relationship between our wave functions and the path integral formulation.

Properties of the wave function (6).—Equation (5) or the rule (6) is invariant under $\theta \to \theta+2\pi$; i.e., fractional statistics is 2π periodic in θ, in agreement with the well-known periodicity of the Aharonov-Bohm effect in the flux or that of the θ parameter in the topological action.

When $\theta=0$ and π, the rule (6) coincides with the standard symmetric or antisymmetric rule. For intermediate θ it gives a continuous interpolation between the two extreme cases. However, when $\theta \neq 0,\pi$, the many-body wave functions are not of the form of products of single-particle wave functions. So generally we expect that the physical quantities of a system of many particles are not simply related to those for one particle.

When $n=2$, from Eq. (6) it is easy to recover the condition[1,5]

$$\psi'(\phi_{12}\pm\pi)=e^{\pm i\theta}\psi'(\phi_{12}). \quad (15)$$

For $n \geqslant 3$, Eq. (6) exhibits complicated behavior under permutation or interchange of the positions of particles. Complication occurs even when we exchange only two particles in the presence of a third particle. We have to specify along what loop particle 1 moves from $\vec{r}_1$ to $\vec{r}_2$ and particle 2 from $\vec{r}_2$ to $\vec{r}_1$. The resulting phase change will depend on whether the "spectator" 3 is enclosed inside this loop or not. This situation is a reflection of the fact that the configuration space of identical particles is multiply connected. It is the origin of the difficulties pointed out in Refs. 5 and 6 in dealing with more than two particles. The acceptance and direct use of the multivalued wave functions (6) make the many-particle problem accessible to approach, since the complications mentioned above have been simply built into the factors $\prod_{i<j}(z_i-z_j)^{\theta/\pi}$.

Physically, the long-range interactions due to θ statistics are coded in the factors $\prod_{i<j}(z_i-z_j)^{\theta/\pi}$. Moreover, these factors imply the existence of angular momentum barriers between any pair of particles when $\theta \neq 0$. Thus the many-body wave functions are expected to vanish when any two of the particles coincide (if $\theta \neq 0$), although the particles are not fermions for $\theta \neq \pi$.

Three particles, harmonic well.—As an application let us use the many-body wave functions (6) to attack the problem of three identical particles in a harmonic potential. The Schrödinger equation (for n particles with $m=1$) is

$$H\psi=E\psi, \quad H=-\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial z_i \partial z_i^*}+\frac{1}{2}\omega^2\sum_{i=1}^{n} z_i z_i^*, \quad (16)$$

where ψ satisfies the rule (6) with f totally symmetric. (We have omitted the prime on ψ).

The $n=2$ case has been analyzed in Refs. 5 and 12. In our approach we recover the complete set of solutions as follows:

$$\psi=W^{|L|}w^{|l+2\Delta|}L_N^{(|L|)}(2\omega ZZ^*)L_n^{(|l+2\Delta|)}(\tfrac{1}{2}\omega zz^*)\exp[\tfrac{1}{2}\omega(z_1z_1^*+z_2z_2^*)], \quad (17)$$

$$E=(2N+|L|+2n+|l+2\Delta|+2)\omega, \quad (18)$$

where $N,n \geqslant 0$ are principal quantum numbers for the center-of-mass and relative oscillators respectively; L, $l+2\Delta$ are angular momenta in the center-of-mass and relative coordinates. (l must be even.) $L_M^{(m)}(x)$ are the Laguerre polynomials.[13] We have used the following notation for brevity: $Z=\frac{1}{2}(z_1+z_2)$, $z=z_1-z_2$ and

$$W=\begin{cases} Z \text{ if } L>0, \\ Z^* \text{ if } L<0, \end{cases} \quad w=\begin{cases} z \text{ if } l+2\Delta>0, \\ z^* \text{ if } l+2\Delta<0. \end{cases} \tag{19}$$

Since θ appears only in the form of $|l+2\Delta|$, the 2π periodicity of θ is made clear. It is also easy to verify the continuous interpolation between the spectrum (including degeneracies) of bosons and that of fermions when θ varies from 0 to π.[12,14]

For $n=3$, we have obtained the following solutions for $0\leq\theta<\pi$:

$$\psi=[(z_1-z_2)(z_1-z_3)(z_2-z_3)]^{\theta/\pi}\exp\{-\tfrac{1}{2}\omega r^2\}P, \tag{20}$$

$$P=(z_1+z_2+z_3)^L(z_1-z_2)^l(2z_1-z_2-z_3)^m L_{N_1}^{(L)}(\tfrac{1}{3}\omega R^2)L_{N_2}^{(l+3m+6\Delta-5)}(\tfrac{1}{3}\omega\rho^2)+\text{symmetrization}, \tag{20'}$$

$$E=(2N_1+2N_2+L+l+m+6\Delta+3)\omega, \tag{20''}$$

where all N_1, N_2, L, m, l are nonnegative integers, and l, m such that after symmetrization P does not become identically zero. Moreover, $R^2=|z_1+z_2+z_3|^2$, $r^2=\sum_i|z_i|^2$,

$$\rho^2=|2z_1-z_2-z_3|^2+\text{cyclic permutation}.$$

We note that the parity transformation $z_i\leftrightarrow z_i^*$ and $\theta\rightarrow-\theta$ is a good symmetry of the equation (16) and the rule (6). So applying it on the solutions (20) will lead to more solutions (with l, m such that ψ has no singularities at $z_i^*=z_j^*$). We know that this set of solutions does not exhaust those of the problem; e.g., the three-fermion ground state is missing when $\theta=\pi$.

Even so, we are able to see some important features not present in the solutions of two particles. First, for sufficiently small θ, the ground-state energy is $E_0=(3+3\theta/\pi)\omega$. For n particles, it is $E_0=[n+n(n-1)\theta/2\pi]\omega$. Thus, the n dependence of E_0 has a quadratic part which looks like two-body interaction energy. Second, when $\theta=\pi$ the above energy level moves to 6ω, which exceeds the energy of the three-fermion ground state $E_0'=5\omega$. So when θ varies continuously from 0 to π, there must be level crossing and, therefore, the emergence of new ground states at certain values of θ. This effect may lead to interesting phenomena in realistic systems obeying θ statistics when θ can vary under certain circumstances.

To conclude, I stress that though the rule (6) is derived in two concrete mdoels, it is generally true for any fractional statistics in two dimensions, whatever its origins. This will be confirmed in a model-independent formulation in a forthcoming paper.[15]

The author thanks M. Baker, D. Boulware, L. Brown, R. Tao, F. Wilczek, and A. Zee for useful discussions and comments. This work was supported in part by the U. S. Department of Energy.

[1]F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982). See also M. Peshkin, Phys. Rep. **80**, 376 (1982).

[2]F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).

[3]C. N. Yang and C. P. Yang, J. Math. Phys. **10**, 1115 (1969). They have shown how a Bose gas with repulsive two-body delta-function interactions becomes a free Fermi gas for infinite coupling constant.

[4]B. Halperin, Phys. Rev. Lett. **52**, 1583, 2390(E) (1984).

[5]F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).

[6]F. Wilczek and A. Zee, Institute of Theoretical Physics, University of California, Santa Barbara Report No. NSF-ITP-84-25, 1984 (to be published).

[7]A. S. Goldhaber, Phys. Rev. Lett. **36**, 1122 (1976), and **49**, 905 (1982).

[8]T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).

[9]Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[10]We note here that infinite multivaluedness of a wave function can happen only in two dimensions, since $\pi_1(SO(2))=Z$. In three-space only double-valuedness is allowed because $\pi_1(SO(3))=Z_2$.

[11]Here by the notation $\psi(z_i,z_i^*)$, the set $\{(z_i,z_i^*), i=1,\ldots,n\}$ is understood. Special wave functions of this form have appeared in Ref. 4. Here we proved that Eq. (6) is the general form of many-body functions obeying fractional statistics.

[12]J. Leinaas and J. Myrlheim, Nuovo Cimento Soc. Ital. Fis. **B37**, 1 (1977).

[13]See, e.g., *Encyclopedic Dictionary of Mathematics*, edited by S. Iyanaga and Y. Kawada (MIT Press, Cambridge, Mass., 1977), Appendix A, Table 20 VI.

[14]The continuous interpolation between the two-body scattering amplitudes of bosons and those of fermions is being discussed by F. Wilczek and A. Zee (private communication).

[15]Y. S. Wu, to be published.