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CARTESIANNESS: TOPOLOGICAL SPACES, UNIFORM SPACES, AND AFFINE SCHEMES

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Introduction

Let $\mathbf{A}$ be a category with finite limits, or equivalently, finite products and equalizers. An object Y is *cartesian* if the functor $-\times Y: \mathbf{A} \to \mathbf{A}$ has a right adjoint. We begin with a brief discussion of cartesian objects in a general category $\mathbf{A}$. In Sections 2–4 we give complete characterizations in three cases, namely, the category $\mathbf{Top}/T$ of topological spaces over a fixed space T, the category $\mathbf{Un}/T$ of uniform spaces over a fixed uniform space T, and the category of affine schemes over a fixed affine scheme. (This last category is, of course, the dual of the category of commutative K-algebras for some fixed commutative ring K.)

If Y is a cartesian topological space, then the right adjoint to $-\times Y$ can be expressed as a function space $\mathbf{Top}(Y,-)$, where $\mathbf{Top}(Y,Z)$ denotes the set of continuous maps from Y to Z. Cartesian objects in $\mathbf{Top}$ were characterized by Day and Kelly [2] as those spaces Y such that the lattice $\mathbf{O}(Y)$ of open subsets of Y is a continuous lattice (in the sense of Scott [12]). More recently, Hofmann and Lawson showed that every distributive continuous lattice is isomorphic to $\mathbf{O}(Y)$ for some cartesian space Y [6].

In Section 2, we characterize cartesian objects in $\mathbf{Top}/T$. Among corollaries, we show that an object $p: Y \to T$ is cartesian in $\mathbf{Top}/T$ whenever Y is locally compact and T is Hausdorff. We also deduce that the inclusion of a subspace Y of T is cartesian in $\mathbf{Top}/T$ if and only if Y is a locally closed subset of T.

As a consequence of the theorem we obtain for $\mathbf{Un}/T$, we establish a somewhat surprising connection between cartesian uniform spaces over T and covering spaces of T (or more specifically, overlays of T in the sense of Fox [4]). In addition, when T is a one point space, we see that Y is cartesian in $\mathbf{Un}$ if and only if its uniformity has a least member.

We conclude our discussion of cartesianness in Section 4, by showing that a scheme Spec A over Spec K is cartesian in the category of affine schemes over Spec K if and only if A is finitely generated and projective as a K-module.

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1. Cartesian objects

Throughout this section all categories will have finite limits (equivalently, finite products and equalizers). If $\mathbf{A}$ is a category, then $|\mathbf{A}|$ denotes the class of objects of $\mathbf{A}$, and $\mathbf{A}(X, Y)$ denotes the set of morphisms from X to Y in $\mathbf{A}$.

Given $Y, Z \in |\mathbf{A}|$, we define Z^Y to be any object of $\mathbf{A}$ representing $\mathbf{A}(- \times Y, Z)$. An object Y is *cartesian* if Z^Y exists for all $Z \in |\mathbf{A}|$, or equivalently if $- \times Y : \mathbf{A} \to \mathbf{A}$ has a right adjoint. $\mathbf{A}$ is *cartesian closed* if every object of $\mathbf{A}$ is cartesian. If $\mathbf{A}$ is pointed (in particular, abelien), then the only cartesian objects are the zero objects, for if Y is cartesian, then $\mathbf{A}(Y, Y) \cong \mathbf{A}(0 \times Y, Y) \cong \mathbf{A}(0, Y^Y)$, a one element set. It is also not difficult to show that $Z^{Y \times Y'} \cong (Z^Y)^{Y'}$ and $Z^1 \cong Z$, where 1 is the terminal object of $\mathbf{A}$.

If $T \in |\mathbf{A}|$, then the category $\mathbf{A}/T$ of *objects of* $\mathbf{A}$ *over* T is the category whose objects are $\mathbf{A}$-morphisms $p_X : X \to T$, and morphisms $f : p_X \to p_Y$ are commutative triangles in $\mathbf{A}$ of the form

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & T & \end{tikzcd}$$

A product $X \times T$ becomes an object over T via the projection $\pi_2 : X \times T \to T$. This induces a functor $T^* : \mathbf{A} \to \mathbf{A}/T$ which is clearly right adjoint to the forgetful functor $\Sigma_T : \mathbf{A}/T \to \mathbf{A}$.

We note that $\mathbf{A}/T$ has finite limits, since $\mathbf{A}$ does, and they are determined as follows. Equalizers in $\mathbf{A}/T$ are formed as in $\mathbf{A}$, i.e. the forgetful functor $\Sigma_T : \mathbf{A}/T \to \mathbf{A}$ creates them. The terminal object of $\mathbf{A}/T$ is the identity morphism $1_T : T \to T$. The product of $p_X : X \to T$ and $p_Y : Y \to T$ in $\mathbf{A}/T$ is given by $p_X \times p_Y : X \times_T Y \to T$, where $X \times_T Y$ is the pullback of p_X and p_Y in $\mathbf{A}$, and $p_X \times p_Y$ is the obvious projection.

If $p_Y : Y \to T$ is an object of $\mathbf{A}/T$, consider the functor $p_Y^* : \mathbf{A}/T \to (\mathbf{A}/T)/p_Y$. An object $f : p_X \to p_Y$ of $(\mathbf{A}/T)/p_Y$ is completely determined by the $\mathbf{A}$-morphism $f : X \to Y$, i.e. by an object of $\mathbf{A}/Y$. In particular, this correspondence yields an isomorphism of categories $(\mathbf{A}/T)/p_Y \cong \mathbf{A}/Y$. Thus, a morphism $p : Y \to T$ of $\mathbf{A}$ (considered as an object of $\mathbf{A}/T$) induces a functor $\mathbf{A}/T \to \mathbf{A}/Y$, which we shall also denote by p^*. Its left adjoint is given by composition with p, and is denoted (via a similar abuse of notation) by Σ_p.

Given an object $p_Y: Y \to T$ of $\mathbf{A}/T$, we can consider the following functors

$$-\times p_Y: \mathbf{A}/T \to \mathbf{A}/T, \qquad p_Y^*: \mathbf{A}/T \to \mathbf{A}/Y, \qquad -\times_T Y: \mathbf{A}/T \to \mathbf{A}.$$

They are sometimes referred to as the product, change of base, and the pullback functors, respectively. If p_Y is a cartesian object of $\mathbf{A}/T$, then by definition $-\times p_Y$ has a right adjoint. If p_Y^* has a right adjoint, it is denoted by Π_{p_Y}. We shall see that any one of the above three functors has a right adjoint if and only if they all do. First, we prove a general proposition.

If a functor F is left adjoint to G, we write $F \dashv G$. If $f: FX \to Y$ and $g: X \to GY$ correspond via the adjunction, we say that g is the *right adjunct* of f.

Proposition 1.1. *A functor $F: \mathbf{B} \to \mathbf{A}/T$ has a right adjoint if and only if $\Sigma_T \circ F$ has a right adjoint.*

Proof. If F has a right adjoint, then since Σ_T has a right adjoint, so does $\Sigma_T \circ F$.

Conversely, suppose $\Sigma_T \circ F \dashv G'$. If $X \in |\mathbf{B}|$, let $\sigma_X: X \to G'T$ be the right adjunct of FX considered as a morphism $\Sigma_T(FX) \to T$. Then, if $f: X \to X'$ is a $\mathbf{B}$-morphism, the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\sigma_X"'] & & X' \arrow[dl, "\sigma_{X'}"] \\ & G'T & \end{tikzcd}$$

is commutative. In other words, σ is a natural transformation from the identity functor on $\mathbf{B}$ to the constant $G'T$-valued functor. If $p_Y: Y \to T$ is an object of $\mathbf{A}/T$, let Gp_Y be the following equalizer in $\mathbf{B}$

$$Gp_Y \rightarrowtail G'Y \underset{\sigma_{G'Y}}{\overset{G'p_Y}{\rightrightarrows}} G'T.$$

Using naturality of σ, it is clear how $G: \mathbf{A}/T \to \mathbf{B}$ becomes a functor. Applying $\mathbf{B}(X, -)$ we get an equalizer of sets

$$\begin{tikzcd} \mathbf{B}(X, Gp_Y) \arrow[r, tail] & \mathbf{B}(X, G'Y) \arrow[r, shift left, "\mathbf{B}(X, G'p_Y)"] \arrow[r, shift right, "\mathbf{B}(X, \sigma_{G'Y})"'] \arrow[d] & \mathbf{B}(X, G'T) \arrow[d] \\ & \mathbf{A}(\Sigma_T(FX), Y) \arrow[r, shift left, "\mathbf{A}(\Sigma_T(FX), p_Y)"] \arrow[r, shift right, "\varrho"'] & \mathbf{A}(\Sigma_T(FX), T) \end{tikzcd}$$

where ϱ is the map that makes the diagram commute in the obvious sense. By naturality of σ, $\mathbf{B}(X, \sigma_{G'Y})$ takes everything to σ_X. Hence, ϱ takes everything to FX considered as a morphism of $\mathbf{A}$. It follows that $\mathbf{A}/T(FX, p_Y)$ is the equalizer of the bottom row. Therefore, we obtain a natural bijection

$$\mathbf{B}(X, Gp_Y) \cong \mathbf{A}/T(FX, p_Y)$$

showing that $F \dashv G$.

Corollary 1.2. *The following are equivalent for an object $p_Y : Y \to T$ of $\mathbf{A}/T$:*
(a) $- \times p_Y : \mathbf{A}/T \to \mathbf{A}/T$ *has a right adjoint (i.e. p_Y is cartesian in $\mathbf{A}/T$);*
(b) $p_Y^* : \mathbf{A}/T \to \mathbf{A}/Y$ *has a right adjoint;*
(c) $- \times_T Y : \mathbf{A}/T \to \mathbf{A}$ *has a right adjoint.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{A}/T \arrow[r, "- \times p_Y"] \arrow[d, "p_Y^*"'] \arrow[dr, "\times_T Y"] & \mathbf{A}/T \arrow[d, "\Sigma_T"] \\ \mathbf{A}/Y \arrow[r, "\Sigma_Y"] & \mathbf{A} \end{tikzcd}$$

Applying Proposition 1.1 to the top (respectively, bottom) triangle yields the equivalence of (a) and (c) (respectively, (b) and (c)).

If $T = 1$, the terminal object of $\mathbf{A}$, then there is an isomorphism of categories $\mathbf{A}/1 \cong \mathbf{A}$, and hence Corollary 1.2 says that $- \times Y : \mathbf{A} \to \mathbf{A}$ has a right adjoint if and only if $Y^* : \mathbf{A} \to \mathbf{A}/Y$ has a right adjoint.

Corollary 1.3. *If $q_X : X \to Y$ is cartesian in $\mathbf{A}/Y$ and $p_Y : Y \to T$ is cartesian in $\mathbf{A}/T$, then $X \xrightarrow{q_X} Y \xrightarrow{p_Y} T$ is cartesian in $\mathbf{A}/T$.*

Proof. By Corollary 1.2(a) $\Rightarrow$ (b), we know that $p_Y^* : \mathbf{A}/T \to \mathbf{A}/Y$ and $q_X^* : \mathbf{A}/Y \to \mathbf{A}/X$ have right adjoints. Therefore, their composite $q_X^* \circ p_Y^* : \mathbf{A}/T \to \mathbf{A}/X$ has a right adjoint, and so by Corollary 1.2(b) $\Rightarrow$ (a) $X \xrightarrow{q_X} Y \xrightarrow{p_Y} T$ is cartesian in $\mathbf{A}/T$.

Corollary 1.4. *Suppose that $p_Y : Y \to T$ is cartesian in $\mathbf{A}/T$, and $f : S \to T$ is a morphism of $\mathbf{A}$. Then $f^*(p_Y)$ is cartesian in $\mathbf{A}/S$.*

Proof. By Corollary 1.2(a) $\Rightarrow$ (c), it suffices to show that $- \times_S (Y \times_T S) : \mathbf{A}/S \to \mathbf{A}$ has a right adjoint, where $Y \times_T S$ is an object over S via $f^*(p_Y)$. If $q_X : X \to S$ is an object of $\mathbf{A}/S$, then the juxtaposition of pullbacks

$$\begin{tikzcd} X \times_S (Y \times_T S) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \times_T S \arrow[r, "\pi_1"] \arrow[d, "f^*(p_Y)"'] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "q_X"] & S \arrow[r, "f"] & T \end{tikzcd}$$

is a pullback in $\mathbf{A}$, and hence induces an isomorphism $X\times_T Y \cong X\times_S(Y\times_T S)$, which is clearly natural in X. Thus, we have a diagram

$$\begin{tikzcd}
\mathbf{A}/S \arrow[rr, "-\times_S(Y\times_T S)"] \arrow[dr, "\Sigma_f"'] & & \mathbf{A} \\
& \mathbf{A}/T \arrow[ur, "-\times_T Y"'] &
\end{tikzcd}$$

of functors which commutes up to natural isomorphism. Therefore, $-\times_X(Y\times_T S)$ has a right adjoint since Σ_f does, in general, and $-\times_T Y$ does by Corollary 1.2(a) $\Rightarrow$ (c).

2. Cartesian objects of Top/T

Let $p_Y: Y\to T$ be an object of $\mathbf{Top}/T$. Then, Y is the *total space*, T is the *base space*, and p_Y is the *projection*. By a *space* Y *over* T, we shall mean a topological space Y together with a particular projection $p_Y: Y\to T$.

Suppose that $t\in T$. Then the one point space 1 becomes a space over T via the constant t-valued map, which we shall denote by $t: 1\to T$. If $p_Y: Y\to T$ is continuous and A is a subset of Y, let $A_t = A\cap p_Y^{-1}(t)$. The *fiber of* Y *over* t is the set Y_t with the subspace topology. It is not difficult to see that as a set Y_t can be identified with $\mathbf{Top}/T(t,p_Y)$, and as a space Y_t is homeomorphic to the total space of $p_Y\times t$.

Let $p_Y: Y\to T$ and $p_Z: Z\to T$ be objects of $\mathbf{Top}/T$. If $(p_Z)^{p_Y}$ exists in $\mathbf{Top}/T$, we have a bijection

$$\mathbf{Top}/T(p_X\times p_Y, p_Z) \xrightarrow{\ \theta\ } \mathbf{Top}/T(p_X, (p_Z)^{p_Y})$$

which is natural in p_X. Taking p_X to be constant t-valued maps, we see that the total space of $(p_Z)^{p_Y}$ can be identified with the set of pairs (σ, t), where $\sigma: Y_t\to Z_t$ is a continuous map. If $x\in X_t$, then applying naturality to one point inclusions $x: 1\to X$ (considered as morphisms $t\to p_X$) and using the above identification, we obtain

$$[\theta(f)(x)](y) = f(x,y)$$

where $f: X\times_T Y\to Z$ is a continuous map over T, and $y\in Y_{p_X(x)}$.

Now, if $p_Y: Y\to T$ and $p_Z: Z\to T$ are *any* objects of $\mathbf{Top}/T$, motivated by the above, we *define* (p_Y, p_Z) to be the collection of pairs (σ, t) with $\sigma: Y_t\to Z_t$ continuous. We are going to topologize (p_Y, p_Z) so that (,) (with the obvious projection) becomes a functor $(\mathbf{Top}/T)^{\mathrm{op}}\times \mathbf{Top}/T\to \mathbf{Top}/T$. Moreover, when $(p_Z)^{p_Y}$ exists in $\mathbf{Top}/T$, it will turn out that its total space has the same topology as (p_Y, p_Z).

Let H be a subset of the collection $\mathbf{O}(Y)$ of open subsets of Y. We say that H is *saturated* if, $U\in H$, $U\subseteq V \Rightarrow V\in H$, and H has the *finite union property* (fup) if, $\bigcup_A U_\alpha\in H \Rightarrow \bigcup_F U_\alpha \in H$, for some finite $F\subseteq A$. The family of saturated H with fup defines a topology which is known as the *Scott-topology* on the lattice $\mathbf{O}(Y)$ [12].

Lemma 2.1 (Day and Kelly [2]). *If Y is a Hausdorff space, then the compact subsets of Y are precisely those of the form $\bigcap H$, where H is a nonempty Scott-open subset of $\mathbf{O}(Y)$.*

Proof. If C is a compact subset of Y, then clearly the collection of open subsets containing C is saturated with fup. But in a T_1 space, any subset C is of the form $\bigcap H$ where H is the collection of open subsets containing C. This proves one direction.

For the converse, suppose H is a nonempty Scott-open subset of $\mathbf{O}(Y)$. First we show that $\bigcap H$ is closed. If $y \notin \bigcap H$, then $y \notin V$ for some $V \in H$. For every $v \in V$ choose nonintersecting open neighborhoods U_v and V_v of y and v, respectively. Then $\bigcup_{v \in V} V_v \in H$ and hence some finite union of the V_v is in H. Let U be the intersection of the corresponding U_v. Then $y \in U \subseteq Y \setminus \bigcap H$, and it follows that $\bigcap H$ is closed. Now, suppose H is nonempty and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for $\bigcap H$. Then $(\bigcup_A U_\alpha) \cup (Y \setminus \bigcap H) = Y \in H$ since H is nonempty and saturated, so by the fup $(\bigcup_F U_\alpha) \cup (Y \setminus \bigcap H) \in H$ for some finite $F \subseteq A$. Therefore, $\bigcap H \subseteq \bigcup_F U_\alpha$, showing that $\bigcap H$ is compact.

If $p_Y: Y \to T$ is an object of $\mathbf{Top}/T$, let $\mathbf{O}(p_Y)$ denote the collection of pairs (U, t) where $t \in T$ and $U \in \mathbf{O}(Y_t)$, with the obvious projection $\mathbf{O}(p_Y) \to T$. In the following, we shall identify $\mathbf{O}(p_Y)_t$ with $\mathbf{O}(Y_t)$. If $H \subseteq \mathbf{O}(p_Y)$, we say that H is *saturated* or has *fup* if H_t does for all $t \in T$. We say that H is *binding* if $\{t \mid (U_t, t) \in H_t\}$ is open in T whenever U is open in Y. The collection of saturated, binding H with fup is denoted by $\mathscr{H}_{p_Y}$. Thus, every open subset of $\mathbf{O}(Y_t)$ is Scott-open.

Suppose Z is a space over T with projection p_Z. For $H \in \mathscr{H}_{p_Y}$ and W open in Z, define $\langle H, W \rangle \subseteq (p_Y, p_Z)$ by

$$\langle H, W \rangle = \{(\sigma, t) \mid (\sigma^{-1} W_t, t) \in H_t\}.$$

We give (p_Y, p_Z) the topology whose subbase is the collection of all such $\langle H, W \rangle$. It is routine to check that the projection $(p_Y, p_Z) \to T$ is continuous and $(\ ,\ )$ defines a functor $(\mathbf{Top}/T)^{\mathrm{op}} \times \mathbf{Top}/T \to \mathbf{Top}/T$. The latter fact uses the following principle. If $f: Y \to Y'$ is a morphism of spaces over T and $H \in \mathscr{H}_{p_Y}$, define $H' = \{(U, t) \mid U \in \mathbf{O}(Y'_t)$ and $(f^{-1}U, t) \in H\}$. Then $H' \in \mathscr{H}_{p_{Y'}}$.

Suppose $p_X: X \to T$ and $p_Y: Y \to T$ are spaces over T. If $x \in X_t$, let $(x, \text{-}): Y_t \to X_t \times Y_t$ be given by $y \mapsto (x, y)$. Composing with the obvious isomorphism $X_t \times Y_t \to (X \times_T Y)_t$ we obtain a continuous map $Y_t \to (X \times_T Y)_t$ which we shall also denote by $(x, \text{-})$. Define $\eta: X \to (p_Y, P_X \times p_Y)$ by $\eta(x) = ((x, \text{-}), t)$ where $p_X(x) = t$.

Lemma 2.2. *If $p_X: X \to T$ and $p_Y: Y \to T$ are objects of* $\mathbf{Top}/T$, *then $\eta: X \to (p_Y, p_X \times p_Y)$ defined above is continuous.*

Proof. Let $H \in \mathscr{H}_{p_Y}$ and let W be open in $X \times_T Y$. Suppose $\eta(x) \in \langle H, W \rangle$, where $x \in X_t$. Then $(\{y \mid (x,y) \in W\}, t) \in H$. For each such y, there exist open neighborhoods M_y and N_y of x and y in X and Y, respectively, with $M_y \times_T N_y \subseteq W$. Since $\{y \mid (x,y) \in W_t\} \subseteq \bigcup_y (N_y)_t$, by Scott-openness of H_t, the union of a finite number of the $(N_y)_t$ is in H_t (abuse of notation). Let N be the union of the corresponding N_y and M the intersection of the corresponding M_y. Suppose $G = \{s \in T \mid (N_s, s) \in H_s\}$. By the binding property G is open. Furthermore, $t \in G$. Consider $V = M \cap p_X^{-1} G$. Clearly, $x \in V$. If $x' \in V_s$, then $(\{y \mid (x',y) \in W_s\}, s) \in H_s$ since the first coordinate contains N_s. Therefore, $\eta V \subseteq \langle H, W \rangle$, and it follows that η is continuous.

Let $\mathbf{2}$ denote the Sierpinski space, i.e., the two-point space $\{0,1\}$ with $\{1\}$ open but not $\{0\}$. A continuous map $f: Y \to \mathbf{2}$ can be identified with an open subset of Y, namely $f^{-1}(1)$. More generally, $(p_Y, \pi_2 : \mathbf{2} \times T \to T)$ can be identified with $\mathbf{O}(p_Y)$. If H is a subset of $\mathbf{O}(p_Y)$, we let $\bigcap H$ be the subset of Y whose fiber over t is $\bigcap H_t$, where H_t is considered as a subset of $\mathbf{O}(Y_t)$ and it is understood that if $H_t = \emptyset$, then $\bigcap H_t$ is all of Y_t.

Theorem 2.3. *The following are equivalent for an object $p_Y : Y \to T$ of* $\mathbf{Top}/T$

(a) $- \times p_Y \dashv (p_Y, -)$.

(b) *p_Y is cartesian in* $\mathbf{Top}/T$.

(c) *$(\pi_2 : \mathbf{2} \times T \to T)^{p_Y}$ exists in* $\mathbf{Top}/T$.

(d) *Given $y \in U \in \mathbf{O}(Y_t)$, there exists $H \in \mathscr{H}_{p_Y}$ such that $(U,t) \in H$ and $\bigcap H$ is a neighborhood of y in Y.*

(e) *$p_Y^* : \mathbf{Top}/T \to \mathbf{Top}/Y$ has a right adjoint.*

(f) *$- \times_T Y : \mathbf{Top}/T \to \mathbf{Top}$ has a right adjoint.*

(g) *$f \times_T 1_X : X \times_T Y \to X' \times_T Y$ is a quotient map whenever $f : X \to X'$ is a quotient map over T.*

Proof. (a)$\Rightarrow$(b)$\Rightarrow$(c) is clear. (b)$\Leftrightarrow$(e)$\Leftrightarrow$(f) follows from Corollary 1.2. We shall show that (c)$\Rightarrow$(d)$\Rightarrow$(a) and (f)$\Leftrightarrow$(g).

(c)$\Rightarrow$(d). We have seen that if $(p_Z)^{p_Y}$ exists, its total space can be identified with the set (p_Y, p_Z) of pairs (σ, t), where $\sigma : Y_t \to Z_t$ is continuous, and moreover the bijection

$$\mathbf{Top}/T(p_X \times p_Y, p_Z) \overset{\theta}{\cong} \mathbf{Top}/T(p_X, (p_Z)^{p_Y})$$

is the obvious one. Consequently, taking $p_X = (p_Z)^{p_Y}$, we see that the map $\varepsilon : (p_Z)^{p_Y} \times p_Y \to p_Z$ given by $\varepsilon((\sigma,t),y) = \sigma y$ must be continuous since it corresponds under θ to $1_{(p_Z)^{p_Y}}$. In particular, taking $Z = \mathbf{2} \times T$, and using its identification with $(\pi_2 : \mathbf{2} \times T \to T)^{p_Y}$, $\mathbf{O}(p_Y)$ becomes a space over T. In the remainder of the proof of (c)$\Rightarrow$(d) we shall assume that $\mathbf{O}(p_Y)$ has the topology induced by this identification. The continuous map $\varepsilon : \mathbf{O}(p_Y) \times_T Y \to \mathbf{2} \times T$ is clearly given by

$$\varepsilon((U,t),y) = \begin{cases} (1,t) & \text{if } y \in U, \\ (0,t) & \text{otherwise.} \end{cases}$$

First, we show that if $y \in U \in \mathbf{O}(Y_t)$, then continuity of $\varepsilon : \mathbf{O}(p_Y) \times_T Y \to \mathbf{2} \times T$ implies the existence of an open subset H of $\mathbf{O}(p_Y)$ such that $(U, t) \in H$ and $\bigcap H$ is a neighborhood of y in Y. Then it suffices to show that every open subset of $\mathbf{O}(p_Y)$ is in $\mathscr{H}_{p_Y}$.

Suppose $y \in U \in \mathbf{O}(Y_t)$. Then $\{1\} \times T$ is an open neighborhood of $\varepsilon((U,t),y)$ in $\mathbf{2} \times T$. By continuity of ε, there exists H open in $\mathbf{O}(p_Y)$ with $(U,t) \in H$, and V open in Y with $y \in V$ such that $\varepsilon(H \times_T V) \subseteq \{1\} \times T$. It follows that $V \subseteq \bigcap H$ since $\varepsilon((V',t'),y') = (1,t')$ for every $((V',t'),y') \in H \times_T V$, and hence $\bigcap H$ is a neighborhood of y in Y.

Note that since θ takes continuous maps to continuous maps, if W is open in $X \times_T Y$, then the map $X \to \mathbf{O}(p_Y)$ given by $x \mapsto (W[x], t)$ is continuous, where $p_X(x) = t$ and $W[x] = \{y \mid (x,y) \in W\}$.

Let H be any open subset of $\mathbf{O}(p_Y)$. Suppose that $t \in T$, $\{U_\alpha\}_{\alpha \in A} \subseteq \mathbf{O}(Y_T)$ and $(\bigcup_A U_\alpha, t) \in H$. Let $\hat{A}$ be the collection of finite subsets of A together with A itself, and let $V \subseteq \hat{A}$ be open if $A \notin V$ or if $A \in V$ and $V \supseteq \{G \mid G \supseteq F\}$ for some finite $F \in \hat{A}$. Let $\hat{A} \to T$ be the constant t-valued map, and let $W \subseteq \hat{A} \times_T Y \cong \hat{A} \times Y_t$ be defined as follows

$$W = \left\{(G, y) \mid y \in \bigcup_G U_\alpha\right\}.$$

Then W is open in $\hat{A} \times_T Y$, since $y \in \bigcup_G U_\alpha$ implies that $y \in U_\alpha$ for some $\alpha \in A$, and so $G \mapsto (W[G], t)$ is a continuous map $\hat{A} \to \mathbf{O}(p_Y)$ and $(W[A], t) \in H$. Hence, there exists a finite $F \subseteq A$ such that $(W[F], t) \in H$. This shows that H has fup.

For the binding property, let V be open in $Y \cong T \times_T Y$. Then $t \mapsto (V_t, t)$ defines a continuous map $T \to \mathbf{O}(p_Y)$, and so clearly H is binding.

Finally, to see that H is saturated, suppose $U \subseteq V \in \mathbf{O}(Y_t)$ and $(U,t) \in H$. Consider $\mathbf{2}$ as a space over T via the constant t-valued map. Define $W \subseteq \mathbf{2} \times_T Y \cong \mathbf{2} \times Y_t$ by $W = \{0\} \times U \cup \{1\} \times V$. Then W is open and so since $(W[0], t) = (U,t) \in H_t$ it follows that $(W[1], t) = (V, t) \in H_t$, as desired.

(d)$\Rightarrow$(a). Consider $\eta : p_X \to (p_Y, p_X \times p_Y)$ and $\varepsilon : (p_Y, p_Z) \times p_Y \to p_Z$ defined as before. By Lemma 2.2, η is continuous. To see that ε is continuous, suppose W is open in Z and $\varepsilon((\sigma,t),y) = \sigma y \in W$, where $((\sigma,t),y) \in (p_Y, p_Z) \times_T Y$. Then $y \in \sigma^{-1}(W_t)$ which is open in Y_t, so by (d) there exists $H \in \mathscr{H}_Y$ such that $(\sigma^{-1}(W_t), t) \in H$ and $\bigcap H$ is a neighborhood of y in Y. Then

$$((\sigma,t),y) \in \langle H, W \rangle \times_T \bigcap H \subseteq \varepsilon^{-1}(W)$$

Therefore, ε is continuous. One then easily checks that the adjoint functor identities hold relative to ε and η.

(f)$\Rightarrow$(g). If $- \times_T Y : \mathbf{Top}/T \to \mathbf{Top}$ has a right then it preserves coequalizers. But every quotient map over T is a coequalizer. Therefore, $- \times_T Y$ takes quotient maps over T to quotient maps in **Top**.

(g)$\Rightarrow$(f). The functor $- \times_T Y$ preserves coproducts (i.e. disjoint unions) in any case, and it also preserves coequalizers at the set level. Therefore, if it takes quotient maps over T to quotient maps, then it preserves coequalizers in **Top**$/T$, hence all colimits. The result follows from the dual of the special adjoint functor theorem [5; p. 89].

The implication (d)$\Rightarrow$(a) owes much to an unpublished proof of Mitchell of the following corollary.

A space Y is *locally compact* if every element of Y has a basic family of compact neighborhoods. Note that this is not the usual definition unless Y is Hausdorff.

If T is a one point space and Y is locally compact, then it is not difficult to show that the topology on (Y, Z) is the compact-open topology. To see this it suffices to show that $(Y, \mathbf{2})$ has the compact-open topology, since for any space both topologies on $\mathbf{Top}(Y, Z)$ can be described as the coarsest topology such that all maps $\mathbf{Top}(Y, f): \mathbf{Top}(Y, Z) \to \mathbf{Top}(Y, \mathbf{2})$ are continuous, where $f: Z \to \mathbf{2}$ is continuous. But this is straightforward, and hence left to the reader.

Corollary 2.4. *If Y is locally compact and T is Hausdorff, then any continuous map $p: Y \to T$ is cartesian in* **Top**$/T$.

Proof. If C is a compact subset of Y, define $H \subseteq \mathbf{O}(p_Y)$ by $H = \{(U, t) \mid C_t \subseteq U\}$. Now, C_t is compact since T is T_1, and so H_t is Scott-open for all $t \in T$, i.e. H is saturated and has fup. To see that H is binding suppose U is an open in Y. Since $p|C$ is a continuous map from a compact space to a Hausdorff space, it is closed. Therefore, $p(C \setminus U)$ is a closed subset of T. But

$$T \setminus p(C \setminus U) = \{t \mid C_t \subseteq U_t\} = \{t \mid (U_t, t) \in H\}.$$

Hence H is binding. The result follows easily from Theorem 2.3(d) since every point of Y has a basic family of such C as neighborhoods.

Using Lemma 2.1 and condition (d) of the theorem, we obtain the following corollary, which was first proved in [2].

Corollary 2.5. *If Y is locally compact, then Y is cartesian in* **Top**. *On the other hand, if Y is Hausdorff and cartesian in* **Top**, *then Y is locally compact.*

It follows that to exhibit noncartesian objects of **Top**, it suffices to exhibit Hausdorff spaces which are not locally compact, for example, the rational numbers, or the open disc with one boundary point.

A subset Y of T is *locally closed* if $Y = U \cap F$, where U is open in T and F is closed in T.

Lemma 2.6. *A subset Y of T is locally closed if for every $y \in Y$ there is an open neighborhood U of y in T such that $U \cap Y = U \cap F$ for some closed F in T.*

Proof. If F is closed and U is open, then the condition $U \cap Y = U \cap F$ easily implies that $U \cap Y = U \cap \bar{Y}$. Then choosing such a U, say U_y, for each $y \in Y$, we have

$$Y = \bigcup_{y \in Y} (U_y \cap Y) = \bigcup_{y \in Y} (U_y \cap \bar{Y}) = \left(\bigcup_{y \in Y} U_y\right) \cap \bar{Y}.$$

Corollary 2.7. *If Y is a subspace of T, then the inclusion $i: Y \to T$ is cartesian in* **Top**/T *if and only if Y is a locally closed subset of T.*

Proof. For the 'if' direction it suffices to show that if Y is either open or closed in T, then $i: Y \to T$ is cartesian in **Top**/T, since the product of two cartesian objects of any category is cartesian. We shall use Theorem 2.7(d) $\Rightarrow$ (b).

Suppose $y \in U \in \mathbf{O}(Y_t)$. Let $H = \{(\{t\}, t) \mid t \in Y\}$ if Y is open in T, and let $H = \{(\{t\}, t) \mid t \in Y\} \cup \{(\emptyset, t) \mid t \notin Y\}$ if Y is closed in T. Note that when Y is closed this addition is necessary to make H binding. Then $H \in \mathscr{H}_i$, $(U, t) \in H$ and $\bigcap H = Y$.

For the converse, suppose $i: Y \to T$ is cartesian in **Top**/T. By Theorem 2.3(b) $\Rightarrow$ (d), if $y \in Y$, there exists $H \in \mathscr{H}_i$ such that $(\{y\}, y) \in H$ and H is a neighborhood of y in Y. Let G be an open subset of T with $y \in G \cap Y \subseteq \bigcap H$ and consider

$$U = G \cap \{t \mid H_t \neq \emptyset\} \quad \text{and} \quad F = \{t \mid (\emptyset, t) \notin H\}.$$

Then U is open since $\{t \mid H_t \neq \emptyset\} = \{t \mid (Y_t, t) \in H\}$ which is open by the saturated and binding properties of H, and F is closed since its complement is open by the binding property of H. Then $y \in U \cap F = U \cap Y$, and so by the above lemma, Y is locally closed in T.

Lemma 2.8. *Let $\mathscr{U}$ be an open cover of Y. A morphism $p: Y \to T$ is cartesian in* **Top**/T *if and only if $U \xrightarrow{i} Y \xrightarrow{p} T$ is cartesian in* **Top**/T *for every $U \in \mathscr{U}$.*

Proof. If $Y \xrightarrow{p} T$ is cartesian in **Top**/T and $U \in \mathscr{U}$, then $U \xrightarrow{i} Y \xrightarrow{p} T$ is cartesian in **Top**/T by Corollary 1.3, since $U \xrightarrow{i} Y$ is cartesian **Top**/Y by Corollary 2.7.

Suppose $U \xrightarrow{i} Y \xrightarrow{p} T$ is cartesian in **Top**/T for every $U \in \mathscr{U}$, and let V be an open neighborhood of y in Y_t. Then $y \in U$, for some $U \in \mathscr{U}$, and so by Theorem 2.3(b) $\Rightarrow$ (d) there exists $H \in \mathscr{H}_{p \circ i}$ such $(U \cap V, t) \in H$ and $\bigcap H$ is a neighborhood of y in U. Now, $H \in \mathscr{H}_{p \circ i}$ gives rise to $H' \in \mathscr{H}_p$ using the inclusion $U \xrightarrow{i} Y$, i.e. $H' = \{(U', t) \mid (i^{-1}U', t) \in H\}$. Hence, the result easily follows by (d) $\Rightarrow$ (b) of the theorem.

Corollary 2.9. *Every local homeomorphism $Y \to T$ is cartesian in* **Top**/T.

By Corollary 1.4, if $f: S \to T$ is a continuous map, then f^*: **Top**/$T \to$ **Top**/S preserves cartesian objects. In particular, taking $S = 1$ and $f = t$ the constant t-valued map, it follows that if $p_Y: Y \to T$ is cartesian in **Top**/T, then Y_t is cartesian in **Top**.

Similarly, if F is cartesian in **Top**, then $\pi_2 : F \times T \to T$ is cartesian in **Top**$/T$.

Let $p : Y \to T$ be a continuous map. Then p is *locally trivial with fiber* F if there is an open cover $\mathscr{U}$ of T, and homeomorphisms $p^{-1}U \cong F \times U$ such that

$$\begin{tikzcd} p^{-1}U \arrow[rr, "\simeq"] \arrow[dr] & & F \times U \arrow[dl] \\ & U & \end{tikzcd}$$

commutes, for all $U \in \mathscr{U}$.

Corollary 2.10. *Let $p : Y \to T$ be a locally trivial space with fiber F. Then $p : Y \to T$ is cartesian in* **Top**$/T$ *if and only if F is cartesian in* **Top**.

Proof. If $p : Y \to T$ is cartesian over T, then by the above remark $F = Y_t$ is cartesian. Conversely, if $\mathscr{U}$ is an open cover of T such that $p^{-1}U \cong F \times U$ for every $U \in \mathscr{U}$, then by Lemma 2.8 and the above remark $p : Y \to T$ is cartesian in **Top**$/T$.

3. Cartesian objects in Un/T

We begin with a quick review of uniform spaces. For all unfamiliar terminology and unproved statements we refer the reader to [9, Chapter 6].

Let X be a set. For $U, V \subseteq X \times X$, define $U^{-1} = \{(x,y) \mid (y,x) \in U\}$ and $U \circ V = \{(x,y) \mid (x,z) \in V \text{ and } (z,y) \in U \text{ for some } z \in X\}$. Let $\Delta = \{(x,x) \mid x \in X\}$. A *uniformity* on X is a collection $\mathscr{U}$ of subsets of $X \times X$ satisfying

(i) $\Delta \subseteq U$, for all $U \in \mathscr{U}$.
(ii) $U \in \mathscr{U} \Rightarrow U^{-1} \in \mathscr{U}$.
(iii) $U \in \mathscr{U} \Rightarrow V \circ V \subseteq U$, for some $V \in \mathscr{U}$ (*triangle inequality*).
(iv) $U, V \in \mathscr{U} \Rightarrow U \cap V \in \mathscr{U}$.
(v) $U \subseteq V \subseteq X \times X$, $U \in \mathscr{U} \Rightarrow V \in \mathscr{U}$.

The pair $(X, \mathscr{U})$ is a *uniform space*. If $U \in \mathscr{U}$, then we say that U is *uniform* for X.

Let X and Y be uniform spaces. A map $f : X \to Y$ is *uniformly continuous* if $(f \times f)^{-1}(V)$ is uniform for X whenever V is uniform for Y. Let **Un** denote the category of uniform spaces and uniformly continuous maps.

If $\{f_\alpha : X \to X_\alpha\}_{\alpha \in A}$ is a family of maps where each X_α is a uniform space and X is a set, then $\{(f_\alpha \times f_\alpha)^{-1}(U) \mid U \text{ is uniform for } X_\alpha, \alpha \in A\}$ is a subbase for a uniformity on X, called the uniformity *induced* by the f_α's. In particular, if Y is a uniform space and $X \subseteq Y$, then the uniformity induced by the inclusion is called the *relative uniformity* on X, and X is a *sub-uniform space* of Y. Limits in **Un** are formed in **Sets** and given the uniformity induced by the projections.

If $\{f_\alpha : X_\alpha \to X\}_{\alpha \in A}$ is a family of maps where each X_α is a uniform space and X is a set, one can also define the uniformity *coinduced* by f_α's as the uniformity induced by the family of all maps $g : X \to Z$ where Z is a uniform space and $f \circ g$ is uniformly continuous for all $\alpha \in A$. Note that this seemingly awkward definition is necessary to obtain the triangle inequality. If $f: X \to X'$ is a uniformly continuous surjection and X' has the uniformity coinduced by f, we say that f is a *quotient map*. Colimits in **Un** are formed in **Sets** and given the uniformity coinduced by the injections.

Let T be a uniform space, and suppose $p_Y: Y \to T$ is an object of **Un**/T. Then Y is the *total space*, T is the *base space*, and p_Y is the *projection*. By a *uniform space over T*, we shall mean a uniform space Y together with a particular projection $p_Y: Y \to T$. We shall say that a morphism $f: p_X \to p_{X'}$ of **Un**/T is a *quotient map*, if $f: X \to X'$ is a quotient map in **Un**.

Lemma 3.1. *If $p: Y \to T$ is a uniformly continuous map, then $- \times_T Y$*: **Un**/$T \to$ **Un** *preserves quotient maps.*

Proof. The case where T is a one point space appears as an exercise in [8, p. 53, 8(c)]. For a proof one can also see [7, p. 96]. Using this, as well as the fact that $X' \times_T Y$ has the relative uniformity as a subset of $X' \times Y$, it is easy to see that commutativity of

$$\begin{tikzcd} X \times_T Y \arrow[r, "f \times_T 1_Y"] \arrow[d, tail] & X' \times_T Y \arrow[d, tail] \\ X \times Y \arrow[r, "f \times 1_Y"] & X' \times Y \end{tikzcd}$$

implies that $f \times_T 1_Y$ is a quotient map whenever f is.

Suppose $t \in T$. Then the one point uniform space 1 becomes a uniform space over T via the constant t-valued map $t: 1 \to T$. If $p_Y: Y \to T$ is a uniform space over T, then the *fiber of Y over t* is the set Y_t with the relative uniformity. As in **Top**, the underlying set of Y_t can be identified with **Un**/$T(t, p_Y)$, and as a uniform space Y_t is isomorphic to the total space of $p_Y \times t$.

Let $p_Y: Y \to T$ and $p_Z: Z \to T$ be objects of **Un**/T. If $(p_Z)^{p_Y}$ exists in **Un**/T, then arguing as in **Top**, its total space can be identified with the set of pairs (σ, t) where $\sigma: Y_t \to Z_t$ is uniformly continuous, and moreover the bijection

$$\mathbf{Un}/T(p_X \times p_Y, p_Z) \cong \mathbf{Un}/T(p_X, (p_Z)^{p_Y})$$

is the obvious one.

Let $d: Y \times Y \to \mathbb{R}$ be a pseudometric on the set Y. Given $r \in \mathbb{R}^+$, let $U_r = \{(y, y') \mid d(y, y') < r\}$. Then the collection of $U \subseteq Y \times Y$ such that $U_r \subseteq U$ for some positive real number r is a uniformity on Y. When $Y = \mathbb{R}$ and d is the usual metric, this uniformity is called the *usual uniformity*.

Lemma 3.2 (Metrization lemma, Kelley [9, p. 185]). *If Y is a uniform space and U is* *uniform for Y, then (relative to the product uniformity on $Y\times Y$ and the usual uniformity on $\mathbb{R}$) there is a uniformly continuous pseudometric $d\colon Y\times Y\to\mathbb{R}$ such that*

$$\{(y,y')\mid d(y,y')<1\}\subseteq U.$$

Remark. The map $\min:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is uniformly continuous. Therefore, we may assume that $d\colon Y\times Y\to I$ in the lemma since $\min(1,d(y,y'))\colon Y\times Y\to I$ is a pseudometric, where I is the unit interval with the relative uniformity. Actually, we shall not be using the fact that d satisfies the triangle inequality, only that d is uniformly continuous and satisfies $d(y,y)=0$ for all $y\in Y$.

If $p_Y\colon Y\to T$ is an object of $\mathbf{Un}/T$, $t,t'\in T$ and $V\subseteq Y\times Y$, let

$$V_{tt'}=V\cap Y_t\times Y_{t'}.$$

Theorem 3.3. *The following are equivalent for $p_Y\colon Y\to T$ in* $\mathbf{Un}/T$

(a) *p_Y is cartesian in* $\mathbf{Un}/T$.

(b) *$(\pi_2:I\times T\to T)^{p_Y}$ exists in* $\mathbf{Un}/T$.

(c) *There exists U_0 uniform for Y satisfying*

(i) *$G_V=\{(t,t')\mid V_{0tt'}\subseteq V_{tt'}\}$ is uniform for T for all V uniform for Y.*

(ii) *there exists G_0 uniform for T such that the projection $V_{0tt'}\to Y_t$ is a surjection whenever $(t,t')\in G_0$.*

(d) *$p_Y^*:\mathbf{Un}/T\to\mathbf{Un}/T$ has a right adjoint.*

(e) *$-\times_T Y\colon\mathbf{Un}/T\to\mathbf{Un}$ has a right adjoint.*

(f) *$-\times_T Y$ preserves coproducts.*

Remarks. The reader should not the analogy between Theorem 2.3 and Theorem 3.3, i.e., 2.3(b)–(g) correspond to 3.3(a)–(f). In particular, the unit interval I is the 'Sierpinski' uniform space. Of course, 2.3(g) and 3.3(f) are not quite the same, i.e. $-\times_T Y\colon\mathbf{Top}/T\to\mathbf{Top}$ preserves coproducts but not quotients in general, while $-\times_T Y\colon\mathbf{Un}/T\to\mathbf{Un}$ preserves quotients but not coproducts in general. Finally, the absence of an analogue to 2.3(a) is due to the fact that we were unable to define the necessary uniformity on the appropriate 'function space' without some assumption on the exponent.

Proof of Theorem 3.3. (a)$\Rightarrow$(b) is clear. (a)$\Leftrightarrow$(d)$\Leftrightarrow$(e) follows from Corollary 1.2. We shall show that (b)$\Rightarrow$(c)$\Rightarrow$(a) and (e)$\Leftrightarrow$(f).

(b)$\Rightarrow$(c). If $(\pi_2:I\times T\to T)^{p_Y}$ exists in $\mathbf{Un}/T$, then as before, its total space, which we shall denote by $(p_Y,I\times T)$, can be identified with the set of pairs (σ,t) with $\sigma: Y_t\to(I\times T)_t\cong I$ uniformly continuous. Moreover, the bijection

$$\mathbf{Un}/T(p_X\times p_Y,I\times T\to T)\overset{\theta}{\cong}\mathbf{Un}/T(p_X,(I\times T\to T)^{p_Y})$$

is the obvious one. Then map $(p_Y,I\times T)\times_T Y\xrightarrow{\varepsilon} I\times T\xrightarrow{\pi_1} I$ given by $((\sigma,t),y)\mapsto\sigma y$ is uniformly continuous since it corresponds to $1_{(p_Y,I\times T)}$ under θ.

Since $\{(x,x')\in I\times I \mid |x-x'|<1\}$ is uniform for I, by uniform continuity of $\pi_1\varepsilon$ there exists F uniform for $(p_Y, I\times T)$ and V_0 uniform for Y such that

$$|\sigma y-\sigma' y'|<1 \quad \text{whenever } ((\sigma,t),(\sigma',t'))\in F,\ (y,y')\in V_{0tt'}. \tag{1}$$

To show that (i) holds, let V be uniform for Y. By the metrization lemma, there exists a uniformly continuous pseudometric $d: Y\times Y\to I$ with

$$V\supseteq\{(y,y')\mid d(y,y')<1\}.$$

The map $(Y\times T)\times_T Y\to Y\times Y$ given by $((y,t),y')\mapsto(y,y')$ is an isomorphism of uniform spaces since the outer rectangle and the right square of the diagram

$$\begin{tikzcd}
Y\times Y \arrow[r, "1_Y\times p_Y"] \arrow[d, "\pi_2"'] & Y\times T \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & Y \arrow[d] \\
Y \arrow[r, "p_Y"] & T \arrow[r] & 1
\end{tikzcd}$$

are pullbacks, and hence the left square is a pullback. Thus, d induces a uniformly continuous map $\bar{d}: Y\times T\to(p_Y, I\times T)$ defined by $\bar{d}(y,t)=(d(y,\text{-}),t)$, where $d(y,\text{-}): Y_t\to I$. Hence, there exists V' uniform for Y and G uniform for T such that $(\bar{d}(y,t),\bar{d}(y',t'))\in F$ whenever $(y,y')\in V'$ and $(t,t')\in G$. In particular, we have

$$(\bar{d}(y,t),\bar{d}(y,t'))\in F \quad \text{whenever } y\in Y,\ (t,t')\in G \tag{2}$$

since V' contains the diagonal. We claim that $G\subseteq G_V=\{(t,t')\mid V_{0tt'}\subseteq V_{tt'}\}$. If $(t,t')\in G$ and $(y,y')\in V_{0tt'}$, then using (1) and (2) we have

$$d(y,y')=|d(y,y)-d(y,y')|=|\varepsilon(\bar{d}(y,t),y)-\varepsilon(\bar{d}(y,t'),y')|<1.$$

It follows that $(y,y')\in V_{tt'}$, proving (i).

To prove (ii), let F' be symmetric and uniform for $(p_Y, I\times T)$ with $F'\circ F'\subseteq F$. Taking $X=T$ and $p_X=1_T$, we see that the map $T\to(p_Y, I\times T)$ defined by $t\mapsto(1,t)$ (where 1 denotes the constant 1-valued map $Y_t\to I$) is uniformly continuous since it corresponds to $T\times_T Y\cong Y\xrightarrow{1\times p_Y} I\times T$ under θ. Therefore,

$$G'=\{(t,t')\mid((1,t),(1,t'))\in F'\}$$

is uniform for T.

Let $d: Y\times Y\to I$ be a uniformly continuous pseudometric with

$$V_0\supseteq\{(y,y')\mid d(y,y')<1\}.$$

Replacing F with F' in the proof of (i) we get G uniform for T such that $(\bar{d}(y,t),\bar{d}(y,t'))\in F'$ whenever $y\in Y$, $(t,t')\in G$.

We claim that if $(t,t')\in G_0=G\cap G'$, then the projection $V_{0tt'}\to Y_t$ is surjective, proving (ii). Suppose that $(t,t')\in G_0$, $y\in Y_t$ and $(y,y')\notin V_0$ for all $y'\in Y_{t'}$. Then $d(y,\text{-}): Y_{t'}\to I$ is constant 1-valued so $\bar{d}(y,t')=(1,t')$. Therefore, $(\bar{d}(y,t),(1,t'))\in F'$

since $(t,t')\in G$ and $((1,t'),(1,t))\in F'$ since (t,t') G' and F' is symmetric. Hence, $(\bar{d}(y,t),(1,t))\in F'\circ F'\subseteq F$, and it follows that

$$1=|d(y,y)-1|=|\varepsilon(\bar{d}(y,t),y)-\varepsilon((1,t),y)|<1.$$

This contradiction shows that $(y,y')\in V_0$ for some $y'\in Y_{t'}$.

(c)$\Rightarrow$(a). Fix V_0 and G_0 satisfying (c). Let V be symmetric and uniform for Y with $V\circ V\subseteq V_0$. Then $V_{0tt'}=V_{tt'}$ whenever $(t,t')\in G_V$. Thus, replacing G_0 by $G_0\cap G_V$ and V_0 by V in (c), we see that (i) and (ii) still hold, but now we have V_0 symmetric and furthermore

$$V_{0t't''}\circ V_{0tt'}\subseteq V_{0tt''}\quad\text{whenever } (t,t'')\in G_0. \tag{3}$$

Now, suppose that $(t,t')\in G_0$. Let $(y,y'')\in V_{0tt''}$ and use (ii) to obtain $(y,y')\in V_{0tt'}$. If, in addition, $(t',t'')\in G_0$, then by symmetry of V_0 and (3) we see that $(y',y'')\in V_0$. Thus we have shown that

$$V_{0t't''}\circ V_{0tt'}\supseteq V_{0tt''}\quad\text{whenever } (t,t'),(t',t'')\in G_0. \tag{4}$$

Suppose $p_Z:Z\to T$ is an object of $\mathbf{Un}/T$. Let (p_Y,p_Z) denote the set of pairs (σ,t) with $\sigma: Y_t\to Z_t$ uniformly continuous. Given G uniform for T and W uniform for Z, we define $\langle G,W\rangle$ to be the set of pairs $((\sigma,t),(\sigma',t'))\in(p_Y,p_Z)\times(p_Y,p_Z)$ such that $(t,t')\in G$ and $(\sigma y,\sigma y')\in W$ for all $V_{0tt'}$. We claim that the collection of such $\langle G,W\rangle$ gives a base for a uniformity on (p_Y,p_Z). It is easy to see that $((\sigma,t),(\sigma,t))\in\langle G,W\rangle$ for all uniformly continuous $\sigma: Y_t\to Z_t$ since V_{0tt} is the least member of the relative uniformity on Y_t by (i). Also, $\langle G,W\rangle^{-1}=\langle G^{-1},W^{-1}\rangle$ since V_0 is symmetric, and clearly $\langle G,W\rangle\cap\langle G',W'\rangle=\langle G\cap G',W\cap W'\rangle$. It remains to check the triangle inequality. Given $\langle G,W\rangle$, let G' be uniform for T with $G'\circ G'\subseteq G\cap G_0$ and W' be uniform for Z with $W'\circ W'\subseteq W$. Suppose $((\sigma,t),(\sigma',t'))\in\langle G',W'\rangle$ and $((\sigma',t'),(\sigma'',t''))\in\langle G',W'\rangle$. Then $(t,t'')\in G$. If $(y,y'')\in V_{0tt''}$, then since $(t,t'),(t',t'')\in G_0$, by (4) we get $y'\in Y_{t'}$ such that $(y,y')\in V_{0tt'}$ and $(y',y'')\in V_{0t't''}$. It follows that $(\sigma y,\sigma''y'')\in W'\circ W'\subseteq W$, and so we have shown that $\langle G',W'\rangle\circ\langle G',W'\rangle\subseteq\langle G,W\rangle$.

Let $(p_Z)^{p_Y}:(p_Y,p_Z)\to T$ denote the projection $(\sigma,t)\mapsto t$. Then $(p_Z)^{p_Y}$ is uniformly continuous since the inverse image of G is $\langle G,Z\times Z\rangle$. If $f:p_Z\to p_{Z'}$ is a morphism of $\mathbf{Un}/T$, and $(f)^{p_Y}:(p_Z)^{p_Y}\to(p_{Z'})^{p_Y}$ is the obvious map, then

$$((f)^{p_Y}\times(f)^{p_Y})^{-1}\langle G,W\rangle=\langle G,(f\times f)^{-1}(W)\rangle.$$

Thus, we get a functor $(\ )^{p_Y}:\mathbf{Un}/T\to\mathbf{Un}/T$.

Consider $\varepsilon:(p_Y,p_Z)\times_T Y\to Z$ given by $\varepsilon((\sigma,t),y)=\sigma y$. If W is uniform for Z, then we have

$$\langle T\times T,W\rangle\times_T V_0\subseteq(\varepsilon\times\varepsilon)^{-1}(W)$$

and hence ε is uniformly continuous. Now, consider $\eta: X\to(p_Y,p_X\times p_Y)$ given by $\eta(x)=((x,\text{-}),t)$ for $x\in X_t$, where $(x,\text{-}): Y_t\to X_t\times Y_t\cong(X\times_T Y)_t$ is the map $y\mapsto(x,y)$. If $\langle G,U\times_T V\rangle$ is a basic uniform set for $(p_Y,p_X\times p_Y)$, then

$$U\cap(p_X\times p_X)^{-1}(G\cap G_V)\subseteq(\eta\times\eta)^{-1}(\langle G,U\times_T V\rangle).$$

This shows that η is uniformly continuous. Finally, one checks that the adjunction identities for adjoint functors hold relative to ε and η.

(e)$\Rightarrow$(f). If $-\times_T Y$: $\mathbf{Un}/T \to \mathbf{Un}$ has a right adjoint, then it preserves coproducts, i.e., sums over T.

(f)$\Rightarrow$(e). By Lemma 3.1, $-\times_T Y$ preserves quotient maps, and hence coequalizers. Therefore, since it preserves coproducts by assumption, it preserves all colimits. The result follows from the dual of the special adjoint functor theorem [5; p. 89].

Corollary 3.4. *A uniform space is cartesian in* **Un** *if and only if its uniformity has a least member. In particular, a T_0 uniform space is cartesian if and only if it is discrete.*

If T is a uniform space we will always consider T as a topological space via the *uniform topology*, that is, U is open in T if for every $t \in U$ there exists G uniform for T such that $G[t] = \{s \mid (t,s) \in G\} \subseteq U$.

Let Y and T be topological spaces. A continuous map $p: Y \to T$ is an *overlaying*, or Y is an *overlay* of T, if there exists an open cover $\mathscr{U}$ of T and an open cover $\{A_i^U \mid i \in I_U\}$ of $p^{-1}U$ for each $U \in \mathscr{U}$, such that p maps each A_i^U homeomorphically onto U, and such that $A_i^U \cap A_j^V$ is either empty or maps onto $U \cap V$. It follows, in particular, that for fixed U the sets A_i^U are disjoint, so p is a covering map. On the other hand, examples of covering maps which are not overlayings are given in [4].

If T is a uniform space, an open cover $\mathscr{U}$ of T is a *uniform cover* if $\mathscr{U}$ has a refinement of the form $\{G[t]\}_{t \in T}$ for some G uniform for T. In this case we say that G is *subordinate* to $\mathscr{U}$.

Let Y be a topological space and T a uniform space. Then an overlaying $p: Y \to T$ is *uniform* if the open cover $\mathscr{U}$ can be taken to be uniform.

We recall that every uniform space has a base consisting of open (in the product topology on $T \times T$) symmetric neighborhoods of the diagonal [9, p. 179].

In the following, by a uniformity on a space T we shall mean a uniformity on T such that the uniform topology agrees with the given topology.

Theorem 3.5. *Let T be a uniform space, Y a topological space, and let $p: Y \to T$ be a continuous map. Then Y is a uniform overlay of T if and only if Y has T_0 fibers and there exists a uniformity on Y such that p is cartesian in* **Un**$/T$.

Proof. Suppose $p: Y \to T$ is a uniform overlaying. Let $\mathscr{U}$ and $\{A_i^U\}$ be as in the definition. Let $\mathscr{G}$ be the collection of all symmetric open uniform sets for T which are subordinate to $\mathscr{U}$. If $G \in \mathscr{G}$ and $y \in Y_t$, let $G[t] \subseteq U \in \mathscr{U}$. Let i be the index such that $y \in A_i^U$. Define

$$V_G[y] = A_i^U \cap p^{-1}(G[t]). \tag{5}$$

From the definition of overlay it follows immediately that this is independent of the choice of U. Then one easily checks the following conditions.

(i) $(y, y) \in V_G$ for all $y \in Y$.
(ii) $(V_G)^{-1} = V_G$ (using symmetry of G).
(iii) $V_G \cap V_H = V_{G \cap H}$.
(iv) If $H \circ H \subseteq G$ and $H, G \in \mathscr{G}$, then $V_H \circ V_H \subseteq V_G$.

In other words, $\{V_G \mid G \in \mathscr{G}\}$ is a base for a uniformity on Y. If G is uniform for T and $H \subseteq G$ with $H \in \mathscr{G}$, then

$$(p \times p)^{-1}(G) \supseteq (p \times p)^{-1}(H) \supseteq V_H,$$

so p is uniformly continuous. From (5) we see that a basic neighborhood of y in the uniform topology is a neighborhood in the given topology. On the other hand, since T has the uniform topology and p is a local homeomorphism it follows that any neighborhood of y in the given topology contains a subset of the form $V_G[y]$. Thus the uniform topology agrees with the given topology. Finally, we check that (i) and (ii) hold. Fix $G_0 \in \mathscr{G}$ and let $V_0 = V_{G_0}$. Then (ii) is clear and (i) follows from the observation that $G \cap G_0 \subseteq \{(t, t') \mid V_{0tt'} \subseteq V_{Gtt'}\}$ for any $G \in \mathscr{G}$.

For the converse, suppose that there exists a uniformity on Y such that p is cartesian in $\mathbf{Un}/T$. Let V_0 and G_0 be as in Theorem 3.3(c). As in the proof of (c) $\Rightarrow$ (a) of Theorem 3.3 we may assume that V_0 and G_0 are open and symmetric, and moreover

$$V_{0t't''} \circ V_{0tt'} \subseteq V_{0tt''} \tag{6}$$

whenever $(t, t'') \in G_0$. Let G be open and symmetric with $G \circ G \subseteq G_0$ and let $V = V_0 \cap (p \times p)^{-1}(G)$. Moreover, every V uniform for Y contains a V_0 with these properties.

Fix such a V and G. Then $G[t]$ and $V[y]$ are open in T and Y, respectively, since G and V are open in $T \times T$ and $Y \times Y$, respectively. We claim that

$$p^{-1}(G[t]) = \bigcup_{y \in Y_t} V[y].$$

Obviously, the right side is contained in the left side. Take y' in the left side, say $py' = t'$, where $(t, t') \in G$. Then there exists $y \in Y_t$ such that $(y, y') \in V$ since the projection $V_{tt'} \to Y_{t'}$ is surjective. Interchanging the roles of y and y' in the argument shows that each $V[y]$ maps onto $G[t]$. Then using (6) and the fact that $Y_{t'}$ is discrete we see that $p: V[y] \to G[t]$ is injective for all $y \in Y_t$. To see that each $p \mid V[y]$ is a homeomorphism it suffices to show that p is open. For this it suffices to show that for any point $y \in Y$, p maps arbitrarily small neighborhoods of y onto neighborhoods of py. But this is clear since there are arbitrarily small neighborhoods of the form $V[y]$ where V is as above. It remains to show that $V[y] \cap V[y']$ is either empty or mapped onto $G[t] \cap G[t']$. Suppose that $w \in V[y] \cap V[y']$ where $pw = s$. If $u \in G[t] \cap G[t']$, then by (ii) there exist $z, z' \in Y_u$ such that $(y, z), (y', z') \in V \subseteq V_0$. Now, since (w, y) and (w, y') are in $V \subseteq V_0$ and $(s, u) \in G \circ G \subseteq G_0$, using (6) it follows that (w, z) and (w, z') are in $V_{0_{su}}$. Using (6) again, we see that $(z, z') \in V_{0_{uu}}$. Then by (i) and discreteness of Y_u, $z = z'$. Hence, $V[y] \cap V[y']$ is mapped onto $G[t] \cap G[t']$.

If T is a compact Hausdorff space, then T admits a unique uniformity such that the uniform topology agrees with the given topology and moreover, every open cover is a uniform cover [9, p. 197–200]. Hence, we have the following corollary.

Corollary 3.6. *Suppose T is a compact Hausdorff space and $p: Y \to T$ is a continuous map with T_0 fibers. Then p is an overlaying if and only if p is cartesian in* **Un**$/T$ *for some uniformity on Y.*

If T is a paracompact space, then every open cover is even, and the collection of neighborhoods of the diagonal of $T \times T$ form a subbase for a uniformity on T [9, p. 156–157]. Thus, we have the following corollary.

Corollary 3.7. *If $p: Y \to T$ is an overlaying with T paracompact, then uniformities can be chosen for Y and T such that p is cartesian in* **Un**$/T$.

4. Cocartesian rings

The category **Aff** of affine schemes is contravariantly equivalent to the category **Ring** of commutative rings with identity. Thus, to determine cartesian affine schemes it suffices to consider the dual concept in the category **Ring**. More generally, if K is a fixed commutative ring, to determine cartesian objects in **Aff**/Spec$K$, one may consider the dual concept in **Ring**$\setminus K$, the category of commutative rings under K, or equivalently, the category of commutative K-algebras. Thus, if **A** is a category with finite colimits, we say an object A is *cocartesian* if $A \amalg - : \mathbf{A} \to \mathbf{A}$ has a left adjoint, or equivalently if A is cartesian in $\mathbf{A}^{\mathrm{op}}$. Note that the coproduct in **Ring**$\setminus K$ is given by $\otimes_K$.

Let A be a commutative K-algebra. Then $A \otimes_K -$ can be considered either as an endofunctor of **Ring**$\setminus K$ or of the category **Mod K** of K-modules. Moreover, if T: **Ring**$\setminus K \to$ **Mod K** denotes the forgetful functor, then the following diagram is commutative

$$\begin{tikzcd} \mathbf{Ring}\setminus K \arrow[r, "T"] \arrow[d, "A\otimes_K -"'] & \mathbf{Mod\,K} \arrow[d, "A\otimes_K -"] \\ \mathbf{Ring}\setminus K \arrow[r, "T"] & \mathbf{Mod\,K} \end{tikzcd}$$

Now, T has a left adjoint S which associates to each module M the symmetric algebra SM. Note that the counit $\varepsilon_A : STA \to A$ is surjective, and hence a coequalizer in **Ring**$\setminus K$. We shall use the following general lemma.

Lemma 4.1. *Let A and B be categories. Suppose that $S \dashv T$:* $\mathbf{A} \to \mathbf{B}$ *with the counit*

$\varepsilon_A : STA \to A$ a coequalizer for every $A \in |\mathbf{A}|$, and suppose U and V are endofunctors of $\mathbf{A}$ and $\mathbf{B}$ respectively, such that $TU = VT$. If V has a left adjoint, so does U.

Proof. Every object A of $\mathbf{A}$ can be written as a coequalizer between objects of the form SB where $B \in |\mathbf{B}|$, namely

$$STA' \underset{f_2}{\overset{f_1}{\rightrightarrows}} STA \xrightarrow{\varepsilon_A} A$$

where ε_A is the coequalizer of

$$A' \underset{g_2}{\overset{g_1}{\rightrightarrows}} STA$$

and $f_i = g_i \varepsilon_{A'}$. Hence, it suffices to show that $\mathbf{A}(SB, UA)$ is a representable functor of A for every $B \in |\mathbf{B}|$. But if $L \dashv V$, then

$$\mathbf{A}(SB, UA) \cong \mathbf{B}(B, TUA) \cong \mathbf{B}(B, VTA) \cong \mathbf{B}(LB, TA) \cong \mathbf{A}(SLB, A)$$

as required.

We shall abbreviate finitely generated to fg and finitely presented to fp.

Lemma 4.2. *The following are equivalent for any right K-module M over any ring K.*

(a) *M is fg (respectively, fp).*

(b) *For any family E_i of left K-modules, the induced homomorphism $\theta_M : M \otimes_K (\Pi_i E_i) \to \Pi_i (M \otimes_K E_i)$ is an epimorphism (respectively, isomorphism).*

(c) *Condition (b) with $E_i = K$, for all i.*

Proof. The proof of (a) $\Rightarrow$ (b) entails a standard application of the 5-Lemma [1, p. 5] and is left to the reader. Also, (b) $\Rightarrow$ (c) is obvious. We shall provide a proof of (c) $\Rightarrow$ (a). Suppose

$$\theta : M \otimes_K (\Pi_M K) \to \Pi_M (M \otimes_K K) \cong \Pi_M M$$

is an epimorphism. Consider the element $a \in \Pi_M M$ given by $a_m = m$ for all $m \in M$. Then since θ is surjective there exists $(\sum_{i=1}^n m_i \otimes k^i) \in M \otimes_K (\Pi_M K)$ whose image under θ is a. Thus, we have

$$m = \sum_{i=1}^{n} mk_m^i$$

for all $m \in M$, so M is generated by $m_1, \dots, m_n$. Another standard application of the 5-Lemma show that M is fp when θ is also a monomorphism.

Theorem 4.3. *A commutative K-algebra is cocartesian as an object of* **Ring** $\backslash K$ *if and only if it is finitely generated and projective as a K-module.*

Proof. Suppose that the commutative K-algebra A is cocartesian as an object of $\mathbf{Ring}\setminus K$. Then $A\otimes_K$ - : $\mathbf{Ring}\setminus K\to\mathbf{Ring}\setminus K$ preserves limits, in particular, products and monomorphisms. By Lemma 4.2(c) $\Rightarrow$ (a), A is fp as a K-module since $\Pi_i K$ as a K-algebra is the product as a K-module. Now, if M is any K-module, $M\oplus K$ becomes a K-algebra by defining $(m,k)(m',k')=(mk'+m'k,kk')$. Moreover, a monomorphism $M'\to M$ of modules gives a monomorphism $M'\oplus K\to M\oplus K$ of K-algebras. Hence, if $A\otimes_K$ - preserves this monomorphism of K-algebras, it must preserve the monomorphism of K-modules. Therefore, A is flat as a K-module. But a K-module is fp flat if and only if it is fg projective. This proves one direction.

Suppose A is a commutative K-algebra which is fg projective as a K-module. Applying Lemma 4.1 with $T:\mathbf{Ring}\setminus K\to\mathbf{Mod\,K}$ the forgetful functor, we see that it suffices to show that $A\otimes_K$ - : $\mathbf{Mod\,K}\to\mathbf{Mod\,K}$ has a left adjoint. If A is fg and projective, then $A\otimes_K$ - is isomorphic to $\mathrm{Hom}_K(A^*,\text{-})$ as endofunctors of $\mathbf{Mod\,K}$, where A^* is the dual of A. Therefore, $A\otimes_K$ - clearly has a left adjoint as an endofunctor of $\mathbf{Mod\,K}$.

Remark 1. If $\mathbf{Alg\,K}$ denotes the category of all K-algebras (not necessarily commutative), and if A is any K-algebra which is fg and projective as a K-module, then using the tensor algebra in place of the symmetric algebra and applying Lemma 4.1 we see that $A\otimes_K$ - : $\mathbf{Alg\,K}\to\mathbf{Alg\,K}$ has a left adjoint. Note, however, that this does not say that A is cocartesian in Alg K since $\otimes_K$ is not the coproduct.

Remark 2. One can also consider the functor $A\otimes_K$ - : $\mathbf{Ring}\setminus K\to\mathbf{Ring}\setminus A$. However, by the dual of Corollary 1.2(a) $\Leftrightarrow$ (b), this functor has a left adjoint if and only if $A\otimes_K$ - : $\mathbf{Ring}\setminus K\to\mathbf{Ring}\setminus K$ has a left adjoint.

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