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Finite Groups with Standard Components of Lie Type over Fields of Characteristic Two

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0. INTRODUCTION

The purpose of this paper is to classify the finite simple groups which arise as groups of standard type in the Trichotomy Theorem of Gorenstein and Lyons [33]. We prove

THEOREM I. *Let G be a finite simple group of characteristic 2-type in which all proper subgroups are K-groups and $e(G) \geqslant 4$. If G is of standard type with respect to some $(B, x, L) \in \mathscr{S}^*(p)$ for some prime $p \in \beta_4(G)$, then* $G \in \mathrm{Chev}(2)$.

The definitions relevant to Theorem I and the statement of the Trichotomy Theorem referred to above appear in the next section. If G is not of standard type but satisfies the other hypotheses of Theorem I, then the Trichotomy Theorem says roughly that either G contains a 2-local subgroup M with $O_2(M)$ of symplectic type or G possesses a strongly p-embedded maximal 2-local subgroup for various odd primes p. Proving the Trichotomy Theorem is a major step in classifying finite simple characteristic 2-type groups G with $e(G) \geqslant 4$.

The techniques of proof in this paper have appeared before in the solution of odd standard component problems. The articles [23, 35] survey the literature on odd standard component problems and the methods involved. In particular Finkelstein, Frohardt, and Solomon [17–22] have treated almost all the cases of Theorem I in which L is a group of Lie type defined over a field of order 2. We originally intended to restrict ourselves to the remaining cases. However it turned out that this dichotomy was artificial, and so we give a proof of Theorem I which is independent of the work cited above.

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Much of that work is more general than required for the proof of Theorem I: in many cases the hypothesis that all proper subgroups of G be K-groups is avoided.

1. The Main Theorem of Gorenstein and Lyons

The material in this section is taken from [33]. Standard definitions and notations may be found in [27, 28, 31].

The known simple groups are discussed in [28, Chap. II]. A *K-group* is a finite group all of whose simple sections are known. Let X be a finite group. H is a *2-local* subgroup of X if $H = N_X(T)$ from some 2-group $T \subseteq X$, $T \neq 1$. X is of *characteristic 2-type*, if $C_X(H) \subseteq O_2(H)$ for every 2-local subgroup H of X. By definition

$e(X) = \max\{m_{2,p}(X) \mid p$ ranges over all odd primes$\}$, where for any odd prime p

$m_{2,p}(X) = \max\{m_p(H) \mid H$ ranges over all 2-local subgroups of $X\}$

and

$m_p(H)$ is the maximal rank of an abelian p-subgroup of H.

Further

$\beta_k(X) = \{p \mid p$ an odd prime, $m_{2,p}(X) \geqslant k\}$;

$\mathscr{E}^p(X) =$ the set of elementary abelian p-subgroups of X, p an odd prime;

$\mathscr{E}_{k,p}(X) = \{A \in \mathscr{E}^p(X) \mid m_p(A) = k\}$;

$\mathscr{B}_{\max}(X; p) = \{B \mid B \in \mathscr{E}^p(X),\ m_p(B) = m_{2,p}(X)$, and B lies in a 2-local subgroup of $X\}$.

We now define the notion *standard type*. Let G be a finite group and p an odd prime. $\mathscr{S}^*(p)$ is the set of triples (B, x, L) where $B \in \mathscr{B}_{\max}(G, p)$, $x \in B^{\#}$, and L is a component of $C_G(x)$ with the property that $C_G(L)$ has cyclic Sylow p-subgroups. Define $\hat{\mathscr{S}}(p)$ to be set of triples (B, x^*, L^*) with B as before, $x^* \in B^{\#}$, and L^* a p-component of $C_G(x^*)$ with the property that $C_G(L^*/O_{p'}(L^*))$ has cyclic Sylow p-subgroups.

If $(B, x, L) \in \mathscr{S}^*(p)$, a standard *subcomponent* of (B, x, L) is a pair (D, K) such that $x \in D \in \mathscr{E}_2(B)$, $K = L(C_L(D))$ is a single component, and $D = C_B(K)$, with the additional restriction that if $p = 3$ and there exists a pair (D_1, K_1) satisfying these conditions with $K_1 \not\cong U_4(2)$ or A_6, then necessarily $K \not\cong U_4(2)$ or A_6.

This last condition involves a minor technical point and avoids certain generational difficulties.

If $(B, x, L) \in \mathscr{S}^*(p)$, and $(B, x^*, L^*) \in \hat{\mathscr{S}}(p)$, we call (B, x^*, L^*) a *neighbor* of (B, x, L) in G provided the following conditions hold:

(1) (D, K) is a standard subcomponent of (B, x, L), where $D = \langle x, x^* \rangle$ and $K = L(C_L(D))$;

(2) $L^* \subseteq \langle K^J \rangle$, $J = L_{p'}(C_G(x^*))$;

(3) x does not centralize $L^*/O_{p'}(L^*)$

(In the situation in which this notation is used, K will be a subgroup of $L_{p'}(C_G(x^*))$.) We say that (B, x^*, L^*) is a neighbor of (B, x, L) *with respect to* (D, K), if these conditions hold.

In the definition of standard type, L and L^* will be covering groups of Chevalley groups (including twisted groups) defined over $GF(2^n)$ and the integer p will divide either $2^n - 1$ or $2^n + 1$. To describe precisely which of these two integers p divides, we need two further definitions.

Let p be an odd prime, and $\hat{J}$ be a covering group of a Chevalley group J defined over $GF(2^n)$ for some n. (We consider twisted groups to be defined over the fixed field of the field automorphism involved in the twist.) We say that *p is a splitting prime for $\hat{J}$* or *p splits $\hat{J}$*, if and only if one of the following holds:

(1) J is untwisted and $p \mid 2^n - 1$; or

(2) J is twisted and $p \mid 2^n + 1$; or

(3) $J = {}^2D_l(2^n)$ or ${}^3D_4(2^n)$, and $p \mid 2^n - 1$; or

(4) $n = 1$ and $p = 3$.

We say that *p is a half-splitting prime for $\hat{J}$*, or *p half-splits $\hat{J}$*, if and only if one of the following holds:

(1) p splits $\hat{J}$; or

(2) $J = B_l(2^n)$, $D_l(2^n)$, $F_4(2^n)$, $E_7(2^n)$, or $E_8(2^n)$, and $p \mid 2^n + 1$; or

(3) $J = {}^2A_5(2^n)$ or ${}^2E_6(2^n)$ and $p \mid 2^n - 1$; or

(4) $J = A_5(4)$ or $E_6(4)$, $Z(\hat{J}) = 1$, and $p = 5$; or

(5) $J = A_8(2)$ and $p = 7$; or

(6) $J = E_6(2)$ and $p = 7$.

Finally, if G is a group, p is an odd prime, and $(B, x, L) \in \mathscr{S}^*(p)$, we say that G is of *standard type with respect to* (B, x, L) if and only if the following conditions hold:

(1) L is a covering group of a Chevalley group of characteristic two;

(2) p is a splitting prime for L;

(3) every element of B induces an inner $\cdot$ diagonal automorphism on L;

(4) B does not centralize every B-invariant 2-subgroup of $C_G(x)$;

(5) for every neighbor (B, x^*, L^*) of (B, x, L) in G, B normalizes L^*, L^* is a covering group of Chevalley group of characteristic two, and either p half-splits L^* or else x induces a nontrivial field automorphism on $L^*/Z(L^*)$;

(6) for every standard subcomponent (D, K) of (B, x, L), there exists a neighbor (B, x^*, L^*) of (B, x, L) with respect to (D, K). Moreover for all $d \in D^{\#}$, $[K, O_{p'}(C_G(d))]$ has odd order.

We now state the Trichotomy Theorem of Gorenstein and Lyons [33].

TRICHOTOMY THEOREM. *Let G be a finite simple group of characteristic 2 type in which all proper subgroups are K-groups and* $e(G) \geqslant 4$. *Then one of the following holds:*

(I) *G is of type $GF(2)$;*

(II) *G is of standard type with respect to some $(B, x, L) \in \mathscr{S}^*(p)$ for some prime $p \in \beta_4(G)$; or*

(III) *G is of uniqueness type with respect to $\sigma(G)$.*

The definitions we have omitted can be found in [33].

2. PROPERTIES OF GROUPS OF LIE TYPE IN CHARACTERISTIC 2

Throughout this paper, we shall assume that the reader is acquainted with the basic theory of the groups of Lie type, particularly the B, N-structure and the commutator relations [12, 54, 55]. It will also be necessary to view groups of Lie type as classical groups from time to time (e.g., $A_n(q) \cong PSL(n+1, q)$, ${}^2A_n(q) \cong PSU(n+1, q)$; see [12, 54].

In this section we list a number of results about groups over a field $\mathbb{F}$ algebraic over $\mathbb{F}_2$. The finiteness of $\mathbb{F}$ is not important in many cases.

Notation. Suppose that G is a group of Lie type over $\mathbb{F}$, but not type ${}^2C_2$, ${}^3D_4$, ${}^2G_2$ or ${}^2F_4$. We let Σ be the associated root system. For each $\alpha \in \Sigma$, there is an associated root group $X_\alpha \leqslant G$. Let $\mathbb{E}$ be a quadratic extension of $\mathbb{F}$. Then X_α consists of elements $x_\alpha(t)$ or $x_\alpha(t, u)$ which satisfy one of the following sets of relations

(i) $x_\alpha(t)\, x_\alpha(u) = x_\alpha(t+u)$, $t, u \in \mathbb{F}$;

(ii) $x_\alpha(t)\, x_\alpha(u) = x_\alpha(t+u)$, $t, u \in \mathbb{E}$;

(iii) $x_\alpha(t_1, u_1)\, x_\alpha(t_2, u_2) = x_\alpha(t_1 + t_2, u_1 + u_2 + \bar{t}_1 t_2)$, $t_1, t_2 u_1, u_2 \in \mathbb{E}$ and $t_i \bar{t}_i = u_i + \bar{u}_i$ for $i = 1, 2$. (note that $x_\alpha(t, u)^{-1} = x(t, \bar{t}t + u)$ and $[x_\alpha(t_1, u_1), x_\alpha(t_2, u_2)] = x_\alpha(0, t_1 \bar{t}_2 + \bar{t}_1 t_2)$.

If the roots α and β are in the same orbit under the Weyl group, then the same relation is associated to X_α and X_β. Recall that two roots are in the same orbit if and only if they have the same length.

The appropriate relations above are called Steinberg relations of type (A). The commutator relations among elements of distinct root groups are called relations of type (B) and they have one of the following shapes:

$$\begin{aligned}
[x_\alpha(t), x_\beta(u)] &= 1 \\
&= x_{\alpha+\beta}(tu) \\
&= x_{\alpha+\beta}(t\bar{u} + \bar{t}u) \\
&= x_{\alpha+\beta}(tu)\, x_{\alpha+2\beta}(tu\bar{u}) \\
&= x_{\alpha+\beta}(tu)\, x_{2\alpha+\beta}(t\bar{t}u) \\
&= x_{\alpha+\beta}(0, t\bar{u} + \bar{t}u), \\
[x_\alpha(t, u), x_\beta(v, w)] &= x_{1/2(\alpha+\beta)}(tv), \\
[x_\alpha(t, u), x_\beta(v)] &= 1 \\
&= x_{\alpha+\beta}(\bar{u}v)\, x_{\alpha+2\beta}(tv, uv\bar{v}), \\
[x_\alpha(t), x_\beta(u, v)] &= 1 \\
&= x_{\alpha+\beta}(\bar{v}t)\, x_{2\alpha+\beta}(tu, vt\bar{t})
\end{aligned}$$

for various $\alpha, \beta \in \Sigma$, t, u, v, $w \in \mathbb{E}$ or $\mathbb{F}$. See [54] for a more thorough discussion and for the detailed list of relations for each group. The elements $x_\alpha(t, u)$ occur only in ${}^2A_n$, n even. Note that the annoying plus or minus signs vanish in characteristic 2.

If G has type ${}^3D_4$ over $\mathbb{F}$ and $\mathbb{E}$ is a degree 3 extension field of $\mathbb{F}$, then X_α (as above) consist of elements $x_\alpha(t)$ satisfying relations (i) or (ii) above. The commutator relations are of the form

$$\begin{aligned}
[x_\alpha(t), x_\beta(u)] &= 1 \\
&= x_{\alpha+b}(tu) \\
&= x_{\alpha+b}(t\bar{\bar{u}} + \bar{t}u + \bar{\bar{t}}\bar{u}) \\
&= x_{\alpha+\beta}(t\bar{u} + \bar{t}u)\, x_{\alpha+2\beta}(t\bar{u}\bar{\bar{u}} + \bar{t}\bar{\bar{u}}u + \bar{\bar{t}}u\bar{u}) \\
&\quad \cdot x_{2\alpha+\beta}(t\bar{t}u + \bar{\bar{t}}t\bar{u} + t\bar{t}\bar{\bar{u}}) \\
&= x_{\alpha+\beta}(tu)\, x_{\alpha+2\beta}(tu\bar{u})\, x_{\alpha+3\beta}(tu\bar{u}\bar{\bar{u}})\, x_{2\alpha+3\beta}(t^2u\bar{u}\bar{\bar{u}}).
\end{aligned}$$

We shall use the following conventions for root systems. When passing from an untwisted group to a Steinberg variation, the root system changes as follows (see [54]):

$$A_n \to C_{[(n+1)/2]},$$

$$D_n \to B_{n-1},$$

$$E_6 \to F_4,$$

$$D_4 \to G_2.$$

In the left column, there is one root length, while in the right column, there are two. Following Steinberg [54], a root in the right column shall be a set $R = \{r\}$, $\{r, \bar{r}\}$ $\{r, \bar{r}, \bar{\bar{r}}\}$ or $\{r, \bar{r}, r + \bar{r}\}$ of distinct roots $r, \bar{r},...$, where the overbar denotes a symmetry of the Dynkin diagram extended to the root system ($\{r, \bar{r}, \bar{\bar{r}}\}$ occurs only for D_4 and $\{r, \bar{r}, r+\bar{r}\}$ occurs only for A_n, n even, and when it does, sets of the form $\{s\}$, where $s = \bar{s}$, are not considered roots in the twisted system). A root of the form $R = \{r, \bar{r}\}$ or $\{r, \bar{r}, \bar{\bar{r}}\}$ is considered short, and the others are considered long.

The Steinberg relations for twisted groups are trickier than those for untwisted groups; compare [36] and [55]. We point out that the Chevalley commutator relations for untwisted groups look like

$$[x_r(t), x_s(u)] = \prod_{i,j>0} x_{ir+js}(t^i u^j),$$

whereas the analogue of the relations for twisted groups looks like

$$[x_R, x_S] = \prod_{\substack{i,j \geqslant 0 \\ i+j>0 \\ i,j \in 1/2\mathbb{Z},\, i+j \in \mathbb{Z}}} x_{iR+jS}.$$

Let G be a simple group of Lie type perhaps extended by diagonal automorphisms and defined over a finite field $\mathbb{F}_q$ of characteristic 2. When we write $G = C_n(q)$ for example, we mean that $O^{2'}(G)$ is isomorphic to $C_n(q)$. We never consider $G = A_1(q)$, ${}^2C_2(q)$, or ${}^2F_4(q)$. We adopt the convention that if Σ is a root system with all roots the same length, then a long root or a short root of Σ means just a root of Σ.

We need information on the 2-local structure of G. Much of the next few lemmas is in [16, Lemma 4.8]. The lemmas follow from the commutator relations above, or one can compute in $A_2(q)$, $C_2(q)$, ${}^2A_3(q)$, ${}^2A_4(q)$.

LEMMA 2.1. *If $\alpha+\beta$ is not a root, $[X_\alpha, X_\beta]=1$ except for the case α, β long in ${}^2A_l(q)$, l* even.

LEMMA 2.2. *If α, β, $\alpha+\beta$ all have the same length, then*

(i) $1 \neq [g, h]$ *for* $g \in X_\alpha^\#$, $h \in X_\beta^\#$;

(ii) *if* $G \neq {}^3D_4(q)$, $[g, X_\beta] = X_{\alpha+\beta}$ *for* $g \in X_\alpha^\#$;

(iii) *if $G = {}^3D_4(q)$, and α, β, $\alpha+\beta$ are long, then the equation in* (ii) *holds, while if α, β, $\alpha+\beta$ are short, then* $[X_\alpha, X_\beta] = X_{\alpha+\beta}X_{\alpha+2\beta}X_{2\alpha+\beta}$.

LEMMA 2.3. *If α, $\alpha+\beta$ are short, and β is long, $g \in X_\alpha^\#$ and $h \in X_\beta^\#$, then*

(i) *If* $G \neq {}^3D_4(q)$, *then*

$$1 \neq [g, h] \in X_{\alpha+\beta}X_{2\alpha+\beta} - (X_{\alpha+\beta} \cup X_{2\alpha+\beta});$$

(ii) *if* $G = {}^3D_4(q)$, *then*

$$1 \neq [g, h] \in X_{\alpha+\beta}X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta} - (X_{\alpha+\beta} \cup X_{2\alpha+\beta} \cup X_{3\alpha+\beta} \cup X_{3\alpha+2\beta});$$

(iii) $[X_\alpha, X_\beta] = X_{\alpha+\beta}X_{2\alpha+\beta}$ *unless $G = {}^3D_4(q)$ or G is is untwisted and* $q=2$;

(iv) $[X_\alpha, X_\beta] = X_{\alpha+\beta}X_{2\alpha+\beta}X_{3\alpha+\beta}X_{3\alpha+2\beta}$ *if* $G = {}^3D_4(q)$;

(v) $[X_\alpha, Z(X_\beta)] = X_{\alpha+\beta}Z(X_{2\alpha+\beta})$.

LEMMA 2.4. *If α, β are short and $\alpha+\beta$ is long, then*

(i) $[X_\alpha, X_\beta] = 1$ *if G is untwisted;*

(ii) $[g, X_\beta] = Z(X_{\alpha+\beta})$ *for $g \in X_\alpha^\#$ if G is twisted.*

LEMMA 2.5. *If G has type ${}^2A_n(q)$, n even, α, β are long and $\frac{1}{2}(\alpha+\beta)$ is short, then $[g, X_\beta] = X_{\alpha+\beta}$ for $g \in X_\alpha - Z(X_\alpha)$ and $[Z(X_\alpha), X_\beta] = 1$.*

Let H be a Cartan subgroup corresponding to our choice of root groups for G. Possibly $H=1$.

LEMMA 2.6. *Suppose $x \in Z(X_\alpha)^\#$ for some $\alpha \in \Sigma$; and if α is short, suppose G is not twisted. Define*

$$Q = \langle X_\beta \mid (\alpha, \beta) > 0 \rangle,$$

$$L = \langle X_\beta \mid (\alpha, \beta) = 0 \rangle.$$

The following conditions hold:

(i) Q *is a 2-group;*

(ii) L *normalizes* Q*;*

(iii) H *normalizes* Q *and* L*;*

(iv) L *is a central product of groups of Lie type, or* $L = 1$;

(v) $Q = O_2(HLQ)$;

(vi) $LQ = O^{2'}(HLQ)$;

(vii) *if* $g \in G$, *and* $x^g \in Z(X_\alpha)^{\#}$, *then* $g \in HLQ$;

(viii) $N_G(Z(X_\alpha)) = HLQ$:

(ix) $C_G(x) = C_H(x)\, LQ = C_G(Z(X_\alpha))$.

Proof. Conditions (i)–(iii) follow from the commutator relations and the fact that H normalizes every root group. Likewise $[LQ, Z(X_\alpha)] = 1$ and $HLQ \leqslant N_G(Z(X_\alpha))$ whence (vii) implies (viii) and (ix). Further, (iv) is a consequence of the Steinberg–Curtis–Tits presentations discussed above and in Proposition 2.27, and (vi) holds because H has odd order.

It remains to prove (v) and (vii). Pick a fundamental system Π for Σ such that α is the highest root of its length in Σ with respect to Π. Observe that $(\alpha, \gamma) \leqslant 0$ for every $\gamma \in \Pi$. Let U and V be the Sylow 2-subgroups of G corresponding to the positive and negative roots, respectively. Note $U \leqslant LQ$. Every $g \in G$ has a unique representation

$$g = uhn_w u^{-}$$

and one consequence of this fact is $U \cap V = 1$. Check that Π contains a fundamental system for the root system corresponding to L. As a consequence $U \cap L$, $V \cap L \in \mathrm{Syl}_2(L)$ which together with $(U \cap L) \cap (V \cap L) = 1$ forces $O_2(L) = 1$. Now (v) must hold.

To check (vii) suppose g is as above and for some $y \in Z(X_\alpha)^{\#}$, $yg = gx$. Reduce gx to standard form and notice that $yg = gx$ forces $w(\alpha) = \alpha$. By [12, Corollary 2.5.4] w is a product of reflections corresponding to roots orthogonal to α. Thus $n_w \in HL$. Since $U \subseteq LQ$, we have $g \in (LQ)(H)(HL)(LQ) = HLQ$.

The method of proof of the preceding lemma works also for the next two lemmas.

LEMMA 2.7. *Suppose* $x \in X_\alpha^{\#}$ *for some short root* $\alpha \in \Sigma$ *and* $G = {}^2A_n(q)$. *Let* $\{\gamma, -\gamma\}$ *be the unique pair of roots such that* γ *is short and* $\gamma + \alpha$ *is long. Define*

$$Q = \langle X_\beta \mid (\alpha, \beta) > 0 \rangle;$$

$$L = \langle X_\beta \mid (\alpha, \beta) = 0, \beta \neq \pm\gamma \rangle.$$

$L_1 = \langle X_\gamma, X_{-\gamma}\rangle = A_1(q^2)$ and there is an involution $n \in N_{L_1}(H)$ such that n inverts $H \cap L_1 \cong \mathbb{Z}_{q^2-1}$, n normalizes X_α, n permutes X_γ and $X_{-\gamma}$, and n induces a field automorphism on $\langle X_\alpha, X_{-\alpha}\rangle \cong A_1(q^2)$. Further, the following conditions hold:

(i) *Q is a 2-group;*

(ii) *L normalizes Q;*

(iii) *H normalizes Q and L;*

(iv) *$L/Z(C) \cong {}^2A_{n-2}(q)$;*

(v) *$[n, L] = 1$ and $[n, Q] \leqslant Q$;*

(vi) *$Q = O_2(\langle n\rangle HLQ)$;*

(vii) *if $g \in G$ and $x^g \in X_\alpha$, then $g \in \langle n\rangle HLQ$;*

(viii) *$N_G(X_\alpha) = \langle n\rangle HLQ$;*

(ix) *$C_G(X_\alpha) = C_H(X_\alpha) LQ$;*

(x) *there is a subgroup $E \subseteq X_\alpha$, $E \cong E_q$, such that for $x \in E^\#$, $C_G(x) = \langle n\rangle C_H(x) LQ = C_G(E)$.*

LEMMA 2.8. *Suppose $x \in X_\alpha^\#$ for some short root $\alpha \in \Sigma$ and $G = {}^2E_6(q)$. Define*

$$Q = \langle X_\beta \mid (\alpha, \beta) > 0\rangle;$$
$$L = \langle X_\beta \mid (\alpha, \beta) = 0 \text{ and } \beta \text{ is long}\rangle.$$

For any γ such that $(\alpha, \gamma) = 0$ and γ is short, $L_1 = \langle X_\gamma, X_{-\gamma}\rangle = A_1(q^2)$, and there is an involution $n \in N_{L_1}(H)$ such that n inverts $H \cap L \cong \mathbb{Z}_{q^2-1}$, n normalizes X_α and n induces a field automorphism on $\langle X_\alpha, X_{-\alpha}\rangle = A_1(q^2)$. Further the following conditions hold:

(i) *Q is a 2-group;*

(ii) *L normalizes Q;*

(iii) *H normalizes L and Q;*

(iv) *$L/Z(L) \cong A_3(q)$;*

(v) *$[n, Q] \leqslant Q$, $[n, L] \leqslant L$, and n induces a graph automorphism on L;*

(vi) *$Q = O_2(\langle n\rangle HLQ)$;*

(vii) *if $g \in G$ and $x^g \in X_\alpha$, then $g \in \langle n\rangle HLQ$;*

(viii) *$N_G(X_\alpha) = \langle n\rangle HLQ$;*

(ix) *$C_G(X_\alpha) = C_H(X_\alpha) LQ$;*

(x) *there is a subgroup $E \leqslant X_\alpha$, $E \cong E_q$, such that for $x \in E^\#$, $C_G(x) = \langle n\rangle C_H(x) LQ = C_G(E)$.*

LEMMA 2.9. *Let α be a root in Σ with α long if $G = {}^3D_4(q)$, ${}^2D_m(q)$, or ${}^2E_6(q)$. If $x \in Z(X_\alpha)^\#$, then $C_G(x)/LQ$ is cyclic.*

Proof. Let $R = Z(X_\alpha)$ and recall $C_G(x) \leqslant N_G(R) = HLQ$. It suffices to show that $C_H(t)/H \cap L$ is cyclic. If $G = A_n(q)$ or ${}^2A_n(q)$, use matrix representations. In the remaining cases L has a root system of rank one less than that of G. Further, except for $G = E_6(q)$, ${}^2E_6(q)L$ admits no outer-diagonal automorphisms whence $H = (H \cap L) \times C_H(L)$ and the result follows. In the case $G = E_6(q)$ use the representations of H as characters on $Z\Phi$ and calculate directly that $|C_H(r)\colon H \cap L| = 1$ or 3. Finally use the usual embedding of ${}^2E_6(q)$ in $E_6(q^2)$ to prove the same result for $G = {}^2E_6(q)$.

A consequence of the last lemma is the following:

LEMMA 2.10. *If $x \in Z(X_\alpha)^\#$ for some $\alpha \in \Sigma$, with α long if $G = {}^3D_4(q)$ or ${}^2D_n(q)$, then*

(i) $C_G(x) \subseteq \langle X_\beta \mid (\alpha, \beta) \geqslant 0 \rangle\, C_H(x)$.

(ii) *If $a \in \mathrm{Aut}(G)$ with $R \cap R^a \neq 1$, then $R^a = R$.*

Proof. The first assertion is clear, and we know the second holds if a is inner. Since $\mathrm{Aut}(G) = \mathrm{Inn}(G)\, N_{\mathrm{Aut}(G)}(R)$, (ii) is valid for a.

LEMMA 2.11. *Let $R = Z(X_\alpha)$ for some $\alpha \in \Sigma$. Take α to be long if G is any twisted group. Let $Q = O_2(N_G(R))$ and $J = \langle R, Z(X_{-\alpha})\rangle$. The following conditions hold:*

(i) $[Q, Q] \subseteq R \subseteq Z(Q)$;

(ii) *$N_G(R)$ has no central factors on Q/R unless $G = A_2(2)$;*

(iii) $O^{2'}(G) = \langle J, Q\rangle$;

(iv) *if $G \subseteq A \subseteq \mathrm{Aut}(G)$, then $O_2(N_A(R)) \subseteq Q$;*

(v) *if $r \in R^\#$ and $G \subseteq A \leqslant \mathrm{Aut}(G)$, then $C_A(r) \subseteq N_A(R)$ and $|R, O_2(C_A(r))| = 1$.*

Proof. Condition (i) follows from the commutator relations and preceding lemmas which describe the generation of Q. Likewise in the notation of the preceding lemmas, $N_G(R) = \langle Q, L, H\rangle$ is a maximal parabolic in G whence $\langle Q, L, H, J\rangle = G$. But HL normalizes Q and J whence $\langle Q, J\rangle \lhd G$ and (iii) holds.

To verify (iv) suppose $a \in O_2(N_A(R)) - G$ with $a^2 \in G$. Since q is even, a is a field or graph automorphism. Assume that α is the highest root of its length with respect to some fundamental system $\Pi \subseteq \Sigma$ and take σ to be the standard automorphism (with respect to Π) for which $d = \sigma^{-1}a$ is an inner-diagonal automorphism of G. If $G = C_2(q)$ or $F_4(q)$ with σ a graph automorphism, then d maps R to a root group X_β with β and α of different lengths. But using the decomposition

$$d = uhnu_1$$

one sees that d cannot act on G in such a way. Otherwise σ normalizes R, whence d does too. Since $a \in O_2(N_A(R))$, it follows that σ induces an inner-diagonal automorphism on LQ/Q and centralizes HLQ/Q. Examination of cases yields (iv).

For $G = A_n(q)$, ${}^2A_n(q)$, or $C_n(q)$ use matrix representations to prove (ii). (Note that $L(G)$ simple implies $G \neq {}^2A_2(2)$.) In the remaining cases use the results and methods of [16, Sects. 3 and 4].

The first part of (v) is a consequence of Lemma 2.6(viii) and $A = GN_A(R)$. From Lemma 2.6(ix) we deduce $O^2(N_A(R)) \subseteq C_A(r) \subseteq N_A(R)$ whence $O_2(C_A(r)) \subseteq O_2(N_A(R))$. Now the second assertion of (v) follows from (i) and (iv) except when $G = A_2(2)$, $A_3(2)$, or $A_2(4)$. In the first two cases $R = \langle r \rangle$ and the assertion is immediate, while in the last case it may be checked directly.

LEMMA 2.12. *Assume the notation of the preceding lemma, and take* $G = {}^2E_6(q)$ *or* ${}^2A_n(q)$, *and* α *short*

(i) $\langle [Q, Q], R \rangle \subseteq Z(Q)$;

(ii) $N_G(R)$ *has no central factors on* Q/R;

(iii) $O^{2'}(G) = \langle J, Q \rangle$;

(iv) *if* $G \subseteq A \subseteq \mathrm{Aut}(G)$, *then* $O_2(N_G(A)) \subseteq Q$;

(v) *if* $r \in R^\#$ *and* $G \subseteq A \subseteq \mathrm{Aut}(G)$, *then* $C_A(r) \subseteq N_A(R)$.

Proof. A proof similar to the preceding one works. In this case $N_G(R)$ is not a maximal parabolic, but one can show that $\langle N_G(R), J \rangle$ contains a maximal parabolic containing $N_G(R)$.

The next four lemmas are proved by matrix calculations and the methods of [16, Sects. 3 and 4].

LEMMA 2.13. *Let* $R = Z(X_\alpha)$ *for some* $\alpha \in \Sigma$ *with* α *long if* Σ *has roots of two lengths. Let* $Q = O_2(N_G(R))$. *Then* Q/R *is a nontrivial irreducible* $N_G(R)$-*module except when* $G = A_n(q)$ *or* $F_4(q)$.

LEMMA 2.14. *Let* $G = {}^2E_6(q)$, ${}^2A_n(q)$, $n \geqslant 3$, *or* $C_n(q)$, $n \geqslant 2$, *and take* $R = X_\alpha$ *with* α *short. Let* $Q = O_2(N_G(R))$; *then* Q *has a unique subgroup* U *with* $R \subset U \subset Q$ *and* $U \lhd N_G(R)$. *Also* $U = Z(Q)$ *and* U *is generated by* R *together with the two root groups* X_β *for which* α *and* β *have different lengths and* $(\alpha, \beta) > 0$.

LEMMA 2.15. *Let* $G = A_n(q)$, $R = X_\alpha$, *and* $Q = O_2(N_G(R))$. *Assume* $n > 3$ *or* $q > 2$. *There are two subgroups* U *such that* $R \subset U \subset Q$ *and* $U \lhd N_G(R)$. *Both subgroups are generated by various root groups corresponding to roots* β *with* $(\alpha, \beta) > 0$. U/R *and* Q/N *are nontrivial irreducible* $N_G(R)$*-modules.*

LEMMA 2.16. *Let* $G = F_4(q)$, $R = X_\alpha$, *and* $Q = O_2(N_G(R))$. *There is unique subgroup* U *of* Q *such that* $R \subset U \subset Q$ *and* $U \lhd N_G(R)$. U *is generated by* X_α *together with the root groups* X_β *for which* $(\alpha, \beta) > 0$, *and* α *and* β *have different lengths.*

LEMMA 2.17. *Let* $R = Z(X_\alpha)$ *with* α *long if* G *is twisted. Define* Q *as in Lemma* 2.6. *If* J *is a summand of* L, *then* $Q = [Q, J]\, R$.

Proof. Let $U = [Q, J]\, R$ and suppose $U \subset Q$. By Lemma 2.11(i), $[Q, Q] \subseteq R$, and it follows that $U = [Q, JQ]\, R$. As $JQ \lhd N_G(R)$, $U \lhd N_G(R)$ also. The possibilities for U are listed in the preceding lemmas, and it is straightforward to find in each case a root group of G which lies in J and acts nontrivially on Q/V, contradicting $[Q, J] \leqslant V$. In many cases we already know that J must equal L and that Q/R is an irreducible L-module.

The group G may be described as the fixed points of a standard algebraic endomorphism σ of an adjoint algebraic group $\tilde{G}$ defined over the algebraic closure $\mathbb{K}$ of our finite field. Here $C_{\tilde{G}}(\sigma)$ includes all the diagonal automorphisms of G. Let $\tilde{\Sigma}$ be a root system for $\tilde{G}$ with root groups $\tilde{X}_{\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Sigma}$. Given a fundamental system $\tilde{\Pi}$ for $\tilde{\Sigma}$, the choices for σ are listed in [8, Table 1]. In each case σ corresponds to a certain symmetry of the Dynkin diagram and the root system, and also to isomorphisms of the root lattice and the Weyl-group $\tilde{W}$. We denote all these maps by σ.

$C_{\tilde{W}}(\sigma)$ is the Weyl group of G; and with an adjustment when $G = {}^2A_n(q)$, n even, the orthogonal projection of $\tilde{\Sigma}$ onto the fixed points of σ on $\mathbb{R}\tilde{\Sigma}$ gives Σ, the root system of G. With the same exception, root groups of G correspond to orbits of root groups of $\tilde{G}$ under $\langle \sigma \rangle$.

The method of Burgoyne and Williamson [10] is useful in answering questions about classes and centralizers of elements of G of order prime to q (where $\mathbb{F}_q$ is the field of definition of G). We give a sketch of the method.

Denote by Γ the dual lattice to $\mathbb{Z}\tilde{\Sigma}$. Each element $\eta \in \Gamma$ defines a homomorphism $\mathbb{K}^* \to T$, T a fixed Cartan subgroup of $\tilde{G}$, which sends $\lambda \in \mathbb{K}^*$ to $t(\eta, \lambda) \in T$. The element $t(\eta, \lambda)$ is itself defined by giving its corresponding character χ

$$\chi\colon \mathbb{Z}\tilde{\Sigma} \to \mathbb{K}^*,$$
$$\chi(\alpha) = \lambda^{\eta(\alpha)}, \qquad \alpha \in \tilde{\Sigma}.$$

Fix a primitive pth root of unity $\lambda \in \mathbb{K}^*$ for some p with $(p, q) = 1$; and let $t(\eta) = t(\eta, \lambda)$. Every element of T of order p is $t(\eta)$ for some $\eta \in \Gamma$.

The Weyl group $\tilde{W}$ of $\tilde{\Sigma}$ acts on Γ by

$$w\eta(\alpha) = \eta(w^{-1}\alpha)$$

and the semidirect product $\tilde{W}.\, p\Gamma$ acts by

$$(w, p\mu)\,\eta = w\eta + p\mu.$$

The conjugacy classes of elements of order p in $\tilde{G}$ correspond to orbits of $\tilde{W}.p\Gamma$ on Γ. If $[\eta]$ denotes the orbit of η, then the corresponding class intersects the finite group G if and only if $[\eta] = [\sigma\eta]$. $G \cap [\eta]$ is a union of G-classes each corresponding to a pair $[\eta, v\sigma]$ with $v \in \tilde{W}$ and $v\sigma u - u \in p\Gamma$. For any such pair $t(\eta)$ centralizes $I_v\sigma$, where I_v is any inner automorphism of $\tilde{G}$ defined by some element of the coset of $N_{\tilde{G}}(T)/T$ corresponding to v. By Lemma 2.33 $I_v\sigma$ is conjugate to σ by an inner automorphism of $\tilde{G}$. Thus, $G \cong C_{\tilde{G}}(I_v\sigma)$, and in fact the centralizer in G of an element of $[\eta, v\sigma]$ is isomorphic to $C_{\tilde{G}}(t(\eta)) \cap C_{\tilde{G}}(I_v\sigma)$. For example when $p \mid q+1$, it turns out that the classes of p-elements of G which intersect B^*, an elementary abelian p-group of G of maximum rank, are $[\eta, w\sigma]$ where w interchanges positive and negative roots of $\tilde{\Sigma}$.

If $t(\eta) \in [\eta, v\sigma]$, then $G_0 = O^{2'}(C_G(t(\eta)))$ is a central product of groups of Lie type defined over $\mathbb{F}_q$ (the field of definition of G) or finite extensions of $\mathbb{F}_q$. G_0 can be recognized from the action of $v\sigma$ on $\tilde{\Sigma}_0 = \{\tilde{\alpha} \mid \eta(\tilde{\alpha}) = 1\}$. If $v = 1$, then the orbits of $\langle\sigma\rangle$ on $\tilde{\Sigma}_0$ give a system of root groups for G_0 which correspond to a root system $\Sigma_0 \subseteq \Sigma$. In general G_0 does not have a root system which is a subsystem of Σ.

LEMMA 2.18. *Using the notation introduced above, let σ be a standard algebraic endomorphism of $\tilde{G}$ with $O^{2'}(C_{\tilde{G}}(\sigma)) \subseteq G \subseteq C_{\tilde{G}}(\sigma)$, and let $\rho = I_v\sigma$. Suppose $\tilde{\Sigma}_0$ is a root system such that*

$$\tilde{\Sigma}_0 \subseteq \tilde{\Sigma},$$

$$\langle \rho, \sigma\rangle \text{ acts on } \tilde{\Sigma}_0,$$

and such that the restriction of v to $\tilde{\Sigma}_0$ is in the Weyl group of $\tilde{\Sigma}_0$. Further suppose that if $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Sigma}_0$ and $[\tilde{X}_{\tilde{\alpha}}, \tilde{X}_{\tilde{\beta}}] \neq 1$, then all linear combinations of $\tilde{\alpha}$ and $\tilde{\beta}$ in $\tilde{\Sigma}$ lie in $\tilde{\Sigma}_0$.

Let $\tilde{G}_0 = \langle \tilde{X}_{\tilde{\alpha}} \mid \tilde{\alpha} \in \tilde{\Sigma}_0 \rangle$ *and* $G_0 = O^{2'}(C_{\tilde{G}_0}(\rho))$. *The following conditions hold:*

(i) $\tilde{G}_0$ *is a central product of Chevalley groups defined over K and corresponding to the orthogonal summands of* $\tilde{\Sigma}_0$;

(ii) *there is an inner automorphism of* $\tilde{G}$ *which carries* $C_{\tilde{G}}(\sigma)$ *to* $C_{\tilde{G}}(\rho)$ *and* $C_{\tilde{G}_0}(\sigma)$ *to* $C_{\tilde{G}_0}(\rho)$;

(iii) G_0 *is a central product of groups of Lie type defined over* $\mathbb{F}_q$ *or its finite extensions;*

(iv) G_0 *has a root system* $\Sigma_0 \subseteq \Sigma$, *and any choice of root groups for* G_0 *corresponding to* Σ_0 *extends to a system of root groups for G with the convention that when* $G = {}^2A_n(q)$, *n even, an abelian root group of* G_0 *may become the center of a nonabelian root group of G.*

Proof. It suffices to find $z \in \tilde{G}$ such that $(I_z)^{-1} \rho I_z = \sigma$ and I_z normalizes $\tilde{G}_0$.

Let $\tilde{\Sigma}_1$ consist of all roots in $\tilde{\Sigma}$ orthogonal to $\tilde{\Sigma}_0$, and let $\tilde{G}_1 = \langle \tilde{X}_{\tilde{\beta}} \mid \beta \in \tilde{\Sigma}_1 \rangle$. $\tilde{G}_1$ is a central product of Chevalley groups over K, and $[\tilde{G}_0, \tilde{G}_1] = 1$. Choose v_0 in the Weyl group of $\tilde{\Sigma}_0$ so that v restricts to v_0. By [12, Corollary 2.5.4], $(v_0)^{-1} v = v_1$ for some v_1 in the Weyl group of $\tilde{\Sigma}_1$. By Lang's theorem choose $y \in \tilde{G}_0 \tilde{G}_1$ such that $(I_y)^{-1} \rho I_y = I_x \sigma$ for some $x \in TC_{\tilde{G}}(\tilde{G}_0 \tilde{G}_1) = T$.

It is a consequence of Hilbert's Theorem 90 that for any $q = 2^m$ and $\lambda \in K^*$ there exists $\mu \in K^*$ with $\mu\mu^{-q} = \lambda$. It follows in a straightforward way that there exists $v \in T$ such that $(I_v)^{-1} I_x \sigma I_v = \sigma$, whence we may take $z = yh$.

As an application of the preceding lemma suppose $\tilde{\Sigma}_0 = \{\pm\tilde{\alpha}\}$ for some $\tilde{\alpha} \in \tilde{\Sigma}$ with $\rho(\tilde{\alpha}) = \pm\tilde{\alpha}$, $\sigma(\tilde{\alpha}) = \tilde{\alpha}$. We see that every root group of

$$C_{\langle \tilde{X}_{\tilde{\alpha}}, \tilde{X}_{-\tilde{\alpha}} \rangle}(\rho)$$

is a root group of G. When $\rho = I_w \sigma$ and w is the element of the Weyl group $\tilde{W}$ of $\tilde{G}$ which interchanges positive and negative roots, then $C_{\tilde{W}}(\rho)$ is transitive on the set of roots $\tilde{\alpha}$ of a fixed length with $\tilde{\rho}(\tilde{\alpha}) = -\tilde{\alpha}$. In fact we may take $\tilde{\alpha}$ to be the highest root of its length whence $\sigma(\tilde{\alpha}) = \tilde{\alpha}$ automatically.

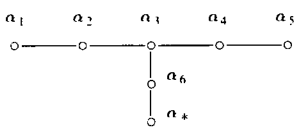
LEMMA 2.19. *Let* $\rho = I_w \sigma$ *with* w *interchanging the positive and negative roots of* $\tilde{\Sigma}$. *If* $\rho(\tilde{\alpha}) = -\tilde{\alpha}$, *then any root group of*

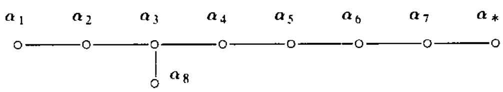
$$C_{\langle X_{\tilde{\alpha}}, X_{-\tilde{\alpha}} \rangle}(\rho)$$

is the center of a root group of G corresponding to a root $\alpha \in \Sigma$. *When* $\tilde{\Sigma}$ *or* Σ *has roots of two lengths, the possibilities are as follows:*

$\tilde{G}$	G	$\tilde{\alpha}$	α
A_n	${}^2A_n(q)$	long	long
C_n	$C_n(q)$	short	short
C_n	$C_n(q)$	long	long
D_n	${}^2D_n(q)$		long
D_4	${}^3D_4(q)$		long
E_6	${}^2E_6(q)$		long
F_4	$F_4(q)$	short	short
F_4	$F_4(q)$	long	long

We make one more application of Lemma 2.18 to the cases $G = E_6(2)$ and $E_8(2)$ with $p = 7$. Choose extended fundamental root systems of type E_6 and E_8 as follows:





where α_* is the lowest root in both cases. Let w_i and w_* be the involutions of the Weyl group corresponding to roots α_i and α_*, respectively. Define v in the Weyl group of E_6 by

$$\begin{aligned} v\colon \alpha_1 &\to \alpha_* \\ \alpha_2 &\to \alpha_6 \\ \alpha_3 &\to \alpha_3 \\ \alpha_4 &\to \alpha_2 \\ \alpha_5 &\to \alpha_1 \\ \alpha_6 &\to \alpha_4 \\ \alpha_* &\to \alpha_5 . \end{aligned}$$

Define the endomorphism ρ of the corresponding algebraic groups $E_6(\mathbb{K})$ and $E_8(\mathbb{K})$ by $\rho = I_v \sigma_2$ and $\rho = I_w \sigma_2$, respectively. Here $I_v \in E_6(\mathbb{K})$ corresponds to v and $I_w \in E_8(\mathbb{K})$ corresponds to $w = w_7 w_*$. We consider elements of order 7 in the groups $E_6(2)$ and $E_8(2)$. In the notation of Burgoyne and Williamson [10], $[2\eta_1 + \eta_5, \rho]$ denotes a conjugacy class in $E_6(2)$ each of whose elements has centralizer isomorphic to

$${}^3D_4(2) \times Z_7,$$

while $[n_6 + n_7, p]$ is a class in $E_8(2)$ with corresponding centralizers

$$E_6(2) \times \mathbb{Z}_7.$$

Let $\tilde{G} = E_8(\mathbb{K})$ or $E_6(\mathbb{K})$ and $G = C_G(\rho)$.

LEMMA 2.20. *In terms of the definitions above, if x is a root group of $\tilde{G}$ corresponding to α_3, then $C_{\tilde{X}}(\rho)$ is a root group of G.*

The following lemma is proved by the method of Burgoyne and Williamson.

LEMMA 2.21. *Let x be an element of order 7 in G with $L = L(C_G(x))$ and $J = C_G(x) \cap C_G(L)$. For each G below we list L and J as x ranges over representations of each G-class of elements of order 7. We also give the corresponding class in the algebraic group. These classes do not split in G. The fundmental roots are labeled as above.*

G	L	J	Class in $\tilde{G}$
$E_6(2)$	$A_2(2)$	$\mathbb{Z}_{21}$	$\eta_1 + \eta_2 + \eta_4 + \eta_5$
	$A_2(2) \times A_2(2)$	$\mathbb{Z}_7$	$2\eta_1 + \eta_2 + \eta_3$
	${}^3D_4(2)$	$\mathbb{Z}_7$	$\eta_1 + 2\eta_5$
	${}^3D_4(2)$	$\mathbb{Z}_7$	$2\eta_1 + \eta_5$
$E_8(2)$	$E_6(2)$	$\mathbb{Z}_7$	$\eta_6 + \eta_7$
	${}^3D_4(2) \times A_2(2)$	$\mathbb{Z}_7$	$\eta_1 + \eta_5$

We mention some general results which are contained in [10, Sect. 5.2].

LEMMA 2.22. *If $y \in G$ has odd order, then $O^{2'}(C_G(y))$ is a central product of groups of Lie type defined over fields of characteristic 2, and $C_G(y) \cap C_G(O^{2'}(C_G(y))$ has odd order.*

LEMMA 2.23. *Suppose $y \in G$ has odd order and normalizes $R = Z(X_\alpha)$ whose X_α is a root group of G with $G \neq {}^3D_4(q)$, ${}^2E_6(q)$, ${}^2D_m(q)$ or ${}^2A_n(q)$, n odd, if α is short. Then $O_2(C_{N_G(R)}(y)) \leqslant O_2(N_G(R))$.*

Proof. This follows from the structure of $N_G(R)$ given in Lemma 2.6. The element y acts as an inner-diagonal automorphism on $N_G(R)/O_2(N_G(R))$. Apply the preceding lemma.

For various computations we will need the following information.

LEMMA 2.24. *Let p be an odd prime and $q = 2^r$.*

(i) *If $p^a \| q - \varepsilon$, $a \geqslant 1$ and $\varepsilon = \pm 1$, then $p^{a+1} \| q^p - \varepsilon$;*

(ii) *if* $s=\Sigma a_n q^n$, *and* $\varepsilon=\pm 1$, *then* $(s, q-\varepsilon)=\Sigma\varepsilon^n a_n$;

(iii) *if* $\varepsilon=\pm 1$, $\tau=\pm 1$, *then*

$$\begin{aligned}(2^a-\varepsilon, 2^b-\tau) &= 2^{(a,b)}-1 && \text{if}\quad \varepsilon=\tau=1\\ &= 2^{(a,b)}+1 && \text{if}\quad \varepsilon=(-1)^{a/(a,b)},\ \tau=(-1)^{b/(a,b)}\\ &= 1 && \text{otherwise.}\end{aligned}$$

Proof. To prove (i) let $q-\varepsilon=bp^a$, $p\nmid b$ and expand $(bp^a+\varepsilon)^p=q^p$. Since p is odd, $p\mid\binom{p}{2}$, and the only terms not divisible by p^{a+2} equal $\varepsilon^p+\binom{p}{1}bp^a\varepsilon^{p-1}=\varepsilon+bp^{a+1}$. Thus, $q^p-\varepsilon\equiv bp^{a+1}$ modulo p^{a+2}.

The last two assertions are proved by induction. The induction steps are

$$(s, q-\varepsilon)=(s-a_n q^{n-1}(q-\varepsilon), q-\varepsilon)=\Sigma\varepsilon^n a_n$$

and assuming $a\geqslant b$

$$\begin{aligned}(2^a-\varepsilon, 2^b-\tau) &= (2^a-\varepsilon-2^{a-b}(2^b-\tau), 2^b-\tau)\\ &= (2^{a-b}\tau-\varepsilon, 2^b-\tau)\\ &= (2^{a-b}-\tau\varepsilon, 2^b-\tau).\end{aligned}$$

LEMMA 2.25. *With q and p as above and $m\geqslant 2$*

(i) $|A_{m-1}(q^p)|\nmid|A_{pm-2}(q)|$ *if* $p\mid q-1$, *and*

(ii) $|{}^2A_{m-1}(q^p)|\nmid|{}^2A_{pm-2}(q)|$ *if* $p\mid q+1$.

Proof. Let $\varepsilon=1$ if (i) fails, and $\varepsilon=-1$ if (ii) fails. Cancelling terms which appear in both order formulas and replacing $1/(m, q^p-\varepsilon)$ with $1/(q^p-\varepsilon)$, we obtain

$$r=\frac{q^{mp}-\varepsilon^m}{q^p-\varepsilon}\ \Big|\ \prod_{\substack{j=2\\ p\nmid j}}^{pm-1}(q^j-\varepsilon^j). \qquad (*)$$

Note that r is an integer by the preceding lemma. Likewise replace each factor $(q^j-\varepsilon^j)$ in $(*)$ by its greatest common divisor with $q^{mp}-\varepsilon^m$ and conclude that r divides some power of $q^m-\varepsilon^m$. Let

$$s=(\varepsilon q)^{m(p-1)}+(\varepsilon q)^{m(p-2)}+\cdots+1. \qquad (**)$$

We have $s(q^m-\varepsilon^m)=q^{mp}-\varepsilon^m$ divides $q^p-\varepsilon$ times some power of $q^m-\varepsilon^m$. By Lemma 2.24(i, ii), $p\,\|\,s$ and $(s, q^m-\varepsilon^m)=p$. Note that $p\mid q-\varepsilon$ implies $p\mid q^m-\varepsilon^m$, so Lemma 2.24(i) is applicable.

Our conditions force $s \mid p(q^p - \varepsilon)$ whence $s \leqslant p(q^p - \varepsilon)$. The summands in $(**)$ are of decreasing magnitude and are all positive or alternate in sign. It follows that $q^{m(p-1)} \leqslant s$ whence $q^{m(p-1)} \leqslant p(q^p + \varepsilon) \leqslant pq^{p+1}$. Thus $q^{(m-2)(p-1)+p-3} \leqslant p$. As $p \geqslant 3$, $2^{(m-2)(p-1)} \leqslant q^{(m-2)(p-1)} \leqslant p$ forces $m = 2$, and likewise $p = 3$.

Now $q^4 + q^2 + 1 = r \leqslant p(q^p + \varepsilon) = 3(q^3 + \varepsilon)$ implies $q = 2$, $\varepsilon = -1$, but $r = 21$ does not divide $p(q^p + \varepsilon) = 27$.

LEMMA 2.26. *In the following table* $|A| \nmid |B|$.

A	B
$G_2(q)$ or ${}^3D_4(r)$, $r^3 = q$	$A_3(q)$ or ${}^2A_3(q)$
${}^3D_4(q)$ or $A_2(q^3)$	$D_5(q)$ or $A_5(q)$
${}^3D_4(q)$ or ${}^2A_2(q^3)$	${}^2D_5(q)$ or ${}^2A_5(q)$

Proof. Similar to that of the preceeding lemma but easier.

PROPOSITION 2.27 (The Steinberg Relations). *Let Σ be an indecomposable root system of rank at least 2 and let $<$ be an ordering on Σ* [54]. *To each $\alpha \in \Sigma$, let there be associated a group X_α (a "root group"). Suppose that for any pair of roots α, β with $\alpha \neq -\beta$ the following holds; whenever $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$, there are elements $x_\gamma \in X_\gamma$ for every $\gamma \in \Sigma$ of the form $\gamma = i\alpha + j\beta$, i, j nonnegative integers or half-integers such that*

$$[x_\alpha, x_\beta] = \prod_\gamma x_\gamma, \qquad (*)$$

where the order of the product is given by $<$.

Let G be the group generated by all X_α, $\alpha \in \Sigma$, subject to the relations in X_α (relations of type (A)) *and all relations* $(*)$ (*relations of type* (B)).

Suppose that $\bar{G}$ is a quasisimple group of Lie type over $\mathbb{F}$ generated by a usual set of root elements $\bar{x}_\alpha$, for $x_\alpha \in X_\alpha$, $\alpha \in \Sigma$ such that there is a homomorphism $G \xrightarrow{\phi} \bar{G}$ satisfying $x_\alpha \mapsto \overline{x_\alpha}$, for all x_α. Then $\ker \phi \leqslant Z(G)$.

Proof. See Steinberg [51, 53, 54]. Our hypotheses imply that $\bar{G}$ is a Chevalley group or a Steinberg variation, or in the family ${}^2F_4$.

LEMMA 2.28. *Let Σ be an indecomposable root system and W the Weyl group. Let $\Theta = \{\frac{1}{2}, 1, 2\}$, $\Theta\Sigma = \{\lambda r \mid \lambda \in \Theta, r \in \Sigma\}$, and define the following equivalence relation on the set of unordered pairs in $\Sigma \times \Sigma$: $\{r, s\} \sim \{r', s'\}$ if and only if*

(i) *the set of lengths in* $\{r, s\}$ *equals that for* $\{r', s'\}$.

(ii) $\langle r, s = \langle r', s'$, *where* $\langle v, w$ *denotes the undirected angle between the nonzero vectors* v *and* w.

(iii) $r + s \in \Theta\Sigma$ *if and only if* $r' + s' \in \Theta\Sigma$.

Let Ω *be the set of equivalence classes. Then each member of* Ω *is an orbit under* W *with the following exceptions:*

(a) Σ *has type* A_n, $n \geqslant 2$; *the equivalence classes of* (r, s), $<r, s = \pi/3$ *and* $2\pi/3$;

(b) Σ *has type* D_n, $n \geqslant 3$; *the equivalence class of* (r, s), $<r, s = \pi/2$.

In any case, if $\{r, s\} \sim \{r', s'\}$, *then the rank* 1 *or* 2 *root systems they generate are conjugate under* W.

Proof. Exercise.

LEMMA 2.29. *Suppose that* Σ *is an indecomposable root system of rank at least* 2 *for the twisted group* $K \in \mathrm{Chev}(2)$. *Let* W *be the usual subgroup of* K *isomorphic to the Weyl group of* Σ *and let* $W^* = \{w \in W \mid$ *when* w *is expressed as the product of fundamental reflections, the number of short roots is even*$\}$.

Then (i) W^* *is transitive on the sets of roots of the same length*; (ii) *for* $w \in W^*$ *and* $\alpha \in \Sigma$ *such that* $\alpha^w = \alpha$, $x_\alpha(t)^w = x_\alpha(t)$ *for all* t (iii) *when* K *has type* ${}^2A_{2n}(q)$, $w \in W^*$ *and* α *is a long root satisfying* $\alpha^w = \alpha$, $x_\alpha(t, u)^w = x_\alpha(t, u)$ *for all appropriate* t, u.

Proof. See [36].

PROPOSITION 2.30. (a) *Suppose that* $G = \langle K, W\rangle$ *where* $K \in \mathrm{Chev}(2)$, W *is the Weyl group of a root system* Σ *of rank at least* 2.

Suppose further that (i) $\Sigma_1 \subseteq \Sigma$ *is a root system for* K; *and* (ii) $W_1 = W \cap K$ *is the Weyl group of* K *in its action on* Σ_1; (iii) $\Sigma_1 \times \Sigma_1$ *contains representatives of every* W-*orbit on* $\Sigma \times \Sigma$; *and* (iv) $W_\alpha :=$ $\{w \in W \mid \alpha^w = \alpha\}$ *normalizes* X_α, *for* $\alpha \in \Sigma_1$ *and* X_α *the root group of* K *associated to* α. *Then* $G \in \mathrm{Chev}(2)$.

(b) *The hypothesis* (a(iii)) *follows if* Σ_1 *is indecomposable and contains roots of all lengths which occur in* Σ, *and* (i) Σ_1 *has rank* 3 *and* Σ *has only one root length, or* (ii) Σ_1 *has rank* 4 *when* Σ *has type* B_n *or* C_n, $n \geqslant 4$ *or* (iii) $\Sigma_1 = \Sigma$ *when* Σ *has type* F_4.

Proof. (a) For $\alpha \in \Sigma$, let $W_\alpha = \{w \in W \mid \alpha^w = \alpha\}$. Set $\tilde{W} = W$ if K is untwisted and let $\tilde{W} = W^*$, the group of Lemma 2.29, when K is twisted. By Lemma 2.29, $\tilde{W}$ is transitive on roots of the same length in Σ. Set $\tilde{W}_\alpha = W_\alpha \cap \tilde{W}$.

We define root elements for $\beta \in \Sigma$ by the formula $x_\beta(t) = x_\alpha(t)^w$ or $x_\beta(t, u) = x_\alpha(t, u)^w$, where $\alpha \in \Sigma$, and $w \in \tilde{W}$ satisfy $\beta = \alpha^w$. To check that this is a good definition, we need to have elements of $\tilde{W}_\alpha$ centralize X_α. Since W_α is generated by the reflections in it and since the set of such corresponding to long (short, respectively) roots fall into W_α-conjugacy classes, it suffices to check a representative from each such W_α-class. But we may take such w_γ in $W_\alpha \cap W_1$ since $\Sigma_1 \times \Sigma_1$ meets every W-orbit on $\Sigma \times \Sigma$. If K is untwisted or α is long or γ is long, $[X_\alpha, w_\gamma] = 1$. If α, γ are short and $\alpha + \gamma \notin \Sigma$, $[X_\alpha, w_\alpha] = 1$. If α, γ are short and $\alpha + \gamma \in \Sigma$, then w_γ induces a "field automorphism" on X_α. Thus, $[X_\alpha, \tilde{W}_\alpha] = 1$, and the well-definedness of the root elements follows.

Let $\mathbb{F}$ be the field of definition of K and $\mathbb{E}$ a quadratic extension, if appropriate. Call $w \in \tilde{W}$ an *even* element if $w \in W$ and *odd* if $w \in W - \tilde{W}$. We have the relations

$$x_\alpha(t)^w = x_{\alpha^w}(\bar{t}) \qquad \text{if } \alpha \text{ is short and } w \text{ is odd,}$$

$$\left.\begin{aligned} x_\alpha(t)^w &= x_{\alpha^w}(t) \\ x_\alpha(t, u)^w &= x_{\alpha^w}(t, u) \end{aligned}\right\} \quad \text{otherwise}$$

for all appropriate t, u and $\alpha \in \Sigma$, when $t \to \bar{t}$ generates $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$.

We verify the Steinberg relations for these root elements; see Proposition 2.27. The relations of type (A) follows easily by conjugation under W. Now for type (B). Take $\alpha, \beta \in \Sigma$ with $\alpha \neq -\beta$, $x \in X_\alpha$, $y \in X_\beta$ and $w \in W$ so that $\alpha^w, \beta^w \in \Sigma_1$. Then $[x^w, y^w]$ is a product of certain root elements as in the relations of type (B) for K. The totally of relations thus obtained is a set of Steinberg relations for some group of Lie type. See [54] for a display of the relations for the untwisted groups and [36] for a display of the relations for the twisted groups. Thus, $\langle K, W\rangle \in \mathrm{Chev}(2)$, as required.

The proof of (b) is an exercise.

PROPOSITION 2.31. (a) *Suppose that* $G = \langle K_1, \ldots, K_m, W\rangle$, *where* $K_1, \ldots, K_m$ *are quasisimple with components in* Chev(2), *W is the Weyl group of an indecomposable root system Σ of rank at least* 2.

Suppose further that (i) *for each $i = 1, \ldots, m$, there is a subset $\Sigma_i \subseteq \Sigma$ so that Σ_i is a root system for K_i; and* (ii) *$W_i = W \cap K_i$ is the Weyl group of K in its action on Σ_i; and* (iii) *$\bigcup_{i=1}^m \Sigma_i \times \Sigma_i$ contains representatives for every W-orbit on $\Sigma \times \Sigma$ and that for every i, j such that $\Sigma_i \cap \Sigma_j \neq \varnothing$, X_α is the same group for $\alpha \in \Sigma_i$ as for $\alpha \in \Sigma_j$, and that if α, $\beta \in \Sigma_i \cap \Sigma_j$, the commutator of elements in X_α and X_β is independent of taking $\alpha, \beta \in \Sigma_i$ or $\alpha, \beta \in \Sigma_j$.* (iv) *$W_\alpha := \{w \in W \mid \alpha^w = \alpha\}$ normalizes X_α, for each $\alpha \in \bigcup_{i=1}^m \Sigma_i$.*

Then $G \in$ Chev(2).

Proof. Imitate the proof of Proposition 2.30. A bit of care is needed to see that root elements are well defined for $m \geqslant 2$.

Tables B, C *and* P. The following three tables are critical to this paper. They contain information about elementary abelian p-subgroups and elements of order p in $K \in \text{Chev}(2)$ whose centralizers are in standard form.

(2.32) *Table* B. In the table which follows, we list simple groups $K \in \text{Chev}(2)$, certain odd primes p, subgroups $B \in \mathscr{B}_{\max}(\tilde{K}; p)$, where $\text{Inn}(K) \leqslant \tilde{K} \leqslant \tilde{\tilde{K}}$, and $\tilde{\tilde{K}}$ is the full group of inner-diagonal automorphisms of K. Every such B contains at least one subgroup $\langle x \rangle$ of order p such that $C_B(L(C_K(x))) = \langle x \rangle$ except in case K has type $A_1(q)$, ${}^2C_2(q)$ or type $A_2(q)$ such that $p = 3$ divides $q - 1$ or $(p, q) = (3, 2)$ and K has type $A_n(2)$, $n \leqslant 2$, in which cases there are none.

We also list $B^* \in SC_p(\tilde{K})$, where $B^* \geqslant B$. Then $|B^* : B| = 1$ or p and B^* is unique up to conjugacy in $C_{\tilde{K}}(B)$. Finally, we list $m(B)$, $m(B^*)$, $A(B)$, $A(B^*)$.

The method of verification of these assertions involves standard techniques from the theory of groups of Lie type, and is omitted. We do single out two results, Theorem 2.33 and Lemma 2.34, as relevant tools.

We construct B^* in this manner. Either B^* is available in a standard Cartan subgroup or we do the following. Let G be the ambient algebraic group containing K. Thus, G has an algebraic endomorphism σ with $K = L(C_G(\sigma))$. Choose $L \in \text{Chev}(2)$ such that $K < L < G$ such that L has the same type as G (so L is untwisted), p divides the order of H, a standard Cartan subgroup of L and $L = L^\sigma$. Let W be the standard copy of the Weyl group in L. Thus, $W \leqslant N_L(H) = HW$. For $w \in W$, if β is the corresponding inner automorphism, $\beta\sigma$ is conjugate to σ by an inner automorphism of G. So, by choosing w appropriately, a conjugate of $\Omega_1(O_p(C_H(\sigma\beta)))$ is our desired B^*.

Once we have B^*, the possibilities for B may be read off from the maximal parabolics of K, as $И_K(B; 2) \neq \{1\}$.

Uniqueness of B^* up to conjugacy in $\text{Aut}(K)$ is shown in Lemma 2.35. Thus, the B^* constructed above is essentially the only one.

Actually, the table contains a few cases where p neither splits nor half-splits K. See Section 1 for a full discussion.

LEMMA 2.33 (Lang's theorem). (i) *Let G be a connected linear algebraic group and σ an endomorphism of G onto G such that $|C_G(\sigma)|$ is finite. Then $x \mapsto x^{-1}x^\sigma$ is a surjective map $G \to G$.*

(ii) *If in addition, α is an endomorphism of G such that $\alpha = \sigma\beta$ for some inner automorphism β of G, there is $\gamma \in \text{Inn}(G)$ such that $\gamma^{-1}\sigma\gamma = \alpha$.*

TABLE B

K, p	$m(B)$	$m(B^*)$	$A(B)$	$A(B^*)$
$A_n(q), p\mid q-1$	n	n	Σ_{n+1}	Σ_{n+1}
	or $n-1, p\mid n+1$	or $n-1, p\mid n+1$	Σ_{n+1}	Σ_{n+1}
$p\mid q+1$	$\left[\frac{n+1}{2}\right]$	$\left[\frac{n+1}{2}\right]$	$2^{[(n+1)/2]}\Sigma_{[(n+1)/2]}$	$2^{[(n+1)/2]}\Sigma_{[(n+1)/2]}$
$p=7, n=8, q=2$	2	3	$3^2\Sigma_2$	$3^3\Sigma_3$
$C_n(q), q\mid q-1$	n	n	$2^n\Sigma_n$	$2^n\Sigma_n$
$p\mid q+1$	$n-1$	n	$2^{n-1}\Sigma_{n-1}$	$2^n\Sigma_n$
$D_n(q), p\mid q-1$	n	n	$2^{n-1}\Sigma_n$	$2^{n-1}\Sigma_n$
$p\mid q+1$	$\{n-1$, n even $\{n-1$, n odd	n $n-1$	$2^{n-1}\Sigma_{n-1}$ $2^{n-1}\Sigma_{n-1}$	$2^{n-1}\Sigma_n$ $2^{n-1}\Sigma_{n-1}$
${}^2D_n(q), p\mid q-1$	$n-1$	$n-1$	$2^{n-1}\Sigma_{n-1}$	$2^{n-1}\Sigma_{n-1}$
$p\mid q+1$	$\{n-1$, n even $\{n-1$, n odd	$\{n-1$ $\{n$	$2^{n-1}\Sigma_{n-1}$ $2^{n-1}\Sigma_{n-1}$	$2^{n-1}\Sigma_{n-1}$ $2^{n-1}\Sigma_n$
${}^2A_n(q), p\mid q-1$	$\left[\frac{n+1}{2}\right]$	$\left[\frac{n+1}{2}\right]$	$2^{[(n+1)/2]}\Sigma_{[(n+1)/2]}$	$2^{[(n+1)/2]}\Sigma_{[(n+1)/2]}$
$p\mid q+1$	$\{n-1$ $\{$or $n-2, p\mid n+1$	$\{n$ $\{$or $n-1, p\mid n+1$	Σ_{n-1} Σ_{n-1}	Σ_{n+1} Σ_{n+1}

Table continued

TABLE B (*continued*)

K, p	$m(B)$	$m(B^*)$	$A(B)$	$A(B^*)$
$F_4(q), p\|q-1$	4	4	W_{F_4}	W_{F_4}
$p\|q+1$	3	4	W_{C_3}	W_{F_4}
$E_6(q), p\|q-1$	$\begin{cases}6\\ \text{or } 5 \text{ if } p=3\end{cases}$	$\begin{cases}6\\ \text{or } 5 \text{ if } p=3\end{cases}$	W_{E_6}	W_{E_6}
$p\|q+1$	4	4	W_{F_4}[a]	W_{F_4}[a]
$p=7, q=2$	2	3	$3^2\Sigma_2$	$3^{1+2}SL(2,3)$
$^2E_6(q)\ p\|q-1$	4	4	W_{F_4}	W_{F_4}
$p\|q+1$	$\begin{cases}5\\ \text{or } 4 \text{ if } p=3\end{cases}$	$\begin{cases}6\\ \text{or } 5 \text{ if } p=3\end{cases}$	W_{A_5}	W_{E_6}
$E_7(q)\ p\|q-1$	7	7	W_{E_7}	W_{E_7}
$p\|q+1$	6	7	W_{D_6}	W_{E_7}
$E_8(q)\ p\|q-1$	8	8	W_{E_8}	W_{E_8}
$p\|q+1$	7	8	W_{E_7}	W_{E_8}
$p\|q^2+q+1, p\neq 3$	3	4	$3^{1+2}SL(2,3)$	$\hat{U}_4(2)$[b]
$^3D_4(q)\ p\|q-1$	2	2	W_{G_2}	W_{G_2}

[a] If M is the 2-local containing B, $A_M(B) \cong W_{C_4}$; the Levi factor for M has type $D_5(q)$.
[b] $\hat{U}_4(2)$ is a covering group of $U_4(2) = W'_{E_6}$.

Table continued

TABLE B (*continued*)

K, p	$m(B)$	$m(B^*)$	$A(B)$	$A(B^*)$
$p \mid q+1$	1	2	Z_2	W_{G_2}
$p \mid q^2+q+1, p \neq 3$	1	2	$\mathbb{Z}_2$	$SL(2,3)$ (an element of order 3 fixes nontrivial elements of B^*)
${}^2F_4(q), p \mid q-1$	2	2	D_{16}	D_{16}
$(q=2^k, k \text{ odd})\ p \mid q+1$	2	2	$GL(2,3)$	$GL(2,3)$
$G_2(q)\ p \mid q-1$	2	2	W_{G_2}	W_{G_2}
$p \mid q+1$	1	2	$\mathbb{Z}_2$	W_{G_2}
${}^2C_2(q)\ p \mid q-1$	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$(k=2^k, k \text{ odd})\, p \mid q+r+1, r^2=2q$	0	1	1	$\mathbb{Z}_4$
$p \mid q-r-1, r^2=2q$	0	1	$\mathbb{Z}_4$	$\mathbb{Z}_4$
${}^2G_2(q)\quad p \mid q-1$	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$(q=3^k, k \text{ odd})\, p \mid q+1$	0	1	0	$\mathbb{Z}_2$
$p \mid q^2+r+1, r^2=3q$	0	1	1	$\mathbb{Z}_2$
$p \mid q^2-r+1, r^2=3q$	0	1	1	$\mathbb{Z}_2$

Proof. (i) See [6, E, 2.2] or [54]. We deduce (ii) from (i) as follows. For $g \in G$, we have $g^{\alpha} = g^{\sigma\beta} = y^{-1}g^{\sigma}y$ for some $y \in G$. Write $y^{-1} = x^{-1}x^{\sigma}$ for some $x \in G$. Then $g^{x^{-1}\sigma x} = (xgx^{-1})^{\sigma x} = x^{-1}x^{\sigma}g^{\sigma}(x^{-1}x^{\sigma})^{-1} = y^{-1}gy = g^{\alpha}$. Take γ to be conjugation by x.

LEMMA 2.34. *Let $G \in \mathrm{Chev}(2)$, H a standard Cartan subgroup, and $g = uhn_w u' \in G$ an element in standard form (see [6, 53]). For $k \in H$, $g \in C_G(k)$ if and only if*

(i) *$w \in W$ fixes k,*

(ii) *when u and u' are written as products of root elements $\prod_{\alpha} x_{\alpha}$, where the product is taken in an appropriate ordering, we have that each x_{α} centralizes k.*

Proof. Exercise.

LEMMA 2.35. *Suppose $K \in \mathrm{Chev}(2)$ and p is an odd prime.*

(i) *If $K \subseteq \tilde{K} \subseteq \mathrm{Aut}(K)$ with $\tilde{K}$ acting as inner-diagonal automorphisms on K and if $B^* \in SC_p(\tilde{K})$ with $m_p(B^*) \geqslant m_{2,p}(\tilde{K})$, then B^* is the unique elementary abelian p-group of its rank in a Sylow p-subgroup of $\tilde{K}$ except for $K = A_2(q)$, $p = 3 \mid q-1$, $K = {}^2A_2(q)$, $p = 3 \mid q+1$, $K = G_2(q)$ or ${}^3D_4(q)$ with $p = 3$;*

(ii) *if K, p appears in Table* B, *and B realizes the 2-local p-rank of $\tilde{K}$, then B is unique up to conjugacy in $\tilde{K}$;*

(iii) *if $\hat{K}$ is a covering group of K and E is an elementary abelian p-group acting as inner-diagonal automorphisms on $\hat{K}$ with $m_p(E) \geqslant m_{2,p}(E\hat{K})$ and $E \in SC_p(E\hat{K})$, then E projects onto a group B^* of* (i) *except for $K = A_2(q)$ or ${}^2A_2(q)$ as in* (i);

(iv) *suppose G is of standard type with respect to $(B, x, K) \in \mathscr{S}^*(p)$ and $e(G) \geqslant 4$. Then $m_{2,p}(BK) = m_{2,p}(G)$ except perhaps when $K = {}^2E_6(q), p = 3 \mid q+1$, $m(B^*) = 7$ or $K = {}^2A_n(q)$, $p \mid (q+1, n+1)$, $m(B^*) = n+1$. In any case $m_{2,p}(B^*K) \geqslant 4$.*

Proof. (i), (ii), (iii) We may assume that the Lie rank of K is at least 3, by inspection of the low rank cases.

Case 1. $p \mid q-1$, then $B = B^*$. We may take $B = \Omega_1(O_p(H))$, H is a standard Cartan subgroup. Let A be another elementary abelian p-group in K such that $A \in SC_p(K)$. Then, as $B = B^*$, we may assume that $\langle A, B \rangle \leqslant P \in \mathrm{Syl}_p(K)$. Then we may take $z \in \Omega_1(Z(P)) \cap A \cap B$, $z \neq 1$. Set $C = C_K(z)$. From [54] we get the shape of C. Set $E = \langle C_{X_{\alpha}}(z) \mid \alpha \in \Sigma \rangle$, where Σ is a root system for K with root groups X_{α}. If $E \neq 1$, by induction we may assume that $E \cap A = E \cap B$. Thus, A and B both stabilize all the normal subgroups of E, whence A induces inner diagonal automorphisms on each normal subgroup E_i of E. If $p \nmid |Z(E_i)|$, then we may apply induction. If $p \mid |Z(E_i)|$ we may apply induction to $E_i|Z(E_i)$ as long as E_i does not contain a p-group Q with the property that $1 \neq Q' \leqslant Z(E_i)$ but $m(Q/Q \cap Z(E_i)) \geqslant m(A/A \cap Z(E_i))$. Since E_i is the family A_n or E_6, the only possibility is E_i of type $A_2(q)$, for $p = 3$. Note that when this occurs in our induction situation, $E_i \cong SL(3, q)$. Thus, $A \cap E_i$ and $B \cap E_i$ are conjugate. Moreover, induction actually gives us that the images of A and B in $\prod_i \operatorname{Aut} E_i$ are conjugate by an element of $\prod_i \operatorname{Inn} E_i$. So, assume $E = 1$. Then $A \cap B = \langle z \rangle$. Without loss, $A \nleqslant B$. Pick $a \in N_A(B) - C_A(B)$. If $O_p(N_G(H)/N_G(H) \cap C_G(B)) = 1$, then for some $n \in N_G(H)$, $\langle a, a^n \rangle$ acts as $SL(2, p)$ on $V = \langle z, z^n \rangle$. But $E \neq 1$ for some and hence all $v \in V^{\#}$. Thus, $O_p(N_G(H)/N_G(H) \cap C_G(B)) \neq 1$ and the Weyl group is S_3 or D_{12}. Using $E = 1$, we have $p = 3$ and $K/Z(K) = A_2(q)$. The assertion of the lemma can be checked directly in this case. (Use the fact that the centralizer of a 3-central element of order 3 in ${}^3D_4(q)$ is isomorphic to $SL(3, p)$.)

Case 2. $p \mid q + 1$. Let G be the ambient algebraic group over $\bar{\mathbb{F}}_2$. Then B^* lies in a Cartan subgroup H of L, where $K \leqslant L$, L is finite and untwisted in the same family as G, and σ has order 2 on L, where $K = O^{2'}(C_G(\sigma))$. Say $A \leqslant K$, $A \cong B^*$.

Suppose $m(B^*) = m_p(H)$. We quote Case 1 to get $g \in L$ with $A^g = B^*$. We argue that we may arrange for $g \in C_G(\sigma)$. We have that σ and σ^g centralize B^*. Since $m(B^*) = m_p(H)$, $C_{L\langle\sigma\rangle}(B^*) = H\langle\sigma\rangle$. Also, $\sigma^g \in H\sigma$ and $\sigma^2(\sigma^g)^2 \in C(H)$. Since H has odd order and σ has order 2 on H, Sylow's theorem applied to $H\langle\sigma\rangle/\langle\sigma^2\rangle$ implies that there is $h \in H$ with $\sigma = \sigma^{gh}$. Thus, $gh \in C_L(\sigma)$ and $A^{gh} = (B^*)^h = B^*$, as required.

Now to prove uniqueness of B up to conjugacy in case $m(B^*) = m_p(H)$. Without loss, $B < B^*$. Suppose $A < K$, $A \cong B$ and A lies in a 2-local of K. Let $A \leqslant A^* \in SC_p(K)$ By the above, we may assume $A < A^* = B^*$.

From Table B, we see that, with a few exceptions, $B = [B^*, \tilde{W}]$, where $W = A_K(B^*)$ is generated by a set of fundamental reflections and $\tilde{W}$ is generated by a subset of fundamental reflections. If $\tilde{W}$ is unique up to conjugacy in W, uniqueness of B follows; this is the case, except for $(W, \tilde{W}) = (W_{F_4}, W_{C_3} = W_{B_3})$ and $(W_{D_n}, W_{A_{n-1}})$. But here, the uniqueness of B via conjugacy in $\operatorname{Aut}(K)$ may be checked, case by case.

Suppose $m(B^*) < m_p(H)$. According to Table B, K has type $A_n(q)$ or $E_6(q)$. By inspecting the standard module for $A_n(q)$, it is easy to check the statement. Finally, let K have type $E_6(q)$.

Let $A \leqslant K$ satisfy $A = B^*$. We may assume that $\langle A, B \rangle \leqslant P \in \operatorname{Syl}_p(K)$. Since $C_K(B^*)$ is abelian, $A_G(B^*) \cong W_{F_4}$ implies that P is abelian for $p \geqslant 5$. So, for $p \geqslant 5$, $A = B$ and we may assume that $p = 3$. Let $\langle z \rangle = \Omega_1(Z(P))$. Then $L(C_K(z)) \cong SU(3, q) \circ SU(3, q)$, $|C_K(z){:}\ L(C_K((z))| = 3$ and elements of

$C_K(z) - O^{2'}(C_K(z))$ induce outer diagonal automorphisms. It is now an easy exercise to see that A and B are conjugate in $C_K(z)$.

Case 3. $p \mid q^2 \pm q + 1$. These few special cases are left as an exercise with Lang's theorem (2.33).

(iv) By definition of standard type, B acts nontrivially on a 2-group $T \subseteq C_G(x)$. If $[T, K] \neq L$, then the first assertion of (iv) is clear. Otherwise $[T, K] = 1 \neq [B, T]$ forces some $b \in B$ to induce an outer automorphism on K, and the assertion follows from checking the possibilities for L on Table B. Now $m_{2,p}(B^*K) \geqslant 4$ except perhaps if $K = {}^2A_n(q)$, $p \mid (q+1, n+1)$. But then $m(B^*) = 4$ and $\langle x \rangle = C_{B^*}(K)$ imply $n \geqslant 3$ whence $p \mid n+1$ forces $n \geqslant 4$ and (iv) holds.

(2.36) *Table* P. In the next table, we list all triples (K, p, L) where L is quasisimple with $L/Z(L) \in \mathrm{Chev}(2)$, K is a standard component in $\tilde{\tilde{L}}$ for the prime p, where p half-splits K or L, $L \leqslant \tilde{\tilde{L}} \geqslant \tilde{L}$, and $\tilde{L}$ is the group of inner-diagonal automorphisms on L, unless L has type $D_4(q)$ in which case $\tilde{L}$ is the group of inner-diagonal-graph automorphisms on L. The restrictions on n are for making $m_p(\tilde{L}) \geqslant 3$ and the G–L restriction consists of an additional condition to make (D, K) a standard subcomponent of (B, x, L); see Section 1 for these notations. In particular, no case with $m_p(K) = 1$ is listed.

Note that in most cases, but not all, p half-splits both K and L.

The completion of this table requires straightforward applications of standard techniques from the theory of groups of Lie type. See (2.31) and (2.32). Tables of a similar nature were compiled by Burgoyne and Williamson; see [10] and [33, Appendix to Part I].

(2.37) *Table* C. In the following table, we record all instances of the following: $K \in (\mathrm{Chev}\ 2)$, $B \leqslant K$ as in Table B, and K_1, K_2 such that (i) $G_i = L(C_K(z_i))$, (ii) $(B, z_i, K_i) \in \mathcal{S}^*(p)$ (with respect to G_1), $i = 1, 2$, (iii) $K = \langle K_1, K_2 \rangle$, (iv) p splits G_1 or G_2 and half-splits both. In the event that there is $L \in \mathrm{Chev}(2)$ such that $B \leqslant K < L$ and $G_i = L(C_L(z_i))$, $(B, z_i, G_i) \in S_L^*(p)$, for $i = 1$ and 2, we subscript $\langle K_1, K_2 \rangle$ with an *. For each occurence of an *, K_1, K_2, K and L are listed at the end of the table.

The table is used by choosing some K_1 down the left column, choosing an admissible K_2 above the solid line in row K_1, then reading $\langle K_1, K_2 \rangle$ just below K_2. Directly below $\langle K_1, K_2 \rangle$ is the subcomponent $L_0 = L(K_1 \cap K_2)$. Restrictions are written above K_2.

LEMMA 2.38. *Let $G_1 = \langle K_1, K_2 \rangle$ be any entry in Table* C *except*

$$A_{n+2}(2) = \langle A_n(2), A_{[(n+1)/2]}(4) \rangle, \qquad p = 3,$$

$$A_{11}(2) = \langle A_8(2), A_3(8) \rangle, \qquad p = 7,$$

TABLE P

K, p	L	$G - L$ restriction
$A_n(q), p\mid q-1, n\geqslant 2$	$A_{n+1}(q)$	
	$A_{n+2}(q), p\mid n+3$	
	$C_{n+1}(q)$	
	$D_{n+1}(q)$	
	${}^2D_{n+2}(q)$	
	$E_{n+1}(q), n=5, 6, 7, (n, p)\neq(7, 3)$	
	$E_8(q), n=8, p=3$	
	${}^2A_5(q^{1/2}), n=2, p\mid q^{1/2}-1$	
	$A_{2n+1}(q^{1/2}), A_{2n+2}(q^{1/2}), n\geqslant 3$	$(p, q)=(3, 4)$
$A_n(q), p\mid q+1, n\geqslant 3$	$A_{n+2}(q),$	$(p, q)=(3, 2)$
	$E_6(q), n=5$	$(p, q)=(3, 2)$ or $(5, 4)$
$A_2(8), p=7$	$A_8(2)$	
$A_2(16), p=5$	$A_5(4)$	$Z(K)=1$
$A_8(2), p=7$	$A_{11}(2)$	
${}^2A_n(q), p\mid q-1, n\geqslant 3$	${}^2A_{n+2}(q), n\geqslant 3$	
	${}^2D_4(q), n=3$	
	${}^2E_6(q), n=5$	
${}^2A_n(q), p\mid q+1, n\geqslant 2$	${}^2A_{n+1}(q)$	
	${}^2A_{n+2}(q), p\mid n+3$	
	$C_{n+1}(q)$	$(n, q)\neq(3, 2)$
	$D_{n+1}(q)$, n odd[a]	
	$D_{n+2}(q)$, n odd	
	${}^2D_{n+1}(q)$, n even	
	${}^2D_{n+2}(q)$, n even	
	${}^2E_6(q), n=5$	
	${}^2E_6(q^{1/3})$	$(n, p)=(2, 3)$
	$E_7(q), n=6$	
	$E_8(q), n=8, p=3$	
	$E_8(q), n=7, p\neq 3$	
$C_n(q), p\mid q-1, n\geqslant 2$	$C_{n+1}(q)$	
	$F_4(q), n=3$	
$C_n(q), p\mid q+1, n\geqslant 2$	$C_{n+1}(q)$	
	$F_4(q), n=3$	$(n, q)\neq(2, 2)$[b]
$D_n(q), p\mid q-1, n\geqslant 2$	$D_{n+1}(q)$	
	$E_{n+1}(q), n=5, 6, 7$	
	$E_6(q), n=4, p=3$	
$D_n(q), p\mid q+1, n\geqslant 2$	${}^2D_{n+1}(q)$	
	${}^2E_6(q), n=4, p=3$[c]	
	$E_7(q), n=6$	
${}^2D_n(q), p\mid q-1, n\geqslant 3$	${}^2D_{n+1}(q)$	
	${}^2E_6(q), n=4$	

[a]The only standard subcomponent of $D_4(q)$ is ${}^2A_3(q)={}^2D_3(q)$ for $p\mid q+1$.

[b]We record this as an official restriction even though it does not apply since we require $m_{2,p}(C_{n+1}(q))\geqslant 3$, i.e., $n\geqslant 4$.

[c]This applies only when $\tilde{L}=O^{3'}(\tilde{L})$.

Table continued

TABLE P (*continued*)

K, p	L	$G-L$ restriction
${}^2D_n(q), p\mid q+1, n\geqslant 3$	$D_{n+1}(q)$	
	$E_6(q), n=4$	
	${}^2E_6(q), n=5$	
	$E_8(q), n=7$	
${}^3D_4(q), p=3, 3\mid q-1$	$D_4(q^3)$	
${}^3D_4(q), p=3, 3\mid q+1$	$D_4(q^3)$	
${}^3D_4(q), p\mid q^2+q+1,$	$E_6(q)$	$p=7, q=2$
$p\mid q^2-q+1$	${}^2E_6(q)$	$p=7, q=2$
$E_6(q), p\mid q-1$	$E_7(q)$	
$p\mid q+1$	none	$(p,q)=(5,4)$
$p\mid q^2+q+1$	$E_8(q)$	$(p,q)=(7,2)$
${}^2E_6(q), q\mid q-1$	none	
$p\mid q+1$	$E_7(q)$	
$E_7(q), p\mid q-1$	$E_8(q)$	
$p\mid q+1$	$E_8(q)$	
$E_8(q), p\mid q-1$	none	
$p\mid q+1$	none	
$F_4(q), p\mid q-1$	none	
$p\mid q+1$	none	
$G_2(q), p\mid q-1$	$D_4(q), p=3$	
$G_2(q), p\mid q+1$	$D_4(q), p=3$	
${}^2F_4(q), p\mid q\pm 1$	none	

or (*)

$$E_6(2)=\langle A_5(2), A_5(2)\rangle, \qquad p=3.$$

Let R be the center of a root group X_α of $L_0=L(L_1\cap K_2)$ with α long if L_0 is any twisted group. For $J=K_1$, K_2, G_1 or G_0, R is the center of a root group X_β of J with β long if J is twisted except that for the entries

$$^2A_7(q)=\langle {}^2A_5(q), A_3(q^2)\rangle, \qquad p\mid q-1,$$

and (**)

$$^2E_6(q)=\langle {}^2A_5(q), {}^2H_5(q)\rangle, \qquad p\mid q-1$$

β is short if J is twisted.

Proof. First suppose L_0, K_1, K_2, and G_1 or G_0 are all defined over $\mathbb{F}_q$ with $p\mid q-1$. Then G_1 or G_0 is the layer of $C_{\tilde{G}}(\sigma)$ for an algebraic group $\tilde{G}$ and standard endomorphism σ. It turns out that the element of $z_1\in B$ with $K_1=L(C_{G_1}(z_1))$ is in a G_1-class (or G_0-class)

$$[\eta, \sigma]$$

TABLE C

	$(p,q)=(3,2)$ *for $p\|\|n+3$ $A_n(q)$	$C_n(q)$	$(p,q)=(3,2)$ $A_n(q)$	$D_n(q)$	${}^2D_{n+1}(q)$	$n=5$ $p=3$ $D_4(q)$	$D_5(q)$	$n=6,7$ $E_n(q)$	$n=6,7$ $D_n(q)$
$A_n(q)$ $p\|q-1$	$A_{n+1}(q)*$ $A_{n-1}(q)$	$C_{n+1}(q)$ $A_{n-1}(q)$	$D_{n+1}(q)$ $A_{n-1}(q)$	$D_{n+1}(q)$ $A_{n-1}(q)$	${}^2D_{n+2}(q)$ $A_{n-1}(q)$	$E_6(q)$ $A_3(q)$	$E_6(q)$ $A_4(q)$	$E_{n+1}(q)$ $A_{n-1}(q)$	$E_{n+1}(q)$ $A_{n-1}(q)$
	$n=8$ $p=3$ $D_7(q)$	$n=8$ $p=3$ $E_7(8)$	$q=8$ $n=3$ $p=3$ $A_8(2)$	$n=3$ $p\|q^{1/2}-1$ ${}^2A_5(q^{1/2})$	$p=3$ $p\|q^{1/2}+1$ $A_{2n-1}(q^{1/2})$	$p=3$ $p\|q^{1/2}+1$ $A_{2n}(q^{1/2})$			
$A_n(q)$ $p\|q-1$	$E_8(q)$ $A_6(q)$	$E_8(q)$ $A_6(q)$	$A_{11}(2)$ $A_2(8)$	${}^2A_7(q^{1/2})$ $A_2(q)$	$A_{2n+1}(q^{1/2})$ $A_{n-1}(q)$	$A_{2n+2}(q^{1/2})$ $A_{n-1}(q)$			
	$(p,q)=(3,2)$ $A_n(q)$	$n=5$ $(p,q)=(3,2)$ $A_5(q)$	$n=5$ $(p,q)=(3,2)$ $A_5(q)$	$(p,q)=(3,2)$ $A\left[\frac{n+1}{2}\right](q^2)$	$n=5$ $(p,q)=(3,2)$ ${}^2D_4(q)$				
$A_n(q)$ $p\|q+1$	$A_{n+2}(q)$ $A_{n-2}(q)$	$E_6(q)$ $A_3(q)$	$E_6(q)$ $A_2(q^2)$	$A_{n+2}(q)$ $A_{[(n+1)/2]}(q^2)$	$E_6(q)$ $A_3(q)=D_3(q)$				
	$q=2$ $n=8$ $p=7$ $A_3(8)$								
$A_n(q)$ $p\|q^2+q+1$ $p\neq 3$	$A_{11}(2)$ $A_2(8)$								
	$n=5$ $A_3(q^2)$	$n=5$ ${}^2D_4(q)$							
${}^2A_n(q)$ $p\|q-1$	${}^2A_7(q)$ $A_2(q^2)$	${}^2E_6(q)$ ${}^2A_3(q)$							

TABLE C *(continued)*

	*for $p \| n+3$ $^2D_n(q)$	$C_n(q)$	n even $D_n(q)$	n even D_{n+1}	n odd $^2D_n(q)$	n odd $^2D_{n+1}(q)$	$n=5$ $p=3$ $D_4(q)$	$n=5$ $^2D_5(q)$	$n=6$ $^2D_6(q)$	$n=7$ $E_7(q)$
$^2A_n(q)$ $p \mid q+1$	$^2A_{n+1}(q)*$ $^2A_{n-1}(q)$	$C_{n+1}(q)$ $^2A_{n-1}(q)$	$^2D_{n+1}(q)$ $^2A_{n-1}(q)$	$^2D_{n+2}(q)$ $^2A_{n-1}(q)$	$D_{n+1}(q)$ $^2A_{n-1}(q)$	$D_{n+2}(q)$ $^2A_{n-1}(q)$	$^2E_6(q)$ $^2A_3(q)$	$^2E_6(q)$ $^2A_4(q)$	$E_7(q)$ $^2A_5(q)$	$E_8(q)$ $^2A_6(q)$
	$n=8$ $p=3$ $^2D_7(q)$	$n=8$ $p=3$ $E_7(q)$								
$^2A_n(q)$ $p \mid q+1$	$E_8(q)$ $^2A_6(q)$	$E_8(q)$ $^2A_6(q)$								
	*for $n=3$ $A_n(q)$	$C_n(q)$	$n=3$ $C_3(q)$							
$C_n(q)$ $p \mid q-1$	$C_{n+1}(q)*$ $A_{n-1}(q)$	$C_{n+1}(q)$ $C_{n-1}(q)$	$F_4(q)$ $C_2(q)$							
	*for $n=3$ $^2A_n(q)$	$C_n(q)$	$C_3(q)$							
$C_n(q)$ $p \mid q+1$	$C_{n+1}(q)*$ $^2A_{n-1}(q)$	$C_{n+1}(q)$ $C_{n-1}(q)$	$F_4(q)$ $C_2(q)$							
	$A_n(q)$	*for $n=4, p=3$ $D_n(q)$	$n=5,6,7$ $A_n(q)$	$n=5$ $D_5(q)$	$n=6,7$ $E_n(q)$	$n=4$ $p=3$ $A_5(q)$	$n=7$ $p=3$ $A_8(q)$			
$D_n(q)$ $p \mid q-1$	$D_{n+1}(q)$ $A_{n-1}(q)$	$D_{n+1}(q)*$ $A_{n-1}(q)$	$E_{n+1}(q)$ $A_{n-1}(q)$	$E_6(q)$ $D_4(q)$	$E_{n+1}(q)$ $D_{n-1}(q)$	$E_6(q)$ $A_3(q)$	$E_8(q)$ $A_6(q)$			

Table continued

TABLE C (*continued*)

	*for $n = 4, p = 3$ $D_n(q)$	$n = 6$ ${}^2A_6(q)$	$n = 6$ $D_6(q)$	$n = 4$ $p = 3$ ${}^2A_5(q)$	n even ${}^2A_n(q)$	n odd ${}^2A_{n-1}(q)$	$n = 6$ ${}^2E_6(q)$		
$D_n(q)$ $p \mid q+1$	${}^2D_{n+1}(q)*$ ${}^2D_{n-1}(q)$	$E_7(q)$ ${}^2A_5(q)$	$E_7(q)$ ${}^2D_5(q)$	${}^2E_6(q)$ ${}^2A_3(q)$	${}^2D_{n+1}(q)$ ${}^2A_{n-1}(q)$	$E_7(q)$ ${}^2D_5(q)$			
	$A_{n-1}(q)$	${}^2D_n(q)$	$n = 4$ ${}^2A_5(q)$	*for $n = 4, p = 3$ ${}^2D_n(q)$	$n = 4$ ${}^2D_4(q)$				
${}^2D_n(q)$ $p \mid q-1$	${}^2D_{n+1}(q)$ $A_{n-2}(q)$	${}^2D_{n+1}(q)$ ${}^2D_{n-1}(q)$	${}^2E_6(q)$ ${}^2A_3(q)$	${}^2D_{n+1}(q)*$ $A_{n-2}(q)$	$E_6(q)$ $A_2(q)$				
	n odd $A_n(q)$	n even ${}^2A_{n-1}(q)$	*for $n = 4, p = 3$ ${}^2D_n(q)$	$n = 4$ $(p,q) = (3,2)$ $A_5(q)$	$n = 5$ ${}^2D_5(q)$	$n = 7$ ${}^2A_7(q)$	$n = 7$ $E_7(q)$	$n = 7$ $p = 3$ ${}^2A_8(q)$	$n = 4$ ${}^2D_4(q)$
${}^2D_n(q)$ $p \mid q+1$	$D_{n+1}(q)$ ${}^2A_{n-1}(q)$	$D_{n+1}(q)$ ${}^2A_{n-2}(q)$	$D_{n+1}(q)*$ $D_{n-1}(q)$	$E_6(q)$ $D_3(q)$	${}^2E_6(q)$ $D_4(q)$	$E_8(q)$ ${}^2A_6(q)$	$E_8(q)$ $D_6(q)$	$E_8(q)$ ${}^2A_6(q)$	$E_6(q)$ ${}^2A_2(q)$
	$A_6(q)$	$D_6(q)$							
$E_6(q)$ $p \mid q-1$	$E_7(q)$ $A_5(q)$	$E_7(q)$ $D_5(q)$							
$E_6(q)$ $p \mid q+1$	none								

TABLE C (*continued*)

	$q=2$ $p=7$ $E_6(q)$			
$E_6(q)$ $p\mid q^2+q+1$ $p\neq 3$	$E_8(q)$ ${}^3D_4(q)$			
${}^2E_6(q)$ $p\mid q-1$	none			
	${}^2A_6(q)$	$D_6(q)$		
${}^2E_6(q)$ $p\mid q+1$	$E_7(q)$ ${}^2A_5(q)$	$E_7(q)$ ${}^2D_5(q)$		
	$A_7(q)$	$D_7(q)$	$p=3$ $A_8(q)$	$E_7(q)$
$E_7(q)$ $p\mid q-1$	$E_8(q)$ $A_6(q)$	$E_8(q)$ $A_6(q)$	$E_8(q)$ $A_6(q)$	$E_8(q)$ $E_6(q)$
	${}^2A_7(q)$	${}^2D_7(q)$	$p=3$ ${}^2A_8(q)$	$E_7(q)$
$E_7(q)$ $p\mid q+1$	$E_8(q)$ ${}^2A_6(q)$	$E_8(q)$ ${}^2A_6(q)$	$E_8(q)$ ${}^2A_6(q)$	$E_8(q)$ ${}^2E_6(q)$
$E_8(q)$, $F_4(q)$, ${}^2F_4(q)$, $G_2(q)$, $p\mid q^2\pm 1$	none			
${}^3D_4(2)$	none with $m_{2,p}\geqslant 4$			

We now list the pairs $G_1 < G_0$ in order of occurence (marked by *) in Table C: (G_1, G_0) have types $(A_{n+1}(q), A_{n+2}(q))$ for $p\mid q-1$, $({}^2A_{n+1}(q), {}^2A_{n+2}(q))$ for $p\mid q+1$, $(C_3(q), F_4(q))$ for both $p\mid q-1$ and $p\mid q+1$, $(D_5(q), E_6(q))$, $({}^2D_5(q), {}^2E_6(q))$ for $p\mid q-1$ and $({}^2D_5(q), {}^2E_6(q))$, $(D_5(q), E_6(q))$ for $p\mid q+1$. *In all cases G_0 is generated by three components.*

(in the notation of Burgoyne and Williamson defined above); and if $\tilde{\Sigma}$ is a root system for $\tilde{G}$, then K_1 is the layer of the centralizer of σ on the root groups of $\tilde{G}$ corresponding to roots of $\tilde{\Sigma}$ in the kernel of χ, the character associated to $t(\eta)$. We can always find a root $\tilde{\alpha}$ in the kernel of χ and fixed by σ. In fact L_0 is located inside K_1 exactly as K_1 is in G_1 or G_0, and we can choose $\tilde{\alpha}$ to correspond to a root group of L_0. If $\tilde{X}$ is the root group of $\tilde{G}$ corresponding ro $\tilde{\alpha}$, then $C_{\tilde{X}}(\sigma)$ is a root group of L_0, K_1 and G_1 or G_0. Since the subgroups corresponding to roots of a given length are all conjugate in L_0, we may take $R = C_{\tilde{X}}(\sigma)$. The same argument works with respect to L_0, K_2 and G_1 or G_0.

If L_0, K_1, K_2 and G_1 or G_0 are all defined over q with $p \mid q+1$, we proceed as above using Lemma 2.19 and the endomorphism ρ defined there. We pick $\tilde{\alpha}$ with $\rho(\tilde{\alpha}) = -\tilde{\alpha}$. Note that if G_0 or $G_1 = F_4(q)$, $p \mid q^2 - 1$, then the roots of $\tilde{\Sigma}_1$ involved in K_1 or K_2 may form a root system of type B_3 or C_3. However for any field $\mathbb{F}$ of characteristic 2 there is an isomorphism $B_3(\mathbb{F}) \to C_3(\mathbb{F})$ which maps root groups to root groups, so we obtain a root group of $K_i = C_3(q)$ with respect to a root system of type C_3 in either case.

The remaining entries in Table C are $(*)$, $(**)$, and $E_8(2) = \langle E_6(2), E_6(2)\rangle$, $p = 7$. In the cases $(**)$ we proceed along the same lines as above taking $\tilde{\alpha}$ with $(\alpha, \sigma(\tilde{\alpha})) = 0$. In the last case use Lemma 2.20.

LEMMA 2.39. *Consider the entries*

$$A_{n+2}(2) = \langle A_n(2), A_{[(n+1)/2]}(4)\rangle, \qquad p = 3$$
$$A_{11}(2) = \langle A_8(2), A_3(8)\rangle, \qquad p = 7,$$

in Table C *and let* $L_0 = L(K_1 \cap K_2)$, $J = K_1, K_2$, *or* G_0. *Pick a root group* R *of* L_0 *and let* $N = N_J(R)$, $Q = O_2(N_J(R))$. *The following conditions hold:*

(i) $\langle [Q, Q], R\rangle \subseteq Z(Q)$;

(ii) *N has no central factors on Q/R;*

(iii) *if $J = G_0$ and $Z(Q) \subset U \subset Q$ with $U \lhd N_J(R)$, then $\langle L_0, U\rangle = J$;*

(iv) *if y acts on J and centralizes $r \in R^{\#}$, then y normalizes R;*

(v) *if y acts on J and $y \in O_2(N_{\langle J, y\rangle}(R))$, then y induces an inner automorphism.*

Proof. Use the standard matrix representation of J. For (v) proceed as in the proof of Lemma 2.11(iv).

LEMMA 2.40. *In any entry $G_1 = \langle K_1, K_2\rangle$ of Table* C *with $L(K_1 \cap K_2) = A_2(4)$, $C_{G_1}(D)$ contains elements acting as outer diagonal automorphisms on $L(K_1 \cap K_2)$. Further there is no entry with $K_i = A_2(4)$.*

Proof. We always find some $K_i = A_3(4)$, $A_5(2)$, or $A_6(2)$. Look in $C_K(D)$; cf. Table C.

LEMMA 2.41. *Consider the entry*

$$E_6(2) = \langle A_5(2), A_5(2)\rangle, \qquad p = 3,$$

in Table C. *Let* $L_0 = L(K_1 \cap K_2) = A_2(4)$ *and* $J = L_0$, K_1 *or* K_2. *Pick a root group* R *of* L_0 *and let* $N = N_J(R)$, $Q = O_2(N_J(R))$. *The following conditions hold:*

(i) $\langle [Q, Q], R\rangle \subseteq Z(Q)$;

(ii) N *has no central factors on* Q/R;

(iii) *if* y *acts on* J *and centralizes* $r \in R^{\#}$, *then* y *normalizes* R;

(iv) *if* y *acts on* J *and* $y \in O_2(N_{\langle J, y\rangle}(R))$, *then* y *induces an inner automorphism.*

Proof. Use the preceding lemma and the method of proof of Lemma 2.39.

LEMMA 2.42. *For any entry* $G_1 = \langle K_1, K_2\rangle$ *of Table* C, $L(K_1 \cap K_2)$ *has a root system of rank at least two with the following exceptions:*

$$D_5(q) = \langle {}^2D_4(q), {}^2A_3(q)\rangle, \qquad p \mid q+1,$$

$$E_6(q) = \langle {}^2D_4(q), {}^2D_4(q)\rangle, \qquad p \mid q+1,$$

in which cases $L(K_1 \cap K_2) = {}^2A_2(q)$, $q > 2$.

Proof. Check the possibilities for $L(K_1 \cap K_2)$ in Table C and determined that the standard component (B, x, L) has $L = {}^2A_3(q)$, $C_3(q)$, ${}^2D_3(q)$, or ${}^2D_4(q)$. Invoke Table B and $m(B) \geqslant 4$ to eliminate the first three possibilities. Consult Table C again.

LEMMA 2.43. *There are no entries in Table* C *with* $K_i = A_2(2)$ *or* $A_3(2)$, *or with* $L(K_1 \cap K_2) = A_2(2)$. *The entries with* $L(K_1 \cap K_2) = A_3(2)$ *are*

$$G_1 = A_7(2) = \langle A_5(2), A_5(2)\rangle, \qquad p = 3,$$

$$G_1 = D_5(2) = \langle {}^2D_4(2), {}^2D_4(2)\rangle, \qquad p = 3,$$

and

$$G_1 \subset G_0 = E_6(2),$$

$$G_1 = E_6(2) = \langle A_5(2), {}^2D_4(2)\rangle, \qquad p = 3.$$

In all these cases $B = B^*$ *has rank* 4 *and* $B \cap L(K_1 \cap K_2)$ *is conjugate in* $N_{G_i}(B)$ *to* D.

Proof. $B = B^*$ has rank 4 from Table B. Exhibit G_i as $C_{\tilde{G}}(\rho)$ where $\tilde{G}$ is the appropriate algebraic group and $\rho = I_w \sigma_2$. Take $D = \langle t(\eta_1), t(\eta_2)\rangle$; here $t(\eta_i)$ is in the standard form given by [10, Appendix 2]. Now $L(K_1 \cap K_2)$ is $C_{\tilde{K}}(\rho)$ for an algebraic group $\tilde{K}$ generated by root groups of $\tilde{G}$ forming a root system of type A_3. Further $B \cap L(K_1 \cap K_2) = B \cap \tilde{K} = \langle t(\eta_3), t(\eta_4)\rangle$, and reducing $t(\eta_3)$, $t(\eta_4)$ by the algorithm of [10, Appendix 2] gives the last assertion of the lemma.

In the same way we prove

LEMMA 2.44. *In the entry*

$$G_1 = E_6(2) = \langle A_5(2), A_5(2)\rangle$$

of Table C, $L(K_1 \cap K_2) = A_2(4)$, $B = B^*$ *has rank* 4, *and* $B \cap L(K_1 \cap K_2)$ *is conjugate in* $N_{G_1}(B)$ *to* D.

LEMMA 2.45. *Let* G *be a simple group which appears as* G_0 *or* G_1 *on Table* C, *and let* a *be an automorphism of* G *of order* p. *One of the following holds:*

(i) *a is G-conjugate to an inner automorphism induced by an element of* B^*;

(ii) *a is G-conjugate to an automorphism centralizing* B^*, *and a is conjugate by an inner-diagonal automorphism of G to a standard (with respect to some system of root groups of G) field automorphism of G;*

(iii) *one of the following occurs:*

G	$O^{2'}(C_G(a))$
$A_n(q), p \mid q-1, p \mid n+1$	$A_r(q^p), r = \dfrac{n+1}{p} - 1$
${}^2A_n(q), p \mid q+1, p \mid n+1$	${}^2A_r(q^p), r = \dfrac{n+1}{p} - 1$
$E_6(q), p = 3, \quad p \mid q-1$	${}^3D_4(q)$ *or* $A_2(q^3)$
${}^2E_6(q), p = 3, \quad p \mid q+1$	${}^3D_4(q)$ *or* ${}^2A_2(q^3)$
$D_4(q), p = 3, \quad p \mid q-1$	$G_2(q)$ *or* $A_2(q)$
$D_4(q), p = 3, \quad p \mid q+1$	$G_2(q)$ *or* ${}^2A_2(q)$
$D_4(q), p = 3$	${}^3D_4(r), r^3 = q$

Proof. If a is inner-diagonal, then the method of Burgoyne–Williamson yields either (i) or one of the first few cases listed in (iii). Suppose a is not inner-diagonal. By [8, Proposition 1.1] either one of the last three cases of (iii) holds or a is conjugate by an inner-diagonal automorphism of G to a standard field automorphism σ.

Exhibit G as $O^{2'}(C_{\tilde{G}}(\rho))$ as above and take an appropriate pth root λ of ρ such that λ and σ differ by an inner-diagonal automorphism of G. Observe that λ centralizes $C_{\tilde{T}}(\rho)$ where $\tilde{T}$ is an appropriate Cartan subgroup of $\tilde{G}$ containing B^*. It follows that a centralizes some group of inner-diagonal automorphisms isomorphic to B^*. But by the discussion preceding Lemma 2.33 there is just one such group up to conjugacy by an inner automorphism, so (ii) holds

DEFINITION 2.46. Let the quasisimple group K satisfy $|Z(K)|$ odd and $K/Z(K)$ an untwisted group or a Steinberg variation in Chev(2). Let Σ be a root system and X_α, $\alpha \in \Sigma$, root groups for K. If K has type ${}^2A_n(q)$, n even, and α is long, let $w_\alpha = x_\alpha(0, 1)\, x_{-\alpha}(0, 1)\, x_\alpha(0, 1)$. Otherwise, let $w_\alpha = x_\alpha(1)\, x_{-\alpha}(1)\, x_\alpha(1)$. Finally, let H be a standard Cartan subgroup of K and set $N = \langle H, w_\alpha \mid \alpha \in Z \rangle$. If $H \neq 1$, $N = N_K(H)$. We call a complement to H in N *a standard copy* of the Weyl group. The group $W = \langle w_\alpha \mid \alpha \in Z \rangle$ is called *the standard copy of the Weyl group*; it will be shown in Lemma 2.50 that it is isomorphic to the Weyl group.

If B^* is a subgroup of K described in Table B, a complement to $C_K(B^*)$ in $N_K(B^*)$ is called a standard copy of $A_G(B^*)$.

These notions all extend in a natural way to finite central products of groups as above.

LEMMA 2.47. *Let K be a field, H a group, M a KH-module. Then there exists an extension of KH-modules $0 \to M \to N \to T \to 0$ where* (i) *$T \cong \mathrm{Ext}^1_{KH}(K, M) = H^1(H, M)$ is a trivial module*; (ii) *if the extension is restricted to any nonzero submodule of T, it remains nonsplit*; (iii) *if $0 \to M \to N_1 \to T_1 \to 0$ is an extension of KH-modules with T_1 a trivial module having property* (ii), *then there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & M & \to & N_1 & \to & T_1 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & N & \to & T & \to & 0. \end{array}$$

In particular, all vertical arrows are inclusions.

Proof. The existence of $0 \to M \to N \to T \to 0$ follows from the properties of the Baer sum, described in [46, p. 69], for example. We give a sketch. Namely, let $\{f_i \mid i \in I\}$ be a K-basis of $\mathrm{Ext}^1_{KH}(K, M)$. To each f_i, we have an extension $0 \to M \to^{\alpha_i} E_i \to K \to 0$. Define $N = \coprod_i E_i/M_0$, where M_0 is the set of all $(m_i\alpha_i) \in \coprod_i M\alpha_i \leqslant \coprod_i E_i$, $m_i \in M$, $\sum_i m_i = 0$. The "universal property" may be proven using a Zorn's lemma argument.

LEMMA 2.48. *Let r be an odd prime, W an indecomposable Weyl group of rank at least 2 and let M be the nontrivial irreducible constituent of the natural $\mathbb{Z}$-lattice A for W reduced modulo r. Then* $\dim_{\mathbb{F}_r} H^1(W, M) = 1$ *when* $W \cong W_{A_n}$ *and* $r \mid n+1$ *or* $W \cong W_{E_6}$ *and* $r = 3$ *and* $\dim_{\mathbb{F}_r} H^1(W, M) = 0$ *otherwise.*

Proof. By inspection, M is faithful for W. If $O_2(W) \neq 1$, we quote [14] or [50]. Say $O_2(M)$. Then $W \cong W_{A_n}$ or W_{E_6}.

Case 1. $W = W_{A_n}$. Then $n \geqslant 2$. It is easy to check the result for $n = 2$ or 3, so assume $n \geqslant 4$ and that the result is true for $W_{A_{n-1}}$. Let V be a natural copy of $W_{A_{n-1}}$ in W. Consider an extension $0 \to M \to N \to T \to 0$, where T is a trivial $\mathbb{F}_r W$-module and $C_N(W) = 0$.

Suppose $H^1(V, M) = 0$. Then, V has a fixed point in any nonzero coset of M in N. Thus, $N = M \oplus U$, as $\mathbb{F}_r V$-modules. Let $t \in W$ be a transvection not in V. We have $\dim C_N(t) = \dim N - 1$. Thus, $W = \langle V, t \rangle$ and $C_N(W) = 0$ imply that $\dim T = \dim U = 0$ or 1. We, thus, get $\dim H^1(W, M) = 1$ in case $r \mid n+1$ by inspecting the permutation module over $\mathbb{Z}$ reduced modulo r. In case $r \mid n+1$, the restriction $H^1(W, M) \to H^1(V, M)$ is a monomorphism, whence $H^1(W, M) = 0$, as required.

We now argue that $H^1(V, M) = 0$. Suppose otherwise. Then $r \mid n$ and $\dim H^1(V, M_1) = 1$ by induction, where M_1 is the nontrivial $\mathbb{F}_r$-constituent within M (we also need $H^1(V, \mathbb{F}_r) = \mathrm{Hom}(V, \mathbb{F}_r) = 0$). Since $r \nmid n+1$, $\dim M = n$. By Lemma 2.47 and $\dim H^1(V, M_1) = 1$, $C_M(V) \neq 0$. Since M is irreducible for W, this means that M is a quotient of the natural permutation module S for $\mathbb{F}_r W$, whence $M \cong S_0$, where $S \cong S_0 \oplus K$. Thus, M is isomorphic to the natural permutation module $\mathbb{F}_r V$. Then Lemma 2.45 implies that $H^1(V, M) = 0$, as required.

Case 2. $W \cong W_{E_6}$, $H^1(W, M) \neq 0$. Then $r \mid |W| = 2^7 3^4 5$, whence $r = 3$ or 5. We claim that $r = 3$. Say $r = 5$. Since W contains a natural copy of W_{A_5} with index prime to 5, case 1 gives $H^1(W, M) = 0$. Thus, $r = 3$.

Let N be the natural $\mathbb{Z}$-lattice for W reduced modulo 3. Since the quadratic form on the lattice given by the Cartan matrix has determinant 3, we have a submodule N_0, the radial of the $\mathbb{F}_3$-valued form. Thus, $\dim N = 6$, $\dim N_0 \geqslant 1$. Since W contains a natural W_{D_5}-subgroup, we have $\dim N/N_0 \geqslant 5$, whence $\dim N_0 = 1$ and $M = N/N_0$. From [57], there is some $\mathbb{Z}$-lattice in $\mathbb{Q} \otimes A$, stable under W, whose reduction modulo 3, E, is indecomposable. Therefore, $\mathrm{Ext}^1(\mathbb{Z}_3, M) \neq 0$. Since M is self-dual, either statement gives $H^1(W, M) \neq 0$. Consider an extension $0 \to M \to M_1 \to T \to 0$ with T a trivial module and $C_{M_1}(W) = 0$ and $T = H^1(W, M)$; see Lemma

2.45. Regard E as a submodule of T. Let V, V_1 be natural W_{A_5}, W_{D_5}-subgroups of W, respectively. Since $\dim H^1(V, E) \leqslant 1$ by Case 1, $\dim C_{M_1}(V) \geqslant \dim T$. Since $M_1 = C_{M_1}(V_1) \oplus [M, V_1]$ and $\dim[M_1, V_1] = 5$, $\dim C_{M_1}(V_1) \geqslant \dim T$. Without loss, $V \cap V_1$ contains $\langle h \rangle$, a group of order 5. Since $M_1 = [M_1, h] \oplus C_{M_1}(h)$ and $\dim[M_1, h] = 4$, $\dim C_{M_1}(h) = 1 + \dim T$. Since $C_{M_1}(h) \geqslant C_{M_1}(V)$ and $C_{M_1}(V_1)$, we get $C_{M_1}(V) \cap C_{M_1}(V_1) \neq 0$ if $\dim T \geqslant 2$. So, if $\dim T \geqslant 2$, $W = \langle V, V_1 \rangle$ has a fixed point on M_1, a contradiction. Therefore, $\dim T = 1$, $E = M_1$ and $\dim H'(W, H) = 1$, as required.

LEMMA 2.49. *Let G_1, K_1, K_2, L_0 be as in Table* C *with G_1 of type $D_n(q)$, ${}^2D_n(q)$ or $C_n(q)$. If $D = C_{B^*}(L_0)$ and $K = L(C_{G_1}(z))$ for some $z \in D^\#$ and $\langle z \rangle = C_B(K)$, then $K = K_1$ or K_2.*

Proof. We sketch the proof. Let M be the standard $2n$-dimensional module over $\mathbb{F}_q$. By Table C, one of the following holds: L_0 has type $D_{n-2}(q)$, ${}^2D_{n-2}(q)$ or $C_{n-2}(q)$ and centralizes a 2-dimensional nonsingular subspace; or $p \mid q-1$, L_0 has type $A_{n-2}(q)$ or $A_{n-3}(q)$ and leaves invariant a pair of maximal totally singular subspaces meeting trivially; or $p \mid q+1$, L_0 has type ${}^2A_{n-2}(q)$ or ${}^2A_{n-3}(q)$ and $[M, L_0]$ may be regarded as the natural $n-1$- or $n-2$-dimensional $\mathbb{F}_{q^2}$-module for L_0. By the action of $\langle B^*, L_0 \rangle$ on M, any such K must be of type $D_{n-1}(q)$, ${}^2D_{n-1}(q)$, $A_{n-1}(q)$ or ${}^2A_{n-1}(q)$ in a natural representation as above. By inspecting the possibilities, one gets the lemma.

LEMMA 2.50. *Assume the notations of Definition* 2.46. *Let K be defined over $\mathbb{F}_q$.*

(i) *$w_\alpha = w_{-\alpha}$ is an involution, for all $\alpha \in \Sigma$, and $W = \langle w_\alpha \mid \alpha \in \Sigma \rangle$ is isomorphic to the Weyl group of K.*

(ii) *If H is a standard Cartan subgroup of K and $V \leqslant K$ so that $HV = HW$ and $H \cap V = 1$, then there is a inner-diagonal automorphism β in $C_{\mathrm{Aut}(K)}(H)$ such that $V^\beta = W$, unless possibly K has type ${}^2A_n(q)$ or ${}^2E_6(q)$. In the latter cases, there are $\beta, \gamma \in$ Aut K such that $V^{\beta\gamma} = W$, where β is as before and $\gamma \in$ Inn(K) and γ centralizes $H_1 = \{ y \in H \mid y^{q+1} = 1 \}$.*

(iii) *Let B^* be the subgroup of K described in Table* B, *$p \mid q^2 - 1$. Then there is a standard copy W^* of $A_K(B^*)$ in K and any two such are conjugate by an element of $C_{\mathrm{Aut}\,K}(B^*)$ in the group of inner-diagonal automorphisms, with the exception described in* (ii) *when B^* lies in a standard Cartan subgroup of K, where K has type ${}^2A_n(q)$ and $p \mid q-1$.*

Let W be as in (ii). *Then, replacing W or W^* by a conjugate in* Aut K, *we have the following containment relations:*

$A_n(q), p\mid q-1$	$W = W^*$	${}^2D_n(q), p\mid q-1$	$W = W^*$
$p\mid q+1$	$W > W^*$	$p\mid q+1$	$W \leqslant W^*$
$C_n(q), p\mid q-1$	$W = W^*$	$E_6(q), p\mid q-1$	$W = W^*$
$p\mid q+1$	$W = W^*$	$p\mid q+1$	$W > W^*$
$D_n(q), p\mid q-1$	$W = W^*$	${}^2E_6(q), p\mid q=1$	$W = W^*$
$p\mid q+1$	$W \geqslant W^*$	$p\mid q+1$	$W < W^*$

$F_4(q), E_n(q), n = 7, 8, p\mid q \pm 1$	$W = W^*$
$A_n(q), p\mid q^2+q+1, p \neq 3$	$W > W^*$
$E_6(q), p\mid q^2+q+1, p \neq 3$	$W > W^*$
$E_8(q), p\mid q^2+q+1, p \neq 3$	$W > W^*$

(iv) *Suppose $L \leqslant K$, $O_2(L) = 1$ and L is generated by a nonempty, proper subset of $\{X_\alpha, Z(X_\alpha) \mid \alpha \in \Sigma\}$. Then $W \cap L$ is a standard copy of the Weyl group for L. Furthermore, the standard copy of the Weyl group for L is contained in one for K,*

(v) *Let L be as in* (iv). (a) *If $p\mid q^2-1$ and B_L^* is a subgroup of L as in Table* B, *B_L^* is contained in a K-conjugate of B^*. Furthermore, if B^{**} is such a K-conjugate, then $B^{**} \leqslant C_K(L)\, C_K(C_K(L))$, unless L has type $A_n(q)$, K has type $A_{n+n'}(q)$ with $p\mid q+1$, n even and n' odd or L has type $A_n(q)$, n even, $p\mid q+1$ and K has type $D_{n'}(q)$, n' even or type ${}^2D_{n''}(q)$, n'' odd, type $C_{n''}(q)$, ${}^2A_{n'''}(q)$ or $F_4(q)$.* (b) *If W_L^* is a standard copy in L of $A_K(B^*)$, then W_L^* lies in a K-conjugate of W^*.* (c) *Let $B \leqslant B^*$ as in Table B, $p \mid q^2-1$, and let (B, x, L) be a standard subcomponent. Say $W^* = W_K^*$, as in* (b). *Then $W^* \cap L$ is a standard copy of $A_L(B^*)$.*

Proof. (i) Since our field has characteristic 2 the structure of the $\langle X_\beta, X_{-\beta}\rangle$ implies that w_α^2 centralizes $K = \langle X_\beta \mid \beta \in \Sigma\rangle$, whence $|w_\alpha| = 2$. To show that W is isomorphic to W_Σ, the Weyl group of Σ, we verify the appropriate relations among the w_α.

Let $m_{\alpha\beta}$ be the order of $\bar{w}_\alpha \bar{w}_\beta$ where bars denote images under $W \to W_\Sigma$. Set $u_{\alpha\beta} = (w_\alpha w_\beta)^{m_{\alpha\beta}}$. We want to show that $u_{\alpha\beta} = 1$. We can use induction on the Lie rank of K to reduce to the case of rank 2, where the root system is possibly decomposable. If decomposable, K is a central product and the result is clearly true. If not decomposable, then K has type A_2, C_2, G_2, ${}^2A_3$, ${}^2A_4$, or ${}^3D_4$. By dropping to the fixed point subgroup of the field automorphism, which contains W, it suffices to treat the cases A_2, C_2, G_2. These cases may be done by inspection.

(ii) The statement that V and W are H-conjugate would follow from the assertion $H^1(W, H) = 0$. This follows from [14] or [50] unless W has type A_n or E_6. The remaining statement follows from Table B, Lemmas 2.48 and 2.49, and the structure of Aut K.

(iii) Let us first suppose that B^* lies in a standard Cartan subgroup H of K. Then the statements follow from (i) and (ii). Thus, we may suppose otherwise.

We have that B^* lies in H_1, a standard Cartan subgroup of K_1, where $K \leqslant K_1 \in \mathrm{Chev}(2)$ and $K = L(C_{K_1}(\alpha))$ where α is a field or field-graph automorphism of order 2 or 3 of K_1. We shall do the case where $|\alpha| = 2$ in detail, and leave $|\alpha| = 3$ as an exercise.

Let W_1 be the standard copy of the Weyl group for K_1, $W_1 \leqslant N_{K_1}(H_1)$. By Table B and the accompanying discussions, we may take $\alpha = w_1\sigma$, where σ is the standard field or field-graph automorphism of K_1 and $w_1 \in W_1$ (considered as a subgroup of Aut K_1), $|w_1| = 2$, $\sigma w_1 = w_1\sigma$. Since $W_1^\alpha = W_1$, $C_{H_1W_1} = C_{H_1}(\alpha)\, C_{W_1}(\alpha)$. The required copy of $A_K(B^*)$ is the subgroup $C_{W_1}(\alpha)$.

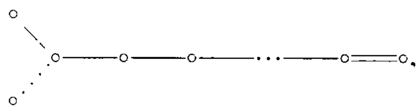
The statements about conjugacy follow as in the proof of (ii). The table is filled by studying the construction of Table B and the standard modules for the groups in Chev(2).

(iv) The statement about $W \cap L$ is clear from the definition of W and the fact that if Σ_1 is a subset of Σ which is itself a root system under the addition of Σ, then $W_{\Sigma_1} = \langle w_\alpha \mid \alpha \in \Sigma_1 \rangle$. If W_L is a standard copy of the Weyl group of L, the last part of (iv) follows unless possibly not all such groups are conjugate in L. In this case, however, L is proper in K and L has type A_n, ${}^2A_n$, E_6 or ${}^2E_6$. Thus, $A_K(L)$ induces the full group of inner-diagonal automorphisms on L, whence all such standard copies of the Weyl group of L are conjugate in $N_K(L)$, and we may proceed as above.

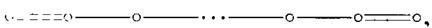
(v) It suffices to treat the case that B^* does not lie in a Cartan subgroup of K.

(a) Suppose that some $Z(X_\alpha)$ lies in L, where α is a root in Σ such that $\alpha^\perp = \{\beta \in \Sigma \mid \alpha \perp \beta\}$has rank one less than the rank of Σ and α is long in case there are two root lengths. Suppose further that K does not have type $B_2(2)$ or ${}^2A_3(2)$ or ${}^2A_4(2)$. Then $S = \langle Z(X_\alpha), Z(X_\alpha)\rangle = A_1(q)$ for some q and $C_K(S)' = L(C_K(S))$ is a central extension of a group in Chev(2). We may arrange for $B_S^* = B^* \cap S$ to have order p. Then $B^* \leqslant C_K(B_L^*) = H_S \cdot C_K(S)$ where H_S is the group of order $q+1$ in S containing B_L^*. If S has rank at least 2, we apply induction to the pair $(L \cap C_K(S)', C_K(S)')$ in place of (L, K). If $L = S$, the result is clear. If L has rank 1 but $L \neq S$, then $L = SU(3, q)$ and K has type ${}^2A_n(q)$. The embedding of B^* in K makes the result clear in his case. Finally, dropping the assumption that K have type $B_2(2)$, ${}^2A_3(q)$ or ${}^2A_4(2)$, we verify (a) directly in these cases.

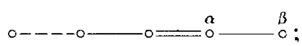
Now suppose that $Z(X_\alpha)$ and α may not be chosen as above. Then either K has type $A_n(q)$, for n, q or else Σ has two root lengths and the root groups in L are associated to only one root length. If K has type $A_n(q)$, the result is clear from the structure of Aut(K). So, assume K does not have type $A_n(q)$. Let l be the Lie rank of K, $l \geqslant 2$. Suppose Σ has type B_n. Since the extended Dynkin diagram looks like



the root lengths for L are short. In Σ, the sum of two orthogonal short roots is a long root. So, K is untwisted, i.e., K has type $B_n(q) = C_n(q)$ and L is a direct factor of $\Pi\langle x_\alpha, x_{-\alpha}\rangle$, where $\{\alpha, -\alpha\}$ sums over all n pairs of distinct short roots. But here, it is clear that L contains a copy of $\mathbb{Z}_{q+1}^n$, as required. (Perhaps a more proper interpretation here is that for this K, the α's should be regarded as long roots in a root system of type C_n.)

Suppose Σ has type C_n. Since the extended Dynkin diagram looks like , the roots for L are short. The structure of Σ shows that we may arrange for L to lie in the natural $A_{l-1}(q)$-subgroup of K, where Σ has rank l and K has type $C_l(q)$ or ${}^2A_r(q^{1/2})$ for $r = 2l - 1$ or $2l$. Our assertions now follow from inspection of the standard module.

Suppose Σ has type F_4, K of type $F_4(q)$ or ${}^2E_6(q)$, $p \mid q + 1$. The extended Dynkin diagram looks like



the three roots on the left are long. In Σ, the sum of two orthogonal short roots is long. So, if $K = {}^2E_6(q)$, the roots for L are short and L has rank at most 2. By properties of Σ, we may assume that $L \leqslant \langle X_{\pm\alpha}, X_{\pm\beta}\rangle$, and the assertions are easily checked. If $K = F_4(q)$ and the previous sentence does not apply, we may use the graph automorphism to invoke symmetry.

(b) By replacing B^* by a conjugate, we may assume that B_L^* is the group of Table B in L. If B^* lies in a Cartan subgroup of K, this is clear. Supposing otherwise, we proceed as follows. Since $L < K$, $N_K(L)$ induces on L the full group of inner-diagonal automorphisms, where all standard copies of $A_L(B^*)$ fuse in $N_K(L)$. We claim that $H^1(A_L(B^*), B^*) = 0$. We have that $B^* = B_1 \times B_2$ as $A_L(B^*)$ modules, where $[B_2, A_L(B^*)] = 1$, B_1 is indecomposable of dimension the rank of $A_L(B^*)$ as a Weyl group. We have already established that $H^1(A_L(B^*), B^*/C_{B^*}(A_L(B^*))) = 0$; see Lemma 2.48. Since $A_l(B^*)$ is generated by elements of order 2, $H^1(A_L(B^*), C_{B^*}(A_L(B^*))) = 0$, whence the claim follows. The claim now implies at once that the standard copy of $A_L(B^*)$ fuses into the standard copy of $A_K(B^*)$ in $N_K(B^*)$.

(c) By (a) and (b), it suffices to treat the case that L is not generated as in (iv). According to Table P, this means that $p|q+1$ and (L, K) is one of

$$(A_2(q), A_5(q)), \qquad ({}^2A_n(q), D_{n+\delta}(q)), \qquad \delta = 1, 2,$$
$$({}^2A_n(q), {}^2D_{n+\delta}(q)), \qquad \delta = 1, 2,$$
$$({}^2A_n(q), C_{n+1}(q)), \qquad ({}^2A_n(q), E_{n'}(q)),$$
$$(D_n(q), {}^2D_{n+1}(q)), \qquad ({}^2D_n(q), D_{n+1}(q)),$$
$$({}^2E_6(q), E_7(q)).$$

The assertion may be verified, case by case.

3. Linear Groups, Presentations and a Fusion Controlling Property of K-Groups

The first several results in this section are mainly concerned with answering the following question: given (B, x, L) and $B \subseteq B^*$ as in Sections 1 and 2, what are the possibilities for $A_G(B^*)$? We know that $A_G(B^*)$ is a subgroup of $GL(m(B^*), p)$ in which the stabilizer of a nonzero vector is essentially $A_L(B^*)$, a Weyl group.

Once we determine $A_G(B^*)$, the action on B^* is essentially unique, i.e., the reduction modulo p of the weight or root lattice when $R(B^*) := \langle \{r \in A_G(B^*) | r \text{ is diagonalizable with eigenvalues } \{-1, 1, 1, ..., 1\}\} \rangle$ is a Weyl group. This is an induction argument when $R(B^*) \cong W_{A_n}$, W_{D_n} or W_{C_n}; an exercise when $R(B^*) \cong W_{E_n}$ (use the natural containments $W_{A_n} \leqslant W_{E_n}$) and $R(B^*) \cong W_{F_4}$ (use $O_2(W_{F_4}) \cong 2_+^{1+4}$).

Lemma 3.1. *Let $p > 0$ be an odd prime, W an indecomposable Weyl group of rank $n \geqslant 3$ and M a nontrivial $F_p W$-module which is a section of the reduction modulo p of the natural $\mathbb{Z}$-free $\mathbb{Z}W$-module of rank n. Let $H \subseteq W$, H a homocyclic group of rank $t \geqslant 1$ and exponent $p^e > 3$. Suppose that $r_1 = \dim M$, $r_0 = \dim C_M(H)$. Then $t + r_0 < r_1$. The same conclusion holds vacuously if W is a Weyl group of type D_4 extended by a group of graph automorphisms.*

Proof. We first do the case that H lies in a subgroup $V \subseteq W$, where V is generated by fundamental reflections and $V \cong \Sigma_{n'}$, where $n' = n$ or $n - 1$. If W has type A_n, we require $V = W$. Note that this case always occurs when

W has type A_n, $B_n = C_n$, D_n, E_8 and E_7. In this case, $r_1 = n'$, $n' - 1$ or $n' - 2$ and $M|_{\mathbb{F}_p V}$ is a section of the natural $\mathbb{F}_p V$-permutation module M_0.

In the natural action of V on $\{1, 2,..., n'\}$, let $0_1, 0_2,..., 0_l$ be the orbits of H. Also, $r_0 = l$ by the structure of M_0. We argue that $n' - l \geqslant t(p^e - 1)$. We first prove the inequalities $n_i - 1 \geqslant t_i(p^e - 1)$, where $n_i = |\theta_i|$ and $t_i \geqslant 1$ is the rank of $\mho^{e-1}(H/C_H(\theta_i))$ for $i \in \{1,..., l\}$ such that $t_i \geqslant 1$. These inequalities follow from the fact that for Σ_{n_i} to contain $Z_{p^e}^{t_i}$ as a semi-regular subgroup we must have $n_i \geqslant t_i p^e$. Now sum these inequalities over i and use the fact that $\sum_{t_i \geqslant 1} t_i \geqslant t$, which follows from $H \subseteq \Sigma_{n'}$.

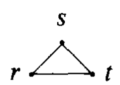
Now to prove that $t + r_0 < r_1$. Suppose $t + r_0 \geqslant r_1$. Then $n' - 2 \leqslant r_1 \leqslant t + r_0 \leqslant t + n' - t(p^e - 1)$, whence $2 \geqslant t(p^e - 2) \geqslant 3t \geqslant 3$, a contradiction.

We have now done a special case, and it remains to treat the case where H does not obviously lie in a suitable V. Thus, W has type E_6, E_7 or E_8 and $p^e = 5$, 7 or 9. (E_4 is out since $p^e > 3$; and W is not an extension of W_{D_4} for the same reason.) If $p^e = 7$, then $n = 7$ or 8 and there is a suitable $V \cong \Sigma_8$ in W. If $p^e = 5$ and $t = 1$, $V \cong \Sigma_6$ works. If $p^e = 5$ and $t = 2$, then $n = 8$ and we have $H \leqslant V_1 \times V_2$, $V_1 \cong V_2 \cong \Sigma_5$ (think of $V_1 \times V_2 \leqslant W_{E_8}$ corresponding to the natural containment $0^-(4, 2) \times 0^-(4, 2) \leqslant 0^+(8, 2)$). Thus, $r_1 = 8$, $r_0 \leqslant 2$, and $t = 2$ satisfy the required inequality. Finally, we look at the case $p^e = 9$. Since W_{E_n} has Sylow 3-group $P \cong Z_3 \sim Z_3$ for $n = 6, 7$ and $P \cong (Z_3 \sim Z_3) \times Z_3$ for $n = 8$, it follows that $t = 1$ and $H = \langle h \rangle \cong Z_9$ satisfies $r_0 \leqslant r_1 - 3$, whence $r_0 + t \leqslant r_1 - 2 < r_1$, as required.

LEMMA 3.2. *If r, s are conjugate reflections in a Weyl group, then $|rs| = 1, 2$ or 3.*

Proof. Let ρ: $W \to (n, R)$ be the natural representation of the Weyl group W. Since the eigenvalues for rs lie in R, $|rs| = 1, 2, 3$ or 6. If $|rs| = 6$, there are associated roots forming an angle of $5\pi/6$, i.e., $W \cong W_{G_2}$. But then r and s are not conjugate.

LEMMA 3.3. *If r, s, t are reflections in a Weyl group W and if*



is satisfied, then $\langle r, s, t \rangle \cong \Sigma_3$ or Σ_4.

Proof. Let $H = \langle r, s, t \rangle$. Since r^W is a class of 3-transpositions in W, any solvable subgroup S of H inverted by r has order 2, 3 or 6. By Lemma 3.2, H is a quotient of $Z^2\Sigma_3$. Let ρ be the natural representation ρ: $W \to O(n, R)$. Then $H^\rho \leqslant O(3, R)$, whence any elementary abelian 3-subgroup of H has order 3. Thus, $3^2 \nmid |H|$. So, if $H \not\cong \Sigma_3$, $O_2(H) \cong Z_2 \times Z_2$ and $H \cong \Sigma_4$.

PROPOSITION CF. *Let F be a field of characteristic $p \neq 2$ and B and F-vector space of dimension $n+1$, $n \geqslant 3$. Suppose that B has a basis $b_0, \ldots, b_n$ and that $H = RS \leqslant \mathrm{Aut}_F(B)$, where $R = \langle r_1, \ldots, r_n \rangle$ is elementary abelian of order 2^n and $b_i^{r_j} = b_i$ if $i \neq j$ and $b_i^{r_i} = b_i^{-1}$, and where $S \cong \Sigma_n$ acts on R and $\{b_1, \ldots, b_n\}$ in the natural way.*

Suppose that $K \leqslant \mathrm{Aut}_F(B)$, K is finite and $H = C_K(b_0)$ or $n = 4$, $C_K(b_0) \cong W_{F_4}$ or $W_{F_4}\langle \gamma \rangle$ where γ is the graph automorphism. Assume that $H^* = N_K(\langle b_0 \rangle) = H \times \langle c \rangle \subseteq K$ where c centralizes $\langle b_1, \ldots, b_n \rangle$. Let r_0 be defined by $b_0^{r_0} = b_0^{-1}$ and $b_i^{r_0} = b_i$ if $i \neq 0$. Let $R^* = \langle R, r_0 \rangle$. Then one of the following holds

(a) $r_0^K \cap \langle H^*, -1_B \rangle \subseteq R^*$ *and either*

(i) $R^* \lhd K$ *and* $K/R^* \cong \Sigma_{n+1}$; *or*

(ii) $p > 0$, $K = O_p(K)\, H^*$, $O_p(K)$ *is elementary abelian and is an $F_p H^*$-submodule of the stability group of* $B \supseteq \langle b_1, \ldots, b_n \rangle \supseteq 1$.

(b) $r_0^K \cap \langle H^*, -1_B \rangle \nsubseteq R^*$ *and either*

(i) $n = 3$, $O_2(K) \cong 2_+^{1+4}$, $K/Z(K) \cong W^*/Z(W^*)$ *where* $W^* \cong W_{F_4}$, *a subgroup of index 2 in* W_{F_4}, *or* $W_{F_4}\langle \gamma \rangle$ *where γ is the graph automorphism of* W_{F_4} *(depending on F, there may be more than one possible K satisfying these conditions if* $W^* \cong W_{F_4}\langle \gamma \rangle$); *or*

(ii) $n = 3$, $p = 3$, $K' \cong A_6$, $K/Z(K) \cong \Sigma_6$ *or* $\mathrm{Aut}\, A_6$ *and K has a subgroup isomorphic to* Σ_6; *if* $K/Z(K) \cong \mathrm{Aut}\, A_6$, $K/K'' \cong D_8$; *in any case* $-1 \in K$.

(iii) $n = 4$, $p = 3$, $H \cong W_{F_4}$ *and* $K \cong Z_2 \times W_{E_6}$ *or H is isomorphic to* $W_{D_4}\langle \theta \rangle$, *where θ is a graph automorphism of order 3,* $-1_B \notin H^*$, $\langle -1_B, H^* \rangle \cong Z_2 \times W_{F_4}$ *and* $K \cong W_{E_6}$.

Proof. We begin by observing that it does no harm to assume that $-1_B \in H^*$. The conclusions where $-1_B \notin H^*$ are easily deduced from those where $-1_B \in H^*$. Also, similar considerations allow us to assume that every element of K has determinant ± 1 on B. So, henceforth, we have $-1_B \in H^*$ and if $k \in K$, $\det k = \pm 1$. Thus, $c = r_0$. Define $K_1 = \langle k \in K \mid \det k = 1 \rangle$.

We first show that if $O(K) \neq 1$, then we are in case (a)(ii). Namely, $O(H^*) = 1$ means that $C_{O(K)}(r_0) = 1$, whence $O(K)$ is abelian. Denying (a)(ii) gives $O_{p'}(K) \neq 1$ whence $|O_{p'}(K)| = 3$ and $\dim_F[B, O(K)] = 2$. Thus, H^* has a 2-dimensional constituent on B, contradiction. So, $O(K) = 1$, and we also get that $Z^*(K) = \langle -1_B \rangle$ since $H^* \subset K$.

First dispose of the special case $n = 4$ and $3^2 \mid |H|$, i.e., H is an extension of W_{D_4} by Σ_3, its group of graph automorphisms or $H \cong W_{F_4}$ or $W_{F_4}\langle \gamma \rangle$ where γ is the graph automorphism of F_4. Work in K_1, so that $H_1 = H^* \cap K_1 \cong H$ satisfies $O_2(H_1) \cong 2_+^{1+4}$ and $H_1/O(H_1) \cong \Sigma_3 \times \Sigma_3$ or

$\Sigma_3 \wr Z_2$. Since $z = r_1 r_2 r_3 r_4 = -r_0 \notin Z^*(K_1)$, there is $t \in z^{K_1} \cap H_1$, $t \neq z$. By looking at traces, $t \notin O_2(H_1)$; also t does not invert $O_{2,3}(H_1)/O_2(H_1)$ since z has trace -3. It follows that $U_1 = C_{O_2(H_1)}(t) \cong Q_8$ or $\mathbb{Z}_2^3$. If Q_8, take $g \in K$ so that $t^g = z$. But then the above remarks about fusion of z force $U^g \cap O_2(H_1) = 1$, a contradiction. Thus, $U_1 \not\cong Q_8$. Now let $U = \langle U_1, t \rangle \cong Z_2^4$, $N = N_{K_1}(U)$. Our fusion information implies that $N \cong W_{D_5} \cong Z_2^4 \Sigma_5$. Fusion in H_1 and in N imply that N' meets two K_1-classes of involutions, i.e., those of z and of $v \in O_2(H_1) - \langle z \rangle$, $|v| = 2$. Also, if $t \in N$ represents a transposition in Σ_5 and t and z have the same set of eigenvalues, then $t \sim_{K_1} z$. Suppose $|K_1 : N|$ is even. Since z is 2-central in K_1, this means $K_1 \cong W_{F_4} \langle \gamma \rangle$. We argue that $\gamma \notin K_1'$. We have that $V = C_{H_1}(\gamma) \cong D_{16}$. Take $g \in K_1$ so that $\gamma^g \in T$, a Sylow group of the subgroup of H_1 corresponding to W_{F_4}. We may assume that $\gamma^g = v$ or z and that $V^g \leqslant T\langle \gamma \rangle$, as x and v are extremal in $T\langle \gamma \rangle$. Since $z^g \neq z$, we have $V^g \cap O_2(H_1) = 1$, a contradiction. So, $|K_1 : N|$ is odd, $H_1 \not\cong W_{F_4}\langle \gamma \rangle$ and so $H_1 \cong W_{D_4} \cdot \Sigma_3$. Now take the standard monomial matrix representation ρ for N. We may assume that

$$z^\rho = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}, \qquad t^\rho = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & 0 & -1 \\ & & & -1 & 0 \end{pmatrix}.$$

Then

$$(zt)^\rho = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

Taking traces, we see that zt does not fuse into N'. Thus, $K' = K_1'$ has index 2 in K_1 and $C_{K'}(z) = 2_+^{1+4}(\Sigma_3 \times Z_3)$ has a Sylow 2-group isomorphic to that of A_8. Since $K \leqslant GL(5, F)$, a theorem of Gorenstein and Harada [29] identifies $K' \cong \Omega(5, 3)$ (and $p = 3$). Thus, $K \cong Z_2 \times O(5, 3) \cong Z_2 \times W_{E_6}$, i.e., conclusion (b)(iii) holds.

We consider another special case, that of $Q = O_2(K) \supset \langle -1_B \rangle$. We may assume that $Q \neq R^*$. Define $R_0 = \langle r_i r_j \mid i, j = 1, 2, ..., n \rangle \cong Z_2^{n-1}$. We claim that $n = 3$. Define $Q_1 = Q \cap K_1$. Letting $Q_0 = N_{Q_1}(R)$, we have that Q_0 stabilizes $C_B(R) = \langle b_0 \rangle$, whence $Q_0 = \langle R, -1_B \rangle$ or $n = 3$. If $n \geqslant 4$, $N_{Q_1}(Q_0)$ preserves $\{\langle b_0 \rangle, ..., \langle b_n \rangle\}$, the eigenspaces for Q_0, whence $Q_1 = Q_0$ and $K = H^*$. So, $n = 3$ and a similar argument gives that $Q_0 = O_2(H^* \cap K_1) \cong 2_+^{1+4}$. Since $C_K(Q_0) \leqslant H^*$, it follows that $Q_0 = Q = C_K(Q)$ and that $K/Z(K) \to \operatorname{Aut} Q$. This leads to case (b)(i).

Having disposed of these special cases, we now have $O(K)=1$, $O_2(K)=\langle -1_B\rangle$, $H\cong W_{C_n}$ and $H^*\cong Z_2\times W_{C_n}$. We now deal with the cases (a) and (b).

(a) Since $Z^*(K)=\langle -1_B\rangle$, $r_0^K\cap H^*\neq\{r_0\}$. An eigenvalue argument shows that $r_0^K\cap H^*=\{r_0, r_1, r_2,..., r_n\}$. Thus, $N=N_K(R^*)$ satisfies $N/R^*\cong \Sigma_{n+1}$.

For $v\in N$, define property (*): if $g\in K$, $v^g\in N$, then $g\in N$. We have (*) for r_0, and $|K:N|$ is odd.

Let $w_i=r_0r_1\cdots r_i$ for $i=0, 1,..., n-1$, $C_i=C_K(w_i)$, $C_i^+=C_{C_i}(\langle b_0,..., b_i\rangle)$, $C_i^-=C_{C_i}(\langle b_{i+1},..., b_n\rangle)$. Since $N_K(\langle b_j\rangle)\leqslant N$ for all j, C_i^+ and C_i^- lie in N for all i. Since $r_0\in C_i^-$ and $r_n\in C_i^+$ for all i, (*) implies that $C_i\leqslant N$ for all i. Take $g\in K$ so that $w_2^g\in N$. Write $w_2^g=rs$ for $r\in R^*$, $s\in S$. Assume $s\neq 1$. The eigenvalues for w_2 restrict s to be (up to conjugacy) t_{01} or $t_{01}t_{23}$. If w_2^g centralizes some r_i, then $r_i\in C_i^g\leqslant N^g$, whence $g\in N$, a contradiction. Therefore, we may assume $n=3$ and $w_2^g=t_{01}t_{23}$. In the group $\bar K=K/\langle -1_B\rangle$, $\overline{R^*}$ is a self-centralizing eights group. Thus, we quote a result of Harada [40] to identify $\bar K$. Since $K\to GL(5,F)$ and $O_2(K)=Z(K)$, the only possibility is $K'\cong A_6$, whence $\langle r_0, K'\rangle\cong\Sigma_6$. But then (a) is violated. It follows that (*) holds for w_2. Now take any w_i and any $g\in K$ so that $w_i^g\in K$. Then w_i^g centralizes some N-conjugate w' of w_2, whence $w'\in C_i^g\subseteq N^g$. Using (*), $g\in N$. Thus, (*) holds for each w_i. Define $\mathscr{D}=\bigcup_{i=0}^{n-1} w_i^K$. Then $\bar K_0=\langle\bar{\mathscr{D}}\rangle$ satisfies a criterion of Aschbacher [2], whence $\bar K_0$ has a strongly embedded subgroup (as $O(\bar K_0)=O_2(\bar K_0)=1$), a contraction.

(b) Here, we must prove that $n=3$. Take $t\in r_0^K\cap H^*$, $t\in R^*$. Then, an eigenvalue argument shows that we may assume $t=t_{12}$, where $t_{ij}\in S\cong\Sigma_n$ is the element interchanging b_i with b_j and fixing the other b_j. Define $C=C_K(t)$, $C^+=C_C([B,t])$, $C^-=C_C(C_B(t))$ as before. Then $C^+\cap H^*=R_1S_1$ where $R_1=\langle r_j\,|\,j\neq 1,2\rangle\cong Z_2^{n-1}$ and $S_1=\langle t_{ij}\,|\,i,j\neq 1,2\rangle\cong\Sigma_{n-2}$ and $C^+\cong Z_2^{n-1}\Sigma_{n-1}$. Let π be the natural projection of C^+ onto Σ_{n-1}. Then $\langle r_0\rangle\times R_2S_1\leqslant C^+$, where $R_2=\langle r_j\,|\,j\neq 0,1,2\rangle\cong Z_2^{n-2}$. Suppose $n\geqslant 5$. Then, $(R_2S_1)^\pi$ must contain a natural copy of Σ_{n-2}. Since r_0^π commutes with this image, $r_0^\pi=1$, i.e., $r_0\in(R^*)^g$ where $g\in K$ satisfies $r_0^g=t$. But $t\in(R^*)^g$ and $C_{C^+\cap(R^*)^g}(R_2S_1)$ is a conjugate of $\langle r_1, r_2r_3\cdots r_n\rangle$, which contains only one element with the eigenvalues of r_0. So $r_0=t$, which is absurd. This leaves the cases $n=3$ and 4.

If $n=4$, then it is easy to see that r_0 is 2-central in H^* and in K. We may then imitate the special argument given at the beginning of the proof to get $K\cong Z_2\times W_{E_6}$. But this is a contradiction since $3^2\nmid|H^*|$ here.

We have $n=3$, $H^*\cong Z_2\times W_{C_3}\cong Z_2\times Z_2\times\Sigma_4$. Since $F^*(K)$ is not a 2-group, a theorem of Harada [40] implies that $F^*(K)\cong A_6$ and $p=3$. This leads to (b)(ii).

The proof of our proposition is complete.

(3.5) PROPOSITION D. *Let F be a field of characteristic $p \neq 2$ and B an F-vector space of dimension $n+1$, $n \geqslant 3$. Suppose that B has a basis $b_0, b_1, \ldots, b_n$ and that $H \subseteq H^* \subset K$ are finite subgroups of $\mathrm{Aut}_F(B)$ with the following properties:*

(i) $H^* = N_K(\langle b_0 \rangle) \leqslant N_K(\langle b_1, b_2, \ldots, b_n \rangle)$.

(ii) *$C_K(b_0)$ contains the subgroup H and $H = C_K(b_0)$ or $n=4$ and $C_K(b_0)$ contains H as a normal subgroup of index 3 (hence $H = O^{2'}(C_K(b_0))$ is characteristic) where $H = RS$, $R = \langle u_{ij} \mid i, j = 1, \ldots, n,\ i \neq j \rangle \cong Z_2^{n-1}$, $S \cong \Sigma_n$, $b_i^{u_{jk}} = b_i^{-1}$ if $i \in \{j, k\}$ and $b_i^{u_{jk}} = b_i$ otherwise, and where S acts naturally on $\{b_1, \ldots, b_n\}$ and on R.*

(iii) *$H^* = \langle C_K(b_0), c \rangle$ where c normalizes H.*

(iv) $C_{H^*}(H) = Z(H) \cdot C_{H^*}(\langle b_1, b_2, \ldots, b_n \rangle)$.

Let m be an integer such that $n = 2m$ or $2m+1$ and let $z = u_{12} u_{34} \cdots u_{2m-1,2m} \in R^{\#}$. Define $R^* = \langle R, -1_B \rangle$. Then one of the following holds.

(a) *$(-z)^K \cap \langle H^*, -1_B \rangle \subseteq R^*$ and either*

(i) *There is $u_{01} \in K$ so that $b_k^{u_{01}} = b_k$ if $k \notin \{0, 1\}$ and $b_k^{u_{01}} = b_k^{-1}$ if $k \in \{0, 1\}$, $\langle R, u_{01} \rangle \lhd K$ and $K \cong W_{D_{n+1}}$, $W_{C_{n+1}}$ or $W_{D_{n+1}} \times \langle -1_B \rangle$,*

(ii) *$K \cong W_{E_6}$ or $W_{E_6} \times \langle -1_B \rangle$ and $n=5$ or $p=3$ and $n=4$,*

(iii) *$p > 0$, $K = O_p(K) H^*$, $O_p(K)$ is elementary abelian and is an $F_p H^*$-submodule of the stability group of $B \supset \langle b_1, b_2, \ldots, b_n \rangle \supset 1$.*

(iv) *$K \cong \Sigma_5$ or $\Sigma_5 \times \langle -1_B \rangle$ and $n=3$.*

(b) *$(-z)^K \cap \langle H^*, -1_B \rangle \nsubseteq R^*$ and either*

(i) *$K \cong W_{E_{n+1}}$ for $n = 6$ or 7,*

(ii) *$K \cong \Sigma_6$ or $\Sigma_6 \times \langle -1_B \rangle$, $p=3$, $n=3$,*

(iii) *$n=3$, $O_2(K) \cong 2_+^{1+4}$ and $K/Z(K) \cong \tilde{W}/Z(\tilde{W})$ where $O_{2,3}(W^*) \subseteq W \subseteq W^* = W_{F_4}$ and $W/O_{2,3}(W^*)$ is the group of order 2 in $W^*/O_{2,3}(W) \cong Z_2 \times Z_2$ satisfying $W \nsubseteq SL(4, F)$ and $R = C_W(R)$.*

Proof. We begin by observing that it does no harm to assume that $-1_B \in H^*$. The conclusions where $-1_B \notin H^*$ are easily deduced from the conclusions where $-1_B \in H^*$. Also, similar considerations allow us to assume that every element of K has determinant ± 1 on B. So, henceforth, we have $-1_B \in H^*$ and if $k \in K$, $\det k = \pm 1$. Define $K_1 = \langle k \in K \mid \det k = 1 \rangle$, and when n is odd, $u_{0n} = -z$.

If $C_K(b_0) \cong W_{C_n}$ or an extension of W_{D_4} by Σ_3, the group of graph automorphisms, then we may quote Proposition CF to identify K. So, we assume that this does not happen.

Our next reduction is to identify K in case $O(K) \neq 1$ (we get (a)(iii)) and $O_2(K) \supset Z(K) = \langle -1_B \rangle$ (we get (a)(i), (b)(iii)). The case $O(K) \neq 1$ is handled as in Proposition CF, so we have $O(K) = 1$. In case $Q = O_2(K) \supset Z(K)$, we argue as in Proposition CF to get $n = 3$ and $O_2(K) \cong 2_+^{1+4}$. The slight changes in the argument are left to the reader.

Suppose $R \ntriangleleft H^*$. The structure of $H \lhd H^*$ then implies $n = 4$ and some 3-group in H^* transitively permutes the three subgroups of $O_2(H)$ which are normal in $O_2(H^*)$ and isomorphic to Z_2^3. We eliminate this situation with a special argument. The difficulty to keep in mind is the fact that the 2-fusion does lead to some simple groups. But we are safe because none of these lies in $GL(5, F)$.

Here is the special argument. Take $h \in H^*$ so that $|h|$ is a power of 3 and $R^h \neq R$. Define $H_0 = C_K(b_0)$. Since $n = 4$ is even and $\langle z \rangle = Z(T) \cap T'$ for $T \in \mathrm{Syl}_2(K)$, clearly $H^* = C_K(z)$ and $T \in \mathrm{Syl}_2(K)$. Since H_0 does not contain W_{C_4}, $O_2(H_0) \cong 2_+^{1+4}$ and $H_0 \cong 2_+^{1+4}(\Sigma_3 \times Z_3)$ is an extension of W_{D_4} by a graph automorphism of order 3. Since $O_2(H_0)$ is absolutely irreducible on $\langle b_1, b_2, b_3, b_4 \rangle$, the structure of $\mathrm{Aut}(2_+^{1+4}) \cong Z_2^4(\Sigma_3 \sim Z_2)$ and the fact that H_0 does not contain W_{C_4} implies that $H_0^* = \langle c \rangle \times H_0$. Since $\det c = \pm 1$, $|c| = 2$. Define $T_1 = T \cap K_1$. Then T_1 is isomorphic to a Sylow 2-group of H_0. Since T_1 is isomorphic to a Sylow 2-group of M_{12}, a look at the conclusions of a theorem of Gorenstein and Harada [29] shows that $K_1 \leqslant GL(5, F)$ implies $K_1 = O(K_1)(H^* \cap K_1)$, i.e., conclusion (a)(iii) holds.

Thus, we have $R \lhd H^*$ from now on. We quote [49] to see that $H^* = H \cdot C_{H^*}(H)$ or n is even and $H^*/C_{H^*}(H) \cong W_{C_n}$. We show that this latter case does not occur. Suppose it does and take $c \in H^*$ so that $C_H(c) \cong Z_2^{n-1}\Sigma_{n-1}$. Then $c^2 \in C(H) \cap C_K(\langle b_1, ..., b_n \rangle)$. Since $\det c = \pm 1$ and c normalizes $\langle b_0 \rangle = C_B(H)$, we get $c^2 = 1$. But now, c or $-zc$ lies in H, whence H contains a copy of W_{C_n}, a contradiction. Therefore, in all cases, $H^* = \langle c \rangle \times H$ where c is trivial on $\langle b_1, ..., b_n \rangle$ and is -1 on $\langle b_0 \rangle$. We keep this structure of H^* in mind during the rest of the proof, which breaks up into treatments of cases (a) and (b). Of course, we also have $O(K) = 1$ and $O_2(K) = Z(K)$.

(a) Let $T \in \mathrm{Syl}_2(H^*)$, $K_1 = \langle k \in K \mid \det k = 1 \rangle$, $T_1 = T \cap K_1$.

Case 1

n is even. Then z is 2-central in H^*, $\langle b_0 \rangle = C_B(Z(T) \cap T_1)$ whence $T \in \mathrm{Syl}_2(K)$ and $C_K(z) \leqslant H^*$. Since (a) holds, an eigenvalue argument shows that $z^K \cap H^* = \{z\}$. Therefore, Glauberman's Z^*-theorem [24] implies that $z \in Z^*(K) = Z(K)$ and so $K = H^*$, a contradiction.

Case 2

n is odd. Then R^* is generated by $\mathscr{B} = \{y \in R^* \mid y$ has two eigenvalues $-1\}$, a set of $\binom{n+1}{2}$ elements which is the union of the two H^*-classes $\{u_{0j} \mid j = 1,..., n\}$ and $\{u_{ij} \mid 1 \leqslant i < j \leqslant n\}$. In either case $(-z)^K \cap R^*$ generates R^*, which forces $T \in \mathrm{Syl}_2(K)$. Let $N = N_K(R^*)$. Then $N/R^* \cong \Sigma_{n+1}$ or Σ_n according to whether $\mathscr{B}$ is in one K-conjugacy class or not.

Subcase. $\mathscr{B} \neq (-z)^K \cap R^*$. We examine $C = C_K(u_{0n})$. Let $B^\varepsilon = \{b \in B \mid b^{u_{0n}} = b^{\varepsilon 1}\}$, $C^\varepsilon = C_C(B^{-\varepsilon})$ where $\{\varepsilon, -\varepsilon\} = \{+, -\}$. Then $B^- = \langle b_0, b_n \rangle$, $B^+ = \{b_1,..., b_n\}$. Since $|K: C \cap N|$ is odd, the action of $C \cap N$ on B^- shows that $\langle u_{01}, u_{0n} \rangle$ maps isomorphically onto a Sylow 2-subgroup of C/C^+. In this subcase, u_{01} does not fuse to $u_{1n} = u_{01} u_{0n}$ modulo C^+, whence C/C^+ is 2-nilpotent. Since $C/C^+ \to GL(2, F)$, u_{01} inverts $O(C/C^+)$ and either $O(C/C^+)$ is cyclic and completely reducible on B^- or $p > 0$ and $O(C/C^+)$ is an elementary abelian p-group.

Suppose that $O(C/C^+) = O(C)\, C^+/C^+$. Set $V = \langle u_{0n}^K \cap C \rangle$. Then $V/O(V)$ is elementary abelian. A theorem of Goldschmidt [26] implies that $M = \langle u_{0n}^K \rangle$ has the property that $M/O(M)$ is elementary abelian. Since $O(K) = 1$, we have $O(M) = 1$ which gives $M = R^*$ and $K = N$, i.e., (a)(i) holds.

We now have that $1 \neq O(C/C^+) \supset O(C)\, C^+/C^+$. Thus, $\mathrm{Out}(C^+)$ contains an element of odd order, whence $n = 5$ and $3 = |O(C/C^+): O(C)\, C^+/C^+|$. Thus, as $C \cap N$ contains a Sylow 2-group of K, it is easy to see that $K_1 \cap N \cong \langle -1_B \rangle \times Z_2^4 A_5$ and $K_1/\langle -1_B \rangle$ is a fusion-simple group with a Sylow 2-group isomorphic to that of A_8. We then quote a theorem of Gorenstein and Harada [30] to conclude that $K_1 \cong U_4(2)$. Therefore, $K \cong Z_2 \times W_{E_6}$, as required.

Subcase. $\mathscr{B} = (-z)^K \cap R^*$. Define $C_{i,j} = C_G(u_{ij})$, and let $C_{i,j}^\varepsilon = C_{C_{ij}}(B^{-\varepsilon})$ where $B_{ij}^\varepsilon = \{b \in B \mid b^{u_{0,n}} = b^{\varepsilon 1}\}$, $\{\varepsilon, -\varepsilon\} = \{+, -\}$. Set $C = C_{0,n}$, $B^\varepsilon = B_{0,n}^\varepsilon$, $C^\varepsilon = C_{0,n}^\varepsilon$. Since $B^+ = \langle b_1, b_2,..., b_{n-1} \rangle$, $B^- = \langle b_0, b_n \rangle$, it follows that $C^+ \leqslant H$, whence $C^+ = \langle u_{ij}, t_{ij} \mid 1 \leqslant i < j \leqslant n-1 \rangle \cong Z_2^{n-2} \Sigma_{n-1}$ and $C^+ \times C^- \lhd C$. Now, $C^- \times R_1 S_1 \leqslant C_{1,2}^+$ where $R_1 = \langle u_{ij} \mid 3 \leqslant i < j \leqslant n-1 \rangle \cong Z_2^{n-4}$, $S_1 = \langle t_{ij} \mid 3 \leqslant i < j \leqslant n-1 \rangle \cong \Sigma_{n-3}$. Since (a) implies $(t_{12} t_{34})^K \cap R^* = \varnothing$, or else such elements would be in $(-z)^N$, it follows that $S_1 \cong S_1^\pi$ is a natural subgroup isomorphic to Σ_{n-3} where π is the quotient map $C_{1,2}^+ \to \Sigma_{n-1}$. Thus, $(C^-)^\pi$ is trivial or is $\langle \tau \rangle$ for a transposition τ. Since $[C^-, R_1 S_1] = 1$, it follows that $|C^-| \mid 4$. On the other hand, C^- contains $\langle u_{0,n}, t_{0,n} \rangle$ and $u_{0,1}$ acts on C^- with centralizer $\langle u_{0,n} \rangle$, as $u_{0,1}$ acts on B^- with eigenvalues $\{-1, 1\}$ and $H^* = N_K(\langle b_0 \rangle)$. Thus, $C/C^+ \cong \langle C^-, u_{0,1} \rangle \cong D_8$. In any case, $C \leqslant N = N_G(\langle R, u_{0,n} \rangle)$. We finish as in the previous subcase by verifying the conditions of Goldschmidt's criterion [26]. This gives (a)(iv).

(b) Here we have to show that $n \in \{6, 7\}$ and that $K \cong W_{E_{n+1}}$.

Case 1

n is even. Then $H^* = C_{K(z)}$ has odd index in K, $t = -z$ is -1 on b_i, 1 on b_j for $j = 1, 2, \dots, n$. Also there is $g \in K$ so that $t^g \in H^* - R^*$. Eigenvalue considerations allow us to assume $t^g = t_{1,2}$. Now, $H^* = C_G(t)$ and $D = C_{H^*}(t_{1,2}) = \langle t, \quad u_{12}, \quad t_{12}, \quad u_{ij}, \quad t_{ij} \mid 3 \leqslant i < j \leqslant n \rangle \cong Z_2 \times Z_2 \times Z_2 \times Z_2^{n-3} \Sigma_{n-2}$. Let π be the natural epimorphism $(H^*)^g \to S^g \cong \Sigma_n$ and let $S_1 = \langle t_{ij} \mid 3 \leqslant i < j \leqslant n \rangle \cong \Sigma_{n-2}$. If $S_1 \cong S_1^\pi$, then $\pi|_{\langle t, u_{12}, t_{12} \rangle}$ has kernel E of order 4. We have $t_{1,2} \notin E$. If $t_{12} u_{12} \in E$, an eigenvalue argument forces $t_{12} u_{12} = t^g = t_{12}$, contradiction. Now, E contains an element with eigenvalues $\{-1, 1, 1, \dots, 1\}$ because $E = \langle t, u_{12} \rangle^g$. The only remaining possibility is $t \in E$. Since $E^{g^{-1}} \leqslant R^*$, we get $t = t^{g^{-1}}$, impossible since $t^g = t_{12}$. We conclude that $S_1 \ncong S_1^\pi$, whence $n - 2 = 4$ or 2.

Suppose $n = 4$. Then $H^* \cong Z_2 \times W_{D_4} \cong Z_2 \times 2_+^{1+4} \Sigma_3$ where an element of order 3 acts fixed point freely on the Frattini factor of the extraspecial group. Proceeding as before with $n = 4$, we observe that K_1 has a Sylow 2-group of type A_8. Since $K_1 \hookrightarrow GL(5, F)$, we get $K_1 \cong W_{E_6}$ [30]. But here, (a) holds, not (b), a contradiction. So $n = 4$ is out.

We have $n = 6$. Then $H^* \cong Z_2 \times W_{D_6}$. Thus, $C_{K_1}(t)$ is isomorphic to the centralizer of a 2-central involution in $Sp(6, 2)$. Since $O_2(K_1) = 1$, [55] implies that $K_1 \cong Sp(6, 2)$, whence $K \cong W_{E_7} \cong Z_2 \times S_p(6, 2)$.

Case 2

n is odd. Then $n \geqslant 5$; for if not, $n = 3$ and every involution of $H^* \cong Z_2 \times \Sigma_4$ outside R^* is conjugate to t_{12} or $-t_{12}$, in conflict with (b). Here, we do not know that H^* has odd index or that H^* contains $C_K(z)$. We use the notation C_{ij}^ε, etc., as in (a). Let $C = C_K(u_{0,n})$. As before, $C^+ = \langle u_{ij}, t_{ij} \mid 1 \leqslant i < j \leqslant n - 1 \rangle \cong Z_2^{n-2} \Sigma_{n-1}$.

The element $u_{0,n} = -z$ fuses to an element of $H^* - R^*$, which an eigenvalue argument shows is conjugate in H^* to $t_{12} t_{34}$. Let $D = C_K(t_{12} t_{34})$ and define D^+, D^- as with C. Then $C^- \times \langle t_{12}, u_{12}, t_{34} u_{34}, R_1 S_1 \rangle \subseteq D^+$, where $R_1 = \langle u_{ij} \mid s \leqslant i < j \leqslant n - 1 \rangle \cong Z_2^{n-6}$ if $n \geqslant 6$ and $R_1 = 1$ if $n = 5$, and $S_1 = \langle t_{ij} \mid 5 \leqslant i < j \leqslant n - 1 \rangle \cong \Sigma_{n-5}$. Let π be the natural projection of D^+ onto Σ_{n-1}.

The first step in our argument is to show that $C \subseteq N$. An eigenvalue argument shows that if $t \in t_{12}^K \cap D$, then t is a transposition. Therefore, $\langle t_{12} u_{12}, t_{34} u_{34}, R_1 S_1 \rangle^\pi$ contains a natural copy of $\Sigma_2 \times \Sigma_{n-5}$. Since $(C^-)^\pi$ commutes with this, we get $(C^-)^\pi$ embedded in a natural copy of $\Sigma_2 \times \Sigma_2$ or in a natural copy of $\Sigma_2 \times \Sigma_2 \times \Sigma_2$ and $n = 7$. Since $C^- \cap \ker \pi \subseteq (R^*)^g$, which consists of elements of determinant 1 only, it follows that $C^- \cap \ker \pi \subseteq \langle u_{0n} \rangle$. Thus, C^- is a 2-group and $|\Phi(C^-)| \leqslant 2$. Since $D^- \supseteq \langle t_{12}, t_{34} \rangle$, C^- contains a fours group. Since $C_{C^-}(u_{01})$ stabilizes $\langle b_0 \rangle = \{ b \in B^- \mid b^{u_{01}} = b^{-1} \}$, $C_{C^-}(u_{01}) \leqslant H^*$, whence $C_{C^-}(u_{01}) = \langle u_{0n} \rangle$.

Thus, $\langle C^-, u_{0n}\rangle$ is a group of maximal class of order at least 8 and at most 16 (since $|\Phi(C^-)| \leqslant 2$). Therefore, C^- contains an involution which conjugates u_{01} t $u_{01}u_{0n} = u_{1n}$ and so interchanges $\langle b_0\rangle$ and $\langle b_1\rangle$ under its action on B. Since C^- centralizes $\langle b_1,..., b_{n-1}\rangle$, it follows that $N = N_K(R^*)$ satisfies $N/R^* \cong \Sigma_{n+1}$ (the eigenspaces $\{\langle b_0\rangle,..., \langle b_n\rangle\}$ for R^* form a single N-orbit). Take $y \in N$ so that $b_0^y = b_1$ and $b_n^y = b_2$. Then $C_1 = (C^-)^y \subseteq H^*$. In fact $C_1 \subseteq \{h \in H^* \mid h$ centralizes b_i for $i \neq 1, 2\}$ $\langle u_{12}, t_{12}\rangle$, a four group. We conclude that $|C^-| = 4$ and $C \subseteq N$.

We now have $C \subseteq N$ and $N \cap D = R_2S_2 \times R_3S_3$, where $R_2 = \langle u_{12}, u_{34}\rangle \cong Z_2^2$, $S_2 = \langle t_{12}, t_{34}, t_{14}t_{23}\rangle \cong D_8$, $R_3 = \langle u_{ij} \mid i,j \in I\rangle \cong Z_2^{n-4}$, $S_3 = \langle t_{ij} \mid i,j \in I\rangle \cong \Sigma_{n-3}$, where $I = \{0, 5, 6,..., n\}$.

Take $k \in K$ so that $(t_{12}t_{34})^k = u_{12}$. Then $(R_3S_3)^k \leqslant C$ and an eigenvalue argument shows that its image in N/R^* is a natural Σ_{n-3} lying in C_{12}/R^*. By replacing k with an element of kC_{12}, we may assume that $R^*(R_3S_3)^k = R^*S_3$. Since $R_3S_3 = \langle t \in R^*S_3 \mid |t| = 2$ and, t has eigenvalues $\{-1, 1,..., 1\}$ on $R^*\rangle$, it follows that k normalizes R_3S_3.

Suppose k normalizes R_3. Since R_3 is an irreducible F_2S_3-module, we may assume that k centralizes R_3. So, $k \in C_K(u_{0n}) = H^* \subseteq N$, which conflicts with $(t_{12}t_{34})^k = u_{12}$. This contradiction would then complete the proof if we knew that k normalizes R_3. If this does not happen, $R_3 \neq O_2(R_3S_3)$, i.e., $n = 5$ or 7. When $n = 5$, $R_3 = \langle u_{05}\rangle = R_3S_3 \cap K_1$ is normalized by k, again, a contradiction. Thus, $n = 7$ and $R_3^k \neq R_3$ is the outstanding subcase.

Subcase $n = 7$. We adopt the notation and situation described in the last paragraph. We must show that $K \cong W_{E_8}$. The first step is to show that $|K: H^*|$ is odd.

Let $z = u_{05}u_{67}$, $T = \langle R^*, t_{12}, t_{34}, t_{14}t_{23}, t_{05}, t_{67}, t_{06}t_{57}, t_{10}t_{25}t_{36}t_{47}\rangle \in \mathrm{Syl}_2(N)$. Then $T/R^* \cong D_8 \wr \mathbb{Z}_2$, a Sylow 2-group of Σ_8, $|T| = 2^{14}$, $\langle -1_B\rangle = Z(T)$, $\langle -1_B, z\rangle/\langle -1_B\rangle = Z(T/\langle -1_B\rangle)$. We show that $T \in \mathrm{Syl}_2(K)$. Define $\mathcal{Q} = \{t \in T|$ has eigenvalues $-1, 1, 1,..., 1\} = \{t_{ij}, t_{ij}u_{ij} \mid \{i,j\} = \{1, 2\}, \{3, 4\}, \{0, 5\}$, or $\{6, 7\}\}$, $\mathscr{B} = \{\langle b\rangle \subseteq B \mid \langle b\rangle = [B, t]$ for some $t \in \mathcal{Q}\} = \{\langle b_ib_j^{-1}\rangle, \langle b_ib_j\rangle \mid \{i,j\} = \{1, 2\}, \{3, 4\}, \{0, 5\}$ or $\{6, 7\}\}$.

Now suppose that S is a 2-group in K containing T properly as a normal subgroup. Then S leaves $\mathcal{Q}$ and $\mathscr{B}$ invariant.

Define $S_0 = \{s \in S \mid s$ is trivial on $\mathscr{B}\}$ and $T_0 = T \cap S = \langle u_{12}, u_{34}, u_{05}, u_{67}, t_{12}, t_{34}, t_{56}, t_{78}\rangle \cong Z_2^8$. Then $s \in S_0$ acts as a scalar on each $\langle b\rangle \in \mathscr{B}$, whence S_0 is abelian. Since $C_K(u_{ij}) \subseteq N$, $S_0 \subseteq N$, whence $S_0 = T_0$. Thus, S/T_1 is embedded in Σ_8 and properly contains T/T_0 of order 2^6, i.e., S/T_1 acts on $\mathscr{B}$ as a full Sylow 2-group of the symmetric group on $\mathscr{B}$. Take $s \in S$ to induce a transposition. Then $s \notin T$ since T induces only even permutations on $\mathscr{B}$. Since $v = u_{01}u_{23}u_{46}u_{57}$ maps to an element of $Z(S/T_1)^\#$ and has orbits of shape $\{\langle b_ib_j\rangle, \langle b_ib_j^{-1}\rangle\}$ in $\mathscr{B}$, we may choose s to interchange $\langle b_1b_2\rangle$ and $\langle b_1b_2^{-1}\rangle$ and fix the other elements of $\mathscr{B}$ (s must preserve the orbits of v). Let $T_2 = N_T(\langle b_1, b_2\rangle)$ and let ψ be the natural map $\langle T_2, s\rangle \to \mathrm{Aut}_F(\langle b_1, b_2\rangle)$. Then $\langle T_2, s\rangle$ permutes $\langle b_1, b_2\rangle$, $\langle b_1 b_2^{-1}\rangle$. Let U be the kernel of this action. Then U^ψ is abelian. Since U^ψ contains $\langle u_{12}, t_{12}\rangle^\psi \cong Z_2 \times Z_2$, U is not cyclic. On the other hand, $\langle T_2, s\rangle^\psi$ must be a group of maximal class since $\{u \in \langle T_2, s\rangle \mid u^\psi$ commutes with $u_{01}\}$ stabilizes $\langle b_0\rangle$, hence lies in $H^* \subseteq N$ and so must have image $Z_2 \times Z_2$ under ψ. Therefore, $\langle T_2, s\rangle^\psi$ has maximal class. We conclude that $U^\psi \cong Z_2 \times Z_2$ and $\langle T_2, s\rangle^\psi = D_8$. Thus, $H = C_K(b_0)$ has order divisible by $|U|/2 = |\langle T_2, S\rangle|/4 = |T_2|/2 = |T|/8 = 2^{11}$, whereas $H \cong W_{D_7}$ has Sylow 2-group of order 2^{10}, a contradiction. We conclude that there is no such S, i.e., $T \in \mathrm{Syl}_2(K)$.

Now that we have $T \in \mathrm{Syl}_2(K)$, we can show that K is an odd-transposition group [4]. Let $\mathscr{D} = t_{12}^k$ be the proposed conjugacy class. Suppose $u, v \in \mathscr{D}$, d is an integer and $|uv| = 2d \geqslant 4$. We obtain a contradiction. Let w be the involution of $\langle uv\rangle$. Then w has eigenvalues $\{-1, -1, 1, 1, 1, 1, 1, 1\}$. If $w \in T$, an eigenvalue argument and the structure of T imply that we may assume $w = t_{12}t_{34}$ or $w = u_{0n}$. Replacing w by a K-conjugate, we may assume $w = u_{0n}$, whence $C_K(w) \subseteq N$. Therefore, u and v each have shape t_{ij} or $t_{ij}u_{ij}$. The structure of $N/R^* \cong \Sigma_8$ and $d \geqslant 2$ imply that $d = 2$ or $d = 3$. The structure of N implies that if u, v commute modulo R^*, then they commute. So, $d = 2$ is out. Since $d = 3$, we may assume $u = \{t_{ij}, t_{ij}u_{ij}\}$, $v \in \{t_{jk}, t_{jk}u_{jk}\}$ for distinct indices i, j, k. Let $l \in \{0, 1, \ldots, 7\} - \{i, j, k\}$. Since $(t_{ij}t_{jk})^3 = 1$ and $t_{ij}^{u_{jl}} = t_{ij}u_{ij}$, $t_{jk}^{u_{jl}} = t_{jk}$, $t_{ij}^{u_{kl}} = t_{ij}$, $t_{jk}^{u_{kl}} = t_{jk}u_{jk}$, it follows that $(uv)^3 = 1$, a contradiction to $|uv| = 2d$. We conclude that for $u, v \in \mathscr{D}$, $|uv|$ is 2 or an odd integer. An inspection of the possibilities shows that $K \cong W_{E_8}$.

The analysis of our subcase is completed and with it the proof of the proposition.

(3.6) PROPOSITION E. *Let F be a field of characteristic $p \neq 2$ and B an F-vector space of dimension $n + 1$ for $n \in \{5, 6, 7, 8\}$. Suppose that B has a basis $b_0, b_1, \ldots, b_n$ and that $H \subseteq H^* \subset K$ are finite subgroups of* $\mathrm{Aut}_F(B)$ *with the following properties:*

$H^* = N_K(\langle b_0\rangle) \subseteq N_K(\langle b_1, \ldots, b_n\rangle)$, $H = C_K(b_0) \cong W_{E_n}$ for $n \geqslant 6$, W_{E_6} for $n = 5$ and $p = 3$, or $Z_2 \times W_{E_6}$ for $n = 6$ or $p = 3$ and $n = 5$. Also $H^* = H \times \langle c\rangle$ and c is trivial on $\langle b_1, \ldots, b_n\rangle$; or $n = 5$ or 6, $H = W_{E_6}$ and c inverts $\langle b_1, \ldots, b_n\rangle$.

Then one of the following holds:

(a) *$p > 0$, $K = O_p(K)\,H^*$ and $O_p(K)$ is an F_pH^* submodule of the stability group of $B \supset \langle b_1, \ldots, b_n\rangle \supset 1$.*

(b) *$H \cong W_{E_n}$ and $K \cong W_{E_{n+1}}$ for $n = 6$ or 7.*

Proof. First some reductions. As in Proposition CF we may assume that $-1_B \in K$ and that every element of K has determinant ± 1. This forces $|c| = 2$. Let $K_1 = \{k \in K \mid \det k = 1\}$.

Suppose $O(K) \neq 1$. Then $C_{O(K)}(H')$ normalizes $\langle b_0 \rangle$. But $O(H^*) = 1$, whence $C_{O(K)}(H') = 1$. Now let $A \neq 1$ be an elementary abelian subgroup of $O(K)$ normalized by H'. Suppose AH^* is irreducible on B. Since $AH' \hookrightarrow GL(n+1, F)$, Clifford's theorem implies that H' must have a proper subgroup of index at most $n+1$, a contradiction. Thus, $p > 0$, A is a p-group, and we get (a).

Suppose $O_2(K) \supset \langle -1_B \rangle$. Then $O_2(K)\, H^*$ acts irreducibly on B. Clifford's theorem and the structure of H^* imply that B is irreducible for $O_2(K)$, i.e., $n = 7$. If $O_2(K)$ had an abelian subgroup $A \supset Z(K) = \langle -1_B \rangle$, invariant under H', the above argument could be applied to AH' to get a contradiction. Therefore, $O_2(K)$ is of symplectic type [27]. Since $|Z(O_2(K))| = 2$, $O_2(K)$ is extraspecial (of order 2^7). But $Sp(6, 2)$ is not involved in $\mathrm{Aut}(O_2(K))$, a contradiction.

Case 1. There is an involution $z^* \in Z(H)$. We set $z = -z^* \in C_K(\langle b_0, \ldots, b_n \rangle)$. Since $Z^*(K) = \langle -1_B \rangle$, the Z^*-theorem [24] implies that $z^K \cap H^* \neq \{z\}$. Take $z_1 \in Z^K \cap H^*$, $z_1 \neq z$. Then, the shape of H^* implies that $z_1 \in H$ and the structure of H shows that $z^K \cap H$ is the natural class of reflections in H. Now take $z_2 \in z^K \cap H$, $z_2 \in C(\langle z, z_1 \rangle) - \{z, z_1\}$. Then we have the following table:

$C(z)$	$C(\langle z, z_1 \rangle)$	$C(\langle t, z_1, z_2 \rangle)$
$Z_2 \times W_{E_6}$	$Z_2 \times Z_2 \times W_{A_5}$	$Z_2 \times Z_2 \times Z_2 \times W_{A_3}$
W_{E_7}	$Z_2 \times W_{D_6}$	$Z_2 \times Z_2 \times W_{C_4}$
W_{E_8}	$Z_2 \times W_{E_7}$	$Z_2 \times Z_2 \times W_{D_6}$

We now prove that z^K is a class of odd transpositions. Suppose false and take $z_3, z_4 \in z^K$ so that $|z_1 z_2|$ is the involution of $\langle z_3, z_4 \rangle$. Since $\langle z_3, z_4 \rangle$ is trivial on $C_B(z_1 z_2) \supseteq \langle b_0 \rangle$, $\langle z_3, z_4 \rangle \subseteq C_H(z_1 z_2)$. By inspecting the above table and (and keeping in mind the class $z^K \cap H$), we see that $\langle z_3, z_4 \rangle = \langle z_1, z_2 \rangle$ is a fours group. Thus, z^K is a class of odd transpositions. We quote [4] to identify $\langle z^K \rangle$, and then K (finally, we see that $H \cong Z_2 \times W_{E_6}$ does not occur).

Case 2. $Z(H) = 1$, i.e., $H \cong W_{E_6}$. It seems that we have to build up the 2-structure. Let L be a natural W_{D_5} subgroup of H, i.e., $L \cong Z_2^4 \Sigma_5$. Let $R = O_2(L)$, $R^* = \langle R, -1_B \rangle$, $Q = C_K(R)$. If $Q = R^*$, the fact that $r \in R$ cannot fuse to $-r$ implies that $\bar{K} = K/\langle -1_B \rangle$ contains the simple group $\bar{K}' = F^*(\bar{K})$having self-centralizing $\bar{R} \cong Z_2^4$. Since $\bar{L}' = N_{\bar{K}'}(\bar{R}) = \bar{R}L$, $\bar{R}_{L_1}$, where $L_1 \cong A_5$ and $\bar{R}$ is a projective $\mathbb{F}_2 L_1$-module, it follows that $\bar{R} = J(\bar{T})$

where $T \in \mathrm{Syl}_2(\bar{L}')$. Since this Sylow 2-group leads to $\bar{K}' \cong A, A_9$ or $U_4(2)$, we have a contradiction.

We have $Q \supset R^*$. If $n = 5$, then QL stabilizes $\langle b_0 \rangle = C_B(R)$, so that $QL \subseteq H^*$, a contradiction. Therefore, $n = 6$. Also, Q is a 2-group. Suppose $\Phi(Q) \supseteq R$. Then $[Q, L] \supset R$ and the action of $[Q, L]\, L$ on $C_B(R)$ forces $[Q, L] \subseteq H$, a contradiction. So, we have $\Phi(Q) \cap R = 1$ and so R is a direct factor of $Q = C_K(R)$. Set $Q_1 = C_Q([B, R])$. The shape of H and the fact that Σ_6 can not act on $X \cong \mathbb{Z}_4^5$ in such a way that $\Omega_1(X)$ is the codimension 1 submodule of the usual $\mathbb{F}_2 \Sigma_6$ permutation module forces $Q = Q_1 R \langle -1_B \rangle$. We claim that Q_1 does not contain an involution inverting $C_B(R)$.

If so, it would lie in H^* and centralize $L \subseteq H$, contradicting the structure of $\mathrm{Aut}\, H \cong H$. Since $Q_1 \to \{x \in GL(2, F) \mid \det x = \pm 1\}$, it follows that $Q_1 = \langle y \rangle$ where is an involution with one eigenvalue -1. Since $Q \cong Z_2^6$, the action of L on Q shows that Q is the only normal subgroup of its isomorphism type in $T \in \mathrm{Syl}_2(QL)$. Thus, Q is characteristic in T. Set $C = C_K(y)$.

Suppose that $T \in \mathrm{Syl}_2(K)$. A check of eigenvalues now shows that since $y \notin Z^*(K) = \langle -1_B \rangle$, y fuses to some $t \in C^+ = \{x \in C \mid x \text{ is trivial on } [B, y]\}$. Since $QL \cap C^+ = \langle -g, L \rangle$ has odd index in C^+, we may assume that $t \in L \leqslant H$. Since $C_H(t) \cong Z_2 \times \Sigma_6$, C^+ contains L properly. We quote Proposition D and use $\Sigma_6 \to C^+$ to get $C^+ \cong W_{D_6}$ or $C^+ \cong W_{E_6} \times Z_2$.

We have $T \notin \mathrm{Syl}_2(K)$. Since Q is characteristic in T, $Q \lhd S \in \mathrm{Syl}_2(N_K(T))$. Notice that $\{y\} = \{x \in Q \mid x \text{ has one eigenvalue } -1\}$. Thus, $N := N_K(Q) \subseteq C_K(y)$. Since $N_K(Q)$ is corefree and 2-constrained and $n = 6$, we get (by Proposition D) $N_K(Q) = \langle y \rangle \times M$, $L \leqslant M \cong W_{D_6} \cong 2^5 \cdot \Sigma_6$. By Proposition D applied to the action of $L \leqslant M$ on $C_B(y)$, $C_K(y) \cong \mathbb{Z}_2 \times W_{E_6}$. The latter case is impossible, as W_{E_6} does not contain a subgroup of shape $2^5 \cdot \Sigma_6$. So, $C_K(y) = N_K(\Omega) = \langle y \rangle \times M$. Also, if $\langle t \rangle = Z(M)$, $C_K(t) = C_K(y)$. Thus, $|K : N_K(\Omega)|$ is odd and $\bar{K} = K/\langle -1_B \rangle$ has an involution with centralizer of the form W_{D_6}. By [59], $\bar{K} \cong Sp(6, 2)$ and so $K \cong W_{E_7}$, as required.

The proof of our proposition is complete.

(3.7) Proposition A. *Let F be a field of characteristic $p \neq 2$ and B an F-vector space of dimension $n + 1$, $n \geqslant 3$. Suppose that $b_0 \in B^{\#}$ and that $H \subseteq H^* \subset K$ are finite subgroups of* $\mathrm{Aut}_F(B)$ *with the following properties:*

(i) $H^* = N_K(\langle b_0 \rangle) \subseteq N_K(H)$, $H^* = H\langle c \rangle$, *where c acts as ± 1 on $[B, H]$;*

(ii) $H \cong \Sigma_{n+1}$, *or* $\Sigma_{n+1} \times Z_2$; *or* $p \mid n + 2$ *and* $H \cong \Sigma_{n+2}$ *or* $\Sigma_{n+2} \times Z_2$; *and*

(iii) *$B/\langle b_0 \rangle$ is isomorphic to $F\Omega/F(\Sigma_{\alpha \in \Omega}\, \alpha)$ or $V/F(\Sigma_{\alpha \in \Omega}\, \alpha)$ where Ω is a set of n objects on which H operates transitively, $F\Omega$ is the permutation module and, when $H \cong \Sigma_{n+2}$, $V = \{\sum_{\alpha \in \Omega} \lambda_\alpha \alpha \mid \sum_{\alpha \in \Omega} \lambda_\alpha = 0\}$.*

Then, one of the following holds:

(a) $H \cong \Sigma_{n+1}$ *and* K *contains a normal subgroup* $K_1 = K_0 \times \langle z \rangle$, *where* z *is* $\pm 1_B$ *and* $K_0 \cong W_{A_{n+1}}$, $p|n+3$ *and* $K_0 \cong W_{A_{n+2}}$ *or* $n=5$ *and* $K_0 \cong W_{E_6}$. *Furthermore,* $K = K_1$ *unless* $K_1 \cong \Sigma_6 \times Z_2$, $K/Z(K) \cong \mathrm{Aut}\, A_6$ *and* $K/K'' \cong D_8$.

(b) $H \cong \Sigma_{n+2}$, $p|n+2$, *and* $n=4$, $p=3$, $K \cong W_{E_6}$ *or* $W_{E_6} \times Z_2$ *or* $n=7, p=3, K \cong W_{E_8}$.

(c) $p > 0$, $K = O_p(K) H^*$, $O_p(K)$ *is elementary abelian and stabilizes the chain* $0 \subset B_1 \subset B$, B_1 *a hyperplane of* B *with* $b_0 \notin B$.

Proof. It suffices to do the case that $K = \langle t_{ij}^K \rangle$ where $t_{ij} \in H$ corresponds to the transposition (ij) under the corresponds to the transposition (ij) under the given isomorphism $H \cong \Sigma_m$, where $m = n+1$ or $n+2$. Thus, t_{ij} effects a reflection on B. Set $t = t_{12}$, $C = C_K(t)$. Then $C \cap H^* = \langle t \rangle \times \langle c \rangle \times C_0$, where $|c| = 1$ or 2 (as $\det c = \pm 1$) and $C_0 = \langle t_{ij} \mid i,j \notin \{1,2\} \rangle \cong \Sigma_{m-2}$. Define $B_0 = C_B(t)$, $C_1 = C_C([B,t])$. Then $\dim_F B_0 = n$, $C_1 = C_0 \langle d_0 \rangle$ for $d_0 = 1$ or $d_0 = ct$ and $C = C_1 \times \langle t \rangle$.

We may assume that $n \geqslant 4$ for the following reasons. If $n = 3$, $H \cong \Sigma_4$ or Σ_5. If $H \cong \Sigma_4 \cong W_{D_3}$, we may quote Proposition D. If $H \cong \Sigma_5$, $p = 5$ and we argue as follows. Let $U = O_2(H)$, $B_1 = [B, U]$, $\langle b_0 \rangle = C_V(U)$. Then $C_K(U)$ stabilizes $\langle b_0 \rangle$, hence lies in H^* and so U is self-centralizing in $H^* \cap SL(B)$. Thus, a Sylow 2-group of $K \cap SL(B)$ has maximal class. Since $K \hookrightarrow GL(4, \overline{\mathbb{F}}_5)$, the classifications [1, 11, 25] give a contradiction.

Now, as $n \geqslant 4$, we may apply induction with C_0, $H^* \cap C_1$, C_1, B_0, b_0 in the roles of H, H^*, K, B, b_0 to get the possibilities for C_1. Suppose that (c) holds for C_1. Then, as $O_0(C_1)$ consists of transvections on B_0 and $B = B_0 \times [B, t]$, with both factors C-invariant, $O_p(C_1)$ consists of transvections on B. Thus, $\langle C_1^K \rangle = \langle (C_1 \cap t^K)^K \rangle = \langle t^K \rangle$ contains transvections on B. Using McLaughlin's theorem [47], we get that (c) holds for K (the other possibilities are eliminated by the shape of H^*). We, therefore, may assume that (a) or (b) holds for C_1.

For simplicity, we first treat the case that C_1 is isomorphic to a Weyl group. If C_1 is isomorphic to the Weyl group of some root system of type other than A, we may quote Proposition CF, D or E. Thus, we may assume that $C_1 \cong \Sigma_r$ for $r = n+1$ or $p|n+2$ and $r = n+2$. Evidently, $C \cap t^K = \{t\} \cup (C_1 \cap t^K)$. We have that $C_1 \cap t^K = t_{34}^{C_1}$. We argue that t^K is a class of odd transpositions. Namely, let $s \in t^K$ so that st has even order. Let u be the involution of $\langle st \rangle$. Then u has eigenvalues $\{-1, -1, 1, 1, 1, \ldots, 1\}$ on B. By the structure of C_1, either $u = tt'$ for $t' \in C_1 \cap t^K$, or $u = t't''$ for distinct t', $t'' \in C_1 \cap t^K$. In any case, $\langle s, t \rangle$ acts faithfully on $[B, u]$ and trivially on $C_B(u)$.

If t'' is in $C_{C_1}(u) \cap t^k$ and t''' does not appear in the above factorization for u, then, as $[B, t'''] \subseteq C_B(u) = C_B(\langle s, t \rangle)$, we get $\langle s, t \rangle \leqslant C_K(t''')$, which is conjugate to C. It follows from $s, t \in t^K$ and the structure of C that $\langle s, t \rangle$ is a four-group. If no such t''' exists, then, as $n \geqslant 4$, we must have u of the form tt'. But then it is obvious that $\langle s, t \rangle$ is a four-group. Consequently, t^K is a class of odd transpositions. Using the list of conclusions in [4] and $C \cong Z_2 \times \Sigma_r$, we get that $K \cong \Sigma_{r+2}$ or $r = 4$, $p = 3$ and $K \cong W_{E_6}$, as required.

Now suppose that $C_1 = C_2 \times \langle z \rangle$, where C_2 is generated by $t^K \cap C_2$ and is isomorphic to a Weyl group of some root system, then we may modify the argument of the previous paragraph, provided that $C \cap t^K = \{t\} \cup (C_2 \cap t^K)$. If this is false, then t fuses in K to some sz, where $s \in C_2$ and s has eigenvalues $\{-1, -1, ..., -1, 1, 1\}$. Since $[B, t]$ is a section of the usual permutation module for $C_2 \cong \Sigma_r$, the number of eigenvalues equal to 1 for s is at least $1 + n/2$, whereas $n \geqslant 4$, a contradiction.

Finally, suppose that C_1 has neither form. Then $n = 4$, $p = 3$ and C_1 is the central extension of Aut A_6 in (a). We have $t^K \cap C \subset C_1 \cup \{t\}$. It suffices to show that every conjugate of t in C_1 lies in C_1^*, the $Z_2 \times \Sigma_6$ subgroup of index 2 in C_1, for then the argument of the previous paragraph may be repeated. So, by way of seeking a contradiction, suppose that there is $s \in C_1 \cap t^K$, $s \notin C_1^*$. Then, the structure of Aut A_6 implies that s must lie in $C_1^* s_1$ where $Y = C_{C_1''}(s_1) \cong D_{10}$ (only two nontrivial cosets in Out A_6 contain involutions). On B_0, Y has two absolutely irreducible constituents, and on each of these s_1 acts as a scalar. So, s_1 has eigenvalues $1, \alpha, \alpha, \beta, \beta \in F$. Since $\det s = \det t = -1$, it follows that $\alpha^2 \beta^2 = -1$. Thus, $|s_1| \geqslant 4$ and, as s_1^2 centralizes C_1'', we have $s_1^2 = \langle z \rangle$. Thus, $\langle C_1'', s_1 \rangle / C_1'' \cong Z_4$, whence s cannot be in $\langle C_1'', s_1 \rangle$. Since $s \notin C_1^*$, it follows from the structure of Aut A_6 that s cannot correspond to an involution in Aut A_6–Inn A_6, a contradiction. The proof is now complete.

LEMMA 3.8. *Let $H \cong A_6$ and X the irreducible* 4-*dimensional $\mathbb{F}_3 H$-module which occurs in conclusion* (b)(ii) *of Proposition* (3.4).

(i) *If H_1 is any subgroup of H ismorphic to A_5, then X occurs in the $F_3 H$-permutation module based on the cosets of H.*

(ii) *If $Y \cong Z_{3^n}^4$ as abelian groups, $n \geqslant 2$ and Y is an H-module, then $Y/\Omega_1(Y) \not\cong X$ as H-modules.*

Proof. (i) By inspecting the Brauer character, one sees that the permutation modules on the cosets of nonconjugate A_5-subgroups are isomorphic $F_3 H$-modules. Take $H_1 \subseteq H$, $H_1 \cong A_5$. Then X is irreducible for H_1 and if $H_2 \subseteq H_1$, $H_2 \cong A_4$, then H_2 fixes the 1-dimensional space $C_X(O_2(H_2))$. Thus, X is the nontrivial absolutely irreducible constituent of the $F_3 H_1$-permutation module for the action of H_1 on the cosets of H_2. Since $X|_{H_1}$ is unique (up to equivalence), so is X.

(ii) This requires some matrix calculations. We suppose that $Y/\Omega_1(Y) \cong X$, then derive a contradiction. Without loss, $n = 2$.

Any subgroup $\langle x \rangle$ of order 3 in H is contained in a subgroup $H_0 \cong \Sigma_4$. Set $T = O_2(H_0)$. Then $Y = [Y, T] \times C_Y(T)$ and the factors are free Z_9-modules of ranks 3 and 1, respectively, and are H_0-invariant. Since $x \in H_0'$, x centralizes $C_Y(T)$. Thus, one may choose a basis for Y so that x has matrix

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, let $\langle s \rangle \times \langle t \rangle \in \mathrm{Syl}_3(H)$ where s and t are conjugate in H. Let y_1, y_2, y_3, y_4 be a basis for Y so that s has matrix as above. Let $\psi: Y \to \bar{Y} = X$ be our H-homomorphism. Since X occurs as a section of the F_3H-permutation module, we may assume that X is generated by elements e_{ij}, $1 \leqslant i, j \leqslant 6$, $i \neq j$, which satisfy the relations $e_{ij} = -e_{ji}$, $e_{ij} + e_{jk} = e_{ik}$, $i \neq k$, and $e_{12} + e_{13} + e_{45} + e_{46} = 0$. We may now choose notation so that $y_1^\psi = e_{14}$, $y_2^\psi = e_{24}$, $y_3^\psi = e_{34}$, $y_4^\psi = e_{56}$, s, t, correspond to the permutations (123), (456), respectively, and the element $g \in H$ acts on e_{ij} by $e_{ij}^g = e_{i',j'}$ where g corresponds to the permutation g', where $i^{g'} = i', j^{g'} = j'$.

We now determine conditions satisfied by the matrix B representing t. Using the basis $e_{14}, e_{24}, e_{34}, e_{56}$ for X, we compute that $e_{14}^t = e_{15} = -e_{24} - e_{34} + e_{56}$, $e_{24}^t = e_{25} = -e_{14} - e_{34} + e_{56}$, $e_{34}^t = e_{35} = -e_{14} - e_{24} + e_{56}$ and $e_{56}^t = e_{64} = e_{14} - e_{24} - e_{34} + e_{56}$, whence t has matrix

$$\beta = \begin{pmatrix} 0 & -1 & -1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} + (c_{ij}),$$

where each c_{ij} is divisible by 3.

Since $st = ts$ and

$$\alpha^{-1} = \alpha^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have

$$\alpha^{-1}\beta\alpha = \begin{pmatrix} -1-c_{31} & -1+c_{32} & c_{33} & 1+c_{34} \\ c_{11} & -1+c_{12} & -1+c_{13} & 1+c_{14} \\ -1+c_{21} & c_{22} & -1+c_{23} & 1+c_{24} \\ 1+c_{41} & -1+c_{42} & -1+c_{43} & 1+c_{44} \end{pmatrix}\alpha$$

$$= \begin{pmatrix} c_{33} & -1+c_{31} & -1+c_{32} & 1+c_{34} \\ -1+c_{13} & c_{11} & -1+c_{12} & 1+c_{14} \\ -1+c_{23} & -1+c_{21} & c_{22} & 1+c_{24} \\ -1+c_{43} & 1+c_{41} & -1+c_{42} & 1+c_{44} \end{pmatrix},$$

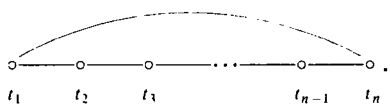
which equals β. Comparing coefficients and noting equalities, we have a, b, c, d, $e \in 3Z_9$ such that $a = c_{11} = c_{22} = c_{33}$, $b = c_{12} = c_{23} = c_{31}$, $c = c_{13} = c_{21} = c_{32}$, $d = c_{41} = c_{42} = c_{43}$ and $e = c_{14} = c_{24} = c_{34}$. Since α and β are conjugate comparing traces gives $c_{44} = 0$.

Now

$$\alpha\beta = \begin{pmatrix} -1+c & a & -1+b & 1+e \\ -1+b & -1+c & a & 1+e \\ a & -1+b & -1+c & 1+e \\ 1+d & 1+d & 1+d & 1 \end{pmatrix},$$

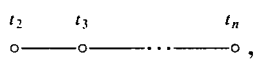
which has trace $-3+3c+1=-2$. As noted in the second paragraph, the trace must be 1 since $\alpha\beta$ represents the element $st \in H$ of order 3. This contradiction completes the proof of (ii).

LEMMA 3.9. *Let the group G be generated by involutions $t_1, \ldots, t_n$, $n \geqslant 2$ subject to the relations*



Then G is a split extension $Z^{n-1}\Sigma_n$, where the normal abelian subgroup A is isomorphic to the submodule $\langle e_i - e_j \mid i, j = 1, \ldots, n\rangle$ of the permutation module $\coprod_{i=1}^n Ze_i$ for the symmetric group $\Sigma_n \cong G/A$ (this isomorphism is given by $t_i \to (i, i+1)$, $1 \leqslant i \leqslant n-1$).

Proof. Let $\varphi: G \to \Sigma_n$ be the epimorphism given by $t_i \to (i, i+1)$ for $1 \leqslant i \leqslant n-1$ and $t_n \to (n, 1)$. Then $a = t_1^{t_2t_3\cdots t_{n-1}}t_n \in \ker\varphi$. Let $A = \langle a^G\rangle \subseteq \ker\varphi$. Then G/A is generated by the images of $t_2, \ldots, t_n$ which satisfy



whence $A = \ker\varphi$. We shall show that A is abelian. For now, assume $n \geqslant 4$.

Let $H = \langle t_2, \ldots, t_n \rangle \cong \Sigma_n$, $C = C_H(t_n) = \langle t_2, \ldots, t_{n-2} \rangle \times \langle t_n \rangle$. We claim that

$$a^{t_n} = a^{-1} \tag{1}$$

$$a^{t_i} = a, \qquad 2 \leqslant i \leqslant n-2. \tag{2}$$

In our calculations, we use the index i for t_i. Thus, $t_i t_j t_k \cdots$ is written $i \times j \times k \cdots$. So, (1) is equivalent to $1 = a\ a^{t_n} = n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times n \times n \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-1 \times n \times n$. Since the right side trivially collapses to the identity, (1) follows. Now for (2). We have $a^{t_i} = t_1^{t_2 t_3 \cdots t_{n-1} t_i} t_n = t_1^{t_2 \cdots t_{i-1} t_i t_{i+1} t_i \cdots t_{n-1}} t_n = t_1^{t_2 \cdots t_{i-1} t_{i+1} t_i t_{i+1} \cdots t_{n-1}} t_n = (t_1^{t_{i+1}})^{t_2 \cdots t_{i-1} t_i t_{i+1} \cdots t_{n-1}} t_n = a$, giving (2).

Define $B = \langle a, a^{t_{n-1} C} \rangle$. We claim that B is abelian. It suffices to prove that $[a, a^{t_{n-1}}] = 1$, i.e., that $1 = n \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times n-1 \times n \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times n-1 \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots n-1 \times n \times n-1 \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times n-1$. But an exercise with relations verifies this requirement (e.g., start right at the middle, using $n-2 \times n-1 \times n-1 \times n-1 \times n-2 = n-2 \times n-1 \times n-2 = n-1 \times n-2 \times n-1$, then move the $(n-1)$'s away from center).

Finally we show $A = B$. We have $H = C \cup Ct_{n-1}C \cup Ct_{n-1}t_1C$. It suffices to show that $a^{t_{n-1}t_n} \in B$. In fact, we show that

$$a^{t_{n-1}t_1} = a^{t_1 t_n} a^{t_{n-1}}; \tag{3}$$

equivalently,

$$1 = a^{t_1 t_n t_{n-1}} a a^{-t_1}.$$

So, we show the triviality of $n-1 \times n \times 1 \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times n \times 1 \times n \times n-1 \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times n \times 1 \times n \times n-1 \times n-2 \times \cdots \times 3 \times 2 \times 1 \times 2 \times 3 \times \cdots \times n-2 \times n-1 \times 1$. Now, cancel $n-1 \times n-1$ and replace both triples $n \times 1 \times n$ by $1 \times n \times 1$. Next move the i's closest to each $2 \times 1 \times 2$ inward, then replace each $1 \times 2 \times 1 \times 2 \times 1$ by 3. What remains is an expression in H. Since $H \cong H^\varphi$, it is routine to show that the expression is trivial by a calculation in $H^\varphi = \Sigma_n$.

As for $n \leqslant 3$, the lemma is trivial for $n = 2$ and the argument for $n = 3$ amounts to showing that $B = \langle a, a^{t_2} \rangle$ is abelian and verifying (3) with a similar calculation.

Now we show $A = B$ is abelian of rank at most $n-1$. Namely, B is generated by a and all a^g where g runs over a right transversal T to $C_C(t_{n-1})$ in C. Taking $T = \{ g \in C \mid g^\varphi = (2, n-1),\ (3, n-1), \ldots,\ (n-2, n-1),$

$(2, n-1)(n, 1), \ldots,$ or $(n-2, n-1)(n-1)\}$ and using the facts that $(n, 1)$ commutes with each $(j, n-1)$, $2 \leqslant j \leqslant n-2$, and t_n inverts a, we get that at most $n-1$ distinct cyclic subgroups are generated by the members of $\{a, a^g \mid g \in T\}$. Thus, B has rank at most $n-1$.

Finally we show that A has rank exactly $n-1$ as follows. Let M be the module $\langle e_i - e_j \mid i, j = 1, \ldots, n\rangle$ for $H \cong \Sigma_n$ as in the statement of the lemma and let MH be the semidirect product. The elements $((e_1 - e_n)\, t_n)^{t_{n-1} t_{n-2} \cdots t_3 t_2}$, $t_2, \ldots, t_n$ satisfy the diagram defing G, and we have a map $\psi \colon G \to MH$. Since $e_1 - e_2 \in G^{\omega}$ and $e_1 - e_2$ generates M as a module, ψ is onto. The shapes of A and M show that ψ is an isomorphism, and we are done.

We will require some results on generation of the known simple groups. Consider the following more general situation.

$$E \subseteq H \subseteq G \qquad \text{and if}$$
$$g \in G \quad \text{with} \quad E^g \cap H \neq 1, \quad \text{then} \quad g \in H,$$

which we refer to by saying that *H controls strong fusion of E in G.* We specify

Hypothesis 3.10. H controls strong fusion of E in G and $E \cong E_{p^2}$, p odd.

LEMMA 3.11. *Suppose H controls strong fusion of e in G and e has prime order p.*

(i) *If* $e \in R \subseteq H$ *and* $R \lhd\lhd X$, *then* $X \subseteq H$.

(ii) *If* $e \in N$, *then* $p \nmid |N \colon N \cap H|$.

(iii) *If* $e \in N$, $V = H \cap N$ *and* $\bar{N} = N/K$ *for some* $K \lhd N$ *with* $K \subseteq V \cap N$, *then* $\bar{V}$ *controls strong fusion of* $\langle \bar{e} \rangle$ *in* $\bar{N}$.

Proof. The proof is straightforward.

LEMMA 3.12. *Suppose* (3.10) *holds.*

(i) *If G is p-solvable,* $H = G$.

(ii) *If H is p-nilpotent,* $H = G$.

Proof. To prove (i) let M be a minimal normal subgroup of G. If $p \nmid |M|$, then $M = \langle C_M(e) \mid e \in E^{\#}\rangle \subseteq H$, while if M is a p-group and $R = EM$, $R \subseteq H$ by Lemma 3.11(i). Thus, $M \subseteq H$ and since H/M controls strong fusion of EM/M in G/M, we have $H = G$ by induction.

For (ii) suppose H is p-nilpotent and pick $P \in \mathrm{Syl}_p(H)$ with $E \subseteq P$. By

Lemma 3.11(i) again, $P \in \mathrm{Syl}_p(G)$. By a result of Glauberman [25, Theorem 12.7] there exists a subgroup $W \subseteq P$ such that

$$\begin{gathered} W \text{ is characteristic in } P, \text{ and} \\ \text{if } z \in P \cap Z(N_G(W)), \text{ then } z \text{ is weakly closed} \\ \text{in } P \text{ with respect to } G. \end{gathered}$$

Let $N = N_G(W)$. If $N = G$, we may apply the induction hypotheses to G/W, so assume $N \neq G$. $H \cap N$ controls strong fusion of E in N, so $N \subseteq H$ by induction. From the structure of H and choice of K we see that every $z \in Z(P)$ is weakly closed in P with respect to G. If $z \in Z(G)$, then we are done by induction, so assume $z \notin Z(G)$ whence $C_G(z) \subseteq H$ by induction. Thus, $z^g \in H$ implies $g \in H$, and in particular $N_G(D) \subseteq H$ for any $D \subseteq P$ with $C_P(D) \subseteq D$. It follows that two elements of P are conjugate in G if and only if they are conjugate in P. But now G is p-nilpotent, and the action of E on $O_{p'}(G)$ forces $O_{p'}(G) \subseteq H$ and $H = G$.

Lemma 3.13. *Suppose* (3.10) *holds and the p-layer of G is* $L_{p'}(G) = K_1 \cdots K_t$ *or* $L_{p'}(G) = 1$.

(i) *Every p-solvable normal subgroup of G lies in H.*

(ii) *If* $R \subseteq H$ *and* $[R, H \cap K_i] \subseteq O_{p'p}(H)$, *then R normalizes* K_i. *In particular* $O_{p',p}(H)$ *normalizes each* K_i.

(iii) *If* $K_i \nsubseteq H$, *then E normalizes* K_i.

Proof. Lemma 3.11(i) yields (i). Next we prove (iii). Assume E does not normalize K_i. By induction on $|G|$, we may assume $O_{p',p}(G) = 1$. Indeed if not, then by (i), $O_{p',p}(G) \subseteq H$ and by induction E normalizes $K_i O_{p',p}(G)$. As $K_i \lhd\lhd G$, K_i is characteristic in $K_i O_{p',p}(G)$ and (iii) holds. We may also assume by induction that E acts transitively on the p-components of G. Since $O_{p',p}(G) = 1$, each K_i is simple.

Let $L = L_{p'}(G)$ and $X = \langle C_L(e) \mid e \in E^{\#} \rangle$. It suffices to show $X = L$. If $N_E(K_i) \neq 1$, then choose $e \in E^{\#}$ to normalize K_i. As $p \mid |K_i|$, $1 \neq C_{K_i}(e) \subseteq X \cap K_i$. Choose $f \in E$ so that $\langle f \rangle$ acts transitively on the components of L. As $C_L(f) \subseteq X$, X projects onto K_i whence $X \cap K_i \lhd K_i$ and $K_i \subseteq X$. It follows that $L = X$ as desired.

If $N_E(K_i) = 1$, then E acts regularly and we can choose $e, f \in E^{\#}$ so that the $\langle e \rangle$-orbit containing K_i and the $\langle f \rangle$-orbit containing K_i have only K_i in common. Letting $X = \langle K_i^{\langle e \rangle} \rangle$ and $Y = \langle K_i^{\langle f \rangle} \rangle$, we have $K_i = [C_X(e), C_Y(f)] \subseteq X$ whence $L = X$.

Now we prove (ii). Just as in the proof of (iii) we may assume $O_{p',p}(G) = 1$ whence K_i is simple. Suppose $r \in R$ does not normalize K_i. If $K_i \subseteq H$, then $K_i (K_i)^r = [K_i, r] \subseteq O_{p',p}(H)$, which is impossible. Thus, $K_i \nsubseteq H$

and by (iii) E acts on K_i. By Lemma 3.12, $H \cap K_i$ is not p-nilpotent, whence $[H \cap K_i, r]$ contains a section which is not p-nilpotent. But $[H \cap K_i, r] \subseteq O_{p',p}(H)$, contradicting the existence of r.

LEMMA 3.14. *Assume the hypothesis of the preceding lemma and assume that each K_i satisfies the Schreier Conjecture; then*

$$L_{p'}(H) = L_{p'}(H \cap K_1) \cdots L_{p'}(H \cap K_t).$$

Further if L_1 is a quasisimple component of H, then $L_1 \subseteq K_i$ for some quasisimple component of G.

Proof. The action of E gives $O_{p'}(G) \subseteq H$. Thus, if L_1 is as above, then $[L_1, O_{p'}(G)] = 1$, whence $[K_i, O_{p'}(G)] = 1$ and K_i is quasisimple.

For the proof of the remainder of Lemma 3.14 we may assume $O_{p',p}(G) = 1$ by induction whence each K_i is simple. Since $L_{p'}(K_i \cap H)$ is clearly a summand of $L_{p'}(H)$, it suffices to show that each p-component of H lies in some K_i.

Let $L = L_{p'}(H)$ and $X = H \cap L_{p'}(G)$. Since $X \lhd H$, $L = L_1 L_2$ where L_1 is the product of all p-components of H lying in X, L_2 is the product of all other p-components of H, and $X \cap L_2 \subseteq O_{p',p}(L_2) \subseteq O_{p',p}(H)$. As $[L_2 \cap X, X] \subseteq L_2 \cap X$, Lemma 3.11(ii) implies that L_2 normalizes each K_i. It follows that L does too.

Now for any $K = K_i$, $L = L_3 L_4$ where L_3 is the product of all p-components of H lying in K and L_4 is the product of the rest. Letting $Y = H \cap K$, we have as before $L_3 \subseteq Y$ and $L_4 \cap Y \subseteq O_{p',p}(L_4)$. By hypothesis, L_4 acts as inner automorphisms on K. Let bars denote images in $\mathrm{Aut}(K)$. $\bar{L}_4 \leqslant \bar{K}$ implies $[\bar{L}_4, \bar{L}_4] \subseteq [\bar{L}_4, \bar{Y}] \subseteq O_{p',p}(\bar{L}_4)$. As L_4 is perfect, $\bar{L}_4 = 1$.

Thus, for any p-component J of H and any $K = K_i$, $J \subseteq K$ or $[J, K] = 1$. As $O_{p',p}(G) = 1$, J acts nontrivially on $L_{p'}(G)$ and (ii) holds.

LEMMA 3.15. *Assume the hypothesis of Lemma 3.13. Suppose $p = 3$ and $P \lhd Q \lhd R \lhd H$ with*

(a) $R/P \cong S_3$;

(b) $P = O_{3'}(R)$;

(c) $Q = O^{3'}(Q)$;

then R normalizes every K_i. Further either $Q \subseteq O_{3',3}(G)$ or there exists $K = K_i$ such that R acts nontrivially on $K/O_{p',p}(K)$ and if Q acts as inner automorphisms on $K/O_{p',p}(K)$, then $Q \subseteq K$.

Proof. Clearly $[R, H] \subseteq Q \subseteq O_{3',3}(H)$ so R normalizes each K_i by Lemma 3.13(ii). If Q acts nontrivially on $O_{3',3}(G)/O_{3'}(G)$, then our hypotheses force $Q \subseteq O_{3',3}(G)$. Thus, we may assume Q acts nontrivially on $K/O_{3',3}(K)$ and induces innerautomorphisms. By induction we may further assume $O_{3',3}(G) = 1$. If $Q \not\subseteq K$, then $[R, H \cap K] \subseteq P$, and letting bars denote images in $\mathrm{Aut}(K/O_{3',3}(K))$, we have $[\bar{R}, \bar{Q}] \subseteq [\bar{R}, H \cap K] \subseteq \bar{P}$ whence $\bar{Q} = 1$, not the case.

The remainder of this section is primarily devoted to determining when certain configurations satisfying Hypothesis 3.10 can occur in the known simple groups. These results will be used in Section 6; and roughly speaking, H will be isomorphic to the centralizer of an element of prime order in a group of Lie type defined over a field of characteristic 2. As a consequence we need only consider configurations satisfying the following more restrictive version of Hypothesis 3.10.

Hypothesis 3.16. (I) H controls strong fusion of E in G and $E \cong E_{p^2}$, p odd.

(II) $K = F^*(G)$ is a known simple group.

(III) Let $Q = O_2(H)$ and $L/Q = L(H/Q)$. The following conditions hold:

(a) The components of L/Q are Chevalley groups or Steinberg variations (i.e., not twisted groups of type B_2 or F_4) over a field of characteristic 2;

(b) L is perfect or $L(H/Q) = 1$;

(c) H/L is solvable;

(d) E acts as inner-diagonal automorphisms on each component of L/Q;

(e) if $L(H/Q) \neq 1$, then either L is 2-constrained, or L is quasisimple with $|Z(L)|$ odd;

(f) if $L(H/Q) = 1$, then either $m_p(K) = 1$, or K has a perfect central extension $\hat{K}$ by a cyclic p-group with $m_p(\hat{K}) = 2$.

(IV) For all $x \in E^{\#}$, $L_{p'}(C_G(x))$ is quasisimple or 1 and each component is a group of Lie type over a field of characteristic 2.

LEMMA 3.17. *If Hypothesis* (3.16) *holds and K is alternating or of Lie type, then one of the following occurs:*

(i) $H = G$;

(ii) $K = A_6$, $p = 3$, $F^*(H \cap K) = E$, *or* $K = A_{5s}$, $s = 2, 3, 4$, $p = 5$, *and* $F^*(H \cap K) = A_5^s$;

(iii) *K is a group of Lie type over a field of characteristic* 2 *and* $p = 3$ *or* 5.

Proof. We may assume $H \neq G$. If $K \subseteq H$, then $\bar{H}$ controls strong fusion of $\bar{E}$ in $\bar{G} = G/K$, and as $\bar{G}$ is always solvable, $H = G$ by Lemma 3.12. Thus, we may assume $K \nsubseteq H$.

Suppose K is alternating, say $K = A_n$, and $e \in E^{\#}$ has cycle structure $1^r p^s$. Assume $s \geqslant p$ and

$$e = (1,..., p)(p+1,..., 2p) \cdots ((p-1)p+1,..., p^2) \cdots.$$

As e is fused in K to

$$f = (1, p+1,..., (p-1)p+1) \cdots (p, 2p,..., p^2) \cdots,$$

we have $\langle a, b \rangle \subseteq H$, where

$$a = (1, 2,..., p) \in C_G(e) \subseteq H$$

and

$$b = (p, 2p,..., p^2) \in C_G(f) \subseteq H.$$

But $\langle a, b \rangle$ is the alternating group of the letters moved by a or b; and it follows easily that $A_r \times A_{sp} \subseteq H$. As $sp \geqslant 9$, conditions III(a)–(c) of Hypothesis 3.16 cannot both be satisfied, and we conclude $s < p$.

A similar argument yields $r \leqslant p$. It follows that E has no regular orbits, and that we can choose e to have cycle structure $1^p p^s$. Control of strong fusion of e forces H to contain $F = A_p \wr A_{s+1}$. In fact, as $e \in F$, $N_G(F) \subseteq H$. Since $N_G(F)$ is maximal in G, $H = N_G(F)$. Applying conditions III and V of Hypothesis 3.16 we obtain conclusion (ii) above.

Next suppose K is of Lie type over a field of characteristic p. If $E \cap K = 1$, then $p = 3$ and $K = D_4(3^{3n})$. Some $e \in E^{\#}$ is a field automorphism with $H \supseteq C_K(e) \cong D_4(3^n)$. By [8, Theorem 1], $C_K(e)$ is maximal in K. By Lemma 3.11, $p \nmid |G: H|$; and as $p \,\|\, |K: C_K(e)|$, it follows easily that $K \subseteq H$. Thus, we may assume $E \cap K \neq 1$. Applying the operator $O_p(N_K(\ ))$ repeatedly to $E \cap K$, we eventually reach $O_p(W)$ for some parabolic subgroup W of K [9]. It follows that $W \subseteq H$ whence $H \cap K$ is a parabolic. Condition III(e) now implies $L(H/Q) = 1$, and in view of Condition III(f) we need only consider the possibilities $m_p(K) = 1$ and $m_p(\hat{K}) = 2$. In the first case K must have Lie-rank 1 whence $H \cap K$ is a maximal subgroup of K. But some $e \in E - K$ is a field automorphism, and it follows easily from $C_K(e) \subseteq H$ that $K \subseteq H$. In the second case, by [38], $p \mid |Z(\hat{K})|$ implies $p = 3$ and

$$K = A_1(9), \qquad B_3(3), \qquad G_2(3), \qquad \text{or} \qquad {}^2A_3(3).$$

Likewise the Sylow 3-subgroup of $Z(\hat{K})$ is elementary whence by Condition

III(f) $|Z(\hat{K})| = 3$. Now $m_3(\hat{K}) \leqslant 2$ implies $m_3(K) \leqslant 3$ by a straightforward argument whence $K = A_1(9) \cong A_1$ and Lemma 3.17(ii) holds.

Finally suppose K is of Lie type over a field of characteristic prime to p. Our conditions imply $K \neq \langle C_K(e)\, e \in E^{\#}\rangle$, and [52, Theorem 1 and Theorem 2] yields conclusion (iv) of Lemma 3.17.

LEMMA 3.18. *If Hypothesis* (3.16) *holds and K is of Lie type over a field of characteristic* 2, *then one of the following holds:*

(i) $p = 3$, $K = C_4(2)$, $H \cap K = O^+(8, 2)$;

(ii) $p = 3$, $K = C_3(2)$, $H \cap K = O^-(6, 2)$;

(iii) $p = 3$, $K = A_2(4)$, $E \subseteq K$, $F^*(H \cap K) = A_6 \cong C_2(2)'$ *or* $H \cap K = {}^2A_2(2)$;

(iv) $p = 5$, $K = {}^2C_2(2^5)$, $H \cap K = Z_4 Z_{25}$;

(v) $p = 3$, $K = A_1(8)$; $H \cap K$ *is dihedral of order* 18.

Further $C_K(E)$ is a p-group, and, in cases (i)–(iii), *E acts on K as inner-diagonal automorphisms. In cases* (i) *and* (ii) *the standard K-module may be decomposed into a direct sum of pairwise orthogonal hyperbolic planes in such a way that E acts nontrivially on each plane and H acts as the orthogonal group preserving the quadratic form which takes the value* 1 *on the nonzero elements of every plane.*

Proof. We sketch the proof, which consists of analyzing all the possibilities for failure of generation presented in [52]. Suppose first that K is classical and E acts as inner-diagonal automorphisms. Let E_0 be a Sylow p-subgroup of the pre-image of E in the universal covering group K_0 of K (or more precisely in K_0 extended by its diagonal automorphisms). Assume first that E_0 is abelian. By [52, (4.1)], $p = 3$, $r^a = 2$, and $K \neq A_n(2)$. Further let V be the standard K_0-module; then

$$V = V_0 \perp V_1 \perp \cdots \perp V_K$$

with $V_0 = C_V(E_0)$, $\dim(V_0) \leqslant 0, 1, 2$ according to whether V is symplectic, unitary, or orthogonal and for $i \geqslant 1$ $V_i = [V_i, E_0]$, $\dim(V_i) = 1$ or 2 according to whether V is unitary or not. Let $D_0 = C_{K_0}(E_0)$, then either $D_0 = E_0$ and D_0 stabilizes the decomposition of V above, or $K_0 = Sp(2n, 2)$ and $D_0 = O^{\varepsilon}(2n, 2)$ preserve the quadratic form which has value 1 on each vector in $V_i^{\#}$, $i \geqslant 1$. Clearly $K \nsubseteq H$ implies $D_0 \neq K_0$.

If $K_0 = SU(n, 2)$, then a lift of $e \in E^{\#}$ has a diagonal matrix representation

$$e \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

with each diagonal entry corresponding to a summand of the decomposition of V. If $\lambda_i = \lambda_j = \lambda_k$ for distinct i, j, k, then D_0 contains a subgroup isomorphic to $SU(3, 2)$ which does not stabilize the decomposition of V. Thus, the multiplicity of any eigenvector is at most 2, whence $n \leqslant 6$. If $n = 6$, then the trace of the matrix of e is 0 for each $e \in E^{\#}$. As each λ_i is a cube root of unity, we may choose the lifts of each e to generate a subgroup $E_1 \subseteq E_0$, $E_1 \cong E$. We identify E with E_1. Let ψ be the corresponding character of E. $\psi(e) = 0$ for $e \in E^{\#}$ and $\psi(1) = 6$ which is impossible as $|E| = 9$. A similar contradiction obtains if $n = 5$.

Consider the case $n = 4$. We have $K_0 = K = SU(4, 2) \cong PSp(4, 3)$. As $H \cap K$ controls strong fusion of $E^{\#}$, $H \cap K$ contains all monomial matrices. The subgroup of such matrices corresponds to a maximal parabolic of $PSp(4, 3)$ whence $H \cap K$ is the monomial subgroup. It follows that the subgroup of diagonal matrices is normal in H. On the other hand there are 3 K-classes of elements of order 3 represented by

$$\begin{pmatrix} \lambda & & & \\ & \lambda^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1} \end{pmatrix}, \quad \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda & \\ & & & 1 \end{pmatrix}.$$

As we saw above, E contains no elements of the third class; it follows easily that E contains elements from the other two classes. But $H \cap K$ contains an element w with matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As $C_V(w)$ has dimension 2, w is fused in K to an element of E. Thus, w is fused in H to E, contrary to the structure of H.

The last two paragraphs show that when E_0 is abelian (and $K \nsubseteq H$) we do not have $K = {}^2A_n(2)$. A similar argument disposes of the other classical groups whenever D_0 stabilizes the decomposition of V.

Suppose E_0 is abelian but D_0 does not stabilize the decomposition of V. $K_0 = Sp(2n, 2)$ and $D_0 = O^{\varepsilon}(2n, 2)$ as discussed above. If the lift of any $e \in E^{\#}$ centralizes two summands of V, then D_0 contains a subgroup isomorphic to $Sp(4, 2)$ which does not preserve the quadratic form. Thus, the centralizers in E_0 of the summands of V are all distinct. It follows that $k \leqslant 4$. As $V_0 = 0$, we have $K = C_3(2)$ or $C_4(2)$ and $H = H \cap K$ contains a subgroup L isomorphic to $O^-(6, 2)$ or $O^+(8, 2)$, respectively. By [52, (2.3)], D_0 does not act on a 2-subgroup of K_0 lest $D_0 = K_0$. Thus, $O_2(H) = 1$, and it follows by arguments on the order of L and H that $L \lhd H$. At this point we have obtained Lemma 3.18(i, ii).

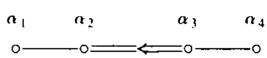
Assume E_0 is nonabelian. By [52, (4.5)] we have $p = 3$ and $K_0 = SL(3^k, 4)$ or $SU(n, 2)$. In the first case if E_0 is reducible, then E_0 acts on a 2-subgroup of K_0 whence $D_0 = K_0$ by [52, (2.3)]. Thus, E_0 is irreducible. As E_0 is extraspecial of order 27, we deduce first that $k = 1$ and then that Lemma 3.18(iii) holds. In the second case suppose some $e \in E$ lifts to $e_0 \in E$ with $|e_0| = 9$. Then $n = 3k$ and with respect to some basis e_0 has matrix

$$\begin{pmatrix} 0 & 0 & \lambda I \\ I & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Further if $k > 1$ and $f \in E - \langle e \rangle$ lifts to f_0, then f_0 acts as a field automorphism on $L(C_{K_0}(e_0))$ whence $2 \,||\, |C_{K_0}(\langle e_0, f_0 \rangle)|$ contrary to [52, (2.3)]. Thus, $k = 1$ and K is solvable, not the case. Finally if every element of $E^{\#}$ lifts to an element of order 3, then E_0 permutes the 3 eigenspaces of any $e_0 \in E_0^{\#}$, and an argument using [52, (2.3)] yields $K = 1$.

Next assume that K is an exceptional group of Lie type and E acts as inner-diagonal automorphisms. The possibilities with $C^0_{K_0}(E_0) \neq K_0$ are listed in [52, (5.1)]. With one exception $p = 3$ and $r^a = 2$ or 4. Notice that as we have seen above $|C_{K_0}(E_0)|$ must be odd. When $p = 3$, it follows that for any $e \in E^{\#}$, $O^{2'}(C_K(e))$ has as possible summands only $A_1(q)$, $A_2(q)$ with $3 | q - 1$, and ${}^2A_2(q)$ with $3 | q + 1$. Thus, for any particular exceptional group K that there are at most a few possibilities for the conjugacy classes of elements in E. In fact when K is of type E_7 or E_8, there are no elements of order 3 whose centralizers have the required structure.

Next suppose $K = F_4(2)$. There is just one possible class for $e \in E^{\#}$. Pick a fundamental system of roots of type F_4



and let $\{\eta_i \mid 1 \leqslant i \leqslant 4\}$ be the dual basis. Let σ be the standard automorphism of the algebraic group $\bar{K}$ with fixed points $F_4(2)$, and let W_0 be the element of the Weyl group interchanging positive and negative roots. $[\eta_3, w_0\sigma]$ describes an element in the K-class of e (where K is taken to be the centralizer on the algebraic group of $I_{w_0}\sigma$ as discussed in Section 2). $C_K(e) = Z_3/({}^2A_2(2) \times {}^2A_2(2))/Z_3$. Let E_1 be the group generated by e and

$$f_1 = [\eta_1 + \eta_2 + \eta_4, w_0\sigma]$$

and check that all elements of E_1 are conjugate in K to e. As $E_1 \subseteq H$, H

controls strong fusion of E_1 in K. E_1 acts on $J = F^*(C_{\tilde{J}}(I_{W_0}\sigma))$ where $\tilde{J}$ is generated by the root groups corresponding to $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$, and $\pm\beta$, β the lowest short root. $J \cong C_4(2)$, so our previous discussion of the case $K = C_4(2)$ yields a subgroup L of H isomorphic to $D_4(2)$. If $O_3(H) \neq 1$, then some 3-element $w \in K$ has a subgroup isomorphic to $D_4(2)$ in its centralizer, which is impossible by inspection. If $O_{3'}(H) \neq 1$, then $O_{3'}(C_K(g)) \neq 1$ for some $g \in E_1^{\#}$, again impossible by inspection. Thus $O_{3',3}(H) = 1$ whence L lies in $E(H)$ and $E(H)$ is a direct product of simple groups. Let L project nontrivially on the summand L_1 of $E(H)$. From the preceding observation $3 \nmid |C_K(L_1)|$ whence all other summands have order prime to 3. As $O_{3'}(H) = 1$, $L_1 = E(H)$. As $3^5 \| |L|$ but $3^6 | |C_K(e)|\ |H|$, $L = L_1$ implies that $N_K(L)$ contains a 3-element w with $C_L(w) \supseteq {}^3D_4(2)$ which is impossible by inspection of the layers of centralizers of elements of order 3. Likewise $L_1 \neq C_4(2)$. On the other hand the conjugates under L_1 of the root involutions of K in L generate L_1, so the possibilities for L_1 are known by Timmesfeld [58]; and $|L|\,|\,|L_1|\,|\,|K|$ gives a contradiction.

The other cases in [52, (5.1)] are dealt with similarly. The failures of generation in which E does not act as inner-diagonal automorphisms are described in [52, (6.1), (6.3), (6.4)]. In (6.1) we find $p = 3$, $K = {}^2C_2(2^5)$ or $A_1(8)$, which leads to Lemma 3.18(iv, v). In (6.3) and (6.4) we have E acting as inner-graph automorphisms on $K = {}^3D_4(2)$, $D_4(2)$, or $D_4(4)$, and we wish to show that $K \not\subseteq H$ leads to a contradiction.

Suppose $K = {}^3D_4(2)$. K has one class of 3-central elements of order 3. Let x be such an element. From [52, Table 3.3], $C_K(x)$ is an extension of $J = SU(3, 2)$ with $|C_K(x){:}J| = 3$ and some 3-element inducing an outer-diagonal automorphism on J. The other class of elements of order 3 in K has centralizer $Z_3 \times A_1(8)$. K has a graph automorphism τ with $C_K(\tau) = G_2(2)$. As $G_2(2)$ contains a 3-central element with centralizer $SU(3, 2)$, we may take τ to centralize J. Thus $C_K(x)\langle\tau\rangle$ contains a Sylow 3-subgroup of $K\langle\tau\rangle \supseteq E$. We take $E \subseteq C_K(x)\langle\tau\rangle$ and $\langle x, \tau\rangle \subseteq S \in \mathrm{Syl}_3(C_K(x)\langle\tau\rangle)$. As $N_K(\langle x\rangle)$ is solvable, $N_K(\langle x\rangle) \subseteq H$ by Lemma 3.12. From [52, (4.3)] we have $C_K(\rho) \not\subseteq N_G(\langle x\rangle)$ for any $\rho \in E - K$. Thus $N_K(\langle x\rangle) \subset H \cap K$ whence $O_3(H) = 1$.

Pick $y \in S \cap K$ with $C_K(y) = \langle y\rangle \times L$, $L \cong A_1(8)$; and $C_{S\cap K}(y) \in \mathrm{Syl}_3(C_K(y))$. Our conditions force $x \in L$ whence (as K has just two classes of elements of order 3) y is inverted in $N_K(\langle y\rangle) \cap C_K(x)$. It follows that $y \in J$. Let $F = O_{3'}(H \cap K)$. If $F \neq 1$, then considering the action of $\langle x, y\rangle$ on F, we have either $O_{3'}(C_K(x)) \neq 1$ or for some $z = yx^i$, $i = 0, 1, 2$, $C_K(z)$ contains a p'-subgroup invariant under $C_K(\langle x, y\rangle) \cong Z_3 \times Z_9$. But as z is conjugate to y in J, we see that neither possibility occurs whence $F = 1$.

We have $O_{3',3}(H \cap K) = 1$ whence $X = E(H \cap K) \neq 1$. Each summand of X is simple with order divisible by 3. As $S \cap K$ has rank 2, X has at most 2 summands. Thus the intersection of S with any summand lies in $Z(S)$. As

$Z(S) = \langle x \rangle$, we have that X is simple and S acts faithfully on X. Further $|X|$ properly divides $|K| = 2^{12} \times 3^4 \times 7^2 \times 13$, and surveying the orders of the known simple groups, we see that no such X exists.

In the cases $K = D_4(2)$ or $D_4(4)$ we can use E to find $A \subseteq H \cap K$ such that H controls strong fusion of A in K, a possibility ruled out earlier.

LEMMA 3.19. *Let QY be a 2-constrained finite group with $Q = O_2(QY)$ a special group of order q^9, $q > 2$ a power of 2, $Y \cong GU^*(4, q) \cong GU(4, q) \times Z_{q-1}$ with Q/Q' the standard module for Y. Then the isomorphism type of QY is unique.*

Proof. Let Y_0 be the subgroup of Y corresponding to $GU(4, q)$. Set $H = QY_0$, $\bar{H} = H/Q'$. Since the Schur multiplier of Y_0 is trivial [38] and $\mathrm{Ext}^1_{Y_0}(F_2, \bar{Q}) = 0$ (because $Z(Y_0) \cong Z_{q+1}$ acts fixed point freely on $\bar{Q}$), we get that the Schur multiplier of H is isomorphic to Q', an elementary abelian group of order q (because the invariants in $\bar{Q} \otimes \bar{Q}$ of Y_0 have dimension 1 over $\mathrm{End}_{Y_0}(\bar{Q}) \cong F_{q^2}$). Therefore, H is a covering group of $\bar{H}$. A result of Schur [51] states that if G is a finite group in which $|G/G'|$ is prime to the order of the multiplier, a covering group is unique up to isomorphism. So, H is uniquely determined. Let $\pi = \pi(q-1)$ and let $\langle y \rangle = O_\pi(Z(Y))$. The action of y on $\bar{H}$ lifts to a unique action on H (see [37, appendix]). Since $H \cap \langle y \rangle = 1$, the isomorphism type of $QY = H\langle y \rangle$ is completely determined.

LEMMA 3.20. *Let K be one of the linear groups in the conclusion of Propositions* A, CF, D *or* E. *Then K does not contain a p-element inducing a quadratic minimal polynomial on B^*, except for $p = 3$ and K essentially W_{I_3}.*

Proof. Let x be a p-element with $[B^*, x, x] = 1$. Then x does not act nontrivially on a nonidentity abelian 2-group. So, if $O_2(K) \neq 1$, it is extraspecial and K is essentially the Weyl group of F_4. In this case, if $R \cong O_2(K)$, then $[R, x] \cong Q_8$, and $C_R(x) = Q_8$.

Suppose $O_2(K) = 1$. Then either K is essentially a Weyl group of type A and B^* is a standard module, in which case the result is obvious, or else K is of type E_6, E_7 or E_8; but then special arguments may be employed. It suffices to do the case $K \cong W_{E_8}$.

If x lies in a subgroup isomorphic to some Σ_n generated by reflections, we are done. Since $W_{A_8} \to W_{E_8}$, the only possibility is $p = 5$ and x has minimal polynomial $(t^5 - 1)/(t - 1)$ on the root lattice, Λ. Say $[\bar{\Lambda}, x, x] = 1$ where $\bar{\Lambda} \cong \Lambda/5\Lambda \cong B^*$. Such an x lies in a diagonal subgroup $S \subseteq S_1 \times S_2$, $S \cong S_1 \cong S_2 \cong W_{A_4}$. The representation theory of $\mathbb{F}_5 A_5$ shows that x is not quadratic on any irreducible module, contradiction.

PROPOSITION 3.21. *Assume Hypothesis* 3.16 *with* K *sporadic. Then* $m_p(K) \leqslant 2$.

PROPOSITION 3.22. *Assume Hypothesis* 3.16 *with* K *sporadic. Then* $p=5$, $K \cong F_{22}$, $H \cong D_4(3)\,\Sigma_3$.

Proof of Proposition 3.22. This follows from Proposition 3.21, [33], and Hypothesis 3.16. By [33, Part I, Section 24] (K, p) must be on the following list when $m_p(K)=2$.

$$\begin{array}{ll} p=3: & K = M_{11}, M_{12}, \\ p=5: & K = HiS, M^cL, F_{22}, \\ p=7: & K = Held, O's, F'_{24}, \\ p=11: & K = J_4. \end{array}$$

We eliminate all but $p=5$, $K=F_{22}$.

Suppose $L(H/Q)=1$. By Hypothesis 3.16(III(f)) and knowledge of Schur multipliers [38], we eliminate all groups on the list above.

Suppose $Q \neq 1$. Then for some $e \in E^{\#}$, $O_2(C(e)) \neq 1$. By checking the properties of the groups on the list, we find that the only possibility is M_{12}, $p=3$, $C(e) \cong 3^{1+2}2$ or $3 \times A_4$ for $e \in E^{\#}$. Thus $|Q|=4$, since E contains elements of both 3-classes, and so $QE \cong 3A_4$. But then, as all elements of $E - Z(P)$ ($E \leqslant P \in \mathrm{Syl}_3(K)$) are fused, we have a contradiction to $C(e) \subseteq H$ for all $e \in E^{\#}$. So, $Q=1$.

We now have that $L = L(H) \cong L(H/Q) \neq 1$. Suppose $O_p(H) \neq 1$. By checking the properties of groups on the list, $L \neq 1$ limits us to the possibilities

$$\begin{array}{llll} HiS: & p=5, & 5 \times A_5, & \\ Held: & p=7, & 7 \times L_3(2), & \text{twice.} \end{array}$$

But in these cases $|K:H| \equiv 0 \pmod p$, a contradiction.

Finally, we get $O_p(H)=1$. Suppose for some $e \in E^{\#}$, $C(e)$ is nonsolvable. We then get the possibilities of the previous paragraph. In particular, $p \geqslant 5$. Since P is nonabelian of order p^3, exponent p, P is the weak closure of E in P. Therefore, L has nonabelian Sylow p-group, whence p divides the order of the Weyl group, whence $|P| \geqslant p^{p-1} \geqslant p^4 > |H|_3 = p^3$, a contradiction. So, $C(e)$ is solvable, for $e \in E^{\#}$. The above argument goes through unless P is abelian or $p=3$. If $p=3$, $L \neq 1$ and $(|K:H|, 3)=1$ imply that $K=M_{11}$ and $L \cong A_6$, $H \cong M_{10}$. But then $N(P) \cong 3^2 \cdot 2^4$ cannot be in H, a contradiction. So, $p=5$ and $K=F_{22}$, as required.

Proof of Proposition 3.21. Until further notice, we assume Hypothesis 3.16(I, II, III). Without loss, $K = F^*(K)$. We let P denote a Sylow p-group containing E.

We assume that the list of Known simple groups in [28, Chapter 2] is complete.

In some of the results which follow, we give information about sporadic groups. It is not possible to give published references in every case. Sometimes, the information is deduced from the character table and class list, copies of which have circulated among the group theorists. The published references are [4, 45, 48]. See also [33].

LEMMA 3.23. *Suppose that $m_r(K) \geqslant 3$ for some odd prime r. Then the possibilities are:*

K	r	$\|K\|_r$	$m_r(K)$
J_3	3	3^5	3
M^cL	3	3^6	$\geqslant 4$
Suz	3	3^7	$\geqslant 5$
$0 \cdot 1$	3, 5	$3^9, 5^4$	$\geqslant 6$ for $r = 3$ 3 for $r = 5$
$0 \cdot 2$	3	3^6	4
$0 \cdot 3$	3	3^7	$\geqslant 5$
F_{22}	3	3^9	$\geqslant 5$
F_{23}	3	3^{13}	$\geqslant 6$
F'_{24}	3	3^{16}	$\geqslant 7$
LyS	3, 5	$3^7, 5^6$	$\geqslant 5$ for $r = 3$ 3 for $r = 5$
$O'S$	3	3^4	4
F_2	3, 5	$3^{13}, 5^6$	$\geqslant 5$ for $r = 3$ 3 for $r = 5$
F_1	3, 5, 7	$3^{20}, 5^9, 7^6$	$\geqslant 7$ for $r = 3$ $\geqslant 4$ for $r = 5$ $\geqslant 3$ for $r = 7$
F_3	3	3^{10}	$\geqslant 5$
F_5	3, 5	$3^6, 5^6$	$\geqslant 3$ for $r = 3, 5$

LEMMA 3.24. *Suppose that $F^*(K)$ is sporadic, $m_p(K) \geqslant 3$ and that $r \in \pi(K)$ is odd.*

(a) *For $x \in K$, $|x| = r$, the possibilities for $C(x)$ are listed below.*

(b) *In* (a), *when $L(C(x)) \neq 1$ and every component lies in* Chev(2), *we mark with an $*$ (an $*?$ indicates a possibility only).*

(c) *When* $x \in K$, $|x| = r$ *and* $O_2(C_{F^*(K)}(x)) \neq 1$, *we mark with a* $\neq$ *(an* $\neq$? *indicates a possibility only).*

(d) *If a Sylow 3-group of K has noncyclic center,* $F^*(K) \cong J_3$, $O'S$, Suz.

Remark 3.25. Since $|\mathrm{Out}(K)| \leqslant 2$, for all sporadic simple groups K, if $|O_2(C_K(x))| \neq 1$ and there is no $\neq$ opposite K for $C(x)$, then $|O_2(C_K(x))| = 2$. This occurs for $K = F_{22} \cdot 2$. In the proof, we may assume $K = F^*(K)$).

Sporadic Group	p	r	Centralizer orders for class of order r	Centralizers	
J_3	3	3	3^5	3^5	
			$2^3\,3^3\,5$	$3 \times A_6$	
		$\geqslant 5$		solvable	
M^cL	3	3	$2^3\,3^6\,5$	$3^{1+4}SL(2, 5)$	
			$2^3\,3^5$	solvable	$\neq$?
		$\geqslant 5$		solvable	
Suz	3	3	$2^7\,3^7\,5\,7$	$3 \cdot U_4(3)$	
			$2^3\,3^4\,5$	$3 \times 3 \times A_6$	*
			$2^3\,3^7$	solvable	$\neq$?
		5	$2^3\,3^2\,5^2$	$5 \times A_6$	*
			$2^2\,3\,5^2$	$5 \times A_5$	*
		7	$2^2\,3\,7$	$3 \times A_4$	$\neq$
		$\geqslant 11$		solvable	
$0 \cdot 1$	3, 5	3	$2^{13}\,3^8\,5\,7\,13$	$3\,Suz$	
			$2^7\,3^9\,5$	$3^{1+4}Sp(4, 3)$	
			$2^6\,3^5\,5\,7$	$3 \times A_9$	
			$2^8\,3^8\,5\,7$	$3 \cdot 3 \cdot U_4(3) \cdot 2$	
		5	$2^7\,3^3\,5^2\,7$	$5 \times HJ$	
			$2^3\,3\,5^4$	$5^{1+2}SL(2, 5)$	
			$2^5\,3^2\,5^3$	$5 \times (A_5 \times A_5)2$	*, $\neq$?
		7	$2^3\,3^2\,5\,7$	$7 \times A_7$	
			$2^3\,3\,7^2$	$7 \times L_2(7)$	
		$\geqslant 11$		solvable	
$0 \cdot 2$	3	3		$3^{1+4}2^{1+4}A_5$	
				$3 \times U_4(2)\,2$	*
		5	$2^2\,3\,5^3$	solvable	
			$2^3\,3\,5^2$	$5 \times A_5 \cdot 2$	*
		7	$2^3\,7$	$7 \times D_8$	$\neq$
		$\geqslant 11$		solvable	

Sporadic Group	p	r	Centralizers orders for class of order r	Centralizers	
$0 \cdot 3$	3	3	$2^3\ 3^4\ 7$	$3 \times L_2(8) \cdot 3$	
			$2^5\ 3^7\ 5$	$3^{1+4}SL(2, 9)\ 2$	
			$2^3\ 3^6\ 5$	$3 \cdot 3^4 \cdot A_5 \cdot 2$	
		5	$2^3\ 3\ 5^3$	solvable	
			$2^2\ 3\ 5^2$	$5 \times A_5$	
		7	$2\ 3\ 7$	solvable	
		11	$2\ 11$	cyclic	$\neq$
		23	23	23	
F_{24}	3	3	$2^{13}\ 3^{14}\ 5^2\ 7\ 13$	$3 \times D_4(3), \Sigma_3$	
			$2^{11}\ 3^{16}\ 5\ 11$	$3^{1+10}U_5(2) \cdot 2$	
			$2^6\ 3^{14}$	solvable	
			$2^9\ 3^{13}\ 5\ 7$	$3 \cdot 3^6 \cdot O^-(6, 3), 2$	
			$2^7\ 3^8\ 7\ 13$	$3 \times 3 \times G_2(3)$	
		5	$2^7\ 3^4\ 5^2\ 7$	$5 \times \Sigma_9$	
		7	$2^4\ 3^2\ 5\ 7^2$	$7 \times \Sigma_7$	
			$2^2\ 3\ 7^3$	solvable	
		11	$2^3\ 3\ 11$	solvable	$\neq$
		$\geqslant 13$		solvable	
F_{23}	3	3	$2^9\ 3^{10}\ 5\ 7\ 13$	$3 \times B_3(3)$	
			$2^{10}\ 3^{13}$	solvable	
			$2^4\ 3^{10}$	solvable	
			$2^7\ 3^{10}\ 5$	$3 \cdot 3^5 \cdot B_2(3)\ 2$	
		5	$2^4\ 3^2\ 5\ 7$	$5 \times \Sigma_7$	
		7	$2^3\ 3\ 5\ 7$	$7 \times \Sigma_5$	*
		11	$2^2\ 11$	solvable	$\neq$
		$\geqslant 13$		solvable	
F_{22}	3	3	$2^8\ 3^7\ 5\ 7$	$3 \times U_4(3)\ 2$	
			$2^7\ 3^9$	solvable	$\neq$?
			$2^3\ 3^7$	solvable	$\neq$?
			$2^6\ 3^7$	solvable	$\neq$?
		5	$2^3\ 3\ 5^2$	$5 \times \Sigma_5$	
					*
		7	$2\ 3\ 7$	solvable	
		11	$2\ 11$	solvable	$\neq$
		13	13	solvable	

Sporadic Group	p	r	Centralizer orders for class of order r	Centralizers	
LyS	3, 5	3	$2^4\ 3^7\ 5$	$3^{2+4}SU^{\pm}(2, 5)$	
			$2^7\ 3^7\ 5\ 7\ 11$	$3M^cL$	
		5	$2^4\ 3^2\ 5^6$	$5^{1+4}SL(2, 9)$	
			$2\ 3\ 5^4$	$(5^{1+2} \times 5) \cdot \Sigma_3$	
		7	$2^4\ 3\ 7^2$	$7 \times SL(2, 7)$	
		$\geqslant 11$		solvable	
$O'S$	3	3	$2^3\ 3^4\ 5$	$3 \times 3 \times A_6$	*
		$\geqslant 5$		solvable	
F_1	3, 5, 7	3	$2^{14}\ 3^{20}\ 5^2\ 7\ 11\ 13$	$3^{1+12}2Suz$	
			$2^{15}\ 3^{11}\ 5^3\ 7^2\ 13\ 19\ 31$	$3 \times F_3$	
			$2^{21}\ 3^{16}\ 5^2\ 7^3\ 11\ 13\ 17\ 23\ 29$	$3F'_{24}$	
		5	$2^8\ 3^3\ 5^9\ 7$	$5^{1+6}2HJ$	
			$2^{14}\ 3^6\ 5^6\ 7\ 11\ 19$	$5 \times F_5$	
		7	$2^4\ 3^2\ 5\ 7^6$	$7^{1+4}2A_7$	
			$2^{10}\ 3^3\ 5^2\ 7^4\ 17$	$7 \times Held$	
		11	$2^6\ 3^3\ 5\ 11^2$	$11 \times M_{12}$	
		13	$2^3\ 3\ 13^3$	$13^{1+2}SL(2, 3)$	
		17	$2^3\ 3\ 7\ 17$	$17 \times L_2(7)$	
		19	$2^2\ 3\ 5\ 19$	$19 \times A_5$	
		23	$2^3\ 3\ 23$		$\neq$
		29	$3\ 29$	29×3	
		31	$2\ 3\ 31$	$31 \times \Sigma_3$	
		41	41	41	
		47	$2\ 47$	47×2	$\neq$
		$\geqslant 59$		solvable	
F_2	3, 5	3	$2^{19}\ 3^{10}\ 5^2\ 7\ 11\ 23$	$3 \times F_{22} \cdot 2$	
			$2^{13}\ 3^{13}\ 5$	$3^{1+8}2^{1+6}U_4(2)$	
		5	$2^{11}\ 3^2\ 5^4\ 7\ 11$	$5 \times HiS \cdot 2$	
			$2^7\ 3\ 5^6$	$5^{1+4}\ 2^{1+4}A_5$	
		7	$2^7\ 3^2\ 5\ 7$	$7 \times 2 \cdot L_3(4) \cdot 2$	$\neq$
		11	$2^3\ 3\ 5\ 11$	$11 \times \Sigma_5$	*
		13	$2^3\ 3\ 13$	$13 \times \Sigma_4$	$\neq$
		17	$2^2\ 17$	$2 \times 2 \times 17$	$\neq$

Sporadic Group	p	r	Centralizer orders for class for order r	Centralizers	
F_3	3	3	$2^4\,3^7\,5$	$3 \cdot 3^4 A_6\,2$	
			$2^6\,3^7\,7\,13$	$3 \times G_2(3)$	
			$2^3\,3^{10}$	solvable	
		5	$2^3\,3\,5^3$	5^{1+2}	
		7	$2^3\,3\,7^2$	$7 \times L_2(7)$	
		$\geqslant 13$		solvable	
F_5	3, 5	3	$2^6\,3^5\,5\,7$	$3 \times A_9$	
			$2^3\,3^6\,5$	$3^{1+4} SL(2, 5)$	
		5	$2^4\,3^2 5^4\,7$	$5 \times U_3(5)$	
			$2^5\,5^6$	solvable	
			$2^3\,3\,5^4$	solvable	
		7		$7 \times A_5$	
		11		2×11	$\neq$
		19	$2\,19$	19×2	$\neq$

Proof. Study the character tables and class lists.

We argue that none of the rows for $p = 3$ and $K = F_{24}'$ or F_{22} deserves a $\neq$. (They tentatively deserve $\neq$'s).

Say $K = F_{24}, |x| = 3, |C(x)| = 2^6 3^{14}$. Without loss, $P \subseteq C(x)$, $P \in \mathrm{Syl}_3(K)$. Say $O_2(C(x)) \neq 1$. Let $2^r = |O_2(C(x))/\Phi(O_2(C(x)))|$, $r \geqslant 1$. Since $\max\{|C(t)|_3 \mid t$ an involution of $F_{24}\} = 3^{10}$, we get $r = 6$, $P/C_P(O_2(C(x))) \cong Z_3 \wr Z_3$ and every element of $O_2(C(x))^{\#}$ has centralizer of the form $2F_{23}$. Since such an involution lies in $F_{24} - F_{24}'$ and $r > 1$, we have a contradiction.

Say $K = F_{23}, x \in K$, $|x| = 3$, $|C(x)| = 2^{10} 3^{13}$, $O_2(C(x)) \neq 1$. Then $|P : C_P(O_2(C(x)))| \leqslant 3^6$. Thus, for $y \in O_2(C(x))$, $|C(y)|_3 \geqslant 3^7$, a contradiction. Say $x \in K$, $|x| = 3$, $|C(x)| = 2^4 3^{10}$, $O_2(C(x)) \neq 1$. Then for $y \in O_2(C(x))$, $|C(y)|_3 \geqslant 3^8$, another contradiction. So, $K \neq F_{23}$.

LEMMA 3.26. *$p \nmid |K : H|$; in fact $N_K(P) \subseteq H$ for $P \in \mathrm{Syl}_p(H)$.*

Proof. If $E \subseteq H_1 \subseteq H$, then $N_K(H_1) \subseteq H$.

LEMMA 3.27. *Suppose $m_p(K) \geqslant 3$, $P_p(H) = 1$ and $L(H) \neq 1$. Then $L(H)$ is quasisimple.*

Proof. Let $L_1, \ldots, L_s$ be the components. We assume $s \geqslant 2$.

We claim that E normalizes each component. If false, take an index i with $E \subset N(L_i)$. Then $L_i \cong L_i^x$ for all $x \in E$, so that $m_r(H) \geqslant 3$ for $r \in \pi(L_i)$. By

Lemma 3.23, $(K, r) = (.1, 3)$, $(.1, 5)$, $(LyS, 3)$, $(LyS, 5)$, $(F_2, 3)$, $(F_2, 5)$, $(F_1, 3)$, $(F_1, 5)$, $(F_1, 7)$, $(F_5, 3)$ or $(F_5, 5)$, for $r \in \pi(L_i)$. In particular, $\pi(L_i) \subseteq \{2, 3, 5, 7\}$. Suppose $r = 7$. Then there are distinct, pairwise commuting conjugates L_i, L_j, L_k and an element $y \in L_i, |y| = 7$ so that $C(y) \supseteq \langle y, L_j, L_k \rangle$. So, $C(y) \cong Z_7 \times$ Held, $L_i \cong L_2(7)$, $s = 3$ and $p = 3$. Take $x \in E - N_E(L_i)$. Then $C_{L(H)}(x) \cong L_2(7)$. But, $C_K(x) \cong 3F'_{24}$, $3^{1+12}2\ Suz$, or $3 \times F_3$, whence $C_K(x)$ cannot be contained in H, a contradiction. So, $r \leqslant 5$ and $\pi(L_i) = \{2, 3, 5\}$. By properties of K-groups, $L_i/Z(L_i) \cong L_2(4)$, $L_2(9) \cong Sp(4, 2)'$ or $U_4(2)$.

Since Lys does not contain a four-group whose centralizer involves a copy of L_i, L_i, $(LyS, 3)$ and $(LyS, 5)$ are out. So, $K \cong \cdot 1$, F_2, F_5 or F_1.

Suppose $K \cong \cdot 1$. The only possibility is $s = 3$, $p = 3$ and $L(H) \cong A_5 \times A_5 \times A_5$. Take $x \in E$, $L_i^x \neq L_i$. Since $|x| = 3$, $C(x) \cong 3^{1+4}Sp(4, 3)$, $3 \times A_9$, $3^2 \cdot U_4(3).2$ or $3.Suz$. Clearly $C(x) \subseteq H$ is impossible, in all these cases.

Similar arguments eliminate the cases F_1, F_2 and F_5. Say $K \cong F_1$. Then $p = 3$, or else $A_5 \times A_5 \times A_5 \times A_5$ is contained in the centralizer of an element of order 5. So $s = 3$ and $p = 3$ and we get a contradiction as above (if $|x| = 3$, $x \in K$, then $C(x) \cong 3F'_{24}, 3 \times F_3$ or $3^{1+12}2\ Suz$). The cases F_2 and F_5 proceed similarly. The claim follows; that is, E normalizes each component.

Let $E \subseteq P \in \mathrm{Syl}_p(H) \subseteq \mathrm{Syl}_p(K)$. We argue that $p \in \pi(L_i)$ for all i. Suppose that $p \notin \pi(L_i)$. Then as L_i is a Chevalley group or Steinberg variation, $p \neq 3$ so that $p = 5$ or 7. Thus, the structure of $\mathrm{Out}(L_i)$ (cyclic Sylow p-groups since $p \notin \pi(L_i)$) implies that $C_E(L_i) \neq 1$ and that if $[L_i, E] = 1$, then some element of E induces a field automorphism on L_i. If $[L_i, E] = 1$, we contradict Lemma 3.24 for $p \geqslant 5$. So, $[L_i, E] = 1$. Thus, L_i is a group defined over some finite field whose degree over the prime field is divisible by $p \geqslant 5$, a contradiction to Lemma 3.24. Therefore $p \in \pi(L_i)$ for all i, as claimed.

We now argue that P has one orbit on $\{L_1, ..., L_s\}$. Suppose otherwise. Since $O_p(H) = 1$, $O_p(Z(L_i)) = 1$ for all i, whence $Z(P)$ is noncyclic, and so K does not contain an element x of order p with $C(x)$ p-constrained, $O_{p'}(C(x)) = 1$ and $O_p(C(x))$ extraspecial. By checking Lemmas 3.23 and 3.24, we eliminate every possibility except $(K, p) = (J_3, 3)$, $(Suz, 3)$ $(F_{22}, 3)$, $(LyS, 3)$, $(O'S, 3)$. From above, there is $x \in Z(P)^{\#}$ with $C(x)$ nonsolvable. Thus, $(K, p) = (Suz, 3)$, $(LyS, 3)$ or $(O'S, 3)$. In all these groups, if y is an element of order p, $C(y)$ does not involve a direct product of two simple groups. By Lemma 3.26, $s = 2$ and $(|\mathrm{Out}(L_i)|, 3) = 1$ for $i = 1, 2$. Therefore, P is decomposable as a direct product $(P \cap L_1) \times C_P(L_1)$. This forces $K = O'S$. However, $N_K(P)$ is transitive on $P^{\#}$, against $P = (P \cap L_1) \times (P \cap L_2)$ and the fact that $N_K(P) \subseteq H$ permutes $\{L_1, L_2\}$.

So, P has one orbit on $\{L_1, ..., L_2\}$, as claimed. Therefore $s > 1$ is a power of p and P involves $Z_p \wr Z_p$. So, if $p \geqslant 5$, then $p = 5$ and $K = F_1$ (look at the list of sporadics in Lemma 3.23). In that case, the centralizer of an element of order 5 in L_1 contains $L_2 \times \cdots \times L_p$, a contradiction to Lemma 3.24. Thus, $p = 3$. Since $L_1 \cong L_2 \cong L_3$, $m_r(H) \geqslant s \cdot m_r(L_i) \geqslant 3$ for $r \in \pi(L_1)$ (we get $s = 3$ and $\pi(L_1) = \{2, 3, 5\}$). Thus, $L_1 \cong A_5, A_6$ or $U_4(2)$ (using properties of K-groups). Consequently, an element y of order 5 in L_1 centralizes $L_2 \times L_3$. Therefore $K = .1$ or F_1 and $C(y) \cong (5 \times A_5 \times A_5)\,2$ or $5 \times F_5$ or $5^{1+6}2HJ$. Since $P \in \mathrm{Syl}_3(K)$ and $|P| = 3^9$ or 3^{20}, we get $|C_P(L(H))| \geqslant |P| \cdot 3^{-7} \geqslant 3^2$, forcing $Z(P)$ to be noncyclic and for $C(z)$ to contain $L_1 \times L_2 \times L_3$, for some $z \in Z(P)^{\#}$. This is clearly impossible since $|C(z)|_5 \leqslant 5^2$, a contradiction which proves the lemma.

LEMMA 3.28. *Suppose that* $m_p(K) \geqslant 3$, $P \in \mathrm{Syl}_p(K)$ *and* $Z(P)$ *is noncyclic. Then* $(K, p) = (J_3, 3)$, $(Suz, 3)$, $(LyS, 3)$ *or* $(O'S, 3)$, *and conversely* $Z(P)$ *is noncyclic for these groups.*

Proof. We may eliminate (K, p) from the list of conclusions if K contains an element x of order p for which $C(x)$ is p-constrained, $O_{p'}(C(x)) = 1$ and $O_p(C(x))$ is extraspecial. What remains are the four pairs above and $(F_{22}, 3)$, which we must eliminate.

Let $K = F_{22}$, $x \in K$ with $|x| = 3$ and $C(x) \cong 3 \times U_4(3) \cdot 2$. Without loss, x is extremal in $P \in \mathrm{Syl}_3(K)$. The structure of $U_4(3)$ implies that $Z(C_P(x)) = \langle x \rangle \times \langle z \rangle$, where $\langle z \rangle = Z(P \cap L(C(x))) \subseteq (P \cap L(C(x)))'$. Since $N_P(C_P(x))$ contains $C_P(x)$ properly, it must act nontrivially on $Z(C_P(x)))$, fixing z. Therefore, $Z(P) = \langle z \rangle \cong Z_3$, as required.

LEMMA 3.29. *There is no* K, H *satisfying our hypotheses with* $O_p(H) = 1$ *and* $L(H) \neq 1$ *quasisimple and* $m_p(K) \geqslant 3$.

Proof. Suppose that there is a pair K, H satisfying our hypotheses with $m_p(K) \geqslant 3$. Let $L = L(H)$, a quasisimple group by Lemma 3.27.

We claim that $p \in \pi(L)$. Suppose false. The structure of $\mathrm{Aut}(L)$ (i.e., cyclic Sylow p-groups) implies that P induces a group of field automorphisms on L. So, $P/C_P(L)$ is cyclic and $E_1 = C_P(L) \cap E \neq 1$. By referring to Lemma 3.24 (the *'s) for the cases $m_r(K) \geqslant 3$, $r \notin \pi(L(C(x)))$ and $L(C(x)) \in \mathrm{Chev}(2)$, we find no possibilities. Since $C_K(E_1) \subseteq H$, we have a contradiction which proves the claim.

If $C_P(L) \neq 1$, then $p \in \pi(L)$ implies that $Z(P)$ is noncyclic and there is $z \in \Omega_1(Z(P))^{\#}$ with $L \subseteq C(z)$, making $C(z)$ nonsolvable. So, by Lemma 3.28, $K = Suz$, LyS or $O'S$. Since $N(P)$ acts irreducibly on $Z(P)$ in the case of LyS and $O'S$, we get a contradiction, since $N(P) \subseteq H \subseteq N(L)$ and $L \cap Z(P) \neq 1$. So, $K = Suz$, $|P| = 3^7$ and there is $z \in Z(P)^{\#} \cap C(L)$ with

$C(z)$ nonsolvable (as $L \subseteq C(z)$). So, by Lemma 3.24, $C(z) \cong 3U_4(3)$. Since $L \in \mathrm{Chev}(2)$, and L is embedded in $U_4(3)$, $L \cong A_5, A_6, U_4(2), L_3(2)$ or $L_3(4)$. In any of these cases $|C_p(L)| \geqslant 3^3$, since P normalizes L. However, $3^3 \nmid |C(y)|$ for an element $y \in K$ of order 5 or 7, a contradiction. Therefore, $C_p(L) = 1$.

Thus, P acts faithfully on L. Since $m_p(P) \geqslant 3$, either the Lie rank of L is at least 3, or it is 2, and some elements of P induce field automorphisms of L. Note that no element of order p may induce a field-diagonal automorphism in the latter case.

Suppose that the Lie rank of L is 2. Then L is defined over F_q, where $q = 2^k$, $k \equiv 0 \pmod p$. Either $E \subseteq L$ or $|E \cap L| = 3$ and there is $x \in E - L$ inducing a field automorphism on L. In the latter case, $L(C(x)) \cong A_2(2^{k/p})$. This possibility does not occur with an * in Lemma 3.24. So, $E \subseteq L$.

Suppose $L \cong A_2(q)$. Thus, (a) all elements of E are conjugate and lie in the center of a Sylow p-group of H, (b) $C(x)$ is solvable, for $x \in E^{\#}$, (c) $Z(P)$ is cyclic if $p = 3$, (d) $m_p(P) = 3$, (e) p divides $2^k - 1$.

Say $p = 3$. By Lemma 3.24, $F^*(K) = J_3$, or F_3. On the other hand, if $y \in P$, $y = 3$ and y induces a field automorphism on L, then $C(y)$ contains a copy of $A_2(2^{k/3})$. Neither J_3 nor F_3 satisfy this condition. So, $p = 5$ or 7. Since $k \equiv 0 \pmod p$, P contains a copy of $Z_{p^2} \times Z_{p^2}$ as a proper normal subgroup. This condition, with $m_p(K) \geqslant 3$, quickly forces $(K, p) = (LyS, 5)$, $(F_2, 5)$, $(F_5, 5)$ (all of which, incidentally, have isomorphic Sylow 5-subgroups) or $(F_1, 7)$. But upon closer inspection we find that none of these pairs has the requisite property.

Suppose that $L/Z(L) \cong {}^2A_n(q)$ for q even and $n \leqslant 4$. If $n = 2$, $m_p(L) \geqslant 2$ implies that $p | q + 1$. If $p \neq 3$, $Z(p)$ has rank at least 3, a contradiction to Lemma 3.24. So, $p = 3$. Thus, $m_3(K) = 3$ whence $K = J_3$ or F_5 by Lemma 3.24. However, for $K = J_3$, $\Omega_1(P)$ is abelian, whereas $\Omega_1(P \cap L)$ is nonabelian, a contradiction.

Say $K = F_5$. Then $|K|_3 = 3^6$. Since an element of P induces a field automorphism on L and $3 | q + 1$, we get $q + 1 \equiv 0 \pmod 9$. Therefore, $|P| \geqslant 3\,|L|_3 \geqslant 3^7$, a contradiction. Consequently, $n = 3$ or 4. If $p = 3$, $p | q - 1$ or $p | q + 1$. In either case, $Z(P)$ is noncyclic, whence $K = Suz$, $O'S$, LyS or J_3. If $3 | q - 1$, $\Omega_1(P) \cong Z_3^3$ whence $K = J_3$. However, for L of type ${}^2A_n(q)$, $\Omega_1(P) \nleqslant P'$. So, $3 | q + 1$. Thus, $Z(P)$ is noncyclic since L contains the normalizer of a torus of shape $(Z_{q+1})^n \cdot \Sigma_{n+1}$ modulo a group of order $(n + 1, q + 1)$, $n = 3$ or 4. In particular, $m_p(P) = n + 1 = 4$ or 5, whence $n = 3$ and $K = O'S$, which has abelian Sylow 3-groups, a contradiction. So, $p \neq 3$. By Lemma 3.28, P is nonabelian, whence $n = 3$ or 4 implies that $p = 5$, $5 | q + 1$, $n = 4$ and $|P| \geqslant (5^2)^4 \cdot 5^{-1} \cdot 5 \cdot 5 = 5^9$. So, $K = F_1$ and the inequality is an equality. However, the structure of $\mathrm{Aut}(L)$ implies that P is metabelian, which conflicts with the structure of F_1.

Suppose that the Lie rank of L is 2 but L does not have type A_n or ${}^2A_n$.

Since L is a Chevalley group or a Steinberg variation, by (III), L has type $B_2(q)$. Thus $m_p(L)=2$ implies that $p|q^2-1$. Also $P\cap L$ is abelian and $Z(P)\cap L$ is noncyclic. So, K has type J_3, $O'S$, LyS or Suz. Since $m_3(P)=3$ (consider $\mathrm{Aut}(L)$), $K=J_3$. In J_3, $P'=\Omega_1(P)$, which is not the case in H, a contradiction.

The Lie rank l' of L is, therefore at least 3. We have $p=3$, 5 or 7. Let l be the rank of the largest subgroup of type A in L generated by root groups. We have $l\geqslant l'-1\geqslant 2$. From Lemma 3.24 we see that no element of E induces a field automorphism on L ($C_K(x)\leqslant H$ for all $x\in E$ and, if x induces a field automorphism, $L(C_K(x))$ has Lie rank 3). The same goes for field-graph automorphisms in case L has type $D_4(q)$. Suppose $x\in E$ induces a graph automorphism on $L\cong D_4(q)$. Then $|x|=3$ and $L(C(x))\cong G_2(q)$ or $G_2(2)'\cong U_3(3)$. Lemma 3.24 shows that this is impossible. So, E induces inner-diagonal automorphisms on L.

Say $31\,|\,|L|$. Then $K=O'S$, F_2, F_1 or F_3. The structure of $O'S$ and F_1 and the fact that $127=2^7-1$ imply that the Lie rank of L is at most 6. We claim that $p=3$. If $p=7$, the facts that the Lie rank of L is at most 6 and $|K|\leqslant 7^6$ imply that a Sylow 7-group of L is abelian or has an abelian subgroup of index 7. But then $K=F_2$ or F_3 and $m_7(K)\leqslant 2$, a contradiction. Therefore, $p=5$ or 3. If $p=5$, $m_5(K)\geqslant 3$ implies that $K=F_1$ or F_2. Since the Lie rank of L is at most 6, P' is abelian, a contradiction. (See Lemma 3.24.) So, $p=3$, as claimed. Say $K\neq O'S$. For $K=F_1, F_2$ or F_3, when $x\in K$, $|x|=3$ and $L(C(x))\neq 1$, $L(C(x))\notin \mathrm{Chev}(2)$. So, for $x\in E^{\#}$, $L(C(x)=1$ whence $C(x)$ is solvable (since x is a semisimple element in L). The only possibility is $K=F_3$ with $|C(x)|=2^3\cdot 3^{10}$ for $x\in E^{\#}$. In particular $|P|=3^{10}$. If $q=2^f>2$, then an element of order 31 lies in a cyclic group of order $(q^5-1)/(q-1)>31$, a contradiction. So, $q=2$. Therefore, L has type $A_4(2)$, ${}^2D_6(2)$, $D_5(2)$, $B_5(2)$. Since $C_P(L)=1$, $|P|\leqslant 3^6$, whereas $|K|_3=3^{10}$, a contradiction. Thus $K=O'S$. As above, L has type $A_4(2)$, ${}^2D_6(2)$, $D_5(2)$ or $B_5(2)$. Since P is abelian, L has type $A_4(2)$. But then $C_P(L)\neq 1$, a contradiction.

We have shown that $31\nmid|L|$. Thus, $(q^5-1)/(q-1)$ does not divide $|L|$; in particular, L does not involve $PSL(5,q)$, whence L has lie rank at most 4.

We claim that $p=3$. If $p\geqslant 5$, the Sylow p-subgroup is abelian, unless $p=5$ and L has type ${}^2A_n(q)$ for $4\leqslant n\leqslant 10$, $5|q+1$ or ${}^2E_6(q)$ for $5|q+1$. If the Sylow 5-group is abelian, we have a conflict with Lemma 3.24. Suppose L has type ${}^2A_n(q)$. Since $5|q+1$, $q=4$ or $q=2^s\geqslant 64$. Since $n\geqslant 4$, $q^{10}\,|\,|L|$, forcing $q=4$ since $|K|_2\leqslant 2^{46}$. If $n\geqslant 5$, $q^6+1=4097=17\cdot 241$ divides $|K|$, a contradiction. So $n=4$. But since $|{}^2A_4(q)|$ is divisible by $(q^5+1)/(q+1)$, we get $1025/5=205=5\cdot 41$ as a divisor of $|K|$. Therefore, $K=F_1$ which is impossible since $31\nmid|K|$.

Suppose that the Lie rank of L is 4. Then Table P tells us that there is an element $x\in L$, $|x|=3$ with $L(C_L(x))\in\mathrm{Chev}(2)$ and $L(C_L(x))$ of Lie rank at least 3. According to Lemma 3.24, there is no example of an element of order 3 in K with a such group involved in $C_K(x)$.

Suppose that L does not have type ${}^2A_n(q)$, for $n=5$ or 6. We have that the Lie rank of L is exactly 3. Then $m_3(P)=3$ or $m_3(P)=4$ and some element of P induces a field automorphism on L. In the latter case, an element x of order 3 in P has $L(C_L(x))$ of Lie rank 3, a contradiction to Lemma 3.24. Thus, P has rank 3 and a normal homocyclic abelian subgroup P_1 of index 3 and rank 3. A look at the groups in Lemma 3.24 reveals no such possibility.

We have that L has type ${}^2A_5(q)$ or ${}^2A_6(q)$. If L has type ${}^2A_6(q)$, there is an element x of order 3 in H with $L(C_L(x))$ of type ${}^2A_5(q)$ or ${}^2A_4(q)$, a contradiction to Lemma 3.24. The same argument applies to L of type ${}^2A_5(q)$ unless $H/C_H(L) \cong {}^2A_5(q) \cdot k$, where $(k,3)=1$, and if $P_1 = P \cap L \cong Z_3^4$. Then $N_L(P_1)/C_L(P_1) \cong \Sigma_6$. So, $|P| = 3^6$. Thus, $K = McL$, $\cdot 2$ or F_5. Since $q^{15} = |{}^2A_5(q)|_2$ but $2^{15} \nmid |McL|$ or F_5, we get $K = \cdot 2$. However, if $K \cong \cdot 2$, $\langle z \rangle = Z(P)$, then $C(z) \cong 3^{1+4}SL(2,5)$ and if $x \in P$ represents the other class of elements of order 3 then $C(x) = 3 \times U_4(2).2$. Thus, $q=2$ and $E^{\#} \subset x^H$ since $C_K(e) = C_H(e)$ is the centralizer of a semisimple element in L. Also, $C(z) \not\subseteq H$.

We eliminate this last possibility. In the usual matrix representation of $SU(6,2)$, we may assume that $x \in E^{\#}$ has shape

$$x \to \begin{pmatrix} \omega &&&&& \\ & \omega^{-1} &&&& \\ && 1 &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}$$

Let $y \in E - \langle x \rangle$. By adjusting with scalars, we have

$$y \to \begin{pmatrix} 1 &&&&& \\ & \alpha &&&& \\ && \beta &&& \\ &&& \gamma && \\ &&&& \delta & \\ &&&&& \varepsilon \end{pmatrix}.$$

if $\alpha = 1$, we may assume $\beta = \omega$, $\gamma = \omega^{-1}$, $\delta = 1$, $\varepsilon = 1$. But then $C_L(xy) \cong SL(2,2) \times SL(2,2) \times SL(2,2)$, whence $C_K(xy) \not\subseteq H$. Suppose

$\alpha \neq 1$; without loss, $\alpha = \omega$. We may assume that $\beta = \omega^{-1}$, $\gamma = \delta = \varepsilon = 1$. Then xy^2 is congruent to

$$\begin{pmatrix} \omega & & & & & \\ & \omega & & & & \\ & & \omega & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

modulo scalars. Since $C_L(xy^2) \cong (SU(3,2) \cdot SU(3,2).2$, $xy^2 \notin x^L$, a contradiction. So, we have eliminated the possibility.

This completes the proof of Lemma 3.29.

LEMMA 3.30. *Let $G \in$ Chev(2), $g \in G$, g odd, $C = C_G(g)$, C^0 the intersection of C with the connected component of the identity of $C_{\bar{G}}(g)$, where $\bar{G}$ is the ambient algebraic group over $\bar{\mathbb{F}}_2$ containing G. Then C/C^0 is abelian of odd order and C^0 is generated by conjugates of root elements of G.*

Proof. See Burgoyne and Williamson [10].

In the next series of results, we assume (I), (II), (III), (IV) and $O_p(H) \neq 1$.

LEMMA 3.31. $p \neq 7$.

Proof. If so, $K = F_1$ and $C(x) \cong 7 \times$ Held or $7^{1+4} \cdot 2 \cdot A_7$. Since $L(C(x))$ must be semisimple and have components in Chev(2) for $x \in E^{\#}$, we have a contradiction.

LEMMA 3.32. $p \neq 5$.

Proof. Suppose so. Then $K = .1$, LyS, F_2, F_1 or F_5. By looking for *'s in Lemma 3.24, we find that $K = .1$ or F_5 are the only possibilities.

Say $K = .1$. Then for $x \in E^{\#}$, $C(x) \cong 5 \times (A_5 \times A_5).2$. Since $C(x)$ operates on $O_5(H)$, we get $\langle x \rangle = O_5(H)$ for all $x \in E^{\#}$, which is absurd.

Say $K = F_5$. Let $Z = Z(P)$. Then $N_K(Z) \cong 5^{1+4}.2^{1+4}.5.4$. Since $|E| = 5^2$, $E \cap O_5(N_K(Z)) \neq 1$. Therefore, $N_K(Z) \subseteq H$. Since $1 \neq O_5(H) \cap N_K(Z) \lhd N_K(Z)$, we get $H = N_K(Z)$. If there is $x \in E - O_5(H)$, the fact that $O_5(H)/Z$ is an indecomposable $\langle x \rangle$-module means that $|C_p(x)| \leqslant 5^3$, a contradiction. Therefore, $E \subseteq O_5(H)$. Since $\pi(N_K(Z)) = \{2, 5\}$, $\pi(C_K(x)) \subseteq \{2, 5\}$ for $x \in E^{\#}$. The only possibility $E^{\#} \subseteq x^K$, where $|C(x)| = 2^2 \cdot 5^4$. We eliminate this possibility by showing that there does not exist $E \subseteq P$, $E \cong Z_5 \times Z_5$ with $E^{\#} \subseteq x^K$.

Suppose such an E exists. In the notation of [41], x lies in the class 5B. Let χ be an irreducible character of K of degree 133 [41]. Then $\chi(x) = 3$[41].

We have $\sum_{g \in E} \chi(g) = 133 + 24(3) = 210$. However, this sum should be congruent to 0 mod $25 = |E|$, a contradiction.

COROLLARY 3.33. *$p = 3$ and $O_3(H) \neq 1$.*

LEMMA 3.34. *If $K = J_3$, $H = N_K(Z(P)) = N_K(P)$.*

Proof. Suppose $K = J_3$. Since $E \subseteq \Omega_1(P) \cong Z_3^3$, $E \cap Z(P) \neq 1$. We claim that $E = Z(P)$. If not, $|E \cap Z(P)| = 3$, and if $x \in E - Z(P)$, $C(x) \cong 3 \times A_6$. Thus, $H \supseteq \langle N(P), N(\langle x \rangle)\rangle$, a group of order divisible by $2^4 3^5 5$ and containing a copy of A_6. Since a Sylow 3-normalizer in A_6 acts irreducibly on its Sylow 3-group, $L(C(x)) \cap P = Z(P)$. It follows that $O_p(H) = 1$, a contradiction to our temporary hypothesis. So, $E = Z(P)$ as claimed. Thus, $N(P) = N(Z(P)) \subseteq H$.

Since $O_3(H) \neq 1$, by hypothesis, the facts that $Z(P)$ is weakly closed in P with respect to K and $N(P)$ operates irreducibly on $Z(P)$ imply that $H \subseteq N(Z(P))$. So, $H = N(Z(P))$ as required.

LEMMA 3.35. *$K \neq LyS$.*

Proof. Say $K = LyS$. If $x \in K$, $|x| = 3$, $C(x) \cong 3^{2+4}.SL(2, 5)$ or $3.McL$. So, Hypothesis 3.16(IV) is not satisfied.

LEMMA 3.36. *$K \neq O'S$.*

Proof. Since $N(P)$ acts on $P \cong Z_3^4$ irreducibly, $O_3(H) \neq 1$ implies that $P = O_3(H)$, whence $H = 3^4 \cdot 2^{1+4} \cdot D_{10}$. However, if $x \in P^{\#}$, $C(x) \cong 3 \times 3 \times A_6$, so that $C(x) \nsubseteq H$, a contradiction.

LEMMA 3.37. *$K \neq Suz$.*

Proof. In Suz, there are three classes of elements of order 3, called $3U$, $3V$, $3W$, with centralizers of shape $3.U_4(3)$, $3 \times 3 \times A_6$, $3.3^{1+4}SL(2, 3)$.

Since at least one of these classes is represented by an element of $E^{\#}$, Hypothesis 3.16(IV) forces $E^{\#}$ to be disjoint from the first class. We claim that $E^{\#}$ meets class 3V. If false, $E^{\#}$ lies in class 3W and, given $e \in E^{\#}$ there is $y \in 3U$ such that $C(e) \subseteq C(y) \cong 3 \cdot U_4(3)$. In $C(y)$, $E \cap \langle y \rangle = 1$ and $\langle N_{C(y)}(\langle e_1 \rangle) | e_1 \in E^{\#} \rangle = C(y)$ (property of $U_4(3)$, since $N_{C(y)}(e_1)$ maps to a maximal parabolic in $U_4(3)$). Therefore, $C(y) \subseteq H$. Since $O_3(H) \neq 1$, $O_3(H) = \langle y \rangle$, contradicting $L \in \text{Chev}(2)$. So, $E^{\#} \cap 3V$ contains e, say, and $L(C(e)) \subseteq L$. Take $z \in O_3(H) \cap Z(P)^{\#}$. Then $C(z) \cong 3 \cdot U_4(3)$ and $L \in \text{Chev}(2)$ implies that $L = L(C(e)) \cong A_6$. Then, $|P| = 3^7$ implies that $|C(L)|_3 = 3^5$ and P is decomposable, a contradiction to the shape of $C(z) = 3 \cdot U_4(3)$.

LEMMA 3.38. *Without loss* (i) $|Z(P)| = 3$; (ii) $L_3(H) = 1$ *and* H *is* 3-*constrained*; (iii) *if* $x \in E^{\#}$, $C(x)$ *is solvable.*

Proof. If $|Z(P)| > 3$, then K has type J_3, *Suz*, *LyS* or *O'S*. These possibilities have been treated. Since $O_3(H) \neq 1$, we get (ii) from (i) unless $O_{3',3}(L_{3'}(H)) > O_{3'}(L_{3'}(H))$. Suppose that this happens. Let $\langle z \rangle = \Omega_1(Z(P))$. Then $\langle z \rangle$ maps onto $Z(L_{3'}(H)/O_{3'}(H))$ and $N_K(\langle z \rangle)$ covers $H/O_{3'}(H)$. Since $O_3(H) \neq 1$, $z \in O_3(H)$, whence $O_{3'}(H) \leqslant C(z)$. Therefore, $H \leqslant N_K(\langle z \rangle)$ and so $H = N_K(\langle z \rangle)$. By consulting Lemmas 3.24 and 3.25 we see that there are no such possibilities for $m_3(K) \geqslant 3$.

Suppose $x \in E^{\#}$ and $C(x)$ is nonsolvable. By Lemma 3.30 and Hypothesis 3.14(IV), $L(C(x)) \neq 1$. An application of the $P \times Q$ Lemma (see 5.3.4 of [27]) and the definition of $L(C(x))$ implies that $[O_3(H), L(C(x))] = 1$. Therefore, $L(C(x))$ is a $3'$-group, since $Z(P)$ is cyclic. By inspecting the *'s in Lemma 3.25, we find no such possibility. So, (iii) follows.

LEMMA 3.39. *K does not exist.*

Proof. We have $p = 3$, $O_3(H) \neq 1$ and $L_{3'}(H) = 1$. Then Hypothesis 3.16(IV) gives a contradiction, as $m_p(K) \geqslant 3$ and $L(H/O_2(H)) = 1$.

This completes the proof of the fusion controlling result, Proposition 3.21.

4. THE FIELD AUTOMORPHISM CASE

We begin the proof of the Main Theorem by establishing some notation which will be used throughout the rest of this paper. Take G to be of standard type with respect to (B, x, L) in $\mathscr{S}^*(p)$ and fix a standard subcomponent (D, J) of (B, x, L). Let $x = z_1$ and let $\langle z_2 \rangle \cdots \langle z_r \rangle$ be the distinct subgroups of order p in D for which those exist neighbors, $(B, z_2, K_2),\ldots,$ (B, z_r, K_r) of (B, x, L) with respect to (D, J). Let $(B, x, L) = (B, z_1, K_1)$.

By Table B, B lies in an elementary abelian p-group B^* such that B^* contains every element of order p in its centralizer and $|B^*: B| \leqslant p$. We define $N = N_G(B^*)$. In this section we prove

PROPOSITION 4.1. $O_p(A_G(B^*)) = 1$.

COROLLARY 4.2. *No element of* B^* *involves a field automorphism on any* K_i.

Proof of Corollary 4.2. Suppose false; then by the structure of $\mathrm{Aut}(K_i)$, $A_{K_i}(B^*)$ contains an element a of order p with $|B^*: C_{B^*}(a)| = p$. Let $F = \langle a^{A_G(B^*)} \rangle$. As $O_p(A_G(B^*)) = 1$, McLaughlin's theorem [47] together with the structure of $A_L(B^*)$ forces $F = SL(B^*)$ or $Sp(B^*)$. Indeed the only other possibility is $\langle x \rangle = C_{B^*}(F) \lhd N_G(B^*)$; but $\langle x \rangle \ntriangleleft N_{K_i}(B^*)$. Now all subgroups of order p in B^* are conjugate in $N_G(B^*)$, and in particular $\langle x\rangle$ is conjugate to $\langle z_2\rangle$. We have a contradiction as no element of B^* acts as a field automorphism on L (by definition of standard type) while such an action does occur on K_i.

Now we begin the proof of Proposition 4.1. We assume $P = O_p(A_G(B^*)) \neq 1$, and we define $B_1 = C_{B^*}(P)$, $M = N_G(B_1)$, $C = C_G(B_1)$. Pick $U \in \mathrm{Syl}_p(C)$. We will show that U normalizes one of the K_j's, inducing field automorphisms. Letting U_1 be the subgroup of U which induces inner-diagonal automorphisms on K_j, we show that U_1 is abelian and weakly closed in a Sylow p-group of G, and that $N_G(U_1)$ has a quotient of order p. By a theorem of Wielandt on transfer [39, 43] G is not simple, a contradiction. We conclude $P = 1$.

By the structure of B^* as a module for $A_L(B^*)$ (see Table B), we may define B_2 by $|B^*: B_2| \leqslant p$, $x \in B_2$, $B_2/\langle x\rangle$ is an absolutely irreducible module for $W = A_l(B^*)$ and $B^*/\langle x\rangle$ an indecomposable one.

LEMMA 4.3. $B^* = B_1 \times \langle x\rangle$.

Proof. Let $P_1 = C_P(\langle x\rangle)$, and pick $R \subseteq C_G(x)$ projecting onto P_1. R projects into $O_p(A_{N_G(L)}(B^*))$ whence $[R, R] = 1$ by Table B. Thus $[R, B] \subseteq C_{B^*}(L) = \langle x\rangle$. Since R centralizes $\langle B^* \cap L, x\rangle$, we have $P_1 = 1$ or $|B^*: \langle B^* \cap L, x\rangle| = p$, $\langle x\rangle = [B^*, P_1]$, and $|P_1| = p$. In particular as $\langle x\rangle \not\trianglelefteq N_G(B^*)$, we have $P_1 \neq P$ and $x \notin B_1$.

Since $B_1 \cap \langle x\rangle = 1$, B_1 projects nontrivially into $B/\langle x\rangle$ whence $B_1 \cap B_2 \neq 1$. B_1 is W-invariant where $W = A_L(B^*)$, so $B_2 = (B_1 \cap B_2) \times \langle x\rangle$, by Dedekind's law. If $B^* = \langle B_1, x\rangle$, we are done; so assume $B_2 = B_1 \times \langle x\rangle$ has index p in B^*. In particular B_1 is an absolutely irreducible W-module.

As $W = O^p(W)$ and B^*/B_2 is a trivial W-module, we see that $[B^*, B] \subseteq B_1$. Let $P_0 = C_P(B^*/B_1)$. We have that $[P, W] \subseteq P_0$ since $[B^*, W, P] = 1$ and $[P, B^*, W] \leqslant [B^*, W] \leqslant B_1$. Also, we have that $[P, W] \neq 1$ lest P normalize $\langle x\rangle = C_{B^*}(W)$ and $P = P_1$. Thus $P_0 \neq 1$. For any $b \in B^*$, $f_b(p) = [p, b]$ defines a map from P_0 to B_1. Our conditions imply $f_b(pq) = [pq, b] = [p, b]^q [q, b] = f_b(p) f_b(q)$, and further for any $w \in W$, $f_b(p)^w = [p^w, b^w] = [p^w, b[b, w]] = f_b(p^w)$ as $[b, w] \in B_1$. Thus, f_b is a homomorphism of groups and $P_0/\ker(f_b)$ is elementary abelian, i.e., $P_0/C_{P_0}(b)$ is elementary abelian for every $b \in B^*$ whence P_0 is also elementary abelian since we have an embedding $P_0 \to \prod_{b \in B^*} P_0/C_{P_0}(b)$. Further f_b is a homomorphism of W-modules. Since B_1 is absolutely irreducible, f_b is either trivial or an isomorphism. As $|B^*: B_1| = p^2$, P_0 is isomorphic to B_1 or $B_1 \times B_1$ as a W-module. In the latter case $C_{P_0}(x) = \ker f_x \cong B_1$ as a group, contradicting $|P_1| \leqslant p$. Thus $P_0 \cong B_1$ and we have $C_{P_0}(x) = 1$. Further by absolute irreducibility of P_0, we have $|B^*: C_{B^*}(P_0)| = p = |C_{B^*}(P_0): B_1|$.

Now for $d \in C_{B^*}(P_0) - B_1$, $f_d(p) = [p, d]$ defines a map from P to B_1 because $[d, P] \subseteq B_1$. We see exactly as before that $P/C_P(d)$ and $[P, d] \leqslant B_1$ are isomorphic W-modules. Since B_1 is irreducible, $|B_1| \geqslant p^2$ and $||P, d|| \leqslant p$, we get $[P, d] = 1$ whence $P = C_P(d)$ and we are done.

LEMMA 4.4. *The following hold*:

(i) $C_P(x) = 1$, $[B^*, P, P] = 1$, *and* P *is elementary abelian*;

(ii) *Either* $[P, x] = B_1$ *and* P *is isomorphic to* B_1 *as an* $A_L(B^*)$-*module, or* $B_2 < B^*$, $[P, x] = B_1 \cap B_2$ *and* P *is isomorphic to* $B_1 \cap B_2$ *as an* $A_L(B^*)$-*module*;

(iii) $D \cap B_1 = \langle z_i \rangle$ *for some* $i > 1$, B_1 *acts as inner-diagonal automorphism on* K_i, *and* $\langle x \rangle$ *induces a field automorphism on* K_i.

Proof. Parts (i) and (ii) follow from Lemma 4.3. Note that $|B_1 \cap B_2| \geqslant p^2$ in all cases. Now for (iii). Let $D \cap B_1 = \langle d \rangle$, and choose a p-group $R \leqslant C_G(d) \cap N_G(B^*)$ projecting onto P. By definition of standard type, J lies in a p-component K of $C_G(d)$. Suppose R does not normalize K. Since (D, J) is a subcomponent and $m_p(B) \geqslant 4$, $B^* \cap J$ acts nontrivially on J and so projects nontrivially into $\langle K^R \rangle / O_{p'p}(\langle K^R \rangle)$. This latter group is semisimple with at least 3 direct factors, whence $[B^*, R, R] \neq 1$, a contradiction by (i). Thus R must normalize K. As $[R, D] \geqslant [R, \langle x \rangle] \geqslant B_1 \cap B_2$, R does not normalize $D = C_{B^*}(J) = C_{B^*}(JO_{p'}(K)/P_{p'}(K))$. It follows that $K \neq JO_{p'}(K)$, $\langle d \rangle = \langle z_i \rangle$ for some i, and $K = K_i$.

It remains to prove the last two assertions of (iii). Suppose B^* acts as inner-diagonal automorphisms on K, and let R_1 be the subgroup of R which is inner-diagonal on K. In particular, the possibilities for K are given by Table P. By Table B, R_1 centralizes K unless perhaps $K = A_2(q)$ or $G_2(q)$, not the case by Table P. We have $[R_1, B^*] \subseteq C_{B^*}(K) = \langle d \rangle$. It follows that P_1, the projection of R_1 on P, has order at most p. If $G \neq D_4(q)$, then R/R_1 is cyclic whence $|P| \leqslant p^2$. As $m_p(B^*) \geqslant 4$, we must have $B_1 \cap B_2 \subset B_1$. In particular $B_2 \subset B$ whence $L = A_n(q)$ or ${}^2A_n(q)$ with $p | n+1$, or $L = E_6(q)$ or ${}^2E_6(q)$ with $p = 3$. As $n \geqslant 3$ in the first case, we have $m_p(B^*) \geqslant 5$ in all cases. But now Lemma 4.4(ii) implies $m_p(P) \geqslant 3$, a contradiction. If $G = D_4(q)$, then a similar argument yields $|P| \leqslant p^3$ against $m_p(B) = 5$ and $P \cong B_1 \cap B_2 = B_1 \cong E_p^4$.

Reasoning as in Remark 5.1, we see that no element of B^* acts on K as a graph or graph-field automorphism. Thus $B^* = A \times \langle a \rangle$ with A inner diagonal on K and b acting as a (standard) field automorphism. If $A = B_1$, we are done, so we may assume $a \in B_1 - A$. As $B_1 \cap B_2 \subseteq [P, B^*] \subseteq A$, $B_1 = (B_1 \cap B_2) \times \langle a \rangle$.

Let $Y = A_G(B^*)$ and $\bar{Y} = Y/C_Y(B_1)$. Since $|B^* : B_1| = p$ and $B_1 = C_{B^*}(P)$, it follows that $O^{p'}(C_y(B_1)) = P$, $C_Y(B_1)/P$ is cyclic of order dividing $p-1$, and $O_p(\bar{Y}) = 1$. Repeating the argument of the proof of Corollary 4.2, we apply McLaughlin's theorem and obtain that $\langle \bar{a}^{\bar{Y}} \rangle$ acts irreducibly on B_1. We conclude $B_1 = [P, B^*] \subseteq A$, and Lemma 4.4 is proved.

Recall that $M = N_G(B_1)$, $C = C_G(B_1)$ and $B^* \leqslant U \in \mathrm{Syl}_p(C)$. Note that $N_G(B^*) \leqslant M$. Further U_1 is the subgroup of U which acts as inner-diagonal automorphisms on K. Let $V = C_U(K)$. Clearly $V \leqslant U_1$; also $\Omega_1(C_U(x)) \leqslant C_U(B^*)$, whence $\Omega_1(C_U(x)) \leqslant B^*$.

LEMMA 4.5. *The following conditions hold*:

(i) $V \ntrianglelefteq N_M(U)$;

(ii) *for any* $r \in N_M(U) - N_G(V)$, $V \cap V^r = 1$;

(iii) V *is cyclic*;

(iv) $\langle V^{N_M(U)} \rangle \subseteq U_1$;

(v) $U_1 \lhd N_M(U)$.

Proof. To prove (i) pick $t \in N_L(B^*)$ with $\langle d \rangle \neq \langle d^t \rangle$. As $N_G(B^*) \subseteq M = CN_M(U)$, these exists $r \in N_M(U)$ with $d^r = d^t$. Since $\langle d \rangle = C_{B^*}(K) = V \cap B^*$, $r \notin N_G(V)$ and (i) holds.

For any r as in (ii) let $W = V \cap V^r$. If $\langle r \rangle$ normalizes $\langle d \rangle$, then $\langle r \rangle$ acts on K and normalizes $V = C_K(U)$. Thus $\langle d \rangle \neq \langle d^r \rangle$ whence $W \cap B^* = 1$. If $W \neq 1$, then as $W \lhd U$, $1 \neq \Omega_1(C_W(x)) \subseteq B^*$. Thus $W = 1$ and (ii) is valid.

Suppose $n \in N_M(U) - N_G(V)$; we claim $V^n \cap U_1$ is cyclic. Since x acts as a field automorphism on K, x centralizes $\Omega_1(U/V)$. $V \cap V^n = 1$ implies $[x, \Omega_1(V^n)] = 1$ and $\Omega_1(V^n) \subseteq B^*$. Thus $\Omega_1(V^n \cap U_1) = V^n \cap B_1 = (V \cap B_1)^n = \langle d^n \rangle$, and our claim is proved.

To prove (iii) it suffices to prove (iv), so assume $n \in N_M(U)$ and $V^n \nsubseteq U_1$. Some element of V^n induces a field automorphism on K. It follows that $[U_1, V^n]\, V/V$ is abelian of rank at least 2 except perhaps when $J = A_2(q)$ or ${}^2A_2(q)$. In these cases we find by checking the possibilities for L (cf. Table P) that $J \cong SL(3, q)$ or $SU(3, q)$ whence $m([U_1, V^n]\, V/V) \geqslant 2$ in all cases. However $[U_1, V^n] \subseteq V^n \cap U_1$ yields a contradiction by the preceding paragraph.

It remains to prove (v); suppose $U_1 \neq U_1^n$ for $n \in N_M(U)$, and let $E = U_1 U_1^n$, $A = U_1 \cap U_1^n$. We know $V \cap V^n = 1$ and $VV^n \subseteq A$. Since U_1/V is abelian of exponent p^s for some $s \geqslant 2$, so is A. Pick $w \in U_1^n - U_1$ with $w^p \in U_1$ and if possible $w \in U_1(U_1^n)$. We have $[E, w] \subseteq \langle w^p, V^n \rangle$ and $[E, w] \subseteq V^n$ if $w \in U_1(U_1^n)$. On the other hand w acts as a field automorphism on K, and the considerations of the preceding paragraph yield that $[E, w]\, V/V$ is abelian of rank at least 2. We conclude first that $[E, w] \nsubseteq V^n$ whence $|E \colon U_1| = |U_1 : A| = p$ and secondly that $m(\Omega_s(U_1/V)) = 2$, whence from Table P, $m(B^*) = 4$ and, consequently,

$B_2 = B^*$. Since U/U_1 is cyclic, E is independent of the choice of n, which implies $E \lhd N_M(U)$.

Our conditions imply $\Omega_1(E/V) = B^*V/V$. Thus $\Omega_1(E) \subseteq B^*V$; and as V is cyclic, $B^* = \Omega_1(E) \lhd Y = N_M(U)$. Thus $Y \subseteq N_G(B^*) \subseteq M$, and since $M = YC$, B_1 is an irreducible Y-module. Further let $D = N_C(B^*)$. D acts on B^* and centralizes B_1 whence $O^{p'}(D)$ projects to $P = O_p(A_G(B^*))$. As $U \subseteq D$, U also projects to P, and we have by Lemma 4.4 that $U/C_U(B^*)$ is an irreducible Y-module with $U/C_Y(B^*) \cong B_1 \cong E_{p^3}$.

Let $F = C_E(B^*)$; as $E \lhd Y$ and U/E is cyclic, we must have $U = EC_U(B^*)$ and F/E is isomorphic to $U/C_U(B^*)$ as a Y-module. The structure of E/V implies $[E, E]\, V/V \subseteq \Omega_1(Z(E/V))$, and likewise for E/V^n. As $V \cap V^n = 1$, it follows that $[E, E] \subseteq \Omega_1(Z(E))$; and as p is odd, taking pth powers is an endomorphism of E. By the same argument, $\mho_1(E) \subseteq Z(E)$. As $E/\Omega_1(E) = E/B^* \cong \mho_1(E)$, $|E: \mho_1(E)| = p^4$. Thus $\langle x, \mho_1(E)\rangle \subseteq F$ forces $F = \langle x\rangle \times \mho_1(E)$ and $\mho_1(E) = Z(E)$. Since taking pth powers commutes with the action of Y on E, $\mho_1(E)/\mho_2(E)$ is isomorphic to E/F as a Y-module, and we see that $\mho_1(E)$ is homocyclic of rank 3. As E/V has exponent p^s, so does E whence $\mho_1(E) \cong (Z_{p^{s-1}})^3$. In particular $|E| = p^{2s+1}$ and $|U_1| = p^{3s}$. It follows that $U_1/V \cong (Z_{p^s})^2$ and $V \cong Z_{p^s}$. However (iv) above implies $V \subseteq U_2$ where U_2 is the largest subgroup of U_1 normal in Y. Thus $U_2 \not\subseteq F$ and we must have $E = U_2F$. But then $|U_2| = |U_1|$ implies $U_1 \lhd Y$. This contradicts $U_1 \neq U_1^n$ and completes the proof of Lemma 4.5.

LEMMA 4.6. *We have*

(i) U_1 *is abelian of exponent* p^s *and after perhaps replacing* x *by another generator of* $\langle x\rangle$, $u^x = u^{1+p^{s-1}}$ *for all* $u \in U_1$;

(ii) U_1 *contains a homocyclic subgroup of rank* $m(U_1)$ *or* $m(U_1) - 1$ *and exponent* p^s;

(iii) $U_1 = J(U)$, *the Thompson subgroup of* U;

(iv) $B_1 = \Omega_1(U_1)$ *and* $B^* = \Omega_1(U)$;

(v) $O_p(M/C) = 1$.

Proof. By Lemma 4.5, pick $r \in N_M(U) \subseteq N_M(U_1)$ with $V \cap V^r = 1$; the structure of U/V implies (i). $\Omega_1(U/V) = B^*V/V$ forces $\Omega_1(U) \subseteq B^*V$. As V is cyclic, (iv) holds.

To prove (ii) we repeat an argument from the proof of Lemma 4.5. By (iv), $N_M(U) \subseteq N_G(B^*) \subseteq M$; so $M = CN_M(U)$ and Lemma 4.4 imply that $U_1(C_{U_1}(B^*)$ is isomorphic to B_1 or $B_1 \cap B_2$ as an $A_L(B^*)$-module. As $\Omega_s(U_1)$ has rank at least 2, it projects nontrivially on $U_1/C_{U_1}(B^*)$ whence $m(\Omega_s(u_1)) \geqslant m(B_1 \cap B_2) \geqslant m(B_1) - 1$ and (ii) holds.

Assertion (iii) follows easily from (i) and (ii). To prove (v) suppose first that $O_p(M/C) \neq 1$. Let $Y = N_G(B^*)$ and $Z = C_G(B^*)$. By (iv), $M = YC$ so

$O_p(Y/C_Y(B_1)) \neq 1$. On the other hand $C_Y(B_1)/Z$ is an extension of $O_p(Y/Z) \cong P$ by a group which is cyclic of order dividing $p-1$. It follows that $O_p(Y/Z)$ covers $O_p(Y/C_Y(B_1))$ whence $O_p(Y/C_Y(B_1)) = 1$.

LEMMA 4.7. (i) *$N = N_G(B^*)$ covers M/C*; (ii) *B_1 is weakly closed in B^* with respect to G*; (iii) *U_1 is weakly closed in R with respect to G, where $U \leqslant R \in \mathrm{Syl}_p(M)$*, (iv) *$R \in \mathrm{Syl}_p(G)$*.

Proof. (i) and (ii) A Frattini argument implies that N covers M/C. From this it follows easily that B_1 is weakly closed in B^* with respect to G: for if $g \in G$, $B_1^g \leqslant B^*$, then $B_1 \leqslant (B^*)^{g^{-1}}$, whence there is $c \in C$ such that $(B^*)^{g^{-1}c} \leqslant U$, implying $g^{-1}c \in N$ and $g^{-1} \in Nc^{-1} \subseteq M$, as required.

To prove (iii) and (iv) it suffices to show $U_1 = J(R)$. Assume not and pick A abelian of maximum order in R with $A \neq U_1$. By the Thompson Replacement Theorem [27, Theorem 8.2.5], we may assume $[U_1, A, A] = 1$. As $U_1 \lhd R$, we have $[U_1, A] \subseteq C_{U_1}(A) = A \cap U_1$. Thus $[u^r, a^s] = [u, a]^{rs}$ for $u \in U_1$ and $a \in A$. Let $A_1 = \mho_{s-1}(A/A \cap U_1)$; we have $[A_1, \mho^{s-1}(U_1)] = 1$.

We claim $[A_1, B_1] = 1$. If not, then by, Lemma 4.6 (ii), $|B_1 : B_1 \cap \mho^{s-1}(U_1)| = P$ and A_1 induces transvections on B_1. Let $F = \langle A_1^{N_M(U)} \rangle$. We know that $N_M(U) \subseteq N_G(B^*) \subseteq M$ and $M = CN_M(U)$. As B_1 is an indecomposable $A_L(B^*)$-module, Lemma 4.6(v) and McLaughlin's theorem imply that $N_M(U)$ acts irreducibly on B_1 forcing $B_1 \subseteq \mho^{s-1}(U_1)$ and establishing our claim.

Now $A_1 \subseteq C_R(B_1) = U$. As U/U_1 is cyclic of order dividing s, so is $A_1/A \cap U_1$. It is easy to see that $p \geqslant 3$ and $s \geqslant 2$ imply $p^{s-1} > s$ whence $|A_1 : A \cap U_1| < p^{s-1}$. It follows that $A = A_1 \subseteq U$, and we are done by Lemma 4.6(iii).

LEMMA 4.8. *Proposition* 4.1 *holds.*

Proof. Let $N_0 = N_G(U_1)$, $C_0 = C_G(U_1)$. By Lemma 4.7, U_1 is weakly closed in $R \in \mathrm{Syl}_p(G)$, so by the Hall–Wielandt theorem [43, Theorem 14.4.2], G has a quotient of order p if N_0 does. As G is simple, it suffices to show $x \notin [N_0, N_0]$ to complete the proof by contradiction of Proposition 4.1.

Let $H = C_G(x)$, $Y = C_H(L)$, and $U_2 = C_{U_1}(x) = \Omega_{s-1}(U_1)$. From the structure of $\mathrm{Aut}(L)$, $F = H \cap C$ covers H/LY (recall $C = C_G(B_1)$). By definition of standard type Y has cyclic Sylow p-subgroups. If $\langle w \rangle \in \mathrm{Syl}_p(Y)$, $\Omega_1(\langle w \rangle) = \langle x \rangle \subseteq Z(Y)$ implies $Y = \langle w \rangle O_{p'}(Y)$. Pick $\langle w \rangle$ so $\langle w, U_2 \rangle \subseteq P \in \mathrm{Syl}_p(H)$; then U_2 normalizes $\langle w \rangle$. It follows that $\Omega_1(B^*\langle w \rangle) = B^*$ whence $\langle w \rangle \subseteq N_G(B^*) \subseteq N_G(B_1)$. Thus $[w, B_1] \subseteq \langle w \rangle \cap B_1 = 1$ and $w \in H \cap C$. Consequently F covers $H/LO_{p'}(Y)$. Since $LO_{p'}(Y) \subseteq [H, H]\, O^p(H)\, U_2$ and $U_2 \subseteq F$, $x \notin [F, F] O^p(F) U_2$ implies $x \notin [H, H] O^p(H) U_2$. But F acts on K with x acting as a field automorphism and $U_2 \subseteq U_1$ inducing inner-diagonal automorphisms, so $x \notin [H, H]\ O^p(H) U_2$.

Let $H_0 = H \cap N_0$ and $X = [H_0, H_0] O^p(H_0) U_2$. As U_1 is abelian and is a Sylow p-subgroup of C_0, $C_0 = U_1 \times O_{p'}(C_0)$. Thus $H_0 \cap C_0 = U_2 \times O_{p'}(H_0 \cap C_0) \subseteq X$, and if $N_0 = H_0 C_0$, N_0 has the desired quotient of order p by the preceding paragraph.

Assume $H_0 C_0 \neq N_0$. By the action of x on U_1, $\langle x \rangle C_0/C_0 \subseteq Z(N_0/C_0)$, and it follows that $\bar{B}^* \lhd \bar{N}_0 = N_0/O_{p'}(C_0)$. If $n \in N_0$ and $\bar{x}^n = \bar{x}\bar{b}$ with $\bar{b} \in \mho^{s-1}(\bar{U}_1)$, then for some $u \in U_1$, $[\bar{x}, \bar{u}] = \bar{b}$ whence $\bar{n} \in \bar{U}_1 C_{\bar{N}_1}(\bar{x})$. As H_0 covers $C_{\bar{N}_0}(\bar{x})$, we have $n \in H_0 C_0$. Thus, the assumption $H_0 C_0 \neq N_0$ implies that U_1 is not homocyclic. Lemma 4.6 yields $|B_1 : \mho^{s-1}(U_1)| = p$. As we have seen before, we must have $B_1 \cap B_2 = \mho^{s-1}(U_1) \subset B_1$.

Consider the action of N_0 on $\bar{B}^*/\overline{B_1 \cap B_2} \cong E_{p^2}$. Let $N_1 = C_{N_0}(\bar{B}^*/\overline{B_1 \cap B_2})$ and $\tilde{N}_0 = N_0/N_1$. As $\langle \bar{x} \rangle$ covers $\bar{B}^*/\bar{B}_1$, $[N_1, \bar{B}^*] \subseteq \bar{B}_1$. If $n \in N_0$ normalizes $\bar{B}_2$, then the analysis of the preceding paragraph gives $n \in H_0 C_0$. As $[B^*, U_1] = [\langle x \rangle, U_1] \subseteq \mho^{s-1}(U_1)$, $\tilde{C}_0 = 1$ and we have $\tilde{n} \in \tilde{H}_0$. In other words $N_{\tilde{N}_0}(\tilde{B}_2) \subseteq \tilde{H}_0$. Picking elements in $\bar{B}_1$ and $\bar{B}_2$ as a basis for $\bar{B}^*/\overline{B_1 \cap B_2}$, we see that $\tilde{N}_0$ is represented by matrices of the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$$

and $\tilde{N}_0 \cong Z_r$ or $Z_r \cdot Z_p$ with $r | p-1$. In the latter case $N_{\tilde{N}_0}(\tilde{B}_2) \cong Z_r$ is maximal in $\tilde{N}_0$ whence $\tilde{H}_0 \cong Z_r$. Thus in either case $p \nmid |\tilde{H}_0|$.

Since $H_0 \cap C_0 \subseteq X$, $H_0 C_0/XC_0$ is an abelian p-group, and $x \notin XC_0$. As $p \nmid |H_0 C_0 : N_1|$, $H_0 C_0/XC_0 \cong N_1/(XC_0 \cap N_1)$, and N_1/X_1 is an abelian p-group, where X_1 is the largest subgroup of $X_0 C_0 \cap N_1$ normal in N_0. Further $X_1 \supseteq C_0$, so $[N_0, x] \subseteq X_1$. Thus, N_0/N_1 acts on N_1/X_1 and centralizes $\langle x \rangle X_1/X_1$. If N_0/N_1 is a p' group, then N_0/X_1 has a quotient of order p as desired, so assume $\tilde{N}_0 \cong Z_r \cdot Z_p$.

It suffices to show $X_1 = XC_0 \cap N_1$. Suppose $X_1 = X_0 C_0 \cap N_1$ and pick $h \in H_0$ with $\langle \tilde{h} \rangle \cong Z_r$. We may take h to be of p' order whence $h \in X$ and $[h, N_1] \subseteq XC_0 \cap N_1 = X_1$. Thus $\tilde{h}$ and $O_p(\tilde{N}_0) = [\tilde{h}, \tilde{N}_0]$ centralize N_1/X_1. Letting $W/N_1 = O_p(\tilde{N}_0) \cong Z_p$, we see that W/X_1 is an abelian p-group (as $N_1/X_1 \subseteq Z(W/X_1)$ and W/N_1 is cyclic). Now as before N_0/W acts on W/X_1 with fixed points and N_0/X_1 has a quotient of order p.

It remains to show $X_1 = XC_0 \cap N_1$. Let $X_2 = X_1 \cap H_0$. As $XC_0/(XC_0 \cap N_1) \cong H_0 C_0/N_1 \cong Z_r$, $|X : X_2| = sp^a$ for some $s | r$. Further $X_2 \lhd H_0$. Suppose $N_H(B^*) = H_0 C_G(B^*)$. Examination of the possibilities for L yields $A_H(B^*) = O^p(A_H(B^*))$ whence $H_0 = C_{H_0}(B^*) X$. Further X/X_2 is an extension of an abelian p-group by Z_s, and the structure of $A_H(B^*)$ yields $|H_0 : C_{H_0}(B^*) X_2| \,|\, s$.

Let $F = C_{H_0}(u_1) = C_0 \cap H_0$. We claim $X \cap C_{H_0}(B^*) = X_2 \cap C_{H_0}(B^*) = F$ whence $|X : X_2| = |XC_{H_0}(B^*) : X_2 C_{H_0}(B^*)| \,|\, s$, and it follows that $X_1 = X_0 C_0 \cap N_1$ as desired. $C_{H_0}(B^*) = H \cap C \cap N_0 = N \cap N_C(U_1)$ has a

Sylow P-subgroup Q which is an extension of U_2 by a cyclic group with $x \in Q - U_2$. As $C_{H_0}(B^*)$ acts on U_1 as a p-group, and (as we saw above) $F = U_2 \times O_{p'}(F)$, we see that $C_{H_0}(B^*)/F$ is a cyclic p-group with $x \in C_{H_0}(B^*) - F$. Now $C_0 \subseteq X_1$ implies $F = C_0 \cap H_0 \subseteq X_2 \subseteq X$, and $x \notin X$ implies $X \cap C_{H_0}(B^*) \subseteq F$ as desired.

5. Construction of $G_0 \leqslant G$, $G_0 \in \mathrm{Chev}(2)$

We let $z_1,..., z_r, K_1,..., K_r$ $r \geqslant 2$, B, B^*, etc., have the same meaning as in Section 4. Set $G_0 = \langle K_1,..., K_r \rangle$. The object of this section is to show that $G_0 \in \mathrm{Chev}(2)$; see Proposition 5.20. In Section 6, the problem of showing $G_0 = G$ will be handled.

Before discussing our plan, we establish some further notation. Set $C_i = C_G(z_i)$, $N_i = N_G(\langle z_i \rangle)$, $A_i = A_{K_i}(B^*)$, $A_i^* = A_{N_i}(B^*)$, $i = 1,..., r, r \geqslant 2$. For any distinct pair $i, j \in \{1,..., r\}, L_0 = L(K_i \cap K_j)$. Set $A_0 = A_{L_0}(B^*)$.

The main step in identifying G_0 is to identify $G_1 = \langle K_1, K_2 \rangle$, where G is of standard type with respect to $(B, z_1, K_1) \in S^*(p)$. We know that p half-splits K_2 but we do not know whether G is of standard type with respect to (B, z_2, K_2). Thus, the roles of K_1 and K_2 in our argument are not usually symmetric.

Our method is to first identify $A = \langle A_1, A_2 \rangle$ as a prelude to identifying G_1. Recall that B^* contains B with index 1 or p. Eventually one needs to produce a "Weyl group" for G_1, and the most sensible method appears to be to work in $A_G(B^*)$ rather than in $A_G(B)$. The results of Section 3, plus further special arguments, enable us to determine the possibilities for A. We have $O_p(A) = 1$ as a consequence of Section 4. From there we proceed to identify G_1 by analyzing various cases for A, A_1, A_2. Tables B and P are used heavily to study how subgroups fit together. Finally Proposition 2.30 is used in the various cases we consider to identify G_1. The identification of G_0 is then a relatively easy consequence of the preceding work.

Before embarking on the proof of our main result (Proposition 5.11) we recall that $4 \leqslant m(B) \leqslant m(B^*) \leqslant m(B) + 1$. Table B and Lemma 2.35(iv) imply that $m_{2,p}(K_1) \geqslant 3$ and $m_p(K_2) \geqslant 3$, whence the Lie rank of each K_j is at least 2 and is at least 3 if K_j is not of type ${}^2A_4(q)$. Familiarity with the "splitting prime" and "half-splitting prime" situation is assumed; see Section 1.

Remark 5.1. In Section 4, we showed that x does not induce a field automorphism on any K_j. We argue that B^* induces a group of inner-diagonal automorphisms on each K_j. If false, choose j so that x induces a graph or a graph-field automorphism on K_j. Then $p = 3$ and K_j has type $D_4(q)$. By [10], $C_{K_j}(x) \cong G_2(q)$, $SL(3, q)$ if $3 | q - 1$, or $SU(3, q)$ if $3 | q + 1$.

Since $\langle z_j \rangle \cap K_j = 1$ and $m_3(C_{K_j}(x)) = 2$, we get $m(B) = 4$. If $3 \mid q-1$, the fact that $m_{2,3}(D_4(q)) = 4$ implies $m(B) = 5$, a contradiction. So, $3 \mid q+1$, whence $m(B^*) = 5$, $m(B^*) = 4$. However, the shape of $C_{K_j}(x)$, which must contain B^*, forces $m(B^*) = 4$, a contradiction.

A *reflection* shall mean a linear transformation on a finite dimensional vector space of characteristic not 2 such that the eigenvalues are $-1, 1, 1, \ldots, 1$.

Remark 5.2. In every case, $N_{K_i}(B) \leqslant N_{K_i}(B^*)$, a great convenience. The groups $K \in \mathrm{Chev}(2)$ which appears in the following arguments usually have $|Z(K)|$ odd. If some $|Z(K_i)|$ is even, there may be a complication in a generator and relations argument. We comment when a relevant $|Z(K_i)|$ might be even and otherwise say nothing.

We now begin the identification of G_1 and G_0. Since p splits K_1, A_1 is isomorphic to a Weyl group (though not necessarily the Weyl group on a (B, N)-pair for K_1). Thus we consider what happens when the A_i are various Weyl groups.

LEMMA 5.3. *Let bars denote images under $N_G(B^*) \to A_G(B^*)$. Suppose that $K \leqslant G$, $K \in \mathrm{Chev}(2)$, that $B^* \cap K$ lies in the "B^*" column of Table* B *and $B^* = C_{B^*}(K)(B^* \cap K)$. Let $t_1, t_2 \in N_K(B^*)$ be involutions so that $\bar{t}_1$ and $\bar{t}_2$ are distinct reflections on B^*. If $k \in \mathbb{Z}$ and $|\bar{t}_1 \bar{t}_2| = k$, then $(t_1 t_2)^k \in O_2(K)$. If $m_{2,p}(K) \geqslant 3$, then such involutions t_1, t_2 always exist and may be arranged to satisfy $(t_1 t_2)^k = 1$.*

Proof. We first assume $O_2(K) = 1$. A look at Table B* and properties of K imply that $A_K(B^*) \cong A_K(C_K(B^*))$ and that if $s_1, s_2 \in N_K(B^*)$ induce distinct reflections on B^*, then $[C_K(B^*), s_1] \cap [C_K(B^*), s_2] = 1$. Set $u = (t_1 t_2)^k$. Then $\bar{u} = 1$, whence $u \in C_K(B^*)$ and u is inverted by t_1 and t_2, whence $u = 1$, as required.

Now, drop the assumption that $O_2(K) = 1$. Then $|Z(K)|$ even and the fact that $m_{2,p}(K) \geqslant 3$ implies that K has type ${}^2A_5(2)$ or ${}^2E_6(2)$. It suffices to prove the statements for $p \mid q+1$ because of the embedding of the natural subgroup isomorphic to $A_K(B^*)$ for $p \mid q-1$ into that for $p \mid q+1$; see Lemma 2.50. When K has type ${}^2A_5(q)$, the elements $\bar{t}_i$ are images of unitary transvections under $SU(6, q) \to K$ (to see this, regard the $\bar{t}_i$ as images of transpositions from the standard group of permutation matrices and $C_K(B^*)$ as a subgroup of the full diagonal group). The facts that we may arrange $|t_i| = 2$ and $(t_1 t_2)^2$ or $(t_1 t_2)^3$ is 1 may be read off from the generators and relations for the covering group of ${}^2A_5(2)$ [36]; in fact, we define $\langle t_j \rangle$ as a conjugate of $[Y_\alpha, Y_\beta]$, where Y_γ is the preimage under $K \to {}^2A_5(2)$ of a root group for a short root and $\alpha + \beta = \gamma$ is a long root. The argument for the case K of type ${}^2E_6(2)$ is reduced to that of ${}^2A_5(2)$ because $A_K(B^*) \cong W_{E_6}$

and all pairs $\bar{t}_i, \bar{t}_j$ are conjugate in $A_K(B^*)$ to a pair in the image of $A_L(B^*) \to A_K(B^*)$, where L is a natural ${}^2A_5(2)$ subgroup of K.

DEFINITION. The involutions $t_i \in N_K(B^*)$ representing the fundamental reflections are called *special involutions* if $|Z(K)|$ is odd or if $|Z(K)|$ is even and K has Lie rank at least 3 and the t_i are chosen as in the proof of (5.2).

Note that each K_i has Lie rank at least 3 or $|Z(K_i)|$ odd; see Section 1 and use the facts about Schur multipliers in [36, 38].

The next result verifies the extra hypothesis (iv) of Lemma 2.30.

LEMMA 5.4. *Let $G^* = \langle G^*, W\rangle$, K_1 and B^* as above, W the Weyl group of root system Σ of rank at least* 2. *Assume hypotheses* (i), (ii), *and* (iii) *of Lemma* 2.30. *Set $A_\alpha = C_{B^*}(\langle Z(X_\alpha), Z(X_{-\alpha})\rangle)$, $\langle b_\alpha\rangle = B^* \cap \langle Z(X_\alpha), Z(X_{-\alpha})\rangle$, $\alpha \in \Sigma_1$. Suppose that W normalizes B^* and satisfies $N_W(\{\pm\alpha\}) = N_W(\langle b_\alpha\rangle) = N_W(A_\alpha)$ and $N_W(\alpha) = C_W(b_\alpha)$. Then* (iv) *holds, i.e., $W_\alpha := \{w \in W \mid \alpha^w = \alpha\}$ normalizes X_α, for $\alpha \in \Sigma_1$ and X_α is the root group of K associated to α.*

Proof. Note that $A = A_\alpha$ is a hyperplane of B^* and $B^* = Ax\langle b_\alpha\rangle$.

We consider the possibility that there is a standard subcomponent $(D, \hat{K})$ of (B, x, K_1) with the properties (1) $b_\alpha \in D$, (2) $D = \langle b_\alpha, b_\beta\rangle$ where b_α and b_β are conjugate by an element of W, and (3) $b_\beta \in A$.

Since $m(B^*) \geqslant 4$, a study of Tables B and P shows that such a $(D, \hat{K})$ may be obtained whenever W does not have type A_l when $p \mid l+1$.

Suppose that (1), (2), (3) are achieved. We have $C_G(A) \leqslant C_G(b_\beta) \tilde{_G} C_G(b_\alpha)$, the structure of which implies that

$$L(C_G(A)) = \langle Z(X_\alpha), Z(X_{-\alpha})\rangle \qquad \text{if} \quad |Z(X_\alpha)| > 2;$$

$$O^{2'}(C_G(A)) = \langle Z(X_\alpha), Z(X_{-\alpha})\rangle \qquad \text{if} \quad |Z(X_\alpha)| = 2.$$

In any case, $S := \langle Z(X_\alpha), Z(X_{-\alpha})\rangle \cong SL(2, q)$ for $q = |Z(X_\alpha)|$ and W_α acts on S, centralizing $\langle b_\alpha\rangle$. Thus, $[S, W_\alpha] = 1$, as required.

Suppose W has type A_l, when $p \mid l+1; l \geqslant 4$ since $m(B^*) \geqslant 4$. Then $p \mid q-1$. Choose β so that α, β span a root system of type A_2. Set $W_{\alpha,\beta} = W_\alpha \cap W_\beta$. Since $l \geqslant 4$, we may take a root α orthogonal to α and β. Since $b_\gamma \sim b_\alpha \sim b_\beta$ via W, we may look in $C_G(b_\alpha)$ to get the structure of $C_G(A_0)$ where $A_0 = C_{B^*}(\langle X_\alpha, X_{-\alpha}, X_\beta, X_{-\beta}\rangle)$. We get $J = L_{p'}(C_G(A_0)) = J_0 O_{p'}(J)$, where $J_0 = \langle X_{\pm\alpha}, X_{\pm\beta}\rangle$ is of type $A_2(q)$, $q = |X_\alpha|$. Since $W_{\alpha,\beta}$ centralizes $B^* \cap L(C_G(A_0)) = \langle b_\alpha, b_\beta\rangle$ and $W_{\alpha,\beta}$ is generated by involutions, $W_{\alpha,\beta}$ centralizes $L(C_G(A_0)) J_0$. Since W_α is generated by the $W_{\alpha,\beta}$, for all possible choices of β, we are done in this case. The lemma is proven.

LEMMA 5.5. *For $i = 1, 2$, let W_i be a subgroup of K_i normalizing B^* and as described in Lemma* 2.50(v)(c). *Let bars denote images under $N_G(B^*) \to A_G(B^*)$. Suppose $W_i \cong \bar{W}_i = A_i$ for $i = 1, 2$ and that $\overline{W_1 \cap W_2} = A_1 \cap A_2$ and that A is generated by reflections $r_1, r_2, \ldots, r_{n+1}$ which satisfy the relations of a Dynkin diagram and $A_1 = \langle r_1, \ldots, r_n \rangle, A_2 = \langle r_2, r_3, \ldots, r_{n+1} \rangle$. Then $\langle W_1, W_2 \rangle \cong A$.*

Proof. Let $t_1 \in W_1, t_2, t_3, \ldots, t_n \in W_1 \cap W_2, t_{n+1} \in W_2$ be the special involutions for which $\bar{t}_i = r_i, i = 1, \ldots, n+1$. If $|r_1 r_{n+1}| = k$, then we can get $(t_1, t_{n+1})^k = 1$ by the argument of Lemma 5.3 provided we know that distinct t_j in A have commutators on $C_G(B^*)$ meeting trivially. Since this is true with A_1 or A_2, it suffices to check the statement for t_1, t_{n+1}. Since the Dynkin diagram has no loops, t_1 and t_{n+1} commute. We assume that $[B^*, t_1] = [B^*, t_{n+1}]$. Since $n \geqslant 3$, we may choose an index $j, j \neq 1, n+1$, such that $|r_j r_1| > 2$ and $r_j r_{n+1} = 2$. By conjugating with t_j, we get $[B^*, t_1^{t_j}] = [B^*, t_{n+1}]$, whence $[B^*, t_1 t_j] = 1$. But since A_i acts faithfully on B^*, this is a contradiction.

LEMMA 5.6. *Suppose A_1, A_2 are Weyl groups of type A. Then one of the following occurs*: (a) $A_1 \cong A_2 \cong W_{A_n}$ *and A is a Weyl group of type A_{n+1} or D_{n+1}*, (b) $p = 3$, *one of A_1 or A_2 is isomorphic to W_{A_5} and $A \cong W_{E_6}$*, (c) $p = 3$, *one of A_1, A_2 is isomorphic to W_{A_8} and $A \cong W_{E_8}$.*

Also, in (a), *A_0 is the usual* 1*-point stabilizer for the symmetric groups $A_1 \cong A_2$ and in* (b) and (c), *A_0 is a 2-point stabilizer.*

Proof. See Proposition A.

LEMMA 5.7. *Assume the hypotheses of Lemma* 5.6 *and that A is a Weyl group of type A. If K_1 (equivalently, K_2) has type $A_n(q)$, ${}^2A_n(q)$, respectively, then G_1 has type $A_{n+1}(q)$ or ${}^2A_{n+1}(q)$.*

Proof. Suppose K_1 and K_2 have type $A_n(q)$. Then $G_1 = \langle K_1, W \rangle$ is identified as a group of type $A_{n+1}(q)$ by Proposition 2.30.

Suppose K_1 and K_2 have type ${}^2A_n(q)$. Then $p \mid q+1$, $n \geqslant 4$ and $G_1 = \langle L_0, W \rangle$. Let $\phi: W \to \Sigma_{n+2}$ be an isomorphism so that the involutions of Lemma 5.3 inducing reflections on B^* go to transpositions. Let $\tau \in W$ so that $\tau^\phi = (12)(34) \cdots (2l-1, 2l)$, where $l = [(n+1/2]$. We may alter ϕ so that $C_{W_1}(\tau) \cong W_{C_{l-1}}$ and $C_{W_1}(\tau)$ is a standard copy of the Weyl group of L_0; see Lemma 2.50(v)(c). By Proposition 2.30, we can identify G_1 as a group of type ${}^2A_{n+1}(q)$ if $l \geqslant 4$, i.e., $n \geqslant 7$. So we may assume $4 \leqslant n \leqslant 7$.

Let us look a bit more carefully at Proposition 2.30. We can use W to define root elements for any $n \geqslant 4$. The problem is verifying relations between elements of the shape $x_\alpha(t)$, $x_\beta(u)$, (or $x_\alpha(t, t')$, $x_\beta(u, u')$) where $\alpha, \beta \in \Sigma$, our root system, and α, β are both short and form an angle of $\pi/3$

or $2\pi/3$ or $n=6$ and they are orthogonal, or α, β are of unequal length and orthogonal and $n \leqslant 5$.

Choosing epimorphisms $SU(n+1, q) \to K_i$, $i=1, 2$, which agree on a subgroup isomorphic to $SU(n, q)$ mapping onto L_0, we assume that $SU(n+1, q)$ is a matrix group relative to an orthonormal basis $\{e_j \mid 1 \leqslant j \leqslant n+1\}$ and that $B^* \cap K_i$ is the image of a diagonal group. We then define S_{ij} to be the subgroup of K_1 or K_2 corresponding to the $SU(2, q) \cong SL(2, q)$ subgroup associated with the ith and jth basis vectors.

Using the action of W on the S_{ij}, we see that $[S_{ij}, S_{i'j'}]=1$ whenever $\{i, j\} \cap \{i', j'\} = \varnothing$. Thus, if $n=6$, Σ_1 and Σ_2 are orthogonal sets of roots in Σ such that both have type C_2, then $[x, y]=1$ whenever x, y are root elements associated to roots in Σ_1, Σ_2, respectively. Therefore, $[x_\alpha(t), x_\beta(u)]=1$ whenever α, β are short roots, orthogonal, and $\alpha+\beta \notin \Sigma$. If we take $\Sigma_1=\{r, -r\}$ for r long and $\Sigma_2=\{s \in \Sigma \mid s$ is orthogonal to $r\}$, then we get that root elements associated with orthogonal roots of unequal length commute.

Suppose $n=5$ and α, β are short roots generating a subsystem of type A_2. We may arrange for $V \leqslant W$, $V \cong W_{C_3}$ so that $V \leqslant K_1$ and $V \cap L_0$ is a standard copy of the Weyl group of L_0 (see (2.50(iv))). Then each $x_\alpha(t)$, $x_\beta(u)$ is a natural root element in K_1, we can define the commutator relations between these elements and complete the verification of Steinberg relations for G_1.

We are now left with the case $n=4$, α, β short generating a subsystem of type A_2. We observe that $p \neq 3$; for $p=3$, then $A \cong \Sigma_6$ has a nontrivial fixed point on B^*, rank 5, and $C_{B^*}(A)=C_{B^*}(A_1)$, a contradiction. So $p \neq 3$ and $q>2$.

We have $B^*=(B^* \cap K_1)(B^* \cap K_2)$ and we replace the hyperplane B by a conjugate in $N_G(B^*)$ so that $B \cap L_0$ has index p in $B^* \cap L_0$. We have $C_{L_0}(B)=\langle S, H\rangle$ where S induces $SU(2, q)$ in a natural way on a two-dimensional summand U_0 of the standard module U for $L_0 \cong SU(4, q)$, and $H_1 \cong Z_{q+1} \times Z_{q+1} \times Z_{q+1}$. Let Z be a Sylow 2-group of S. Define $M_i=N_{K_i}(Z)$, $Q_i=O_2(M_i)$, $i=1$, 2, $M_0=N_{L_0}(Z)$, $Q_0=O_2(M_0)$. We have $Q_1 \cap Q_2=Q_0$. Also, $J=\langle N_{W_1}(B), N_{W_2}(B)\rangle \cap N_G(Z)$ (the W_i are as in (5.4)) induces Σ_4 on B and permutes X_1, X_2, X_3, X_4 is a natural way under conjugation, where $Q_1=ZX_1X_2X_3$, $Q_0=ZX_1X_2$, $Q_2=ZX_1X_2X_4$, and the X_j are K_i-conjugates of nonabelian root groups for a long root in a root system for K_i, $i=1$ or 2. (Think of X_j as follows: let $\{e_k\}$ be our orthogonal basis, $U_0=\operatorname{span}\{e_1, e_2\}$, $X_j \in \operatorname{Syl}_2(S_{12j})$, where S_{12j} induces the special unitary group on $\operatorname{span}\{e_1, e_2, e_j\}$, is trivial on $\operatorname{span}\{e_k \mid k \neq 1, 2, j\}$ and $X_j \geqslant Z$.) It follows that $Q=Q_1Q_2$ is a special 2-group. A Levi factor in M_i contains a unitary transvection acting nontrivially on X_1 and trivially on the other X_j in Q_i. Now, using the action of J, we get that $[M_1, X_4]=[M_2, X_3]=1$. Take $h_1 \in H$, $|h_1|=q+1$, so that h_1 acts as a scalar on the above-mentioned two-dimensional space U_0 and h_1 acts trivially on the orthogonal complement to U_0 in U. Take $h_2 \in N_S(Z)$, $|h_2| = q-1$ and set $h = h_1 h_2$. Then the actions of h and $\langle M_1, M_2 \rangle$ on Q commute (this follows from the structures of K_1 and K_2). Considering the $\langle h_2 \rangle$-action, we see that commutation gives a $\mathbb{F}_q$ bilinear form on Q/Z, whence $\langle M_1, M_2 \rangle$ gives a subgroup of $Sp(8, q)$. Now, considering the $\langle h_1 \rangle$-action, we see that $\langle M_1, M_2 \rangle$ induces a subgroup of $GU^*(4, q)$ (the subgroup of $GL(4, q)$ fixing a nondegenerate Hermetian form up to a scalar) on Q/Z. Since this subgroup contains two distinct copies of $SU(3, q)$, it is not difficult to see that it must be isomorphic to $GU^*(4, q)$ (for example, one can show that the nonsingular 1-dimensional subspaces form a system of imprimitivity for the action of $PGU(4, q)$). By Lemma 3.19, the isomorphism type of $\langle M_1, M_2 \rangle = QY$, where $Y = C_{\langle M_1, M_2 \rangle}(h) \cong GU^*(4, q) \cong GU(4, q) \times Z_{q-1}$, is uniquely determined, hence is necessarily isomorphic to the parabolic subgroup of ${}^2A_5(q)$ corresponding to the subset ∘══∘ of the Dynkin diagram ∘——∘══∘ for ${}^2A_5(q)$.

We now verify the required commutator relations. We have $M_0 \leqslant QY$ and, as $L_0 \cong {}^2A_3(q)$, M_0 contains representatives of each L_0-conjugacy class of root groups (these are the root groups for the system of type A_5). Let $V_i \leqslant K_i$, V_i a standard copy of the Weyl group of K_i derivable from the system of root groups already chosen, $i = 1, 2$. Then $V_i \cong \Sigma_5$, $V = \langle V_1, V_2 \rangle \Sigma_6$ and $V \cap QY \leqslant Y$, $V \cap QY \cong \Sigma_4$. Let H be a Cartan subgroup in Y associated with the given root groups and let $H^* = HZ(Y)$ (recall that $Z(Y)$ acts as the multiplicative group of $\mathbb{F}_{q^2}^\times$ on Q/Z). We claim that H^* has exactly four irreducible subspaces in its action on Q/Z, i.e., H has exactly four irreducible $\mathbb{F}_{q^2}$-subspaces in its action on Q/Z. This follows from viewing $H \cong Z_{q^2-1} \times Z_{q^2-1}$ as a group of matrices preserving the Hermitian form with matrix

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

and letting generators for direct factors of H act via

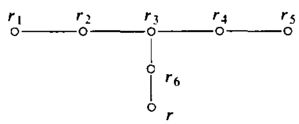
$$\begin{pmatrix} \lambda^{-1} & & & \\ & \lambda^q & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \lambda^{-1} & \\ & & & \lambda^q \end{pmatrix},$$

where $\mathbb{F}_{q^2}^{\times} = \langle \lambda \rangle$. Let Q_j/Z be these four subspaces, $j = 1, 2, 3, 4$. Then, as the one-dimensional spaces above are singular under the bilinear form, each Q_j is abelian. By their uniqueness and the isomorphism of QY with the parabolic subgroup of ${}^2A_5(q)$, $[Q_j, H]$ corresponds to a root group for a short root. It follows that QY contains a pair of root groups X_α, X_β in our system with α, β both short and forming an angle of $\pi/3$ and a pair of root groups for the angle $2\pi/3$. Using the isomorphism of QY with the parabolic subgroup, we get the desired commutator relations.

This completes the argument for the case $n = 5$ and with it the proof of the lemma.

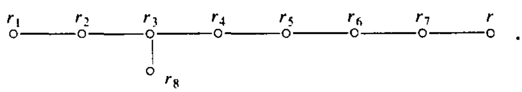
LEMMA 5.8. *Assume the hypotheses of Lemma 5.6 and that $\langle A_1, A_2 \rangle \cong W_{E_6}$ or W_{E_8}. Then there is some z_i and a q so that K_i has type $D_4(q)$ in the first case and type $E_7(q)$ in the second case.*

Proof. Suppose $\langle A_1, A_2 \rangle \cong W_{E_6}, A_j \cong W_{A_5}$, $p = 3$, $m(B^*) = 5$. We have $A_0 \cong W_{A_3}$ and L_0 has type $A_3(q)$ or ${}^2A_3(q)$ for some q, $3 | q-1$ or $3 | q+1$, respectively. In fact there is a reflection $r \in A_j$ with $C_{A_j}(r) = \langle r \rangle \times A_0$. Thus, we may choose reflections $r_1, \ldots, r_6$ in $\langle A_1, A_2 \rangle$ so that



is satisfied and $A_0 = \langle r_2, r_3, r_4 \rangle$. Then $\langle z_1, z_2 \rangle = C_{B^*}(A_0)$ and clearly, $C_{B^*}(\langle r_2, r_3, r_4, r_6 \rangle) = \langle z \rangle$ has order 3. Since $\langle r_2, r_3, r_4, r_6 \rangle \cong W_{D_4}$, z is in fact one of the z_i's. By Tables B and P, $L(C_G(z))$ must have type $D_4(q)$.

Suppose $\langle A_1, A_2 \rangle \cong W_{E_8}, A_j \cong W_{A_8}, p = 3$, $m(B^*) = 8$. We have $A_0 \cong W_{A_6}$ and there is a reflection $r \in A_j$ with $C_{A_j}(r) = \langle r \rangle \times A_0$. Thus we may choose reflections $r_1, \ldots, r_8$ to satisfy



Thus, $C_{B^*}(A_0) = \langle z_1, z_2 \rangle$ and $C_{B^*}(r_1, r_2, \ldots, r_7 \rangle) = \langle z \rangle$ has order p. Since $\langle r_1, r_2, \ldots, r_7 \rangle$ acts irreducibly as W_{E_7} on $B^*/\langle z \rangle$, Table B tells us that $L(C_G(z))$ has type E_7.

LEMMA 5.9. *Suppose that $A_i \cong W_{E_n}$ for some $n \in \{6, 7, 8\}$ or $A_i \cong W_{F_4}$. Then $n = 6$ or 7 and $A = \langle A_1, A_2 \rangle \cong W_{E_{n+1}}$ or $A_i \cong W_{F_4}$ and $A \cong W_{E_6}$. Also, $\langle K_1, K_2 \rangle$ has type $E_7(q)$ or type $E_8(q)$ for some q with $p | q^2 - 1$ and the case $A_i \cong W_{F_4}$ does not occur.*

Proof. Suppose A_i has type W_{E_n}. By Proposition E, $A \cong W_{E_{n+1}}$ for $n = 6$ or 7 and $m(B^*) = n + 1$. Since $K_i \in \mathrm{Chev}(2)$, Table B implies that K_i has type ${}^2E_6(q)$ for $p|q+1$ and $n = 6$ or $E_n(q)$ for some $n = 6, 7$ and q such that $p|q^2 - 1$.

If K_i has type $E_n(q)$, Proposition 2.30 then enables us to identify $\langle K_1, K_2\rangle$ as a group of type $E_{n+1}(q)$. (Note that, in $E_m(q)$, the standard copy of $A_G(B^*)$ for $p|q+1$ is also one for $p|q-1$, if $m = 7$ or 8).

Now suppose K_i has type ${}^2E_6(q)$, $p|q+1$, $n = 6$. We have to be alert to any possible "exceptional" coverings of groups in Chev(2) when $q = 2$. We have $W_i \cong W_{E_6}$ and $W \cong W_{E_7}$. Let $K \leqslant K_i$ be a natural subgroup of type $D_4(q)$, i.e., K is generated by all the root groups for long roots in the root system for K. Then $W \cap K$ is a natural W_{D_4} subgroup of K; see Lemma 2.50. We note that K is simple (see the description of the exceptional covering of ${}^2E_6(2)$ in [36]). Thus we may use W and the Steinberg relations to construct a group $Y = \langle K, W\rangle$ of type $E_7(q)$. Then $Y \geqslant B^*$ since W acts irreducibly on B^* and $K \cap B^* \neq 1$. Also, since $A_Y(B^*)$ contains a copy of W, Table B implies that $Y \cap K_i = K_i$. Since $L_0 \leqslant Y$, a similar argument with Table B implies that $K_j \leqslant Y$, as Table P tells us that the possibilities are that L_0 has type ${}^2A_5(q)$ or ${}^2D_4(q)$, whence, by Table P, K_i, has types ${}^2A_6(q)$, $C_6(q)$, $D_6(q)$, ${}^2E_6(q)$ or $D_6(q)$, ${}^2E_6(q)$, respectively. Therefore, $G_1 = \langle K_1, K_2\rangle \leqslant Y = \langle K, W\rangle = \langle K, W_1, W_2\rangle \leqslant G_1$, whence G_1 has type $E_7(q)$, as required.

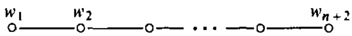
Suppose $A_i \cong W_{F_4}$. By Proposition CF, $m(B^*) = 5$, $p = 3$ and $A = \langle A_1, A_2\rangle \cong W_{E_6} \times Z_2$. There are three orbits of A on $(B^*)^{\#}$ with stabilizers $W_{F_4}, \Sigma_2 \times \Sigma_6$ and a 3-local subgroup of index 40 in A. Let $\{j, j'\} = \{1, 2\}$. Since p half splits K_j, A_j is therefore W_{F_4} or W_{A_5}. Suppose $A_j \cong W_{A_5}$ so that K_j has type $A_5(q)$ or ${}^2A_5(q)$. Then L_0 has type $A_3(q)$ or ${}^2A_3(q)$. By Table P, K_j, cannot have type $F_4(q')$ or ${}^2E_6(q')$ for any q', contradiction. Therefore, $A_j \cong W_{F_4}$. Consequently, $A_0 \cong W_{C_3}$ and L_0 has type $C_3(q)$, ${}^2D_4(q)$ or ${}^2A_5(q)$ for some q. Set $Q = O_2(A_0) \cong A_2^3$. Then $R = [Q, A_0]$ is a four group in $O_2(A_j), j = 1, 2$. Also $\langle t\rangle = Z(A_0)$, where t has eigenvalues $\{1, 1, -1, -1, -1\}$. By inspecting the maximal 2-locals of $W_{E_6} \times Z_2$ and noting that no involution of A_0 can have more than three eigenvalues -1, hence cannot be conjugate to any t_j, we see that $C_A(R)$, hence $N_A(R)$, must lie in the 2-local $A_j \times \langle -1_{B^*}\rangle$ for both $j = 1$ and 2. Thus, $C_A(R) = Q \times \langle t_j\rangle \times \langle -1_{B^*}\rangle$, $j = 1, 2$, and $|C_A(R)| = 2^5$. But then $Z(N_A(R)) = \langle t_1, t_2 - 1_{B^*}\rangle \cong Z_2^3$, which is incompatible with the structure of A_1 and A_2. This contradiction completes the proof of Lemma 5.9.

LEMMA 5.10. *Suppose that K_i has type $A_n(q), p|q+1, n \geqslant 7$. Then G_1 has type $A_{n+2}(q)$.*

Proof. We have $k = [(n+1)/2] = m(B^*) - 1$. By Proposition CF,

$A=\langle A_1,A_2\rangle\cong W_{C_{k+1}}$ or W_{F_4}. In this case, $4\leqslant m(B)=m(B^*)-1$, by 2.35(iv), whence $A\cong W_{C_{k+1}}$ for $k\geqslant 4$. Note that $K_1\cong K_2\cong A_n(q)$; by Table P, $K_j=E_6(q)$ may be possible, but is out by Proposition CF.

Choose standard copies W_i^* in K_i of $A_{K_i}(B^*)$ as in Lemma 5.5. Thus $\langle W_1^*, W_2^*\rangle\cong A$ by extensions of the natural isomorphisms $W_i^*\cong A_i$. Now, choose standard copies W_i of the Weyl groups of each K_i such that $W_i\geqslant W_i^*$, $i=1$, 2 and $W_1\cap W_2$ is a standard copy of the Weyl group for $L_0=L(K_1\cap K_2)$; see Lemma 2.50(iii). We want to show that $\langle W_1, W_2\rangle\cong W_{A_{n+2}}$. Choose fundamental reflections $w_1, w_2,\ldots$ so that $W_1=\langle w_1, w_2,\ldots, w_n\rangle$, $W_2=\langle w_3, w_4,\ldots, w_{n+2}\rangle$ and $w_1, w_2,\ldots$ satisfy the relations



are satisfied. We wish to show that $[\langle w_1, w_2\rangle, \langle w_{n+1}, w_{n+2}\rangle]=1$.

The embedding $W_i^*\leqslant W_i$ can be described by regarding W_i^* as the centralizer in $W_i\cong\Sigma_{n+1}$ of an element corresponding to $(12)(34)\cdots(k-1,k)$. Even though $\langle w_1, w_2\rangle$ does not normalize B^*, we know that $B_1=C_{B^*}(\langle w_1, w_2\rangle)$ has index p^2 in B^*. Also, if $B_2=C_{B^*}(\langle w_{n+1}, w_{n+2}\rangle)$, $|B^*:B_2|=p^2$. Furthermore, B_1, B_2 and $B_0=B_1\cap B_2$ are direct products of the A-transforms of $\langle z_1\rangle$ (or $\langle z_2\rangle$) which they contain and $|B^*:B_0|=p^4$. It follows that $J_0=L(C_G(B_0))$ has type $A_7(q)$ or $A_8(q)$. Also, it contains $J_i^*=L(C_G(B_i))$, which has type $A_3(q)$ or $A_4(q)$, $i=1,2$. The action of B^* on J_0 and the fact that B_1 and B_2 fuse in $N_A(B_0)$ imply that $J_1^*\cong J_2^*$. Let J_i be the natural $A_3(q)$-subgroup of J_i^* which contains $B^*\cap J_i^*$, $i=1,2$. The structure of $J_0=B_{i'}\cap J_i^*\cong Z_p\times Z_p$, where $\{i,i'\}=\{1,2\}$ and the action of $(B_1\cap J_2)\times(B_2\cap J_1)$ force $[J_1, J_2]=1$, which gives us the desired relation.

Set $W=\langle W_1, W_2\rangle\cong W_{A_{n+2}}$. Then $G_1=\langle K_1, W\rangle\cong A_{n+2}(q)$, by Proposition 2.30.

LEMMA 5.11. *Suppose that* $A_i\cong W_{C_n}$. *Then* $A=\langle A_1,A_2\rangle\cong W_{C_{n+1}}$ *or* $n=3$, $A\cong W_{F_4}$ *or* $n=p=3$ *and* $A''\cong A_6$, *the alternating group. Also, there is a* q *so that* G_1 *has type* $C_{n+1}(q)$ *or* $p|q-1$ *and* G_1 *has type* $A_{n'}(2)$, $[(n'+1)/2]=n+1$, ${}^2D_{n+2}(q)$, ${}^2A_7(q)$, ${}^2E_6(q)$ *or* $F_4(q)$. *Moreover, the case* $A''\cong A_6$ *does not occur.*

Proof. Proposition CF gives the possibilities for A. The possibilities for K_i, since p half-splits K_i, are groups of type $C_n(q)$, $D_{n+1}(q)$ with $p|q+1$, ${}^2D_{n+1}(q)$ with $p|q-1$, ${}^2A_5(q)$ with $n=3$, $p|q-1$ or $A_{2n-1}(2)$, $A_{2n}(2)$ with $p=3$, $n\geqslant 3$, or $A_5(4)$ with $p=5$, $n=3$. We deal with these cases individually.

Suppose K_i has type $D_{n+1}(q)$. Then $m(B^*) = n$ if n is even, $n+1$ if n is odd. By Table B, L_0 has type ${}^2D_n(q)$, whence K_1 has type $D_{n+1}(q)$, ${}^2E_6(q), E_6(q)$ or $E_8(q)$. But p does not split $D_{n+1}(q), E_6(q)$ or $E_8(q)$ since $p \mid q+1$, whence K_1 has type ${}^2E_6(q)$. Thus, $m(B^*) = 7$ or $p = 3$ and $m(B^*) = 6$; moreover, $A \cong W_{E_7}$. But then we have a contradiction to Proposition CF with regard to the containment $A_i < A$.

Thus K_i has type $C_n(q)$ or $p \mid q-1$ and K_i has type ${}^2D_{n+1}(q)$ or ${}^2A_5(q)$. Since we have eliminated $D_{n+1}(q)$, we observe that K_i can involve no "exceptional" covering as $n \geqslant 4$ or $p \mid q-1$.

Suppose $A \cong W_{C_{n+1}}$ and suppose that K_i does not have type $A_l(q')$, for some l and q'. Then $W_i := W \cap K_i$ is a standard copy of the Weyl group of K_i (see Table B and Lemma 2.50), we use Proposition 2.30 to show that $G_1 = \langle K_1, K_2 \rangle$ has type $C_{n+1}(q)$, ${}^2D_{n+2}(q)$, ${}^2A_7(q)$ when K_i has type $C_n(q)$, ${}^2D_{n+1}(q)$, ${}^2A_5(q)$ respectively (note that $n \geqslant 3$ implies that K_i has Lie rank at least 3).

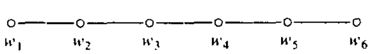
Suppose that K_i has type $A_n(2)$. Then $p = 3$. If $n \geqslant 7$, Lemma 5.10 gives the desired conclusion. Say $n < 7$. Then $m_3(K) \geqslant 3$ implies that $n = 5$ or 6 and $B = B^*$ has rank 4. The possibilities for L_0 are $A_{n-2}(2)$ or $SL(\lfloor (n+1)/2 \rfloor, 4) = SL(3, 4)$. From Table P, we see that the possibilities for the type of K_1 are

$$A_n(2), \qquad n = 5, 6 \qquad \text{for } L_0 \cong A_{n-2}(2)$$

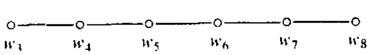
$$\left.\begin{array}{l} A_n(2) \text{ or } A_{n-1}(2) \\ A_3(4) \\ C_3(4) \\ D_3(4) \\ {}^2D_4(4) \\ {}^2A_5(2) \end{array}\right\} \text{for } L_0 \cong SL(3,4).$$

Say $L_0 \cong A_{n-2}(2)$. Then $m_{2,3}\ (K_1) \geqslant 3$ implies that K_1 has type $A_4(2)$, $A_6(2)$ or $E_6(2)$. By Lemma 5.9, $K_1 \cong A_5(2)$ or $A_6(2)$. We have $A_1 \cap A_2 \cong W_{C_2}$.

We treat the case $K_1 \cong A_6(2)$ in detail and leave the $A_5(2)$ case as an exercise. For $i = 1, 2$, choose standard copies W_i of the Weyl group of K_i in K_i so that $W_i = W_i \cap N_{K_i}(B^*)$ is a standard copy of $A_{K_i}(B^*)$. Then $W_1 \cong \Sigma_7$, $W_1 \cap L_0 \cong \Sigma_5$, $W_2 \cong \Sigma_7$. Let $w_1, \dots, w_8$ satisfy $W_1 = \langle w_1, \dots, w_6 \rangle$, $w_2 = \langle w_3, \dots, w_8 \rangle$, $W_1 \cap W_2 = \langle w_3, \dots, w_6 \rangle$,



and



We wish to verify relations

$$\underset{w_1}{\circ}\!-\!\!-\!\underset{w_2}{\circ}\!-\!\!-\!\underset{w_3}{\circ}\!-\!\!-\!\underset{w_4}{\circ}\!-\!\!-\!\underset{w_5}{\circ}\!-\!\!-\!\underset{w_6}{\circ}\!-\!\!-\!\underset{w_7}{\circ}\!-\!\!-\!\underset{w_8}{\circ} .$$

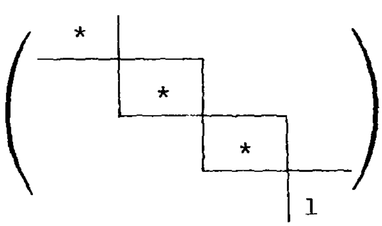
Let ϕ: $K_1 \cong GL(7, 2)$ satisfy

$$w_i^{\varphi} = i_{i+1} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0\,1 & & & \\ & & & 1\,0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}, \qquad i = 1,\ldots, 6$$

with respect to the basis $v_1,\ldots, v_7$. Set $\langle z_2 \rangle = D \cap K_1$; then (B, z_2, K_2) is a neighbor since z_2 fuses to z_1 in $N_G(B)$ (each $\langle z_i \rangle$ is the commutator of B with a fundamental reflection in $O_2(A_G(B))$). We may arrange for

$$z_2^{\phi} = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

and for $(B \cap K_1)^{\phi}$ to have shape



with each block 2×2. This is compatible with preceding arrangements since $N_{W_1}(B) = C_{W_1}(t)$, where

$$t^{\phi} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & \\ & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \\ & & & 1 \end{pmatrix}.$$

Similarly we may arrange for an isomorphism ψ: $K_2 \cong GL(7, 2)$ to satisfy

$$w_i^{\psi} = \begin{array}{r} \\ \\ i \\ i+1 \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0\,1 & & & \\ & & & 1\,0 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \quad i=3,\ldots,8$$

with respect to the basis $v_3, \ldots, v_9$, and for $(B \cap K_2)^{\psi}$ to have shape

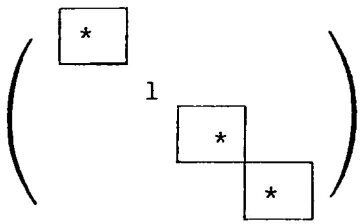
$$\left(\begin{array}{c|c|c|c} * & & & \\ \hline & * & & \\ \hline & & 1 & \\ \hline & & & * \end{array}\right)$$

Let $M = \tilde{M}\phi^{-1}$, $\tilde{M} = \{ g \in GL(7, 2) \mid g$ fixes $v_i, i \neq 5, 6, 7$ and leaves the span of $\{v_5, v_6, v_7\}$ invariant$\} \cong GL(3, 2)$.

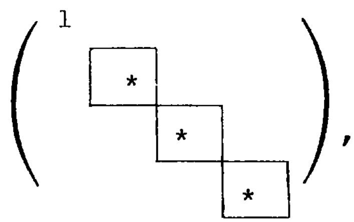
We claim that $L(C_G(M)) \cong GL(6, 2)$. Since $B \cap M \cong Z_3$ fuses in $N_G(B)$ to z_1, we get $L(C_G(M)) \hookrightarrow GL(7, 2)$. If $M_0 \leqslant M$ corresponds under ϕ to a natural $GL(4, 2)$ subgroup and $|M_0 \cap B| = 9$, we have $L(C_G(M_0)) \cong GL(6, 2)$. Thus, $L(C_G(M)) \cong GL(6, 2)$, $GL(7, 2)$ or is 1 (this happens if M centralizes a maximal parabolic of $L(C_G(B \cap M))$). Assume $L(C_G(M)) = 1$. In this case, taking $z_2 \in L(C_{K_1}(M)) \cong GL(4, 2)$ and considering the natural action of $GL(6, 2)$ on its standard module, we find that z_2 centralizes a subgroup of the shape $2^4 \cdot GL(4, 2) \times Z_3$ in $C_G(M) \cap L(C_G(B \cap M))$; but this violates the shape of $C_{K_i}(M) \cong GL(4, 2)$ and its embedding in $C_G(z_2)$. So the claim holds.

We change bases slightly. Let $\mathscr{B} = \{v_1 - v_2, v_2 - v_3, \;\; v_1 + v_2 + v_3,$

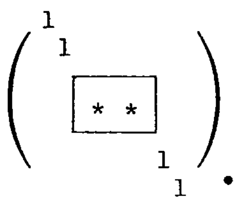
$v_4, v_5, v_6, v_7\}$ and let $\phi': K_1 \to GL(7,2)$ be an isomorphism differing from ϕ by this basis change. We take $A \leqslant K_1 B$, a conjugate of B, so that $z \in A$ and $(A \cap K_1)^{\phi'}$ has shape



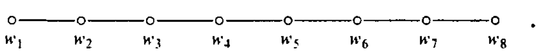
and $(A \cap K_2)^{\psi'}$ has shape



where $C = \{v_1 + v_2 + v_3,\ v_4, v_5, v_6, v_7, v_8, v_9\}$ and $\psi': K_2 \to GL(7,2)$ is a representation of K_2 with respect to this basis. We replace M by the conjugate M^* such that $(M^*)^{\phi'}$ has shape



Let S, T be the subgroup of K_1, K_2 inducing the general linear group on span $\{v_1 - v_2, v_2 - v_3\}$, span$\{v_6, v_7, v_8, v_9\}$ and fixing $\{v_1 + v_2 + v_3, v_4, v_5, v_6, v_7\}, \{v_3, v_4, v_5\}$, respectively. Let V be the natural module for $L(C_G(M^*)) \cong GL(n,2)$, $n = 6$ or 7. Since $[V, T] = [V, T \cap A]$ and $C_V(T) = C_V(T \cap A)$, $[S, N_T(T \cap A(] = 1$ (seen in $N_G(A)$) implies that $[S, T] = 1$. Since $\langle w_1, w_2 \rangle \leqslant S$ and $\langle w_7, w_8 \rangle \leqslant T$, we get the desired relations



At this point, Proposition 2.30 identifies $G_1 = \langle K_1, W \rangle$ as $A_8(2)$, as required. Of course, when $K_1 \cong A_5(2)$, we get $G_1 \cong A_7(2)$.

If $L_0 \cong SL(3,4)$, it is still the case that (B, z_i, K_i) is a standard component and so the above arguments apply to give $G_1 \in \mathrm{Chev}(2)$.

Suppose that K_i has type $A_5(4)$, $p=5$. Then p only half-splits K_i. By Table P, for p to split K_1, we must have $L_0 \cong SL(3,16)$.

Then K_1 is a group defined over $\mathbb{F}_{16}$ and in fact is of type $A_4(16)$, $A_5(16)$, $C_3(16)$, $D_3(16)$, ${}^2D_4(16)$ or K_1 has type ${}^2A_5(4)$. If K_1 is defined over $\mathbb{F}_{16}$, we get $G_2 := \langle K_1, W\rangle \in \mathrm{Chev}(2)$ (W as in Lemma 5.5) by a previous part of this lemma and Lemmas 5.7. There is no possibility for $K_i \cong A_5(4)$ to be compatible with $C_{G_2}(z_i)$ since G_2 is defined over $\mathbb{F}_{16}$ (one must check the cases to see this). If K_1 has type ${}^2A_5(4)$, Lemma 5.7 implies that G_2 has type ${}^2A_6(4)$; in this case $L_0 \cong PSL(3,16)$, not $SL(3,16)$, a contradiction.

Suppose $A \cong W_{F_4}$. Then, we can proceed as in the case $A \cong W_{C_{n+1}}$ to construct $F_4(q)$ or ${}^2E_6(q)$. There is a special problem in that one of K_1 or K_2 will not contain a W-conjugate a given pairs of roots. So, we use both K_1 and K_2 and Proposition 2.31. Thus, an examination of Tables B and P shows that when K_i has type $C_3(q)$, ${}^2D_4(q)$, ${}^2A_5(q)$, respectively, we construct G_1 of type $F_4(q)$, ${}^2E_6(q)$, ${}^2E_6(q)$, respectively.

Finally, we treat the case $A'' \cong A_6$; that is A contains a copy of $\Sigma_6 \times Z_2$ and $Z(A) = \langle -1_{B^*}\rangle$, $A/Z(A) \cong \Sigma_6$ or $\mathrm{Aut}(\Sigma_6)$, $p-3$ and $m(B) = m(B^*) = 4$. Also K_i has type $C_3(q)$, or type ${}^2A_5(q)$ with $3 \mid q-1$, or type ${}^2D_4(q)$, or type $A_n(2)$, $n=5$ or 6 and $p=3$. Since $B=B^*$ and 3 splits $K_1 \in \mathrm{Chev}(2)$, the possible types for K_1 are $A_3(q_1)$ or $C_3(q_1)$ for $3 \mid q-1$ or type ${}^2D_4(q_1)$ for some $q_1 \in \{q, q^2\}$ or type $A_n(2)$ for $n=5$ or 6, $p=3$. Also $3 \mid q-1$ or $q_1 = q$ and K_1 has type ${}^2D_4(q)$.

For now, let us suppose that $3 \mid q_1 - 1$. In any case, K_1 contains a natural subgroup K of type $A_3(q_1) = D_3(q_1)$ such that $V = K \cap W \cong W_{A_3}$, V is a standard copy of the Weyl group for K, and K is generated by appropriate root groups; see Lemma 2.50(iv). Take $\tilde{W} \leqslant W$ so that $\tilde{W} \geqslant V$ and $W \cong \Sigma_6$. By Proposition 2.30, $Y = \langle K, W\rangle \cong A_5(q_1)$. Also, $Z(Y) = 1$.

The following argument is an adaption of the argument in result (5.11) of Finkelstein and Frohardt [17]. Define $N = N_G(B)$, $P \in \mathrm{Syl}_3(N)$, $P_1 = C_p(B_1)$ where $B_1 = \langle z_1, z_0\rangle$ and $\langle z_0\rangle = B \cap Z(P)$, $N^* = N_G(B_1)$, $C^* = C_G(B_1)$ and $L = L(C_G(B_1)) = L(C_{K_1}(z_0)) \cong SL(3, q_1)$ with $3 \mid q_1 - 1$ (check the possible K_1). We have $N_1/C_1 \cong Z_2 \times \Sigma_3$.

We argue that $B = J_e(P)$, where J_e denotes the Thompson subgroup $\langle \tilde{B} \mid \tilde{B} \leqslant P,\ m(\tilde{B}) = m(P),\ \tilde{B}$ elementary abelian$\rangle$. Let $\tilde{B} \leqslant P$ with $\tilde{B} \cong B$, $\tilde{B} \neq B$. Note that $C_G(B)$ has homocyclic abelian rank 4 Sylow 3-subgroups. In fact, Lemma (3.8) implies that $B \in \mathrm{Syl}_3(C_G(B))$. If $|\tilde{B} \cap B| = 3^2$, then $\tilde{B}$ covers $P/C_P(B)$, whence $|Z(P)| = 3^2$, a contradiction. Thus, $|\tilde{B} \cap B| = 3^3$. It follows that A contains a tranvection on B. However, every 3-element of A normalizes but does not centralize a four-group of A, hence cannot be a tranvection on B, contradiction. Therefore $B = J_e(P)$, and we have also shown that B is the unique group of its isomorphism type in P. Consequently,

$P \in \mathrm{Syl}_3(G)$, $|P| = 3^6$ and N controls G-fusion in B, by a Burnside-type argument.

Set $R_1 = P \cap L \lhd P \leqslant C^*$, $R_2 = C_P(R_1)$ and $R = R_1 R_2$. Then $|R_2| = 27$ and $R_1 \cap R_2 = Z(P) = \langle z_0 \rangle \cong Z_3$. We argue that $R_2 \cong 3^{1+2}$. First, note that if $B \cap R_2 \leqslant Z(R_2)$, then R_2 contains an element inducing a transvection on B, against Lemma 3.20. Thus, R_2 is nonabelian, so it suffices to show that $\exp R_2 = 3$. Suppose false. Then $|R\colon \Omega_1(R)| = 3$ and $\Omega_1(R) B$ is a characteristic subgroup of P lying strictly between B and P. However, the structure of A_6 implies that $A_G(P/B)$ contains a cyclic group of order 4, whence $A_G(P/B)$ is irreducible on P/B, contradiction. Thus, $R_2 = \Omega_1(R_2)$ and $R = \Omega_1(R) \cong 3^{1+4}$. We also have that $[R_2, B] = R_2 \cap B$, or else we would have $[R_2, B] \leqslant Z(P)$ and so elements of $P - B$ would have quadratic minimal polynomial on B, against Lemma 3.20. Thus, in its action on $R/Z(P)$, an element of $P - R$ has a matrix similar to

$$\begin{pmatrix} 11 & \\ 01 & \\ & 11 \\ & 01 \end{pmatrix}.$$

Finally, we determine $A_G(R/Z(P))$. It must be a subgroup of $Sp^{\pm}(4, 3)$. From K_1, we get a copy S of $GL(2, 3)$ in $A_G(R/Z(P))$ which satisfies $[R_2/Z(P),\ O_2(S)] = 1$. Since $R_2/Z(P) = [R/Z(P),\ O_2(S)]$, we must have $O_2(A_G(R/Z(P))) \neq O_2(S)$, or else we could contradict the action of $N_G(P)$ on P/R as above. Thus, the structure of $Sp^{\pm}(4, 3)$ and $|A_G(R/Z(P))|_3 = 3$ implies that $O_2(A_G(R/Z(P))) \cong Q_8 \wr Z_2$, $[O_2(A_G(R/Z(P))), O_{2,3}(A_G(R/Z(P)))] \cong Q_8 \times Q_8$ and $A_G(R/Z(P))/O_2(A_G(R/Z(P)) \cong \Sigma_3$.

Define Q_1 to be a complement to R_1 in $N_{K_1}(R_1)$. Then $Q_1 \cong Q_8$ and $[R_2, Q_1] = 1$. We let $Q_2 = Q_1^h$, where $h \in N_G(P)$ and h interchanges R_1 and R_2 under conjugation. Let $\langle t \rangle = Z(Q_2)$. Then $[R_1, t] = 1$ implies that $[L, t] = 1$. Since the maximal subgroups of R_2 form an orbit under Q_2, we may assume that t normalizes Z_1. Also, we may assume that t inverts z_1 by replacing Q_1 with a conjugate by an element of R_1.

The possible structures of Aut K_1 and $[L, t] = 1$ imply that $[K_1, t] = 1$ or $K_1 \cong {}^2D_4(q_1)$, t acts as an orthogonal tranvection on K_1 and $L(C_{K_1}(t)) \cong C_3(q_1)$. Now, take $y \in R_1, y \widetilde{_{N(R)}} z_1$. Then $[y, t] = 1$ and $y \in L(C_{K_1}(t))$ since $B = \langle z_1 \rangle \times (B \cap L(C_{K_1}(t)))$. We have that $L(C_G(\langle z_1, y \rangle)) \cong SL(2, q)$ and y is a noncentral element of order 3 in a natural $A_2(q_1)$ subgroup of $L(C_{K_1}(t))$. Now suppose that K_1 does not have type $A_n(2)$. If $L(C_{K_1}(t))$ contains a copy of $C_3(q_1)$, then $C_G(\langle t, y \rangle) \geqslant \langle t \rangle \times C_{L(C_{K_1}(t))}(y) \cong Z_2 \times GL(2, q_1) \times Y_1$ where $Y_1 \cong Sp(2, q_1) \cong SL(2, q_1)$ or $Y_1 \cong {}^2D_2(q_1) \cong SL(2, q_1^2)$. Since Out K_1 has abelian Sylow 2-subgroups and t lies in a quaternion group in $C_G(y)$, t must induce an inner antomosphism on $L(C_G(y)) \cong K_1$. But $K_1 \cong C_3(q_1)$ or

${}^2D_4(q_1)$ implies that t centralizes $L(C_G(y))$, a contradiction since t induces a reflection on $B \leqslant \langle y, L(C_G(y))\rangle$. The case where $L(C_{K_1}(t))$ does not contain a copy of $C_3(q_1)$ is the case $K_1 \cong A_3(q_1)$. But then $i = 2$ and we merely reverse the roles of z_i and z_1 in the above argument and use the fact that K_2 does contain a natural $C_3(q_1)$-subgroup.

If K_1 has type $A_n(2)$, $n = 5, 6$, we may argue as above to get a contradiction. The only change occurs at the end, namely, $C_G(\langle t, y\rangle) \geqslant \langle t\rangle \times C_{K_1}(y) \cong Z_2 \times GL(2, 4) \times GL(n-3, 2)$.

Thus, the case $A'' \cong A_6$ is eliminated, and the proof of the lemma is complete.

LEMMA 5.12. *Suppose that $A_i \cong W_{D_n}$. Let $A = \langle A_1, A_2\rangle$. Then there is a q so that one of the following holds.*

(i) *$A \cong W_{D_{n+1}}$ and G_1 has type $D_{n+1}(q)$,$p | g-1$ or n is even $n \geqslant 4$, $p \mid q+1$; or G_1 has type ${}^2D_{n+1}(q)$, n odd, $n \geqslant 5$, $p | q+1$.*

(ii) *$A \cong W_{E_{n+1}}$ and (K_i, G_1) have types*

$$\begin{array}{ll} (D_5(q), E_6(q)) & p|q-1 \\ ({}^2D_5(q), {}^2E_6(q)) & p|q+1 \\ (D_4(q), E_6(q)) & p=3, 3|q-1 \\ (D_6(q), E_7(q)) & p|q^2-1 \\ (D_7(q), E_8(q)) & p|q-1 \\ ({}^2D_7(q), E_q(q)) & p|q+1. \end{array}$$

(iii) *$A \cong \Sigma_5$, and G_1 has type $A_4(q)$, $p|q-1$. K_i has type $D_3(q)$*

Proof. The possibilities for A are given by Proposition D. Thus, $A \cong W_{D_{n+1}}$, $W_{C_{n+1}}$ or $n = 5, 6, 7$ and $A \cong W_{E_{n+1}}$ or $n = 4$, $p = 3$ and $A \cong W_{E_6}$ or $n = 3$ and A is a group of small index in $W_{F_4}\langle y\rangle$, y a graph automorphism (i.e., $W_{D_4} \leqslant A \leqslant W_{F_4}\langle y\rangle$) or $n = 3$, $p = 3$ and $A \cong \Sigma_6$, $\Sigma_6 \times Z_2$, $A_6 \cdot D_8$, Σ_5 or $\Sigma_5 \times Z_2$. Let $\{i, i'\} = \{1, 2\}$.

We can eliminate several possibilities for A with a few observations. Suppose $A/A' \cong Z_2 \times Z_2$. We claim that A_1 or A_2 must be associated with a Dynkin diagram with two root lengths. If false, the fact that A_0 is generated by reflections and covers each of A_1/A_1' and A_2/A_2' gives a contradiction. Thus, some A_j is isomorphic to W_{C_n} or W_{F_4}. Since the diagram for A_i has one root length, the same is true for A_0 (see Table P), whence $A_j \cong W_{F_4}$. Thus $A_j \cong W_{C_n}$. By Lemma 5.11, $A \cong W_{C_{n+1}}$ or W_{F_4} and $G_1 \cong C_{n+1}(q)$ or $p|q-1$ and G_1 has type ${}^2D_{n+2}(q)$, ${}^2A_7(q)$, ${}^2E_6(q)$ or $F_4(q)$. As $G_1 \in \mathrm{Chev}(2)$, we look at Table B and see that $A_i \cong W_{D_n}$ is impossible. We conclude that $A/A' \cong Z_2$, whence $A \cong W_{D_{n+1}}$ or $n = 5, 6, 7$ and $A \cong W_{E_{n+1}}$ or $n = 4$, $p = 3$ and $A \cong W_{E_6}$ or $n = 3$ and $A \cong \Sigma_5$ or $n = 4$ and $p = 3$ and $A \cong \Sigma_6$.

Suppose $A \cong W_{E_6}$. Then $n = 5$ or $p = 3$ and $n = 4$. If $n = 5$, K_i has type $D_5(q)$, $p|q-1$, or type ${}^2D_5(q)$, $p|q+1$, and we show that G_1 has type $E_6(q)$ or ${}^2E_6(q)$, respectively, using Proposition 2.30. For K_i of type $D_5(q)$, this is easy. For ${}^2D_5(q)$, it is almost as easy once we see that the natural containment ${}^2D_5(q) \hookrightarrow {}^2E_6(q)$ corresponds to a natural containment $W_{C_4} \hookrightarrow W_{F_4}$.

If $p = 3$ and $n = 4$, K_i has type $D_4(q)$ and A is isomorphic to W_{E_6}. We have that L_0 has type $A_3(q) = D_3(q)$. Since A_i is a Weyl group for a root system with one root length, Table P implies that $A_i \cong W_{A_4}$ or W_{A_5} or W_{D_4}. If $A_{i'} \cong W_{A_4}$ or W_{D_4}, then A is generated by five reflections. If $A \cong W_{E_6}$, this is impossible (look at the usual representation in $O(6, \mathbb{R})$). Thus, $A_{i'} \cong W_{A_5}$. The orbits of W_{E_6} on the 121 one-dimensional subspaces of B^* have lengths 40, 36 and 45, whence $A_{i'}$ is a natural W_{A_5} subgroup of W_{E_6}. We therefore may use Proposition 2.30 to get G_1 of type $E_6(q)$. Note that this forces $C_A(b_i)$ to be an extension of W_{D_4} by the graph automorphism of order 3 and $N_A(\langle b_i \rangle) \cong W_{F_4}$.

Suppose $A \cong \Sigma_6$, $n = 3$, $p = 3$. Then $A_i \cong W_{D_3}$ implies K_i has type $D_3(q)$, $p|q-1$ or ${}^2D_3(q)$, $p|q+1$, whence $A_0 \cong \Sigma_3$. Since $A \cong \Sigma_6$, A is not generated by four reflections, whence A_i' has type W_{A_4} (rather than W_{A_3}). Since $A_{i'}$ must fix a nontrivial element of B^*, we have $p = 5$, a contradiction. So, $A \not\cong \Sigma_6$.

If $A \cong A_6 \cdot D_8$, we quote the last line of Lemma 5.11.

If $A \cong \Sigma_5$, we quote Lemma 5.6, the case $n = 3$. Thus $p|q-1$ and G_1 has type $A_4(q)$ (G_1 cannot be ${}^2A_4(q)$ as $m_{2,p}(G_1) \geqslant 4$).

If $A \cong W_{E_7}$, K_i has type $D_6(q)$, $p|q^2-1$, and if $A \cong W_{E_8}$, K_i has type $D_7(q)$, $p|q-1$, or ${}^2D_7(q)$, $p|q+1$. In both cases, A_i is a natural W_{D_n} subgroup of $A \cong W_{E_{n+1}}$; see Proposition D. In the first case, we let K be a natural $D_5(q)$ subgroup and show that $G_1 = \langle K, W \rangle$ has type $E_7(q)$. In the second case, we let K be a natural $D_6(q)$ subgroup and prove that $G_1 = \langle K, W \rangle$ has type $E_8(q)$.

Finally suppose that $A \cong W_{D_{n+1}}$. Then K_i has type $D_n(q)$ and $p|q-1$, type $D_n(q)$ n is even and $p|q+1$, or type ${}^2D_n(q)$ and n is odd, $p|q+1$. We must show that G_1 has type $D_{n+1}(q)$, ${}^2D_{n+1}(q)$, $D_{n+1}[q]$, respectively. As usual, we need Proposition 2.30 but we have to be slightly careful about choosing the subgroup K of that proposition. If $p|q-1$, take $K = K_i$ and if $p|q+1$, let K be a natural subgroup of type $D_n(q)$, ${}^2D_{n-1}(q)$, $D_{n-1}(q)$, respectively. In the first and third cases, we want the Lie rank of K to be at least 3. Since $m(B) \geqslant 4$, if $p|q-1$, $n \geqslant 3$ and if $p|q+1$, $n \geqslant 4$, so there's no problem. In the second case, there are two root lengths, so we want K to have Lie rank at least four, i.e., $n-2 \geqslant 4$ or $n \geqslant 6$. Since n is even here, it remains to treat the case $n = 4$. It is no problem to verify the Steinberg relations for a pair of root elements which can be conjugated by an element of W to a pair of elements in K.

Let us tabulate the possible configurations up to W_{B_4}-conjugacy of pairs of linearly independent roots in the system Σ of type B_4. Here s, s' (r, r') denote typical short (and long) roots respectively and $\langle r_1 r_2$ is the angle between the roots $r_1, r_1 \in \Sigma$.

(1) $\langle s, s' = \pi/2$,

(2) $\langle r, r' = \pi/3$,

(3) $\langle r, r' = 2\pi/3$,

(4) $\langle r, r' = \pi/2$, $\mathbb{R}r + \mathbb{R}r'$ contains a short root,

(5) $\langle r, r' = \pi/2$ $\mathbb{R}r + \mathbb{R}r'$ does not contain in a short root

(think of a root system of type B_k as all $\pm e_\alpha$, $\pm e_\alpha \pm e_\beta$, $\alpha \neq \beta$, where $e_1, \ldots, e_k$ is an orthonormal basis for $\mathbb{R}^k$). Pairs of root elements corresponding to pairs (1) and (4) are W-conjugate to pairs of root elements of K or type ${}^2D_3(q)$. The pairs (2), (3) and (5) involve only long roots, and, as K_i(type $D_4(q)$) contains K as a natural subgroup, root elements for long roots in K one root elements in K_i, and the verification of the relations is immediate.

The proof of Lemma 5.12 is now complete.

COROLLARY 5.13. *G_1 is described by one of the preceding five lemmas.*

Proof. If A_i is a Weyl group of type $B = C$, D, E or F, this is clear. Otherwise, A_i has type A. In fact, we may assume that both A_1 and A_2 have type A. Then Lemmas 5.6 and 5.7 apply, and we are left with the case $A \cong W_{D_{n+1}}$, $A_1 \cong A_2 \cong W_{A_n}$, $A_1 \subset A_2 \cong W_{A_{n-1}}$. Then $p \mid q-1$, $n \geqslant 3$ and $A_1 \cong A_2$ has type $A_n(q)$. Since $n \geqslant 3$, it is easy to see that Proposition 2.30 may be applied to get $G_1 \cong D_{n+1}(q)$.

LEMMA 5.14. *If $K_j \nleqslant G_1$, then $A_j \nleqslant A$.*

Proof. We may assume that $A_j \leqslant A$. Define $L = L(C_{G_1}(z_j))$. We have $L_0 \leqslant L < K_j$. We claim that $L_0 = L$; assume otherwise.

Set $L_{00} = \langle L_0^{W_j} \rangle \leqslant K_j \cap G_1$. Then $L_{00} \leqslant L$, and, since $W_j \nleqslant N_{K_j}(L_0)$, $L_0 < L_{00}$. Consequently $W_j \cap L_{00} \lhd W_j$ and $W_j \cap L \lhd W_j$. Since, by Lemma 2.50, $W_j \cap Y$ is a standard copy of $AY(B^*)$ for $Y = L_{00}$ and L, either $W_j \cap Y = W_j$ or $W_j \cap Y$, W_j are isomorphic to W_{D_n}, W_{C_n} or W_{D_4}, W_{F_4}, respectively. The last case is out, by Proposition CF applied to $A_j < \langle A_1, \ldots, A_r \rangle$. Thus, $W_j \cap Y$ and W_j are "almost equal," i.e., $|W_j : W_j \cap Y| \leqslant 2$.

If $W_j \leqslant L$, then $L \neq K$ and Tables B and P show that $W_j \cong W_{C_n}$ for some n (we have eliminated $W_j \cong W_{F_4}$). Thus, whether W_j lies in L or not, $W_j \cong W_{C_n}$. Therefore, $\langle A_1, \ldots, A_r \rangle \cong W_{C_{n+1}}$ or W_{F_4}, by Proposition CF. We replace K_2 by K_1 in the preceding part of this section to get $G_1^* = \langle K_1, K_j \rangle \in \mathrm{Chev}(2)$. We use Table B to get the possibilities for G_1^*.

Suppose G_1^* has type $F_4(q)$. Then, by Table C, K_1 and K_j both have type $C_3(q)$ and L_0 has type $C_2(q)$. It is clear that $K_2 \leqslant G_1^*$, K_2 has type $C_3(q)$ and that G_1 has type $C_4(q)$ or $F_4(q)$. Since $K_j \nleqslant G_1$, G_1 has type $C_4(q)$ and $K_j \cap G_1$ must have Lie rank 3, hence $G_1 \cap K_j \cong A_3(q)$. But then $A_j \cong W_{C_3}$, $A_j \nleqslant A$, a contradiction.

Suppose that G_1^* has type $C_{n+1}(q)$, we may repeat the above argument unless K_1 and K_2 have type $D_n(q)$ or ${}^2D_n(q)$. Then G_1 has type $D_{n+1}(q)$ or ${}^2D_{n+1}(q)$. Since $K_j \cong C_n(q) \ncong K_i$, $i = 1, 2$, we must have L_0 of type $A_{n-1}(q)$ or ${}^2A_{n-1}(q)$, whence K_i has type $A_n(q)$, ${}^2A_n(q)$, $D_n(q)$ or ${}^2D_n(q)$. However, viewing $DL_0 \leqslant G_1^*$, we see that $K_i \ncong D_n(q)$ or ${}^2D_n(q)$, $i = 1, 2$. From the usual matrix representation of $C_{n+1}(q)$, we see that at most one of $\{K_1, K_2\}$ can be $A_n(q)$ or ${}^2A_n(q)$, a contradiction.

Finally, if G_1^* has type $D_{n+1}(q)$ or ${}^2D_{n+1}(q)$, the preceding argument may be modified to show that $G_1 = G_1^*$, a contradiction.

LEMMA 5.15. *If $A_j \nleqslant A$, then $(A_j, A, \langle A, A_j\rangle)$ is one of the following:*

$$(W_{A_n}, W_{A_{n+1}}, W_{A_{n+2}}), \qquad p \mid n+3,$$
$$(W_{A_4}, W_{D_5}, W_{E_6}), \qquad p = 3,$$
$$(W_{C_3}, W_{C_4}, W_{F_4}).$$

Furthermore, in these cases $\langle A, A_j\rangle$ contains all A_k, $k = 1, 2,..., r$.

Proof. Letting $A_\infty = \langle A_1,..., A_r\rangle$, we quote Propositions A, CF, D and E and use $O_p(A_\infty) = 1$ and the fact that A_∞ is generated by reflections. Compare this result with the last few lines of Table C.

PROPOSITION 5.16. *Suppose that $K_j \nleqslant G_1$ for some $j \geqslant 3$. Then we are in one of the following situations:*

L_0	K_i	K_2	K_3	G_1	G_0	p
$A_{n-1}(q)$	$A_n(q)$	$A_n(q)$	$A_n(q)$	$A_{n+1}(q)$	$A_{n+2}(q)$	$p \mid n+3, p \mid q-1$
${}^2A_{n-1}(q)$	${}^2A_n(q)$	${}^2A_n(q)$	${}^2A_n(q)$	${}^2A_{n+1}(q)$	${}^2A_{n+2}(q)$	$p \mid n+3, p \mid q+1$
$A_2(q)$	$C_3(q)$	$C_3(q)$	$C_3(q)$	$C_4(q)$	$F_4(q)$	$p \mid q-1$ ($p \nmid q+1$ since p splits some K_i)
$A_3(q)$	$D_4(q)$	$D_4(q)$	$D_4(q)$ ${}^2A_5(q)$	$D_5(q)$	$E_6(q)$	$p = 3, 3 \mid q-1$
${}^2A_3(q)$	$D_4(q)$	$D_4(q)$	$D_4(q)$ ${}^2A_5(q)$	${}^2D_5(q)$	${}^2E_6(q)$	$p = 3, 3 \mid q+1$
$A_2(q)$	${}^2D_4(q)$	${}^2D_4(q)$	${}^2D_4(q)$	${}^2D_5(q)$	${}^2E_6(q)$	$p = 3, 3 \mid q-1$
${}^2A_2(q)$	${}^2D_4(q)$	${}^2D_4(q)$	${}^2D_4(q)$	$D_5(q)$	$E_6(q)$	$p = 3, 3 \mid q+1$

In particular, $r=3$ unless $\langle A_1,\ldots,A_r\rangle \cong W_{E_6}$ in which case $r \geqslant 6$ or $\langle A_1,\ldots,A_r\rangle \cong W_{F_4}$ and $r=4$.

Proof. By Lemmas 5.13 and 5.14, we have $A_j \nleqslant A$ and the possibilities for $A=\langle A_1,\ldots,A_r\rangle = \langle A_1, A_2, A_3\rangle$. We identify the group $G_0^* = \langle G_1, W_3\rangle$ by Proposition 2.30 and Lemma 2.50, except for the case G_1 or type $C_4(q)$. Once G_0^* is identified, we get G_0^* by checking components. If G_1 has type $C_4(q)$, L_0 has type $A_3(q)$ or ${}^2A_3(q)$. If we choose L_0 differently in this case, i.e., L_0 of type $C_2(q)$, the components generate a group G_0^{**} of type $F_4(q)$. But its evident that our K_1, K_2 and K_j all lie in G_0^{**} (by Table P, for instance, and the structure of $L(C_{G_0^{**}}(z))$ for $z \in B^*$), whence $G_1 < G_0 = G_0^{**}$, as required. The last statement in the proposition is an exercise.

COROLLARY 5.17. *Define* $A^* = \langle A_1,\ldots,A_r\rangle$, $A^{**} = A_G(B^*)$. *Then* $A^{**} = A^*A_0^*$ *where* $A_0^* = \{\alpha \in A^{**} \mid \alpha$ *induces a scalar transformation on* $B^*\}$, *or we are in one of the following cases*:

(a) $A^*A_0^* \lhd A^{**}$ *and either*

(i) $A^* \cong W_{A_5}$, $|A_0^*| = 2$, $A^{**}/A_0^* \cong \mathrm{Aut}(A_6)$, $p=3$ *and* $m(B^*) = 4$; *or*

(ii) $A^* \cong W_{D_n}$, *n even*, $A^{**} = A_0^*A_1^*$, *where* $A_1^* \cong W_{C_n}$; *or*

(iii) $A^* \cong W_{D_4}$, $A^{**}/A^*A_0^*$ *is a subgroup of* Σ_3; *or*

(iv) $A^* \cong W_{F_4}$, $A^{**}/A^*A_0^* \cong Z_2$; *or*

(v) $A^* \cong W_{E_6}$, $A^{**}/A^*A_0^* \cong Z_2$.

(b) $A^*A_0^* \ntriangleleft A^{**}$ *and*

(i) $A^* \cong W_{C_4}$, $A^{**} = A_0^*A_1^*$, $A_1^* \cong W_{F_4}$ *or* $W_{F_4}\langle\gamma\rangle$ *where* γ *induces the graph automorphism on* $A_2^* \cong W_{F_4}$, $A_2^* \leqslant A_1^*$; *or*

(ii) $A^* \cong W_{A_n}$, $A^{**} = A_0^* \times A_1^*$ *where* $A_1^* \cong W_{A_{n+1}}$, $p \mid n+2$ *or* $n=4$ *and* A^{**} *is the group of* (a)(i); *or*

(iii) $A^* \cong W_{D_5}$, $A^{**} = A_0^* \cong A_1^*$, $A_1^* \cong W_{E_6}$.

Proof. Use Propositions A, CF, D and E.

LEMMA 5.18. *Define* $A^* = \langle A_1,\ldots,A_r\rangle$, $A^{**} = A_G(B^*)$. *In the notation of Lemma* 5.17, $A_0^* \leqslant \langle -1_{B^*}\rangle$ *and* $A^* \lhd A^{**}$.

Proof. The structure of Aut K_i, $K_i \in \mathrm{Chev}(2)$ implies that $A_0^* \leqslant \langle -1_{B^*}\rangle$. We show that $A^* \lhd A^{**}$ by eliminating each of the cases in conclusion (b) of Lemma 5.17. However, the structures of A^* and A^{**} gives contradictory values for r. For example, if $A^* = W_{C_4}$, $r \leqslant 3$ whereas if $A^{**} \cong W_{F_4}$, then $r=4$.

COROLLARY 5.19. *The A^*-conjugacy class of $\langle z_i \rangle$ is invariant under A^{**} unless possibly*

(i) $A^* \cong W_{D_n}, A_i \cong W_{A_{n-1}}$ *and* $A^{**} \cong W_{C_n}$,

(ii) $A^* \cong W_{F_4}, A_i \cong W_{C_3}, A^{**} = A^*\langle \gamma \rangle$ *where* γ *induces a graph automorphism on* A^*, $\gamma^2 \in Z(A^*)$.

Proof. The only opportunity for the statement to fail occurs when A^{**} induces a noninner automorphism on A^*. In this case, $A^* \cong W_{D_n}$ or W_{F_4}. After an examination of the cases and the using fact that A^* and A^{**} are both irreducible linear groups, we get (i) and (ii).

COROLLARY 5.20. $\langle N_G(B), N_G(B^*) \rangle \leqslant N_G(G_0)$.

Proof. Let $g \in N_G(B)$ or $N_G(B^*)$. If an element of the coset $C_G(B)\, N_{G_0}(B)\, g$ or $C_G(B^*)\, N_{G_0}(B^*)\, g$ leaves invariant each of the z_i, then the entire coset lies in $N_G(G_0)$, as $G_0 = \langle K_1, ..., K_r \rangle$. For $g \in N_G(B^*)$, this does happen with the exception of Corollary 5.19. Let us consider those two cases.

Assume $A^* \cong W_{D_n}$. Then G_0 has type $D_n(q)$, ${}^2D_n(q)$, $n \geqslant 4$. Since $N_G(B^*)$ preserves every $N_{G_0}(B^*)$—class of subgroup $\langle b \rangle$ of order p in B^* in which $L(C_{G_0}(b))$ is quasisimple, we get $N_G(B^*) \leqslant N_G(G_0)$ because $g \in N_G(B^*) \leqslant C_G(B^*)\, G_0$.

Assume $A^* \cong W_{F_4}$. Say $g \in N_G(B^*)$ induces a graph automorphism of order 2, normalizing W, the standard copy of $A_{G_0}(B^*)$. If g normalizes a standard subcomponent, we are done as above. So, we may assume that L_0 has type $A_2(q)$. But it is clear from studying components that G_0 is the "G_0" for L_0^g, whence $g \in N_G(G_0)$, as required.

Finally, we turn to the case $g \in N_G(B)$, $B < B^*$. The definition of G_0 implies that $C_G(B) \leqslant N_G(G_0)$. The structure of $C_G(z_i)$ implies that if $P \in \mathrm{Syl}_p(C_G(B))$, then $P_1 = \Omega_1(P)$ contains B^* and if $B^* < P_1$, then $P_1 - B^*$ contains an element inducing a field automorphism on each K_i. The existence of such an element would contradict the definition of standard type. Thus $B^* = P_1$, and so a Frattini argument implies that $N_G(B) \leqslant C_G(B)\, N_G(B^*) \leqslant N_G(G_0)$. The proof is now complete.

We summarize the main result of this section.

PROPOSITION 5.21. $G_0 := \langle K_1, K_2, ..., K_r \rangle \in \mathrm{Chev}(2)$.

6. $G_0 = G$

We now know that the following hypotheses are valid.

I. G is a simple K-group of characteristic 2–type.

II. $B \cong E_{p^n}$, $n \geqslant 4$, and B realizes the 2-local p-rank of G.

III. $B \subseteq B^*$ with $m(B^*) = m_p(G)$.

IV. For some $x \in B^\#$, G is of standard type (as defined in Section 1) with respect to (B, x, L).

V. For some $D \subseteq B$ and standard subcomponent (D, J) the set of neighbors of (B, x, L) with respect to (D, J) together with L generates a group G_0 of Lie type over a field of characteristic 2. The possibilities for G_0 are listed as G_0 or G_1 in Table C of Section 2.

VI. B^* acts as inner-diagonal automorphisms on G_0.

VII. $\langle N_G(B), N_G(B^*)\rangle \subseteq N_G(G_0)$.

In this section we prove

PROPOSITION 6.1. $G_0 = G$

Notice that Hypotheses I–IV appear in Section 1, and Hypotheses V and VII are Proposition 5.21 and Corollary 5.20, respectively. Hypothesis VI follows from Corollary 4.2. Note that if $G_0 = D_4(q)$ and $p = 3$, no $b \in B^*$ can act as a graph or nonstandard field automorphism because the p-rank of the centralizer of b would be too small.

We fix a choice of B, B^*, (B, x, L), and (D, J); and we define $M = N_G(G_0)$. Our initial goal (which we attain by proving Lemmas 6.9 and 6.13) is to show that M controls strong fusion of D in G.

LEMMA 6.2. *$C_G(D) \subseteq M$ and $J = L(C_G(D))$.*

Proof. Since $C_G(D)$ normalizes J, it normalizes L and every neighbor. Hence $C_G(D)$ normalizes G_0. The second assertion follows from the first together with $J = L(C_{G_0}(D))$.

LEMMA 6.3. *The following conditions hold*:

(i) $p \nmid |C_G(G_0)|$;

(ii) $2 \nmid |C_G(G_0)|$;

(iii) *No element of $N_G(B^*)$ induces a transvection on B^*.*

Proof. From the definition of standard type $C_G(L)$ has cyclic Sylow p-subgroups. Since $\langle x\rangle$ acts nontrivially on each neighbor, $\langle x\rangle$ acts nontrivially on G_0 and (i) holds.

B^* acts on $C_G(G_0)$ and by (i) B^* normalizes some $T \in \mathrm{Syl}_2(C_G(G_0))$. Assume $T \neq 1$. By Hypothesis I, G_0 acts nontrivially on $Q = O_2(N_G(T))$. Thus for some $d \in D^{\#}$, J acts nontrivially on $C_Q(d)$. If (B, d, K) is a neighbor, then $J \subseteq K \subseteq G_0$ forces K to act nontrivially on $C_Q(d)$ contrary to $K \lhd \lhd C_G(d)$. Similarly $L \lhd \lhd C_G(x)$ forces $d \notin \langle x \rangle$. The remaining possibility is that J covers $F/O_{p'}(F)$ for some p-component F of $C_G(d)$. But then $[J, C_Q(d)] \subseteq O_{p'}(F)$ forcing $|[J, O_{p'}(C_G(d))]|$ even and contradicting the definition of standard type.

Finally suppose $a \in N_G(B^*)$ induces a transvection on B^*. By Proposition 4.1 and Hypothesis VII we have $O_p(N_M(B^*)/C_M(B^*)) = 1$. Let A be the normal closure of a in $N_M(B^*)$. By a result of McLaughlin [47] the image of A in $\mathrm{Aut}(B^*)$ is a product of linear and symplectic groups, but Table B supplies a contradiction.

For any $d \in D^{\#}$ we define

$$K_d = \langle J^{L(C_{G_0}(d))} \rangle.$$

Of course $K_d = J$ or $K_d = L$ or (B, d, K_d) is a neighbor of (B, x, L).

LEMMA 6.4. *$|Z(J)|$ is odd, and for all $d \in D^{\#}$, $|Z(K_d)|$ is odd.*

Proof. Since $|Z(G_0)|$ is odd, Lemma 2.22 gives the desired conclusion.

LEMMA 6.5. $K_d \lhd \lhd C_G(d)$.

Proof. If not, then by definition of standard type $K_d = J$ and lies in a p-component A of $C_G(d)$ with $A = JO_{p'}(A)$ and $[J, O_{p'}(A)] \neq 1$.

Choose $R = Z(X_\alpha)$ for some root group X_α of J with α long if J is any group whose root system has roots of two lengths. By Lemma 2.6, $N_J(R)$ is a parabolic subgroup of J. $N_J(R)$ is a maximal parabolic except when $J/Z(J) = A_n(q)$.

Choose $r \in R^{\#}$ and let $E = O_{p'}(A)$, $F = C_E(r)$, $P = O_2(C_G(r))$, $Q = O_2(C_J(r))$.

Suppose $|R| \geqslant 4$. We claim $R \subseteq P$. It will follow that $[R, F] \subseteq F \cap P = 1$ whence $C_E(s) = C_E(r)$ for $s \in R^{\#}$ and $E = \langle C_E(s) | s \in R^{\#} \rangle = F$. But then $J = [J, r]$ centralizes E as desired.

Since G is of characteristic two type, $R \subseteq P$ will follow from $[R, P] = 1$ which in turn will follow from $[R, C_P(e)] = 1$ for all $e \in D^{\#}$. Let A_e be the p-component of $C_G(e)$ containing K_e. As $C_P(e)$ centralizes $r \in R \subseteq K_e$, $C_p(e)$ acts on A_e. By Lemma 2.11(v), $[R, C_P(e)]$ centralizes $A_e/O_{p'}(A_e)$ whence $[R, C_P(e)] \subseteq O_{p'}(A_e)$. Let $Y = [A_e, O_{p'}(A_e)]$. As $A_e = K_e Y$ and A_e has no proper normal subgroups covering A_e/Y, $Y = [K_e, O_{p'}(A_e)]$ whence $|Y|$ is odd by definition of standard type. By Lemma 6.4, $|O_{p',p}(A_e)/Y|$ is odd, so

$|O_{p'}(A_e)|$ is odd. But $[R, C_p(e)]$ is a 2-subgroup of P, so $[R, C_p(e)] = 1$ as desired.

We may assume $|R| = 2$. In particular Lemma 2.38 implies that for each $e \in D^{\#}$, R is the center of a root group X_β of K_e with β long for all twisted groups. We define $Q_e = O_2(N_{K_e}(R))$ and $P_e = C_p(e)$.

We claim $Q \subseteq P$. It suffices to show that for every $e \in D^{\#}$ either $Q_e \subseteq P_e$ or $[Q_e, P_e] \subseteq R$. Indeed $Q \subseteq Q_e$ by Lemma 2.23 so assume $[Q_e, P_e] \subseteq R$ for all $e \in D^{\#}$. It follows that $[Q, P] \subseteq R$ whence $Q \subseteq P$ by definition of P.

We will show that the desired condition holds. $R \subseteq Q_e \cap P_e$, so when Q_e/R is an irreducible $N_{K_e}(R)$-module either $Q_e \cap P_e = Q_e$ or $R \supseteq Q_e \cap P_e \supseteq [Q_e, P_e]$. In the contrary case we have, by Lemma 2.13, $K_e/Z(K_e) = A_n(2)$, $F_4(2)$ or $K_e/Z(K_e) = C_n(2)$ with $R = X_\beta$, β short. A check of Table C shows that $F_4(2)$ does not occur and $p = 3$ in all cases. As $Q_e \cap P_e \lhd N_{K_e}(R)$, Lemmas 2.14 and 2.15 determine $Q_e \cap P_e$. Of course we may assume $R \subset Q_e \cap P_e \subset Q_e$.

Let $T_e = Q_e P_e$. When $K_e/Z(K_e) = C_n(2)$, we have $n \geqslant 3$ by Table B (as $m(B) \geqslant 4$) whence $T_e = Q_e C_{T_e}(A_e/O_{p'}(A_e))$ by Lemma 2.11(iv). By part (i) of the same lemma $[Q_e, P_e] \subseteq RO_{p'}(A_e)$. As we have seen above $|O_{p'}(A_e)|$ is odd; and it follows that $[Q_e, P_e] \subseteq R$. When $K_e/Z(K_e) = A_n(2)$ the same argument works except possibly in the case $n = 3$ when P_e might not act as inner automorphisms on $A_e/O_{p'}(A_e)$. However in this case some element of P_e acts as a graph automorphism. By Lemma 2.15 there are two $N_{K_e}(R)$-invariant subgroups U with $R \subset U \subset Q_e$, and it is easy to check that they are interchanged by a graph automorphism normalizing Q_e. Thus $P_e \cap Q_e \lhd P_e N_{K_e}(R)$ forces $P_e \cap Q_e = R$ or Q_e contrary to the assumption above.

We have shown $Q \subseteq P$ in all cases. Suppose Q contains R^g for some $g \in J - N_J(R)$. $[R^g, F] \subseteq P \cap F = 1$ implies $F = C_E(R^g)$. When $N_J(R)$ is a maximal parabolic, $J = \langle N_J(R), g \rangle$ normalizes F, and it follows that r inverts or centralizes any section of F on which J acts irreducibly. Consequently $J = [J, r]$ centralizes F. When $N_J(R)$ is not a maximal parabolic, $J/Z(J) = A_n(2)$ and it is easy to check that every involution in Q is conjugate in J to r whence $[Q, F] = 1$ which forces $F = E$ and $[J, E] = 1$ as above.

When a root system Σ of J has a root γ of the same length as α but not orthogonal to α, then we may take $(\alpha, \gamma) > 0$ and $R^g = Z(X_\gamma)$. We are left with the cases $J/Z(J) = {}^2A_n(2)$, $n \geqslant 3$, and $J/Z(J) = C_n(2)$, $n \geqslant 3$ and α long. In these cases all roots in Σ of the same length as α are orthogonal to α. Pick a root β with $(\alpha, \beta) > 0$ and $X_\beta \subseteq Q$. Let

$$J_0 = \langle X_\alpha, X_{-\alpha}, X_\beta, X_{-\beta} \rangle.$$

J_0 has a root system Σ_0 of type C_2 and $J_0/Z(J_0) = C_2(2)$, ${}^2A_3(2)$, or ${}^2A_4(2)$. In any case since $Q \subseteq P$ forces $[Q, F] \subseteq F \cap P = 1$, $C_E(X_\beta) \supseteq F$. Likewise if

γ is the other root of Σ_0 with $(\alpha, \gamma) > 0$ and β and γ of the same length, then $C_E(X_\gamma) \supseteq F$. A reflection of the Weyl group of Σ_0 corresponding to the roots orthogonal to α moves $\{\beta, \gamma\}$ to $\{-\beta, -\gamma\}$. By Lemma 2.6 there exists $g \in N_{J_0}(R)$ with $\{X_\beta^g, X_\gamma^g\} = \{X_{-\beta}, X_{-\gamma}\}$. Consequently

$$J_0 = \langle X_\alpha, X_\beta, X_\gamma, X_{-\beta}, X_{-\gamma} \rangle$$

centralizes F. By the argument above J_0 centralizes E whence J centralizes E too, and the proof of the lemma is complete.

LEMMA 6.6. $C_G(J) \subseteq M$.

We prove a preliminary lemma first. Choose $R = Z(X_\alpha)$, X_α a root group of J, with α long.

Define

$$P = O_2(N_G(R)), \qquad Q = O_2(N_{G_0}(R)), \qquad S = N_P(Q),$$
$$J_0 = \langle R, Z(X_{-\alpha}) \rangle.$$

Note $J_0 \subseteq J$.

LEMMA 6.7. *One of the following holds*:

(i) $S = \langle S \cap Q, C_S(J) \rangle$;

(ii) $J = A_3(2)$;

(iii) $G_0 = E_6(2)$, $L = A_5(2)$, $J = A_2(4)$, $p = 3$.

Proof. By the preceding lemma K_d is a component of $C_G(d)$. As $C_S(d)$ normalizes R, $C_S(d)$ acts on K_d. Apply Lemmas 2.38 and 2.11(iv), 2.40, 2.43, 2.12, to deduce that either (ii) holds or $C_S(d) \subseteq K_d(C_G(\langle K_d, d \rangle))$ for all $d \in D^\#$, or we are in one of the cases (*) of Lemma 2.38. In these cases either (iii) holds or Lemma 2.39 applies. Thus we may assume $C_S(d) \subseteq K_d C_G(\langle K_d, d \rangle)$. The lemmas just mentioned assert that $N_{K_d}(R)$ has no central factors on $O_2(N_{K_d}(R)/R)$. As $R \subseteq K_d \cap S$, we have $C_S(d) \subseteq \langle S \cap K_d, C_S(\langle K_d, d \rangle) \rangle$. Since $J \subseteq K_d \subseteq G_0$, we have $S \cap K_d \subseteq O_2(N_G(R)) \cap G_0 = Q$ and $C_S(K_d) \subseteq C_S(J)$; and (i) holds.

We proceed to the proof of Lemma 6.6. In Lemma 6.7(ii, iii), D is conjugate in G_0 to $B \cap J$ by Lemmas 2.7 and 2.8. Thus by Lemma 6.2 we may assume that Lemma 6.7(i) holds, and (ii) and (iii) do not hold. From Lemma 2.38, 2.11(iii) and 2.12(iii) or by Lemma 2.39(iii) we have $G_0 = \langle Q, J_0 \rangle$. Thus $C_S(J)$ acts on G_0, and by Lemma 6.7(i) S acts on G_0. Now $S \subseteq QC_G(G_0)$; and as $|C_G(G_0)|$ is odd by Lemma 6.3, $S \subseteq Q$.

Since $S = N_P(Q)$, we have $P \subseteq Q$. If $P = Q$, then $[R, C_G(J)] = 1$ implies that $C_G(J)$ acts on $\langle Q, J_0 \rangle = G_0$ and Lemma 6.6 holds. In the contrary case

$Z(Q) \subseteq C_G(P) \subseteq P$ forces $Z(Q) \subset P \subset Q$. By Lemmas 2.13, 2.14, 2.15, we have $G_0 = F_4(q)$ or $G_0 = A_n(q), p \mid q-1$, or $A_n(2)$, $p = 3$ or 7. In the first two cases $[Q, Q] \subseteq R$ by Lemma 2.11(i) whence Q centralizes P/R and R which forces $Q \subseteq P$. In the second case $\langle J, P \rangle = G_0$ by Lemma 2.39(iii) and $C_G(J)$ acts on G_0. Lemma 6.6 is proved.

LEMMA 6.8. *The following conditions hold*:

(i) $L(C_G(d)) \subseteq G_0$ *for* $d \in D^{\#}$;

(ii) *D normalizes every component of* $C_G(d)$;

(iii) *every D-signalizer lies in M.*

Proof. Assertion (i) follows from Lemmas 6.5 and 6.6. If (ii) fails, then looking in $C_G(d)$ we find a component of $C_G(D)$ distinct from J contrary to Lemma 6.2.

To prove (iii) let Q be D-invariant of order prime to p. We may assume that Q is an r-group for some prime $r \neq p$. By Lemma 6.2 we may assume $Q = [Q, D]$.

We claim $Q = \langle [C_Q(d), D] \mid d \in D^{\#} \rangle$. Indeed let $P = \langle [C_Q(d), D] \mid d \in D^{\#} \rangle$. If $P \neq Q$, we can find R such that $P \subseteq R \subset Q$, $R \lhd QD$, and Q/R is an irreducible D-module. As $Q = [Q, D], Q/R$ is not a trivial D module; but then $C_D(Q/R) = \langle e \rangle$ and $[C_Q(e), D] \subseteq P$ covers Q/R, a contradiction.

It suffices to show $Q_d = [C_Q(d), D]$ lies in M. By (ii), $Q_d = [Q_d, D]$ normalizes K_d. It follows that Q_d acts as inner automorphisms on K_d whence $Q_d \subseteq K_d C_G(K_d) \subseteq M$ by Lemmas 6.5 and 6.6.

LEMMA 6.9. (i) *If* $(JD)^g \subseteq M$, *then* $g \in M$;

(ii) $N_G(K_d) \subseteq M$ *for* $d \in D^{\#}$;

(iii) $C_G(d) \subseteq M$ *for* $d \in D^{\#}$;

(iv) $C_G(G_0) = 1$.

Proof. For (i) let $K = J^g$, $E = D^g$. It follows from Lemma 6.3 that $K \subseteq G_0$. Now $E = E_{p^2}$ acts on G_0 and centralizes a nontrivial 2-group in K. By the result of Borel and Tits [7] or [9] E normalizes a maximal parabolic subgroup of G_0. The proof of [52, (2.3)] shows that G_0 is generated by 2 E-signalizers. By Lemma 6.8(iii), $G_0 \subseteq M^g$, and it follows easily that $G_0 = (G_0)^g$ as desired.

For (ii) we note that $JD \subseteq B^* K_d C_G(K_d) \lhd N_G(K_d)$ and $B^* K_d C_G(K_d) \subseteq M$ by Lemmas 6.5 and 6.6. Now (i) yields (ii).

Let V be the subgroup of $C_G(d)$ which normalizes all components of $C_G(d)$. From Lemma 6.8(ii), $JD \subseteq V \lhd C_G(d)$. From (i) and (ii) we get (iii).

To prove (iv) pick $T \in \mathrm{Syl}_2(J)$ and let $N = N_G(T)$, $Q = O_2(N)$. The action of D forces $Q \subseteq M$. By Lemma 6.3, $X = C_G(G_0)$ has odd order. Since

$T \subseteq G_0, X \subseteq N$; and since $X \lhd M$, we have $[Q, X] \subseteq Q \cap X = 1$. As G is of characteristic 2-type, we conclude $X = 1$.

Choose $P \in \mathrm{Syl}_p(M)$ with $B^* \subseteq P$. We will analyze fusion in P.

LEMMA 6.10. *The following conditions hold*:

(i) *B^* is the unique elementary abelian subgroup of its rank of P*;

(ii) $P \in \mathrm{Syl}_p(G)$;

(iii) *any two elements of B^* which are conjugate in G are conjugate in* $N_G(B^*)$;

(iv) *No element $y \in P$ induces a field automorphism on G_0 unless* $p = 3$, $G_0 = D_4(q)$, $C_{G_0}(y) = {}^3D_4(q^{1/3})$.

Proof. Suppose $G_0 \neq D_4(q)$. If any $y \in P$ induces a field automorphism of order p on G_0, then as $p \nmid |C_G(G_0)|$, we may assume $|y| = p$. By Lemma 2.45(ii) we may choose y to centralize B^*. But now $y \in B^*$ contrary to Hypothesis VI at the beginning of this section. We conclude that P acts as inner $\cdot$ diagonal automorphisms on G_0 whence by Lemma 2.35, B^* is the unique elementary abelian subgroup of its rank in P. Clearly (i) implies (ii) and (iii).

If $G_0 = D_4(q)$, $p = 3$, and some $y \in P$ induces an outer automorphism on G_0, then the argument above convinces us that we may choose y so that $|y| = 3$, and for any such choice either y induces a graph automorphism or y induces a field automorphism with $C_{G_0}(y) = {}^3D_4(r)$, $r^3 = q$. In any event $C_{G_0}(y)$ has 3-rank 2 by Lemma 2.45(iii) and any elementary abelian subgroup $E \subseteq P$ with $m(E) = m(B^*) = 4$ acts as inner-diagonal automorphisms on G_0. Apply Lemma 2.35 again.

LEMMA 6.11. *If $b \in B^*$, $C_G(b) \subseteq M$ and $y = b^g \in M$, then one of the following holds*:

(i) $g \in M$;

(ii) $G_0 = A_{p-1}(q), p \mid q-1, p \geqslant 5$;

(iii) $G_0 = {}^2A_{p-1}(q), p \mid q+1, p \geqslant 5$;

(iv) $G_0 = D_4(q)$, $p = 3$, $p \mid q-1$, *and y acts on G_0 as a graph automorphism with $L(C_{G_0}(y)) \cong A_2(q)$*.

Further if (i) *does not hold, then y is not conjugate in M to any element of B^*.*

Proof. First suppose $y^m \in B^*$ for some $m \in M$. Since $N_G(B^*) \subseteq M$, Lemma 6.10(iii) ensures that we may choose m so that $y^m = b$ whence $gm \in C_G(b) \subseteq M$ and (i) holds.

Now we assume that (i) fails and show that one of (ii)–(iv) holds. Without loss of generality $\langle y, b\rangle \subseteq P$. Also y is not fused in M to an element of B^*. Using Lemma 6.10(iv) we see that the possibilities for G_0 and y are listed in Lemma 2.45(iii). We choose g so that $(C_M(y))^{g^{-1}} \subseteq C_M(b) = C_G(b)$. Since $C_M(b)/L(C_M(b))$ is solvable, $|L(C_{G_0}(y))\colon Z(L(C_{G_0}(y))|$ divides $|K\colon Z(K)|$ for some component K of $C_{G_0}(b)$. Consider the set of all components of $C_{G_0}(e)$ as e ranges over $(B^*)^{\#}$; the same divisibility condition holds if we take K to be an element of this set which is maximal with respect to inclusion. Apply Lemmas 2.25 and 2.26 and conclude that one of (ii)–(iv) holds. Note that $m_{2,p}(M) \geqslant 4$ rules out the analog of (iv)with $3|q+1$ and $L(C_{G_0}(y)) = {}^2A_2(q)$. Likewise p must be at least 5 in (ii) and (iii).

LEMMA 6.12. *Suppose conclusion* (*i*) *of Lemma* 6.11 *fails and* (ii) *or* (iii) *holds; then* $m(B^*) = p-1$ *and* $O^{p'}(C_{G_0}(b))$ *does not have any summands* $A_k(q)$ *or* ${}^2A_k(q)$ *with* $k \geqslant 2$.

Proof. Take $\varepsilon = 1$ if $G_0 = A_n(q)$, and $\varepsilon = -1$ if $G_0 = {}^2A_n(q)$. By Lemma 6.10, P induces inner · diagonal automorphisms on G_0. Assuming $y = b^g \in P$, we have $P = \langle y\rangle T$ where $T = C_P(B^*)$ is abelian. Otherwise $y \in \Omega_1(T) = B^*$ and $y^r = b$ for some $r \in N_G(B^*) \subseteq M$ contrary to Lemma 6.10.

Let M_0 be the subgroup of M inducing inner-diagonal automorphisms on G_0. $C_G(G_0) = 1$ by Lemma 6.9(iv) whence M_0 is isomorphic to a subgroup of $PGL(n, q)$ or $PSU(n, q)$. Take the usual matrix representations (i.e., matrices we determined up to scalar multiplication) for these groups with the Hermitian form represented by the identity matrix in the second case. For any $m \in M_0$ let $\mathscr{M}(m)$ be the matrix representing m. Arrange things so that $\mathscr{M}(t)$ is diagonal for all $t \in T$ and $\mathscr{M}(y)$ is monomial. Suppose $m \in M_0$ is fused in M to $l \in M_0$ and $\mathscr{M}(m)$ has eigenvalues λ_i, $1 \leqslant i \leqslant p$. There exists a scalar μ and an integer v relatively prime to the order of each the eigenvalues such that the eigenvalues of $\mathscr{M}(l)$ are $\mu(\lambda_i)^v$, $1 \leqslant i \leqslant p$. In particular if the eigenvalues of $\mathscr{M}(m)$ are distinct, so are those of $\mathscr{M}(l)$.

As y is not fused in M to b, the standard module is an irreducible $\langle \mathscr{M}(y)\rangle$-module. Multiplying $\mathscr{M}(y)$ by a scalar if necessary so that $\langle \mathscr{M}(y)\rangle$ is a p-group, we have $(\mathscr{M}(y))^p = \lambda \mathscr{I}$ where $\mathscr{I}$ is the identity matrix, λ is a primitive p^a-root of unity, and $p^a | q-\varepsilon$. Thus the determinant of $\mathscr{M}(y)$ is a primitive p^a-root of unity and y induces an outer-diagonal automorphism on G_0. Since $B^* = \Omega_1(C_P(B^*))$ by Lemma 6.10, it follows that $m(B^*) = p-1$ and we have proved the first part of the lemma.

Assume $O^{p'}(C_{G_0}(b))$ has one of the forbidden summands; it suffices to reach a contradiction. The fusion of y to b can be carried out in steps by means of a conjugation family. Consider the first point at which an element whose matrix acts irreducibly on the standard module is fused to one whose matrix acts reducibly. (Since $P \subseteq M_0$, every element of P has a matrix representation.) Replacing y by an appropriate G-conjugate if necessary, we may assume this point occurs at the first step. Thus there exists $Q \subseteq P$ with the following properties:

$$y \in Q \subseteq P;$$

$$y \text{ is fused in } N_G(Q) \text{ to } e;$$

$$\mathscr{M}(e) \text{ acts reducibly on the standard module.}$$

It follows from the action of $\mathscr{M}(e)$ that e is fused in M to B^*. As e is fused in G to b, Lemma 6.11 implies that e is fused in M to b. Further $\mathscr{M}(e)$ is diagonalizable and we may choose it so that

$$\mathscr{M}(e)^p = \mathscr{I}.$$

If $e \in P - T$, then $\mathscr{M}(e)$ is monomial and $\mathscr{M}(e)^p = \mathscr{I}$ implies that the product of the nonzero entries is 1. We see that the characteristic polynomial of $\mathscr{M}(e)$ is $x^p + 1 = x^p - 1$ whence the eigenvalues of $\mathscr{M}(e)$ are the p distinct pth roots of unity. Considering the summands of $O^{p'}(C_{G_0}(b))$ we see that $\mathscr{M}(b)$ has three identical eigenvalues. Thus e cannot be fused in M to b. We conclude $e \in T$.

Now $|Q: C_Q(e)| \leqslant |P: T| = p$ implies $|Q: C_Q(y)| \leqslant p$ whence $[e, y, y] = 1$. $\mathscr{M}(y)$ is the product of a permutation matrix $\mathscr{P}$ and a diagonal matrix whence

$$[\mathscr{M}(e), \mathscr{P}, \mathscr{P}, \mathscr{P}] = \mathscr{I}.$$

We may assume that conjugation by $\mathscr{P}$ moves each diagonal entry of $\mathscr{M}(e)$ into the next and the last into the first. Picking an appropriate root of unity λ and letting the diagonal entries of $\mathscr{M}(e)$ be

$$\lambda^{i_1}, \ldots, \lambda^{i_p}$$

we find that the commutator condition above amounts to a difference equation of degree 3 for the λ_i's. We must have

$$i_j = kj^2 + lj + m$$

for some integers k, l, m. We know from the fact that e is fused in M to b that for three values of j, i_j is the same. Since i_j is given by a polynomial of degree 2, i_j must be constant whence $e = 1$. But then $b = 1$, $G = C_G(b) \subseteq M$ and Lemma 6.11(i) holds.

The required fusion cannot occur and Lemma 6.12 is valid.

LEMMA 6.13. *If $b \in B^*$, $C_G(b) \subseteq M$ and $y = b^g \in M$, then either $g \in M$ or the following conditions hold*:

(i) $G_0 = A_{p-1}(q)$ *and* $p \mid q-1$ *or* $G_0 = {}^2A_{p-1}(q)$ *and* $p \mid q+1$;

(ii) $p \geqslant 5$;

(iii) $m(B^*) = p-1$;

(iv) $O^{p'}(C_{G_0}(b))$ *has no summands* $A_k(q)$ *or* ${}^2A_k(q)$, $k \geqslant 2$.

In particular if $b \in D$, *then* $g \in M$.

Proof. Notice that if $b \in D$, then $C_G(b) \subseteq M$ by Lemma 6.9. If conditions (i)–(iii) hold, then $J = O^{p'}(C_{G_0}(D)) = A_k(q)$ or ${}^2A_k(q)$ with $k \geqslant 2$. Hence the first part of Lemma 6.13 implies the last assertion. Thus Lemma 6.13 follows from the preceding two lemmas once we rule out the possibility that Lemma 6.11(iv) holds.

Suppose Lemma 6.11(iv) holds. In particular $\langle y \rangle$ acts on G_0 as a graph automorphism of order 3. If M contained elements acting as field or graph-field automorphisms of order 3, then P would contain an element acting as a standard field automorphism contrary to Lemma 6.10(iv). Thus $P = \langle y \rangle (P \cap G_0)$.

Three of the five G_0-classes of elements of order 3 are fused in M. The centralizers in G_0 of the three M-classes are

(1) $A_3(q) \times Z_{q-1}$,

(2) $A_1(q) \times A_1(q) \times A_1(q) \times Z_{q-1}$,

(3) $GL(3, q) \times Z_{q-1}$.

The last centralizer is evident inside the first one. Classes (1) and (3) are 3-central in G_0 and are also the classes appearing in $D^{\#}$. Thus the centralizers in G of elements in these classes lie in M. As class (1) splits into 3 G_0-classes, every element of order 3 which is 3-central in M lies in class (3).

Analyzing the fusion of y to b in G, we may assume $y \in Q \subseteq P$ with $C_P(Q) \subseteq Q$ and $N_G(Q) \not\subseteq M$. There exists an element $z \in Q \cap Z(P)$ with $|z| = 3$ and $C_G(z) \subseteq M$. Applying Lemma 6.11 to z, we see that we may assume z is fused to y in $N_G(Q)$. Thus $C_M(y)$ is isomorphic to a subgroup of $C_G(z) = C_M(z)$. Since $C_{G_0}(y)$ has a section isomorphic to $A_2(q)$, we must have $L(C_{G_0}(y)) = L(C_G(y)) \cong SL(3, q)$. But then $y^n = z$ implies that n conjugates $Z(L(C_{G_0}(y)) = \langle w \rangle$ to $\langle z \rangle = Z(L(C_{G_0}(z)))$. As all elements of order 3 in P which lie in the commutator subgroup of Sylow 3-subgroups of their centralizers in G_0 lie in class (3), $\langle w \rangle$ is fused in G_0 to $\langle z \rangle$ whence $n \in G_0 C_G(z) \subseteq M$ which is impossible.

We have now reached our initial goal: M controls strong fusion of D in G. Before beginning the final phase of the proof of Proposition 6.1 we wish to control strong fusion of other elements of B^*.

LEMMA 6.14. *If* $b \in B^*$ *and* $C_{G_0}(b)$ *has a component* L *with* $m(C_{B^*}(L)) \leqslant 2$, *then* L *is subnormal in* $C_G(b)$.

Proof. If $G_0 = A_n(q)$, $p \mid q-1$, or $G_0 = {}^2A_n(q)$, $p \mid q+1$, then b is fused by $N_{G_0}(B^*)$ to an element of D. M controls strong fusion of b in G by Lemma 6.13 whence $C_G(b) \subseteq M$ and $L \lhd \lhd C_G(b)$.

Otherwise let $N = C_G(b)$, $T = C_M(b)$; if E is any M-conjugate of D lying in T, then T controls strong fusion of E in N. By Lemma 3.14 every component L_1 of T lies in a component K_1 of N. Let L lie in the component K of N. As we may assume $K \not\subseteq T$, K is E-invariant by Lemma 3.13. We will find a configuration satisfying Hypothesis 3.16 inside $\mathrm{Aut}(K)$.

Our conditions imply that $L(T)$ is the central product of the groups $L(T \cap K_1)$ as K_1 ranges over the components of N. Let $H = E(K \cap T)$; we have that $L(H) = L(K \cap T)$ is a product of components of Lie type over a field of characteristic 2. By Proposition 2.22, $|Z(L)|$ is odd. Further from the structure of T we know that E acts on each component of $L(H)$ as inner-diagonal automorphisms and $H/L(H)$ is solvable. Similarly since $C_K(e) \subseteq K \cap T$ for $e \in E^{\#}$, we see that $L(C_K(e))$ is a product of components of Lie type over fields of characteristic 2.

Let $V = C_H(K)$. As $V/V \cap K$ is isomorphic to a subgroup of E and $[V, V \cap K] \subseteq [V, K] = 1$, V is nilpotent. E acts on V and $V \subseteq T$ by Lemma 3.13. Let $W = EK$ and $\bar{W} = W/V$; by Lemma 3.11, $\bar{H}$ controls strong fusion of $\bar{E}$ in $\bar{W}$. All the conditions of the preceding paragraph carry over to $\bar{W}$, and to check that Hypothesis 3.16 holds (with $\bar{W}$ and $\bar{H}$ in place of G and H) it suffices to show that $O_2(\bar{H}) = 1$.

Let $P/V = O_2(\bar{W})$ and $Q = O^{2'}(P)$. Q covers P/V; and as $|W : K|$ is odd, $Q \subseteq K$. Thus $[Q, V] = 1$ whence $Q \subseteq O_2(P)$. We have $Q \subseteq O_2(K \cap T) \lhd \lhd T$.

We need only show $R = O_2(T) = 1$. By Lemma 6.9, M acts faithfully on G_0. By the structure of T, $R \cap G_0 = 1$; and it follows that R cannot acts as inner · diagonal automorphisms on G_0. As $G_0B^* \lhd M$, $[B^*, R] \subseteq [T \cap G_0B^*, R] \subseteq G_0B^* \cap R = 1$. Now check in [52, (8.9), (8.10), §19] that for all choices of G_0 and for any involution $r \in R$ either $O_2(C_{G_0}(r)) = 1$ or $m_p(C_{G_0}B^*(r)) < m(B^*)$.

Lemmas 3.17, 3.18, and 3.22 yield the following possibilities (as $L(\bar{H} \cap \bar{K}) = 1$).

$K/Z(K)$	L
$A_{5s}, p = 5, s = 2, 3, 4$	A_5
$C_4(2), p = 3$	$D_4(2)$
$C_3(2), p = 3$	${}^2A_3(2)$
$A_2(4), p = 3$	A_6
$F_{22}, p = 5$	$D_4(2)$

In the first case $|C_{B^*}(L)\leqslant p^2$ forces $m(B^*)\leqslant 3$, not the case. In the last case G_0 is defined over a field of order q with $p=5$ dividing $q\pm 1$. In particular $q>2$ and as b acts on G_0 as an inner · diagonal automorphism, L is defined over a field of order a power of q which contradicts $L=D_4(2)$. Likewise an examination of the possibilities for G_0 reveals that $L=A_6$ does not occur. In the other cases by examining the possibilities for G_0 we find that we can choose E so that $C_J(E)$ contains a subgroup isomorphic to $A_1(2)$. However the lemmas listed above guarantee that $C_{\bar{K}}(\bar{E})$ is a p-group; and this contradiction completes the proof of the lemma.

LEMMA 6.15. *Suppose $b\in B^*$ and $C_{G_0}(b)$ has a component L with $m(C_{B^*}(L))\leqslant 2$, then M controls strong fusion of b in G.*

Proof. As in the preceding proof we may assume that we do not have $G_0=A_n(q), p|q-1$ or $G_0={}^2A_n(q)$. $p|q+1$. By Lemma 6.13 it suffices to show $C_G(b)\subseteq M$.

Let $E=C_{B^*}(L)$ and $A=B^*\cap L$. Our conditions imply $|B^*:AE|\leqslant p$ with equality unless some element of B^* induces an outer · diagonal automorphism of L. From the preceding lemma $\langle b\rangle \lhd\lhd C_G(b)$. Thus if $D\cap\langle b\rangle L\neq 1$, then $C_G(b)\subseteq M$ as desired. In the contrary case $|E|=p^2$, $AE\neq B^*$, and $B^*=AE\langle d\rangle$ for some $d\in D^{\#}$.

Let $X=C_G(b)\cap C_G(L)$. By Lemma 2.22, $p\nmid|Z(L)|$. Thus $m_p(LX)\leqslant m_p(B^*)$ implies $m_p(X)\leqslant 3$. In fact since some element in B^* induces an outer automorphism on L, the uniqueness of B^* (Lemma 6.10(i)) implies $m_p(LX)<m_p(B^*)$ whence $m_p(X)=2$. We claim that if $Y\lhd X$ and $E\nsubseteq Y$, then $m_p(Y)=1$ and Y has a normal p-component. The second assertion follows from the first as $b\in Z(Y)$, so assume $m_p(Y)\geqslant 2$. Pick a B^*-invariant subgroup $F\subseteq Y$, $F\cong E_{p^2}$ with $\langle b\rangle\subseteq F$. $F\cap B^*\subseteq C_{B^*}(L)=E$, so $F\cap B^*=\langle b\rangle$. It follows that $[B^*,F]\subseteq\langle b\rangle$. As B^* contains all elements of order p in $C_G(B^*)$, F acts as a transvection on B^* contrary to Lemma 6.3.

Take Y to be the largest normal subgroup of X lying in M. Suppose $E\subseteq Y$. For some $f\in E^{\#}$ $\langle f\rangle=C_{B^*}(L_1)$ where L_1 is a component of $C_{G_0}(f)$ containing L. Thus M controls fusion of f in G and $YL\lhd\lhd C_G(b)$ gives $C_G(b)\subseteq G$. We may assume $b\in Y$ but $E\nsubseteq Y$.

By Lemma 3.13(i) Y contains every p-solvable normal subgroup of X. Thus $L(X/Y)\neq 1$, and it follows from the structure of Y that $K=L_{p'}(X)\neq 1$. Every p-component K_1 of K contains an element of order p in $K_1-O_{p',p}(K_1)$ lest K_1 have a normal p-complement by a theorem of Frobenius. As $m_p(K\langle b\rangle)=2$, K must be a single p-component. Further $C_X(K/Y)$ has p-rank 1 and contains b whence $C_X(K/Y)$ has a normal p-complement. The action of D forces $C_X(K/Y)\subseteq M$, and we conclude $C_X(K/Y)=Y$.

We will find a configuration satisfying Hypothesis 3.16 inside $X/Y=\bar{X}$.

We proceed as we did in the proof of Lemma 6.14. Let $T = C_X(b)$; T controls strong fusion of D in X and likewise for $\bar{T}$, $\bar{D}$ and $\bar{X}$. As $T \lhd \lhd C_M(b)$, Lemma 3.14 implies $L_{p'}(T)$ is a p-component of $L_{p'}(C_M(b))$ and (from the structure of $C_M(b)$) $T/L_{p'}(C_M(b))$ is solvable. In fact, if $L_{p'}(T) \neq 1$, then as every p-component of $C_M(b)$ is a component, Lemma 3.14 implies that K is quasisimple. In this case the argument used in the proof of Lemma 6.14 yields that Hypothesis 3.16 holds.

Suppose $L_{p'}(T) = 1$; then T is solvable. Further condition III(f) of Hypothesis 3.16 is satisfied because $m_p(K\langle b\rangle) = 2$. Thus Hypothesis 3.16 holds in this case too. Applying Lemmas 3.17, 3.18, and 3.22 we obtain the following possibilities.

$\bar{K} = K/O_{p',p}(K)$	$\bar{H} = T/O_{p',p}(K)$
$A_6, p = 3$	$F^*(\bar{H}) = \bar{D}$
$A_2(4), p = 3$	$F^*(\bar{H}) = A_6$ or $\bar{H} \cap \bar{K} = {}^2A_2(2)$
${}^2C_2(2^5), p = 5$	$\bar{H} \cap \bar{K} = Z_4 \cdot Z_{25}$
$A_1(8), p = 3$	$H \cap K = Z_2 \cdot Z_9$

Further $b \in \bar{K}$ in the first two cases but not in the last two. Also in the first two cases $\bar{D}$ acts on $\bar{K}$ as inner automorphisms. From the decomposition $B^* = A\langle d\rangle E$ given above it follows that d acts on $\bar{K}$ as an element of $\bar{E}$. Thus $B^* = A_1 \times E$ with $[K, A_1] = 1$. We can find an element of $N_K(E)$ which induces a transvection on B^*, not the case. Likewise in the last two cases d acts nontrivially on a B^*-invarient Sylow p-subgroup Q of K with $Q \cong Z_{p^2}$. Further $E = \langle b, e\rangle$ with $\langle e\rangle = E \cap Q \neq 1$. As $A\langle b\rangle$ centralizes $\bar{K}$, it centralizes Q whence the action of Q on $\langle d, e\rangle$ induces a transvection on B^*, which is impossible. This contradiction establishes the lemma.

LEMMA 6.16. *Suppose* $G_0 = D_4(q), p \mid q-1$, *and* $b \in B^*$ *with* $L(C_{G_0}(b)) = A_1(q) \times A_1(q) \times A_1(q)$; *then M controls strong fusion of b in G.*

Proof. Use the method of proof of the two preceding lemmas. If L is one of the components of $C_{G_0}(b)$, then $L \subseteq K$, a component of $C_G(e)$. If $L \neq K$, then Lemmas 3.17, 3.18 and 3.22 give $q = 4$, $L = A_1(4) \cong A_5$ and $p = 5$. But $p = 3$ in this case, so we have $L = K$ and $L(C_{G_0}(b)) \lhd \lhd C_G(e)$.

By Lemma 6.13 it suffices to show $C_G(b) \subseteq M$, and to do that we need only find $e \in L(C_{G_0}(e))$ with M controlling strong fusion of e in G. Let e be the product of two elements of order p lying in distinct components of $C_G(e)$. $C_{G_0}(e)$ has a single component $K = A_3(q)$ with $C_{B^*}(K) = \langle e\rangle$, so Lemma 6.15 applies.

LEMMA 6.17. *Suppose $b \in B^*$ and $C_{G_0}(b)$ has a component L with p, L and G_0 not listed below. Then M controls strong fusion of b in G.*

L	p	Restriction on G_0
${}^2A_3(2), C_2(2)', D_4(2)$	3	
$A_1(4), D_4(2)$	5	
$A_1(q)$	$p\mid q-1$	$G_0 = A_n(q)$
$A_1(q)$	$p\mid q+1$	$G_0 = {}^2A_n(q)$
$A_k(q^2)$	all p	

In the preceding table q is the order of the field of definition of G_0 (which is the fixed field if G_0 is twisted).

Proof. As in the preceding three proofs $L \subseteq K$ where K is a component of $C_G(b)$ and D acts on K if $K \not\subseteq M$. Likewise Lemmas 3.17, 3.18, 3.22 and the exclusions in the first two lines of the table above force $L = K$. Once we show $C_G(b) \subseteq M$ Lemma 6.13 and the next two lines of the table imply that M controls strong fusion of b in G.

We will find $e \in B^* \cap L$ such that M controls strong fusion of e in G. As $L \lhd \lhd C_G(B)$, we immediately obtain $C_G(b) \subseteq M$.

If $G_0 = A_n(q)$ or ${}^2A_n(q)$, use the standard matrix representations. The restriction in the last line of the table quarantees that L contains an element $e \in B^*$ represented by a matrix of determinant 1 with fixed points of codimention 2 on the standard module (except the codimension is 3 if $G_0 = A_{11}(2)$, $p = 7$). Further e is conjugate in $N_{G_0}(B^*)$ to an element of D, so e is the desired element.

If $G_0 = E_8(2)$, $p = 7$, then we see by Lemma 2.21 that every $b \in B^*$ satisfies the hypotheses of Lemma 6.15. Thus we are done in this case.

In all the remaining cases $p \mid q^2 - 1$. Exhibit $C_G(b)$ as in Section 2 so that

$$G_0 = O^{2'}(C_{\tilde{G}}(\sigma)),$$

$$\sigma = I_{w_0}\sigma_q \qquad \text{or} \qquad I_{w_0}{}^2\sigma_q$$

and $O^{2'}(C_{G_0}(b))$ corresponds to a subsystem $\tilde{\Sigma}_0$ of the root system $\tilde{\Sigma}$ of $\tilde{G}$. Further σ_q and ${}^2\sigma_q$ are standard with respect to some fundamental set of roots. I_{w_0} is an inner automorphism of $\tilde{G}$ corresponding to w_0 in the Weyl group of $\tilde{G}$. If $p \mid q-1$, w_0 is the identity while if $p \mid q+1$ w_0 interchanges positive and negative roots. Letting $\tilde{L}$ be generated by the root groups of $\tilde{G}$ corresponding to roots in $\tilde{\Sigma}_0$ we have

$$O^{2'}(C_{G_0}(b)) = O^{2'}(C_{\tilde{L}}(\sigma)).$$

We see that L corresponds to a subsystem $\tilde{\Sigma}_1 \subseteq \tilde{\Sigma}_0$. $\tilde{\Sigma}_1$ is either a connected component of $\tilde{\Sigma}_0$ or two such components interchanged by σ. However the latter possibility is excluded by the last line in the table above.

Let $\tilde{\alpha}$ be the highest root in $\tilde{\Sigma}_1$ and let $\tilde{J}$ be generated by the root groups corresponding to $\tilde{\alpha}$ and $-\tilde{\alpha}$. Since B^* consists of all elements of order p in $C_{\tilde{T}}(\sigma)$ where $\tilde{T}$ is a maximal torus of $\tilde{G}$ leaving all root groups of $\tilde{G}$ invariant, and since $\langle\sigma\rangle$ acts on $\tilde{J}$ by choice of $\tilde{\alpha}$, B^* acts on

$$J = C_{\tilde{J}}(\sigma) \cong A_1(q).$$

As B^* contains every element of order p in its centralizer, there exists

$$e \in B^* \cap J.$$

As $J \subseteq L$, we need only show that e and $C_{G_0}(e)$ satisfy the hypotheses of Lemma 6.15 or Lemma 6.16 to complete the proof. Let $\tilde{W}$ be the Weyl group of $\tilde{G}$. Since the $\langle\sigma\rangle$-orbits of $\tilde{\Sigma}$ correspond to roots in the root system Σ of G_0 and $C_{\tilde{W}}(\sigma)$ acts as the Weyl group of Σ, $\tilde{\alpha}$ is conjugate by an element of $C_{\tilde{W}}(\sigma)$ to $\tilde{\beta}$, the highest root of its length in $\tilde{\Sigma}$.

Let $\tilde{J}_1$ be generated by the root groups of $\tilde{G}$ corresponding to $\tilde{\beta}$ and $-\tilde{\beta}$. Our conditions imply that $\tilde{J}_1$ is conjugate to $\tilde{J}$ by some element of $C_{\tilde{G}}(\sigma) \cap N_{\tilde{G}}(T)$ which projects to an appropriate element of $C_{\tilde{W}}(\sigma)$. In other words $C_{G_0}(e) \cong C_{G_0}(e_1)$ for some $e_1 \in B^* \cap \tilde{J}_1$. As we have already treated the cases where $\tilde{\Sigma}$ has type A_n, we have that $e_1$centralizes all but one fundamental root group of $\tilde{G}$; that is all root groups corresponding to roots orthogonal to β. It follows that $C_{B^*}(O^{2'}(C_{G_0}(e_1))$ is cyclic. Further except when $\tilde{\Sigma}$ has type D_4, $O^{2'}(C_{G_0}(e_1))$ is either quasisimple or a product of a quasisimple group with a group isomorphic to $A_1(q)$. In either case the hypotheses of Lemma 6.15 are satisfied.

Finally if $\tilde{\Sigma}$ has type D_4, then $m_{2,p}(M) \geqslant 4$ forces $p | q-1$ and $G_0 = D_4(q)$ by Table B in Section 2. $O^{2'}(C_{G_0}(e))$ is a product of three $A_1(q)$'s and Lemma 6.16 applies.

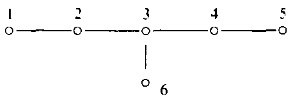
The proof of Lemma 6.17 is complete. We will complete the proof of Proposition 6.1 by showing that M controls strong fusion of $\langle r\rangle$ for some 2-central involution $r \in M$. Of course we are done if $M = G$, so we assume $M \neq G$ and consider the action of G on the cosets G/M. By a result of Holt [42, Theorem 1] G is identified as an alternating group or a Bender group. But one sees easily that these groups do not satisfy the hypotheses on G, so we must have $M = G$ after all.

We proceed to study $C_G(r)$ where r is a root involution of G_0 lying in a long root group if G_0 is any twisted group. We know that r is 2-central in M. Further except in the cases (*) (**) of Lemma 2.38 we may assume $[D, r] = 1$.

In the exceptional cases Lemma 2.38(*), (**) it is necessary to switch from D to E where $D \cong E$, $E \subseteq B^*$, and E centralizes a long root group of G_0. We define $E = \langle d, e\rangle$ for $d \in D^{\#}$ and $e \in B^* - D$. The elements d and e are chosen as follows:

For $G_0 = A_{n+2}(2) = \langle A_n(2), A_{[(n+1)/2]}(4)\rangle$, $p = 3$, consider the standard module V and pick $d \in D^{\#}$ so that $C_V(d)$ has codimension 2 and $C_{G_0}(d) \cong A_n(2) \times Z_3$. Let e be an $N_{G_0}(B^*)$-conjugate of d chosen so that $C_V(de)$ has codimension 4 and $C_{G_0}(de) \cong A_1(4) \times A_{n-2}(2) \times Z_3$. Every $f \in E^{\#}$ is G_0-conjugate to d or ed.

For $G_0 = A_{11}(2)$, $p = 7$ and $G_0 = {}^2A_7(q)$, $p \mid q-1$ we proceed as above. The results are listed below. When $G_0 = E_6(2)$, $p = 3$, or ${}^2E_6(q)$, $p \mid q-1$, label the fundamental system of roots



and take $d = [\eta_6, \sigma]$, $e = [\eta_1 + \eta_5, \sigma]$ where $\tilde{G}$ is the corresponding algebraic group, $G_0 = O^2\,(C_{\tilde{G}}(\sigma))$, and $\{\eta_i \mid 1 \leqslant i \leqslant 6\}$ is the dual basis of the root lattice corresponding to the labelling above.

We list the possible centralizers of elements in $E^{\#}$.

G_0	$C_{G_0}(f), f \in E^{\#}$
$A_{n+2}(2), p = 3, n \geqslant 5$	$A_n(2) \times Z_3$
	$A_1(4) \times A_{n-2}(2) \times Z_3$
$A_{11}(2), p = 7$	$A_8(2) \times Z_7$
	$A_5(2) \times Z_7 \times Z_7$
	$A_1(8) \times A_5(2) \times Z_7$
$E_6(2), p = 3$	$A_5(2) \times Z_3$
	${}^2D_4(2) \times Z_3$
${}^2A_7(q), p \mid Q - 1$	${}^2A_5(2) \times Z_{q^2-1}$
	$A_1(q^2) \times {}^2A_3(q) \times Z_{q^2-1}$
	${}^2A_3(q) \times Z_{q^2-1} \times Z_{q^2-1}$
${}^2E_6(q), p \mid q-1$	${}^2A_5(q) \times Z_{q-1}$
	${}^2D_4(q) \times Z_{q^2-1}$
	${}^2A_3(q) \times Z_{q^2-1} \times Z_{q-1}$

By Lemma 6.17 M controls strong fusion of E in G. We fix

$E = D$ if we are not in one of the cases Lemma 2.38(*)(**);
$E = \langle e, d \rangle$ as above otherwise.

In all cases M controls strong fusion of E in G and E centralizes $R = Z(X)$ where X is a root group of G_0 corresponding to a long root if G_0 is any twisted group. Any $r \in R^{\#}$ is 2-central in M. Fix such an r.

We will show $C_G(r) \subseteq M$. Let $N = C_G(r)$, $V = C_M(r)$, $P = O_2(N)$, and $Q = O_2(V)$. The action of E on P forces $P \subseteq Q$. Since G is of characteristic two type, $Z(Q) \subseteq C_G(P) \subseteq P$.

Let V_0 be the subgroup of V which acts as inner · diagonal automorphisms on G_0. Since $C_G(G_0) = 1$ by Lemma 6.9(iv), the structure of V_0 is given by Lemma 2.6. By Lemmas 2.11 and 6.9, $Q \subseteq V_0$. Let L be defined as in Lemma 2.6 and let J be a summand of L. By Lemma 2.17, $Q = [J, Q]\, Z(Q)$.

Define $\bar{N} = N/P$. From the preceding two paragraphs $[\bar{J}, \bar{Q}] = \bar{Q}$. As $JQ \lhd \lhd V$, we see that $\overline{JQ}$ is a p-component of $\bar{V}$ if J is quasisimple. If J is not quasisimple, then by inspection we see that $p = 3$, $J \cong S_3$, and $L = JJ_1$ where J_1 is quasisimple. Thus in this case $\overline{JQ} \lhd \bar{V}$. $\bar{V}$ controls strong fusion of $\bar{E}$ in $\bar{N}$. Lemma 3.14 is applicable when J is quasisimple and yields that $\overline{JQ}$ lies in a p-component $\bar{K}$ of $\bar{N}$. It is easy to check that Lemma 3.15 is applicable when $J \cong S_3$. We summarize the results so far.

LEMMA 6.18. *Let J be a summand of L and $Y = O^{p'}(J)$. $\overline{JQ}$ normalizes every p-component of $\bar{N}$. Further:*

(i) *If J is quasisimple, then $\overline{JQ}$ is a p-component of $\bar{V}$, and $\overline{JQ}$ lies in a p-component $\bar{K}$ of $\bar{N}$.*

(ii) *If J is not quasisimple, then $p = 3$, $J = S_3$, $\overline{JQ} \lhd \bar{V}$, and one of the following holds:*

(a) $\overline{YQ} \subseteq O_{3',3}(\bar{N})$;

(b) $\overline{YQ} \subseteq \bar{K}$ *for some p-component $\bar{K}$ of $\bar{N}$, and $\bar{J}$ acts on $\bar{K}$;*

(c) $\bar{J}$ *acts on a p-component $\bar{K}$ of $\bar{N}$ and covers a section isomorphic to S_3 in the outer automorphism group of $\bar{K}/O_{3',3}(\bar{K})$.*

Suppose we can show $\overline{JQ} = \bar{K}$ or $\overline{JQ} \subseteq O_{3',3}(\bar{N})$ in all cases. If so, then since $O_{3',3}(\bar{N}) \subseteq \bar{V}$ by Lemma 3.13(i), we have $O^{p'}(LQ) \subseteq X \lhd\lhd N$ and $X \subseteq M$ for an appropriate subgroup X. By Lemma 2.9 there exists $1 \neq e \in E \cap LQ$, whence $N \subseteq M$ by Lemma 3.11(i) and we have shown $C_G(r) \subseteq M$ as desired.

We proceed to consider the various cases. We may assume case (ii)(a) does not hold and $\bar{K} \not\subseteq \bar{V}$. $\bar{E}$ acts on $\bar{K}$ by Lemma 3.13. Let $F/P = O_r(\bar{N})$ for any prime $r \neq p$. As $P = O_2(N)$, $r \neq 2$. $F \subseteq V$ by the action of E. From the structure of V, $[YQ, F] \subseteq Q$. Since $F \lhd V$, $[YQ, F] \subseteq Q \cap F = P$ and we have $[\overline{JQ}, \bar{F}] = 1$. Likewise if $F/P = L(O_{p'}(\bar{N}))$, then $F \subseteq V$ and from the structure of V, $\bar{F} = 1$. Thus $[\bar{K}, F^*(O_{p'}(\bar{N}))] = 1$, which implies that $\bar{K}$ is quasisimple. As we have done before in the proof of Lemma 6.14 we will find a configuration satisfying Hypotheses 3.16 inside $\mathrm{Aut}(\bar{K})$.

Let K be the inverse image of $\bar{K}$ in N and define $W = KE$, $U = C_W(\bar{K})$, $T = V \cap W$. Let $\tilde{W} = W/U$ and denote the projection of any $H \subseteq W$ into $\tilde{W}$ by $\tilde{H}$. By Lemma 3.12, $U \subseteq T$, and, by Lemma 3.11, $\tilde{T}$ controls strong fusion of $\tilde{E}$ in $\tilde{W}$.

We claim $\tilde{Q} = O_2(\tilde{T})$. Let $A_1/U = O_2(\tilde{T})$ and $A = O^{2'}(A_1)$; clearly $Q \subseteq A$. It suffices to show $Q = A$. As $|W: K|$ is odd, $A \subseteq K$ whence $[\bar{U}, \bar{A}] = 1$ and $\bar{A}$ is nilpotent. Thus $\bar{A} = O_2(\bar{A}) \subseteq \bar{Q} = O_2(\bar{T})$. It follows that $A \subseteq Q$ as desired.

Next let X be the product of all the quasisimple summands of L lying in K. By Lemma 6.18 and the structure of V, $\tilde{T}/\tilde{X}$ is solvable. In particular $\tilde{X}\tilde{Q}/\tilde{Q} = L(\tilde{T}/O_2(\tilde{T}))$. It is immediate that conditions I, II and III(a)–(e) of Hypotheses 3.16 hold with $\tilde{W}$, $\tilde{E}$, $\tilde{T}$ in place of G, E, H.

Check that Hypothesis 3.16 (IV) holds as follows: Let $A/U = C_{\tilde{W}}(\tilde{e})$ for some $e \in E^{\#}$. As $W = EK$, $A = E(A \cap K)$. $\bar{U} \cap \bar{K} = Z(\bar{K})$ implies that $\bar{e}$ centralizes $(\bar{A} \cap \bar{K})/Z(\bar{A} \cap \bar{K})$ whence $O^p(C_{\bar{K}}(\bar{e}))$ covers $O^p((\bar{A} \cap \bar{K})/Z(\bar{A} \cap \bar{K}))$. We conclude that $O^p(C_{\bar{W}}(\bar{e}))$ covers $O^p(C_{\tilde{W}}(\tilde{e}))$; and as $C_W(e)$ covers $C_{\bar{W}}(e)$ we have that $O^p(C_W(e))$ covers $O^p(C_{\tilde{W}}(\tilde{e}))$. Now Condition IV follows from the structure of $C_W(e) = C_T(e)$. In particular if the quasisimple summand J of L lies in K, then $L(\tilde{T}/O_2(\tilde{T})) \neq 1$ and we may apply Lemmas 3.17, 3.18 and 3.22. We have

LEMMA 6.19. *Let J, K and T be as above; then $\bar{K}$ is quasisimple, and if J is quasisimple either $\tilde{J}\tilde{Q} = \tilde{K}$ or one of the following occurs:*

$\tilde{K}$	$\tilde{J}\tilde{Q}$
$F_{22}, p = 5$	$D_4(2)$
$C_4(2), p = 3$	$D_4(2)$
$C_3(2), p = 3$	${}^2A_3(2)$

Further in the last two cases $\tilde{T} \cap \tilde{K}$ is isomorphic to $O^+(8, 2)$ or $O^-(6, 2)$, respectively.

Proof. The lemma follows from the preceding remarks. By checking the possibilities for G_0 and E observe that $J = A_6$ never occurs.

Suppose $\tilde{J}\tilde{Q} = \tilde{K}$. We have $\bar{K} \subseteq \overline{JQU} \subseteq \bar{V}$ whence $\bar{K} \subseteq \bar{V}$ and $\bar{K} = \overline{JQ}$ follows immediately. We will show that the possibilities listed in the table above do not occur. Since the field of definition of J is an extension of that of G_0, and since G_0 is defined over a field of order q with $p | q^2 - 1$ (except for some cases when $p = 7$), the first entry on the table does not occur.

We wish to eliminate the last two lines on the table. Assume one of these conclusions holds. Surveying the possibilities for G_0 we find

G_0	L	J
$D_6(2)$	$D_4(2) \times A_1(2)$	$D_4(2)$
${}^2D_5(2)$	${}^2A_3(2) \times A_1(2)$	${}^2A_3(2)$
${}^2A_5(2)$	${}^2A_3(2)$	${}^2A_3(2)$.

To obtain a contradiction it suffices to find $y \in V$ such that $C_{\bar{K}}(\bar{y}) \not\subseteq K \cap V$ and M controls strong fusion of y in G. The centers of the root group containing r and its corresponding negative root group generate a group isomorphic to $A_1(2)$ which commutes with L. Thus we can choose $y \in C_V(L)$ with $|y| = 3$. By Lemma 6.15, M controls strong fusion of y in G. Since V normalizes J, $\bar{y}$ acts on $\bar{K}$. But $\bar{J}$ is too large to lie in $C_{\bar{K}}(\bar{y})$ unless $[\bar{K}, \bar{y}] = 1$ whence $C_{\bar{K}}(\bar{y}) \not\subseteq K \cap V$ as desired.

We have reached the desired conclusion $\overline{JQ} = \bar{K}$ whenever J is a quasisimple summand of L. It remains to consider the possibilities listed in Lemma 6.18(ii)(b, c). By inspection $L = JJ_1$ with J_1 quasisimple, and from the preceding discussion $\bar{J}_1\bar{Q}$ is a component of $\bar{N}$. In particular $\bar{Q} = 1$, $Q = P = O_2(N)$, and $\bar{J} \cong S_3$. Our conditions imply that $\tilde{T}$ is solvable. Also by Lemma 2.9 we have $|B^*: B^* \cap J| \leqslant p^2 = 9$. Further as B^* is the unique elementary abelian subgroup of its rank in any Sylow p-subgroup of G, we must have $m_p(\bar{K}) = 1$. It follows that $p \nmid |Z(\bar{K})|$ and $m_p(\tilde{K}) = 1$. We have verified condition III(f) of Hypothesis 3.16 and checking Lemmas 3.17, 3.18 and 3.22 we find that $\tilde{J} \lhd \tilde{T}$, $\tilde{J} \cong S_3$ is impossible. This contradiction completes the proof that $C_G(r) \subseteq M$.

LEMMA 6.20. *$|M: G_0|$ is odd.*

Proof. Suppose $2 \mid |M: G_0|$. Let $G_0 = O^{2'}(C_{\tilde{G}}(\sigma))$ where $\tilde{G}$ is an algebraic group and σ is standard with respect to some choice of root groups and fundamental set of roots. The roots of G_0 correspond to $\langle\sigma\rangle$-orbits of the roots of $\tilde{G}$. Let t be an involution of $M - G_0$ which induces a standard field, graph, or graph-field automorphism of G_0 with respect to the root system of G_0 corresponding to that of $\tilde{G}$.

We may extend the action of t on G_0 to an action on $\tilde{G}$. If $\sigma = \sigma_q$ and t is a field or graph-field automorphism of G_0, take $t = \sigma_{q/2}$ or ${}^2\sigma_{q/2}$. Otherwise take t to be the standard graph automorphism of $\tilde{G}$.

It is straightforward to calculate $C_{G_0}(t) = C_{\tilde{G}}(\langle t, \sigma\rangle)$ using the methods of [12, Chap. 13]. We see that $G_1 = O^{2'}(C_{G_0}(t))$ is a simple group of Lie type defined over a field of characteristic two. The roots of G_1 correspond to $\langle t, \sigma\rangle$-orbits of roots of $\tilde{G}$.

Let α be the root of $\tilde{G}$ of highest weight. As α is fixed by $\langle t, \sigma\rangle$, it follows that α corresponds to the highest root of G_0 and of G_1. Pick r to be an involution in the centralizer of $\langle t, \sigma\rangle$ on the root group of $\tilde{G}$ corresponding to α; r is also in the highest root groups of G_0 and G_1. In particular $C_{G_1}(r)$ has the description given in Lemma 2.6 and we can check from our knowledge of the root system of G_1 that $O_2(C_{G_1}(r)) \subset O^{2'}(C_{G_1}(r))$.

Let $P_1 = P \cap M$. Clearly G_1 normalizes P_1, and P_1 normalizes $G_1 = O^{2'}(C_{G_0}(t))$. Thus $[P_1, G_1] \subseteq P_1 \cap G_1 = 1$ as $O_2(G_1) = 1$. In particular $C_p(r) \subseteq P_1$ by Lemma 6.19 whence $[C_P(r), G_1] = 1$. Hence $C_{G_1}(r)$ centralizes $C_P(r)$ and it follows that $O^{2'}(C_{G_1}(r))$ centralizes P. As G has characteristic two-type, $O^{2'}(C_{G_1}(r)) \subseteq P$ contrary to the conclusion of the preceding paragraph. Thus we cannot have $2 \| M{:}G_0|$, and Lemma 6.20 is proved.

LEMMA 6.21. *If $r^g \in M$, then $g \in M$.*

Proof. By Lemma 6.20 it suffices to show that r is fused to r^g in M. We will assume r is not fused to $t = r^g$ and obtain a contradiction either by showing $C_M(t)$ is not isomorphic to a subgroup of $C_M(r)$ or by producing x of order p in $C_M(t)$ such that M controls strong fusion of x in G. In the latter case $x^{g^{-1}} \in C_G(t) \subseteq M$ implies $g \in M$.

The centralizers in G_0 of involutions in G_0 are given by Aschbacher and Seitz [3], and the rest of the proof amounts to checking that one of the two conditions above holds for $C_G(t)$ as r runs through representatives of all G_0-classes.

First suppose $G_0 = A_n(q)$ or ${}^2A_n(q)$. In terms of the usual matrix representations t is represented by

$$\mathscr{M}_l = \begin{pmatrix} I_l & 0 & 0 \\ 0 & I_k & 0 \\ I_l & 0 & I_l \end{pmatrix}$$

and the Hermitian form is represented by

$$\begin{pmatrix} 0 & 0 & I_l \\ 0 & I_k & 0 \\ I_l & 0 & 0 \end{pmatrix}$$

where $2l + k = n + 1$. Suppose $p | q - 1$ of $G_0 = A_n(q)$ and $p | q + 1$ if $G_0 = {}^2A_n(q)$. In the first case $n \geqslant 4$, while in the second case $n \geqslant 5$ lest $m_{2,p}(M) < 4$. Let $e_1, \ldots, e_{2l+k}$ be the usual basis elements of the standard module for the matrix representation of G_0. Suppose $k \neq 0$. Unless $p | n + 1$ and $m(B^*) = n - 1$, we may take x to be the element whose matrix (determined up to scalars) acts as follows:

$$e_i \to e_i, \qquad i = l + 1,$$

$$e_{l+1} \to \lambda e_{l+1},$$

where λ is a primitive pth root of unity. As x is fused in G_0 to an element of

D, M controls strong fusion of x in G. When $p|n+1$ and $m(B^*) = n-1$, the same conditions hold if the matrix of x acts as

$$\begin{aligned} &e_1 \to \lambda e_1, \\ &e_{l+1} \to \lambda^{-2} e_{l+1}, \\ &e_{l+k+1} \to \lambda e_{l+k+1}, \\ &e_i \to e_i \qquad \text{for all other } i. \end{aligned}$$

Suppose $k=0$. Unless $p|n+1$ and $m(B^*) = n-1$, take the action

$$\begin{aligned} &e_1 \to \lambda e_1, \\ &e_{l+1} \to \lambda e_{l+1}, \\ &e_i \to e_i \qquad \text{for all other } i. \end{aligned}$$

Again x is fused in G_0 to D. Finally if $k=0$, $p|n+1$, and $m(B^*) = n-1$, take

$$\begin{aligned} &e_1 \to \lambda e_1, \\ &e_2 \to \lambda^{-1} e_2, \\ &e_{l+1} \to \lambda e_{l+1}, \\ &e_{l+2} \to \lambda^{-1} e_{l+2}, \\ &e_1 \to e_i \qquad \text{for all other } i. \end{aligned}$$

Here x is not fused in G_0 to D, but Lemma 6.17 yields that M controls strong fusion of x in G unless $G_0 = A_5(q)$, ${}^2A_5(q)$, or ${}^2A_n(2)$, $n = 6, 7$. Since $n+1 = 2l$ is even and divisible by p, we actually have $p=3$ and $G_0 = A_5(q)$ or ${}^2A_5(q)$. In both these cases (and only in these cases) we show that $C_M(t)$ is not isomorphic to a subgroup of $C_M(r)$. Since $|M: G_0|$ is odd by the preceding lemma, it suffices to show that $H = O^{2'}(C_{G_0}(t))$ is not isomorphic to $K = O^{2'}(C_{G_0}(r))$. Let $P = O_2(H)$ and $Q = O_2(K)$. Suppose $G_0 = A_5(q)$; a similar argument works when $G_0 = {}^2A_5(q)$. $P \cong E_{q^9}$ and $H/P \cong A_2(q)$. As r is in the class represented by $\mathscr{M}_l$ with $l=1$, Q is special of order q^9 and $\bar{K} = K/Q \cong A_3(q)$. Further, commutation induces a nondegenerate bilinear form over F_q on $Q/Z(Q)$ whence any abelian subgroup of Q has order at most q^5. Assuming $H \cong K$, we obtain $|P \cap Q| \leqslant 2^5$ and $|\bar{H}|_2 = |\bar{Q}|\, |A_2(q)|_2 \geqslant q^7 > q^6 = |A_3(q)|_2 = |\bar{K}|$, which is impossible.

The same sort of argument works for $G_0 = A_4(q)$, $p \nmid q-1$ or $G_0 = {}^2A_7(q)$, $p|q-1$, and all the other classical groups. The conjugacy classes of involutions are represented by the matrices $\mathscr{M}_l$ above, and the matrix for x is chosen as above except that instead of being diagonal it has one or two 2×2

(or 3×3 in the case $G_0 = A_{11}(2)$, $p = 7$) blocks along the diagonal. Lemmas 6.15, 6.16 and 6.17 suffice to show that M controls strong fusion of x in G.

Finally consider the exceptional groups of Lie type. Exhibit G_0 as $O^{2'}(C_{\tilde{G}}(\sigma))$ for the standard endomorphism σ. The possibilities for t are given in [3], as products of elements from various root groups of G_0. When $G_0 = {}^2E_6(q)$ we may express each such element as a product of at most two such elements from root groups of $\tilde{G}$. Thus in all cases t is given as a product of elements of root groups of $\tilde{G}$, and by inspection we can find for each t a root $\tilde{\alpha}$ which is orthogonal to all roots involved in t and fixed by σ.

Let $\tilde{T}$ be the maximal torus of $\tilde{G}$ corresponding to our choice of root groups for $\tilde{G}$, and let $\tilde{J}$ be generated by the root groups corresponding to $\tilde{\alpha}$ and $-\tilde{\alpha}$. Clearly $C_{\tilde{J}}(\sigma) \subseteq O^{2'}(C_{\tilde{G}}(\sigma)) = G_0$; and as B^* consists of all elements of order p in $C_{\tilde{T}}(\sigma)$ or $C_{\tilde{T}}(\sigma) \cap G_0$ (when $G_0 = E_6(q)$, $p = 3$, $m(B^*) = 5$), we can pick $x \in B^* \cap \tilde{J}$. Clearly $[\tilde{J}, t] = 1$, and Lemma 6.15 yields that M controls strong fusion of x in G.

When $p | q + 1$, pick α as above so that in addition α is fixed by the element w_0 of the Weyl group of $\tilde{G}$ which interchanges positive and negative roots. Taking

$$\rho = I_{w_0} \sigma,$$

where I_{w_0} is an inner automorphism of $\tilde{G}$ corresponding to w_0, and exhibiting G_0 as $O^{2'}(C_{\tilde{G}}(\rho))$, we see as above that $B^* \cap \tilde{J}$ contains an element y which satisfies the hypotheses of Lemma 6.15. By Lemma 2.18(ii) there is an inner automorphism of $\tilde{G}$ which carries $C_{\tilde{G}}(\rho)$ to $C_{\tilde{G}}(\sigma)$ and $C_{\tilde{J}}(\rho)$ to $C_{\tilde{J}}(\sigma)$. Taking x to be the image of y under this automorphism, we see that x has the desired properties.

We must also consider $G_0 = E_8(2)$, $p = 7$. Here by Lemmas 2.21 and 6.15, M controls strong fusion of all elements of order p in G_0, so it suffices to check in [3] that $|C_{G_0}(t)|$ is always divisible by 7.

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