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On the Number of Solutions to the Complementarity Problem and Spanning Properties of Complementary Cones

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ABSTRACT

The relationship between the number of solutions to the complementarity problem,

$$w = Mz + q,$$

$$w \geqslant 0, \quad z \geqslant 0, \quad w^T z = 0,$$

the right-hand constant vector q and the matrix M are explored. The main results proved in this work are summarized below.

The number of solutions to the complementarity problem is finite for all $q \in R^n$ if and only if all the principal subdeterminants of M are nonzero. The necessary and sufficient condition for this solution to be unique for each $q \in R^n$ is that all principal subdeterminants of M are strictly positive. When $M \geqslant 0$, there is at least one complementary feasible solution for each $q \in R^n$ if and only if all the diagonal elements of M are strictly positive; and, in this case, the number of these solutions is an odd number whenever q is nondegenerate. If all principal subdeterminants of M are nonzero, then the number of complementary feasible solutions has the same parity (odd or even) for all $q \in R^n$ which are nondegenerate. Also, if the number of complementary feasible solutions is a constant for each $q \in R^n$, then that constant is equal to one and M is a P-matrix.

In the cartesian system of coordinates for R^n, an orthant is a convex cone generated by a set of n-column vectors in R^n, $\{A_{\cdot 1}, \ldots, A_{\cdot n}\}$, where for each $j = 1$ to n, $A_{\cdot j}$ is either the jth column vector of the unit matrix of order n (denoted by $I_{\cdot j}$) or its negative $-I_{\cdot j}$. There are thus 2^n orthants in R^n, and they partition the whole space. It is interesting to know what properties these orthants possess if we obtain them after replacing $-I_{\cdot j}$ by some given column vector $-M_{\cdot j}$ for $j = 1$ to n. Orthants obtained in this manner are called *complementary cones*, and their spanning properties are studied.

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1. INTRODUCTION

1.1. The complementary quadratic programming problem is that of finding column vectors $w = (w_i) \in R^n$ and $z = (z_i) \in R^n$ satisfying

$$w = Mz + q,$$

$$w \geqslant 0, \quad z \geqslant 0, \quad w^T z = 0, \tag{1}$$

where $M = (m_{ij})$ is a given $n \times n$ square matrix, $q = (q_i)$ is a given $n \times 1$ column vector, and w^T denotes the transpose of w. R^n is the n-dimensional real Euclidean space.

1.2. Because w, z are nonnegative, the constraint

$$w^T z = \sum_{i=1}^{n} w_i z_i = 0 \Rightarrow w_i z_i = 0 \quad \text{for each} \quad i = 1, \ldots, n.$$

Thus, if one of the variables in the pair w_i, z_i is positive, the other should be zero. Hence the constraint $w^T z = 0$ will be referred to as the *complementarity condition*, and the problem is sometimes known as *the complementarity problem of order n*.

1.3. Consider the quadratic programming problem

$$\begin{aligned} \text{minimize} \quad & w^T z, \\ \text{subject to} \quad & w - Mz = q, \\ & w \geqslant 0, \quad z \geqslant 0. \end{aligned}$$

If Eq. (1) has any solution $(w; z)$, then that solution also solves the quadratic programming problem. Conversely, if the minimum value for the objective function in the quadratic programming problem is zero, then any optimal solution to it also solves (1).

Thus, solving (1) is equivalent to finding out whether the minimum objective value in the quadratic program above is zero or strictly positive. Hence the problem (1) is known as the *complementary quadratic programming problem.*

1.4. Cottle and Dantzig [1] and Lemke [8, 9] have shown that all the problems in linear programming, convex quadratic programming, and also the problem of finding a Nash equilibrium point of a bimatrix game, can be posed in the form of Eq. (1). For other applications of (1) see Scarf [15]. Lemke and Howson [8, 10] have developed a simple algorithm for solving (1) which is based on pivot steps.

Lemke [8] and Cottle and Dantzig [1] have shown that (1) has a solution if all the principal determinants of M are positive or if M is a nonnegative matrix with positive elements in the principal diagonal. Lemke [8] has also given sufficient conditions on M and q under which the number of solutions to (1) is finite. Also see [16, 11, 7, 3].

In this paper our main interest is to examine the relationship of the number of solutions to (1) to the properties of the given matrices M and q. The motivation for this problem was provided by Gale [4] when he asked me to try and prove or construct counterexamples to the following conjectures:

(a) M is a P-matrix if and only if the complementarity problem has a unique solution for each $q \in R^n$.

(b) $M \geqslant 0$ is a Q-matrix if and only if $m_{ii} > 0$ for all $i = 1, \ldots, n$.

(c) If M is a Q-matrix, the complementarity problem has an odd number of solutions whenever q is nondegenerate with respect to M.

The result of the investigation is the present work.

2. NOTATION AND PRELIMINARIES

2.1. If A is any matrix, A^T denotes its transpose. $A_{i\cdot}$ denotes the ith row vector of A and $A_{\cdot j}$ denotes the jth column vector of A. I denotes the unit matrix.

2.2. A square matrix M is called a *P-matrix* if all its principal subdeterminants are strictly positive. The square matrix M is called *nondegenerate* if every matrix A obtained by taking $A_{\cdot j}$ to be either $M_{\cdot j}$ or $I_{\cdot j}$ for each $j = 1, \ldots, n$ is nonsingular. An equivalent definition is that M is a nondegenerate matrix if and only if all its principal subdeterminants are nonzero. M is said to be *degenerate* if it is not nondegenerate. M is called a *Q-matrix* if the problem in Eq. (1) has a solution for all $q \in R^n$.

2.3. Let A be any finite set of column vectors in R^n. The convex cone generated by the column vectors in A is denoted by $\text{pos}\{A\}$. Thus $x \in \text{pos}\{A\}$ if and only if x can be expressed as a nonnegative linear combination of the column vectors in A.

2.4. Suppose $L(q) \subset R^{2n}$ is the linear manifold determined by the linear equality constraints

$$\begin{aligned} & w - Mz = q, \\ & w \in R^n, \quad z \in R^n, \end{aligned} \tag{2}$$

without any nonnegativity constraints. The vector

$$\begin{bmatrix} w \\ \cdots \\ z \end{bmatrix} \in L(q)$$

if and only if it satisfies Eq. (2). For convenience we write down the vector

$$\begin{bmatrix} w \\ \cdots \\ z \end{bmatrix} \in R^{2n} \quad \text{as} \quad (w; z).$$

2.5. The convex polyhedron $K(q) \subset L(q)$ is the set of all *feasible solutions* $(w; z)$ which satisfy

$$\begin{aligned} & w - Mz = q, \\ & w \geqslant 0, \quad z \geqslant 0. \end{aligned} \tag{3}$$

2.6. A *basic feasible solution* is a feasible solution $(w; z) \in K(q)$ such that the column vectors in Eq. (3) of the variables w_j and z_j which are strictly positive are linearly independent. Every basic feasible solution is an extreme point of the convex polyhedron $K(q)$ and vice versa [2, 5].

2.7. A *complementary feasible solution* is a feasible solution $(w; z) \in K(q)$ which satisfies the complementarity condition $w^T z = 0$. A complementary feasible solution is a solution to Eq. (1) and vice versa.

2.8. For each $i = 1, \ldots, n$ the variables w_i, z_i constitute a *complementary pair,* and each of the variables in the pair is the complement of the other. In the system (1) the column vector $I_{.j}$ is associated with the variable w_j and $-M_{.j}$ is associated with z_j. Thus the pair $(I_{.j}, -M_{.j})$ are the *jth complementary pair* of column vectors in (1).

2.9. A *complementary set of column vectors* is a set of column vectors $\{A_{.j}, j = 1, \ldots, n\}$ such that $A_{.j}$ is either $I_{.j}$ or $-M_{.j}$ for each $j = 1, \ldots, n$. Thus any set of column vectors containing exactly one vector from each complementary pair of vectors is a complementary set of column vectors. The corresponding set of variables is called a complementary set of variables. Hence there are 2^n complementary sets of column vectors.

2.10. Each solution to Eq. (1) represents q as a nonnegative linear combination of some complementary set of column vectors.

Conversely, if $\{A_{.j}\}$ is a complementary set of column vectors and if

$$q \in \text{pos}\{A_{.j}, j = 1, \ldots, n\},$$

$$\text{i.e.,} \quad q = \sum_{j=1}^{n} \beta_j A_{.j}, \quad \text{where} \quad \beta_j \geqslant 0 \quad \text{for each } j,$$

then a solution to (1) is obtained by setting the variables associated with the column $A_{.j}$ equal to β_j for $j = 1, \ldots, n$, respectively, and all the other variables in $(w; z)$ not in this complementary set equal to zero.

The pos cone generated by any complementary set of column vectors is known as a *complementary cone.* Thus there are 2^n complementary cones, and the union of all these cones is the set of all q for which (1) has a solution.

All the results related to the number of distinct solutions of the complementarity problem can be interpreted in terms of the spanning and overlapping properties of the complementary cones and vice versa.

2.11. Any set of variables $\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}$, where y_r is either w_r or z_r for each r, is known as a *subcomplementary set* of variables. The column vectors associated with a subcomplementary set of variables constitute a subcomplementary set of column vectors. The complementary pair of variables (w_i, z_i) is the *left-out complementary pair* of variables in the subcomplementary set $\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}$.

2.12. An *almost complementary feasible solution* is a feasible solution $(w; z) \in K(q)$ such that

$$w^T z = w_i z_i \quad \text{for some} \quad i,$$

$$\text{i.e.,} \quad w_j z_j = 0 \quad \text{for all} \quad j \neq i, \quad \text{for some} \quad i.$$

2.13. The set $C_i(q)$ is the *almost complementary set* defined by

$$C_i(q) = \{(w; z) \colon (w; z) \in K(q), \quad w^T z = w_i z_i, \quad \text{i.e.,} \quad w_j z_j = 0 \quad \text{for} \quad j \neq i\},$$

where i is any integer from 1 to n.

2.14. If $x \in R^n$, $x \neq 0$, then the *ray* generated by x is

$$\text{pos}\{x\} = \{y \colon y = \lambda x \quad \text{for some} \quad \lambda \geqslant 0\}.$$

2.15. If $x^1, x^2 \in R^n$, $x^1 \neq 0$, then the set

$$\{y: y = x^2 + \lambda x^1 \quad \text{for some} \quad \lambda \geqslant 0\}$$

is the *half-line* through x^2 parallel to the ray generated by x^1.

2.16. The column vector q is said to be *nondegenerate with respect to* M if and only if, for all $(w; z) \in L(q)$, at most n of the $2n$ variables $\{w_j, z_j\}$ are zero. Equivalently, q is nondegenerate with respect to M if it does not lie in any subspace generated by $(n - 1)$ or less column vectors of $(I \mid -M)$. Otherwise q is said to be *degenerate*. Thus the set of all q which are degenerate belong to a finite number of subspaces of R^n.

2.17. Two basic feasible solutions $(w^1; z^1)$ and $(w^2; z^2)$ are said to be *adjacent extreme points* of $K(q)$ if every convex combination of $(w^1; z^1)$ and $(w^2; z^2)$ has a unique representation as a convex combination of extreme points of $K(q)$. The line segment joining any pair of adjacent extreme points of $K(q)$ is called an *edge* of $K(q)$.

2.18. If $K(q)$ is nonempty and unbounded, any basic feasible solution of

$$w - Mz = 0,$$

$$\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} z_i = 1,$$

$$w, z \geqslant 0,$$

is known as an *extreme homogeneous solution* of Eq. (3). Any half-line through a basic feasible solution in $K(q)$ parallel through the ray generated by an extreme homogeneous solution of (3) lies in $K(q)$. Such a half-line is called an *unbounded edge* (or *extreme half-line*) of $K(q)$ if every point on the half-line has a unique representation as the sum of a convex combination of basic feasible solutions of $K(q)$ and a nonnegative linear combination of extreme homogeneous solutions of (3).

2.19. Consider the set of equality constraints Eq. (2) again:

$$w = Mz + q. \tag{2}$$

The ith constraint in this system is

$$w_i = M_{i\cdot}z + q_i. \tag{2i}$$

A *principal pivot* in the position (i, i) in (2) consists of the following steps:

(i) Solve equation (2i) for the variable z_i in terms of $z_1, \ldots, z_{i-1}, w_i, z_{i+1}, \ldots, z_n$ and replace the ith equation in (2) by this equation, expressing z_i in terms of $z_1, \ldots, z_{i-1}, w_i, z_{i+1}, \ldots, z_n$.

(ii) Substitute the expression obtained for z_i in (i) in each of the other equations in (2).

Thus a principal pivot in position (i, i) in (2) can only be performed if $m_{ii} \neq 0$. The result of this principal pivot is to exchange the variables (w_i, z_i), and we get a transformed system of equations which has the same form as (2), but the left-hand set of variables in it differ from the left-hand set in (2) in one component (the ith). However, the set of the complementary pairs of variables remains unchanged as a result of a principal pivot.

2.20. If a series of principal pivots are performed on the system (2), then it will be transformed into the system

$$u = \tilde{M}v + \tilde{q}, \tag{4}$$

where each pair (u_i, v_i) is a permutation of the complementary pair of variables (w_i, z_i). A complementary feasible solution to (4) is a solution to the system

$$u = \tilde{M}v + \tilde{q},$$

$$u \geqslant 0, \quad v \geqslant 0, \quad u^T v = 0. \tag{5}$$

2.21. We notice that there is a one-to-one correspondence between solutions to (1) and solutions to (5). For example, suppose (5) is obtained from (1) by making only one principal pivot in which w_1, z_1 are exchanged, say. Then

$$\hat{w}; \hat{z} \quad \text{solves} \quad (1) \Leftrightarrow \hat{u} = (\hat{z}_1, \hat{w}_2, \ldots, \hat{w}_n);$$

$$\hat{v} = (\hat{w}_1, \hat{z}_2, \ldots, \hat{z}_n) \quad \text{solves} \quad (5).$$

In general, since u, v in (5) are such that $(u_i; v_i)$ is a permutation of the variables (w_i, z_i), we can construct a solution $(\hat{u}; \hat{v})$ to (5) corresponding to each solution $(\hat{w}; \hat{z})$ to (1) by taking the same permutation, and vice versa.

Thus the *number* of solutions to (1) is invariant under principal pivots.

2.22. Suppose D is a nonsingular principal submatrix of M. Let $\omega^1; \xi^1$ be the variables in problem (1) corresponding to the rows and columns of the principal submatrix D, and let $\omega^2; \xi^2$ denote the rest of the variables. Then (1) can be written in the partitioned form:

$$\begin{aligned} \omega^1 &= D\xi^1 + C\xi^2 + q^1, \\ \omega^2 &= E\xi^1 + F\xi^2 + q^2, \\ \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix} \geqslant 0, &\qquad \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} \geqslant 0, \qquad (\omega^1)^T\xi^1 + (\omega^2)^T\xi^2 = 0. \end{aligned} \tag{6}$$

A *block principal pivot* on the principal submatrix D consists in transforming the problem into the equivalent form:

$$\begin{aligned} \xi^1 &= D^{-1}\omega^1 - D^{-1}C\xi^2 + (-D^{-1}q^1), \\ \omega^2 &= ED^{-1}\omega^1 + (F - ED^{-1}C)\xi^2 + (q^2 - ED^{-1}q^1), \\ \begin{bmatrix} \xi^1 \\ \omega^2 \end{bmatrix} \geqslant 0, &\qquad \begin{bmatrix} \omega^1 \\ \xi^2 \end{bmatrix} \geqslant 0, \qquad (\xi^1)^T\omega^1 + (\omega^2)^T\xi^2 = 0. \end{aligned} \tag{7}$$

The block principal pivot can only be performed if D is nonsingular. The result of the block principal pivot is to transform the problem into an equivalent problem of the same form in which the complementary sets of variables $(\omega^1; \xi^1)$ are exchanged. The set of complementary pairs of variables remains unchanged as a result of the block principal pivot.

When M is nondegenerate, the block principal pivot exchanging the sets of variables $(\omega^1; \xi^1)$ can be performed by a series of principal pivots exchanging each complementary pair of variables in $(\omega^1; \xi^1)$ one at a time.

If $(\tilde{\omega}^1, \tilde{\omega}^2; \tilde{\xi}^1, \tilde{\xi}^2)$ is a solution to (1), then $(\tilde{\xi}^1, \tilde{\omega}^2; \tilde{\omega}^1, \tilde{\xi}^2)$ is a solution to (7) and vice versa. Thus the *number* of solutions to (1) is invariant under block principal pivots.

2.23. Let N be a principal submatrix of M of order s, obtained by striking off from M all the rows excepting the $i_1, \ldots, i_s$th rows and all but the $i_1, \ldots, i_s$th columns. Let

$$\omega = (w_{i_1}, \ldots, w_{i_s})^T, \qquad \xi = (z_{i_1}, \ldots, z_{i_s})^T$$

and $\mathcal{Q} = (q_{i_1}, \ldots, q_{i_s})^T$. Then

$$\begin{aligned} &\omega = N\xi + \mathcal{Q}, \\ &\omega \geqslant 0, \qquad \xi \geqslant 0, \qquad \omega^T\xi = 0, \end{aligned} \tag{8}$$

is known as a *principal subproblem* of (1) in the variables $(\omega; \xi)$.

2.24. Suppose $(\hat{w}; \hat{z})$ is a complementary feasible solution to (1) such that

$$\hat{z}_i = 0 \quad \text{for all} \quad i \neq i_1 \quad \text{or} \quad i_2, \ldots, \quad \text{or} \quad i_s.$$

Let $\hat{\omega} = (\hat{w}_{i_1}, \ldots, \hat{w}_{i_s})^T$ and $\hat{\xi} = (\hat{z}_{i_1}, \ldots, \hat{z}_{i_s})$. Then, from the definition of the principal subproblem (8), we see that $(\hat{\omega}; \hat{\xi})$ solves (8).

2.25. If r is any integer, its *parity* is said to be *odd* if r is an odd integer or *even* if r is an even integer. When considering a set of integers, it is said to be of *constant parity* if all the numbers in the set have the same parity.

2.26. A set of cones in R^n whose union is R^n is said to form a *partition of* R^n if each cone in the set has a nonempty interior and the intersection of the interiors of any two cones in the set is empty.

3. FINITENESS OF THE NUMBER OF COMPLEMENTARY FEASIBLE SOLUTIONS

3.1. Lemke [8] has shown that the number of complementary feasible solutions is finite whenever q is nondegenerate with respect to M. Here we determine the necessary and sufficient conditions under which the number of solutions to (1) is finite for each $q \in R^n$.

3.2. THEOREM. *The number of complementary feasible solutions is finite for all $q \in R^n$ if and only if M is nondegenerate.*

Proof. Suppose there exists a $q \in R^n$ such that (1) has an infinite number of distinct solutions. Each solution to (1) represents q as a nonnegative linear combination of some complementary set of column vectors. There are only 2^n distinct complementary sets of column vectors. Thus, if (1) has an infinite number of distinct solutions, there must exist a complementary set of column vectors $\{A_{\cdot j}, j = 1, \ldots, n\}$ such that

$$\sum_{j=1}^{n} A_{\cdot j} y_j = q,$$
$$y_j \geqslant 0 \quad \text{for each} \quad j = 1, \ldots, n, \tag{9}$$

has an infinite number of distinct solutions. Equation (9) is a square system of n equations in n nonnegative variables. If (9) has an infinite number of solutions, then the set of column vectors $\{A_{\cdot j}, j = 1, \ldots, n\}$

must be linearly dependent. Since $\{A_{\cdot j}\}$ is a complementary set of column vectors, this implies by the definition in 2.2 that M is degenerate.

To prove the converse, suppose M is degenerate. We will show that this implies the existence of a $q \neq 0$ for which (1) has an infinite number of solutions.

Case 1. Suppose one of the column vectors of M, say $M_{\cdot 1}$, is zero. Then let $q = (0, 1, 1, \ldots, 1)^T$. Then $(w; z) = (0, 1, 1, \ldots, 1; \alpha, 0, 0, \ldots, 0)$ is a complementary feasible solution for any $\alpha \geqslant 0$. Thus there are an infinite number of distinct complementary feasible solutions when $q = (0, 1, 1, \ldots, 1)^T$ in this case.

Case 2. Suppose $M_{\cdot 1} \neq 0$. Since M is not nondegenerate, there exists a complementary set of columns, say $\{A_{\cdot j}, j = 1, \ldots, n\}$, which is linearly dependent. So there exists $\alpha = (\alpha_1, \ldots, \alpha_n)^T \neq 0$ such that

$$\sum_{j=1}^{n} A_{\cdot j}\alpha_j = 0.$$

Also $A_{\cdot 1}$ is either $I_{\cdot 1}$ or $-M_{\cdot 1}$, and hence in this case $A_{\cdot 1} \neq 0$.

If $\sum_{j=1}^{n} A_{\cdot j} = 0$, let $q = A_{\cdot 1} \neq 0$. Then every $(w; z)$ obtained by setting the variable associated with $A_{\cdot 1}$ equal to $1 + \alpha$, the variable associated with $A_{\cdot j}$ equal to α for $j \neq 1$, and all other variables in $(w; z)$ equal to zero, is a complementary feasible solution for any $\alpha \geqslant 0$. Hence there are an infinite number of distinct complementary feasible solutions when $q = A_{\cdot 1} \neq 0$ in this case.

If $\sum_{j=1}^{n} A_{\cdot j} \neq 0$, let $q = \sum_{j=1}^{n} A_{\cdot j}$. Let

$$\theta = \min_{\substack{j \text{ such that} \\ \alpha_j < 0}} \left[\frac{1}{-\alpha_j}\right]$$

$$= +\infty \quad \text{if there does not exist any} \quad \alpha_j < 0.$$

So $\theta > 0$. Every $(w; z)$ obtained by setting the variable associated with $A_{\cdot j}$ equal to $1 + \lambda\alpha_j$ for $j = 1, \ldots, n$ and all the other variables in $(w; z)$ equal to zero is a complementary feasible solution for any λ such that $0 \leqslant \lambda \leqslant \theta$. Hence there are an infinite number of distinct complementary feasible solutions when $q = \sum_{j=1}^{n} A_{\cdot j} \neq 0$ in this case.

Hence if M is degenerate there exists a $q \neq 0$ for which (1) has an infinite number of distinct solutions.

3.3. Corollary. *If M is degenerate, there exists a $q \neq 0$ for which there are an infinite number of distinct complementary feasible solutions.*

3.4. Corollary. *If M is degenerate, the set of all q for which* (1) *has an infinite number of solutions is a subset of the union of all complementary cones which have empty interior.*

4. UNIQUENESS OF THE COMPLEMENTARY FEASIBLE SOLUTION

4.1. We now examine the question of when Eq. (1) has a unique solution for each $q \in R^n$, while M is fixed.

4.2. Theorem. *The system* (1) *has a unique solution for each $q \in R^n$ if and only if M is a P-matrix.*

Proof. A proof of this theorem based on mathematical induction is given in [12]. Here we give a much simpler proof due to Gale based on the sign reversal property of matrices discussed in [6]. Suppose M is not a P-matrix. Then by Theorem 2 of [6] there exists an $x \in R^n$, $x \neq 0$, such that $y = Mx$ and, for each $i = 1$ to n, x_i and y_i have opposite signs (i.e., $x_i y_i \leqslant 0$).

Let

$$\begin{aligned} y_i^+ &= y_i && \text{if} \quad y_i > 0, \\ &= 0 && \text{if} \quad y_i \leqslant 0; \\ y_i^- &= 0 && \text{if} \quad y_i \geqslant 0, \\ &= -y_i && \text{if} \quad y_i < 0; \\ x_i^+ &= x_i && \text{if} \quad x_i > 0, \\ &= 0 && \text{if} \quad x_i \leqslant 0; \\ x_i^- &= 0 && \text{if} \quad x_i \geqslant 0, \\ &= -x_i && \text{if} \quad x_i < 0. \end{aligned}$$

Then

$$y_i = y_i^+ - y_i^-, \quad y_i^+ \,\&\, y_i^- \geqslant 0; \qquad y_i^+ y_i^- = 0 \quad \text{for} \quad i = 1 \text{ to } n; \tag{10}$$

$$x_i = x_i^+ - x_i^-, \quad x_i^+ \,\&\, x_i^- \geqslant 0; \qquad x_i^+ x_i^- = 0 \quad \text{for} \quad i = 1 \text{ to } n. \tag{11}$$

Since $x_i y_i \leqslant 0$ for each $i = 1$ to n, we verify that $x_i^+ y_i^+ = x_i^- y_i^- = 0$ for each $i = 1$ to n. So

$$(y^+)^T x^+ = (y^-)^T x^- = 0. \tag{12}$$

Since $y = Mx$, we have

$$y^+ - Mx^+ = y^- - Mx^- = \hat{q}. \tag{13}$$

Since

$$x \neq 0, \qquad x^+ \neq x^-. \tag{14}$$

From (10)–(14) we conclude that, when $q = \hat{q}$, (1) has two distinct solutions, namely,

$$(w; z) = (y^+; x^+)$$

and

$$(w; z) = (y^-; x^-).$$

Thus, if M is not a P-matrix, there exists a $q \in R^n$ for which (1) has two distinct solutions.

Now suppose M is a P-matrix. Then, by Theorem 6 of [1], Eq. (1) has at least one solution for each $q \in R^n$. Suppose there exists a $q \in R^n$ for which (1) has two distinct solutions, namely, $(\bar{w}; \bar{z})$ and $(\hat{w}; \hat{z})$. Then

$$(\bar{w} - \hat{w}) = M(\bar{z} - \hat{z}) \tag{15}$$

and, since these two solutions are distinct, $\bar{z} - \hat{z} \neq 0$. From the complementarity condition

$$(\bar{w})^T \bar{z} = (\hat{w})^T \hat{z} = 0, \tag{16}$$

using (15) and (16), we verify that $(\bar{w}_i - \hat{w}_i)(\bar{z}_i - \hat{z}_i) \leqslant 0$ for all $i = 1$ to n and, since $\bar{z} - \hat{z} \neq 0$, this implies by Theorem 2 of [6] that M is not a P-matrix, which is a contradiction. So the solution to (1) must be unique for each $q \in R^n$.

4.3. Corollary. *Keeping M fixed, if* (1) *has at most one solution for each $q \in R^n$, then M is a P-matrix and* (1) *has exactly one solution for any $q \in R^n$.*

Proof. We have shown in the first part of the proof of Theorem 4.2 that, if M is not a P-matrix, then there exists a $\hat{q} \in R^n$ such that, when $q = \hat{q}$, (1) has at least two distinct solutions. So, under the hypothesis, M has to be a P-matrix and the corollary follows from Theorem 4.2.

4.4. COROLLARY. *If* (1) *has a unique solution for each* $q \in R^n$, *then any principal subproblem of* (1) *has a unique solution for each of its right-hand constant vectors.*

Proof. This follows from Theorem 4.2 and the fact that every principal submatrix of a P-matrix is also a P-matrix.

4.5. COROLLARY. *The set of complementary cones obtained as in Paragraphs* 2.9 *and* 2.10 *forms a partition of* R^n *if and only if* M *is a* P-*matrix.*

Proof. If M is a P-matrix, then by Theorem 4.2, for each $q \in R^n$, (1) has a unique solution. So the union of all the complementary cones in the whole space R^n. Since M is a P-matrix, it is nondegenerate and hence every complementary cone has a nonempty interior. Also, since by Paragraph 2.10 the system (1) has a unique solution for each $q \in R^n$ any $q \in R^n$ which lies in the interior of a complementary cone does not lie in any other complementary cone. Hence the intersection of the interiors of any pair of complementary cones is empty. Therefore the set of complementary cones forms a partition of R^n.

Conversely, if the complementary cones form a partition of R^n, every complementary cone must have a nonempty interior and hence M must be nondegenerate. Also, every $q \in R^n$ which lies in the interior of a complementary cone does not lie in any other complementary cone by the partition property and hence, by Paragraph 2.10, for all such q, (1) has a unique solution. Each complementary set of column vectors is linearly independent and, since the set of complementary cones partition R^n, this implies that (1) has a unique solution corresponding to each $q \in R^n$ lying on the face of any complementary cone. So (1) has a unique solution for each $q \in R^n$ and hence, by Theorem 4.2, M must be a P-matrix.

4.6. *Note.* After Theorem 4.2 was conjectured and proved, a theorem by Samelson, Thrall, and Wesler [14] on the partition of R^n by convex cones which is equivalent to it and to Corollary 4.5 has come to our notice.

Their proof was based on geometric considerations, while the proof given here is based on the properties of P-matrices. However, in Theorems 7.2 and 7.10, we adopt geometric procedures much like theirs.

4.7. As a generalization of Corollary 4.4, it is interesting to check whether every principal submatrix of a Q-matrix is also a Q-matrix. This is not necessarily true, as the following example illustrates. Let

$$M = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}. \tag{17}$$

In Fig. 1 each complementary cone is indicated by a dotted line segment running across its generators. In this case, we verify that the union of the complementary cones is the whole space, R^2.

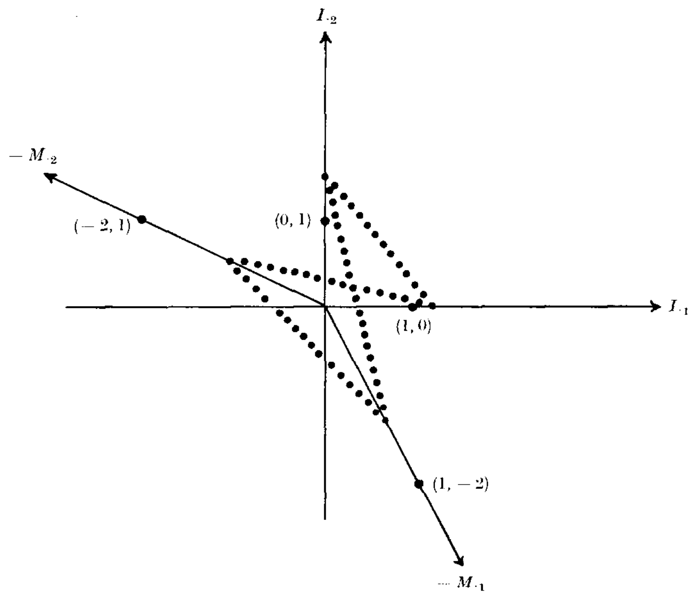

FIG. 1

So, by Paragraph 2.10, the matrix M in (17) is a Q-matrix. Now we examine the principal submatrix obtained by striking off the first row and column.

$$N = (-1).$$

This is not a Q-matrix, because the associated principal subproblem

$$w_2 - (-1)z_2 = q_2,$$
$$w_2 \geqslant 0, \quad z_2 \geqslant 0, \quad w_2 z_2 = 0$$

has no solution if $q_2 < 0$. However, if M is a Q-matrix and $M \geqslant 0$, then every principal submatrix of M is also a Q-matrix. This will be proved under Corollary 5.3.

The property of "uniqueness" of the solution to (1) also affects the nature of the solution. This is discussed below.

4.8. THEOREM. *Suppose M has the property that* (1) *has a unique solution for each $q \in R^n$. Keep $q_2, \ldots, q_n$ fixed but let q_1 vary. Let $z_1(q_1)$ be the value of z_1 in the solution to* (1) *as a function of q_1. Then $z_1(q_1)$ is monotonic decreasing in q_1, and it is strictly monotonic decreasing in the region in which it is positive.*

Proof. Proof by contradiction. Let $\mathscr{Q} = (q_2, \ldots, q_n)^T$, which is held fixed. Pick any value for q_1 and let $\beta > 0$ be arbitrary. Let $(\hat{w}; \hat{z})$ be the solution to (1) when

$$q = \begin{bmatrix} q_1 \\ \mathscr{Q} \end{bmatrix},$$

$(\tilde{w}; \tilde{z})$ be the solution to (1) when

$$q = \begin{bmatrix} q_1 + \beta \\ \mathscr{Q} \end{bmatrix}.$$

Then $z_1(q_1) = \hat{z}_1$ and $z_1(q_1 + \beta) = \tilde{z}_1$. Let

$$\hat{\omega} = (\hat{w}_2, \ldots, \hat{w}_n)^T, \qquad \hat{\xi} = (\hat{z}_2, \ldots, \hat{z}_n)^T,$$
$$\tilde{\omega} = (\tilde{w}_2, \ldots, \tilde{w}_n)^T, \qquad \tilde{\xi} = (\tilde{z}_2, \ldots, \tilde{z}_n)^T,$$

and

$$m_{\cdot 1} = (m_{21}, \ldots, m_{n1})^T.$$

If $\tilde{z}_1 > 0$ we wish to show that $\tilde{z}_1 < \hat{z}_1$. Suppose not; then $\tilde{z}_1 \geqslant \hat{z}_1 > 0$. By complementarity, $\tilde{w}_1 = \hat{w}_1 = 0$. Then, if

$$q = \begin{bmatrix} q_1 + \beta + m_{11}\hat{z}_1 \\ \mathscr{Q} + m_{\cdot 1}\hat{z}_1 \end{bmatrix},$$

(1) has two solutions, namely,

$$w = \begin{bmatrix} \beta \\ \hat{w} \end{bmatrix}, \qquad z = \begin{bmatrix} 0 \\ \hat{\xi} \end{bmatrix},$$

and

$$w = \begin{bmatrix} 0 \\ \tilde{w} \end{bmatrix}, \qquad z = \begin{bmatrix} \tilde{z}_1 - \hat{z}_1 \\ \xi \end{bmatrix},$$

contradicting the uniqueness of the solution to (1) for each $q \in R^n$. Hence, if $z_1(q_1) > 0$, then $z_1(q_1 + \beta) < z_1(q_1)$ for any $\beta > 0$.

It remains to be shown that if $z_1(q_1) = 0$ then $z_1(q_1 + \beta) = 0$ for all $\beta > 0$. Suppose not; then $z_1(q_1) = \hat{z}_1 = 0$ and $z_1(q_1 + \beta) = \tilde{z}_1 > 0$. Then, if

$$q = \begin{bmatrix} q_1 + \beta \\ \mathcal{Q} \end{bmatrix},$$

(1) has two solutions, namely,

$$\tilde{w}; \tilde{z} \quad \text{and} \quad w = \begin{bmatrix} \hat{w}_1 + \beta \\ \hat{w} \end{bmatrix}; \qquad z = \begin{bmatrix} 0 \\ \hat{\xi} \end{bmatrix} = \hat{z},$$

which is again a contradiction.

4.9. We now show that, if M is a Q-matrix and (1) has a unique solution when q is any element of the set $\{I_{\cdot 1}, I_{\cdot 2}, \ldots, I_{\cdot n}; -M_{\cdot 1}, \ldots, -M_{\cdot n}\}$, then M is a P-matrix.

4.10. THEOREM. *Let*

$[z_j]$ = *union of all complementary cones which contain* $-M_{\cdot j}$ *as a generator,*

$[w_j]$ = *union of all complementary cones which contain* $I_{\cdot j}$ *as a generator.*

If $I_{\cdot j} \notin [z_j]$ *and* $-M_{\cdot j} \notin [w_j]$ *for each* $j = 1$ *to* n *and* M *is a* Q*-matrix, then* M *is a* P*-matrix.*

Proof.

4.11. Let N be the principal submatrix of M of order $(n-1)$, obtained by striking off the first row and column of M. We now show that N is also a Q-matrix.

Suppose not. Let $\omega = (w_2, \ldots, w_n)^T$ and $\xi = (z_2, \ldots, z_n)^T$. Consider the principal subproblem in $(\omega; \xi)$, which is to solve

$$\omega - N\xi = \mathscr{Q}$$
$$\omega \geqslant 0, \qquad \xi \geqslant 0, \qquad \omega^T\xi = 0. \tag{18}$$

If N is not a Q-matrix, there exists a $\bar{\mathscr{Q}} \in R^{n-1}$ such that, when $\mathscr{Q} = \bar{\mathscr{Q}}$, (18) has no solution.

Let

$$\bar{q} = \begin{bmatrix} \alpha \\ \cdots \\ \bar{\mathscr{Q}} \end{bmatrix}.$$

If (1) has a solution $(\bar{w}; \bar{z})$ when $q = \bar{q}$, then $\bar{z}_1 > 0$ in it, as otherwise $(\bar{\omega}; \bar{\xi})$ would be a solution to (18). Thus every point on the line

$$\left\{\bar{q}\colon \bar{q} = \begin{pmatrix} \alpha \\ \cdots \\ \bar{\mathscr{Q}} \end{pmatrix}, \quad \alpha \text{ real}\right\}$$

corresponds only to complementary feasible solutions in which $z_1 > 0$. Since there are only a finite number of complementary cones and each one is convex, there must exist an α_0 such that the half-line

$$\left\{q\colon q = \begin{pmatrix} \alpha_0 \\ \cdots \\ \bar{\mathscr{Q}} \end{pmatrix} + \theta I_{\cdot 1}, \quad \theta \geqslant 0\right\}$$

lies entirely in a complementary cone. By the argument above, in every complementary feasible solution corresponding to any point on this half-line we must have $z_1 > 0$. This implies that this half-line lies in a complementary cone for which $-M_{\cdot 1}$ is a generator. This implies that pos$\{I_{\cdot 1}\}$ also lies in the same complementary cone, i.e., $I_{\cdot 1} \in [z_1]$, which is a contradiction. Hence N must be a Q-matrix. By a similar argument we conclude that all principal submatrices of M of order $(n - 1)$ must be Q-matrices.

4.12. Let $\mathscr{N}$ be the principal submatrix of M of order $(n - 2)$ obtained by striking off the first two rows and columns from M. We will now show that $\mathscr{N}$ must be a Q-matrix also.

Suppose not. Then there exists a $\bar{Q} \in R^{n-2}$ such that the subproblem in $(w_3, \ldots, w_n; z_3, \ldots, z_n)$ has no complementary feasible solution when its right-hand constant vector is $\bar{Q}$.

Let

$$\bar{q} = \begin{bmatrix} \lambda \\ \mu \\ \bar{Q} \end{bmatrix}. \tag{19}$$

Then, for any λ, μ real, (1) has no solutions in which both z_1 and z_2 are equal to zero, when $q = \bar{q}$ by paragraph 2.24.

4.13. Fix $\lambda = \lambda_1$, $\mu = \mu_1$, where $\lambda_1 > 0$ and $\mu_1 > 0$, and consider the line

$$\left\{q\colon\ q = \begin{bmatrix} \alpha\lambda_1 \\ \alpha\mu_1 \\ \bar{Q} \end{bmatrix},\quad \alpha \text{ real}\right\}.$$

Points on this line have complementary feasible solutions in which both z_1 and z_2 cannot be zero together. Since the number of complementary cones is finite and each is convex, there must exist an α_0 such that the entire half-line

$$\left\{q\colon\ q = \begin{bmatrix} \alpha\lambda_1 \\ \alpha\mu_1 \\ \bar{Q} \end{bmatrix},\quad \alpha > \alpha_0\right\} \tag{20}$$

is in a complementary cone. Suppose this half-line is in the complementary cone $\text{pos}\{I_{\cdot 1}, -M_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$. Then

$$\lambda_1 I_{\cdot 1} + \mu_1 I_{\cdot 2} \in \text{pos}\{I_{\cdot 1}, -M_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}.$$

Suppose

$$\lambda_1 I_{\cdot 1} + \mu_1 I_{\cdot 2} = \alpha_1 I_{\cdot 1} + \alpha_2(-M_{\cdot 2}) + \sum_{j=3}^{n} \alpha_j A_{\cdot j},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 0$.

If $\alpha_1 \geqslant \lambda_1$, then, if we put $(\lambda, \mu) = (0, \mu_1)$, $\bar{q}$ of (19) will have a complementary feasible solution in which both $z_1 = z_2 = 0$, which is a contradiction to paragraph 4.12.

If $\alpha_1 < \lambda_1$, then $(\lambda_1 - \alpha_1)I_{\cdot 1} + \mu_1 I_{\cdot 2}$ lies in the intersection of $\text{pos}\{-M_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ with $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$. We note that $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$ cannot entirely lie in $\text{pos}\{-M_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ because then $I_{\cdot 1} \in [z_1]$ and $I_{\cdot 2} \in [z_2]$, contradicting the hypothesis. So $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$ and

$\text{pos}\{-M_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ intersect in a half-line and, when $(\lambda, \mu) \geqslant 0$, unless λ, μ are such that

$$\mu = \mu_1,$$
$$\lambda \geqslant \lambda_1 - \alpha_1,$$

we have

$$\lambda I_{\cdot 1} + \mu I_{\cdot 2} \notin \text{pos}\{I_{\cdot 1}, -M_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}.$$

Similarly we see that, for $(\lambda, \mu) \geqslant 0$, if $\lambda I_{\cdot 1} + \mu I_{\cdot 2}$ is contained in a complementary cone $\text{pos}\{-M_{\cdot 1}, I_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$, then $\lambda I_{\cdot 1} + \mu I_{\cdot 2}$ must lie on some half-line in $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$.

Hence, when $(\lambda_1, \mu_1) \geqslant 0$, unless $(\lambda_1 I_{\cdot 1} + \mu_1 I_{\cdot 2})$ lies in the union of a finite number of half-lines in $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$, the half-line in (20) can only be contained in a complementary cone for which both $-M_{\cdot 1}$ and $-M_{\cdot 2}$ are generators. This implies that all the points in $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$ excepting those lying on a finite number of half-lines belong to the union of all complementary cones containing both $-M_{\cdot 1}$ and $-M_{\cdot 2}$ as generators. But this union is a closed cone and, if it contains all points of $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$ excepting those lying on a finite number of half-lines, then it contains all of $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$. This implies that $I_{\cdot 1}$ lies in some complementary cone which has both $-M_{\cdot 1}$ and $-M_{\cdot 2}$ as generators, which is a contradiction to the hypothesis.

So $\mathcal{N}$ must be a Q-matrix. By a similar argument we can show that every principal submatrix of M is a Q-matrix. Hence all the elements in the principal diagonal of M must be strictly positive.

From the hypothesis of the theorem we see that every matrix $\tilde{M}$, obtained from M by performing a series of principal pivots (as in Paragraph 2.20) has the property that all its diagonal elements are strictly positive.

By Tucker's theorem [17] (see also Lemma 6.1 in [13]) this implies that M is a P-matrix.

4.14. *Note.* It may be possible to use Theorem 4.10 to develop an efficient algorithm for testing whether a given real square matrix M is a P-matrix or not.

5. ON THE Q-NATURE OF NONNEGATIVE MATRICES

5.1. Suppose the square matrix M is nonnegative, i.e., $m_{ij} \geqslant 0$ for each i and j. This case is of particular interest because the problem of finding a Nash equilibrium point of a bimatrix game can be formulated as a problem of the form (1) in which $M \geqslant 0$. See [1]. It is of interest to know when such a matrix is a Q-matrix. The following theorem discusses this question.

5.2. THEOREM. *Let $M \geqslant 0$. M is a Q-matrix if and only if $m_{ii} > 0$ for each $i = 1, \ldots, n$.*

Proof of sufficiency. This has been proved by Cottle and Dantzig [1] in a corollary under their Theorem 5.

Proof of necessity. Suppose one of the principal diagonal elements of M, say, $m_{11} = 0$. Then we will construct a $\hat{q} \in R^n$ such that when $q = \hat{q}$, (1) has no solution. Let

$$\hat{q} = (-1, 1, 1, \ldots, 1)^T. \tag{21}$$

If $q = \hat{q}$, (1) becomes

$$w = Mz + \hat{q},$$
$$w \geqslant 0, \qquad z \geqslant 0, \qquad w^T z = 0. \tag{22}$$

In any feasible solution, $w_i = M_{i\cdot}z + \hat{q}_i$ and so $w_i > 0$ since $M_{i\cdot} \geqslant 0$, $z \geqslant 0$, and $\hat{q}_i = 1 > 0$, for each $i = 2$ to n. Hence in any complementary feasible solution of (22) we must have $z_i = 0$ for each $i = 2$ to n.

However, if $z_2 = \cdots = z_n = 0$,

$$w_1 = M_{1\cdot}z + \hat{q}_1$$

implies that $w_1 = -1 < 0$ since $m_{11} = 0$ and $\hat{q}_1 = -1$.

Thus (22) has no complementary feasible solution and hence, when $q = \hat{q}$, (1) has no solution.

The example in (21) is due to Gale.

5.3. COROLLARY. *If $M \geqslant 0$ and M is a Q-matrix, then all principal submatrices of M are also Q-matrices.*

5.4. COROLLARY. *If $M \geqslant 0$, the union of all the complementary cones is the whole space R^n if and only if $m_{ii} > 0$ for all $i = 1$ to n.*

6. ON THE CONSTANT PARITY OF THE NUMBER OF COMPLEMENTARY FEASIBLE SOLUTIONS

6.1. We now examine how the number of solutions to (1) varies as q varies over R^n while M is fixed.

6.2. THEOREM. *If M is nondegenerate, then the number of complementary feasible solutions has the same parity for all $q \in R^n$ which are nondegenerate with respect to M.*

Proof. In the proof of this theorem we use some of the results proved by Lemke in [8].

6.3. *Results* (Lemke). If $\tilde{q}$ is nondegenerate with respect to M:

(i) Then (1) has a finite number of solutions when $q = \tilde{q}$.

(ii) For each $i = 1, \ldots, n$, the almost complementary set $C_i(\tilde{q})$ either is empty or is the union of some edges (bounded or unbounded) of $K(\tilde{q})$.

(iii) The number of unbounded edges in $C_i(\tilde{q})$ differs from the number of solutions to (1) by an even number.

6.4. We now prove that, if M is nondegenerate and $\tilde{q} \in R^n$, then w_i is unbounded on every unbounded edge of $C_i(\tilde{q})$. Suppose F is an unbounded edge of $K(\tilde{q})$ contained in $C_i(\tilde{q})$. Let

$$F = \{(w; z)\colon\ (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2),\quad \theta \geqslant 0\},$$

where $(w^1; z^1)$ is a basic feasible solution and $(w^2; z^2)$ is an extreme homogeneous solution of $K(\tilde{q})$.

Along this unbounded edge F, $w_i = w_i^1 + \theta w_i^2$, and it remains bounded for all $\theta \geqslant 0$ only if $w_i^2 = 0$. If $w_i^2 = 0$, then, if we put $q = \tilde{q} - w_i^1 I_{\cdot i}$, (1) has an infinite number of solutions, namely,

$$w = w^1 - w_i^1 I_{\cdot i} + \theta w^2; \qquad z = z^1 + \theta z^2 \quad \text{for all} \quad \theta \geqslant 0,$$

which is a contradiction to the hypothesis that M is nondegenerate by Theorem 3.2.

Thus, on every unbounded edge of $C_i(\tilde{q})$, w_i is unbounded.

6.5. We now use the result obtained in Paragraph 6.4 to show that, if $\tilde{q} \in R^n$ is nondegenerate with respect to M and α is any real number such that $\hat{q} = \tilde{q} + \alpha I_{\cdot i}$ is also nondegenerate with respect to M, then the number of unbounded edges in $C_i(\tilde{q})$ and $C_i(\hat{q})$ are the same.

Pick any unbounded edge

$$F = \{(w; z) \colon (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2), \quad \theta \geqslant 0\} \subset C_i(\tilde{q}).$$

By 6.4, $w_i^2 > 0$ and w_i is unbounded on this edge. Let

$$\nu = \max\left[0, \frac{w_i^1 + \alpha}{-w_i^2}\right].$$

Then it is easily verified that

$$F^1 = \{(w; z) \colon (w; z) = (w^1 + \theta w^2 + \alpha I_{\cdot i}; z^1 + \theta z^2), \quad \theta \geqslant \nu\}$$

is an unbounded edge in $C_i(\hat{q})$.

Thus we have shown that there exists an unbounded edge F^1 in $C_i(\hat{q})$ corresponding to each unbounded edge F in $C_i(\tilde{q})$. Conversely, by treating $\tilde{q} = \hat{q} + (-\alpha)I_{\cdot i}$ we can establish a correspondence between unbounded edges in $C_i(\hat{q})$ and those of $C_i(\tilde{q})$. This establishes a 1-to-1 correspondence between the unbounded edges in $C_i(\tilde{q})$ and those in $C_i(\hat{q})$. Hence both $C_i(\tilde{q})$ and $C_i(\hat{q})$ must have the same number of unbounded edges.

6.6. Now, to continue the proof of Theorem 6.2, let $\tilde{q}$ and $\bar{q}$ be any two-column vectors in R^n both of which are nondegenerate with respect to M.

By Paragraph 6.5 and (i) and (iii) of 6.3, we conclude that the parity of the number of solutions to (1) does not change if we alter the vector q one component at a time so that it remains nondegenerate with respect to M both before and after the alteration.

It is always possible to alter $\tilde{q}$ by one component at a time, retaining the property of being nondegenerate with respect to M throughout, until it becomes equal to $\bar{q}$.

Hence the number of solutions to (1) has the same parity whether $q = \tilde{q}$ or $\bar{q}$. Thus the number of solutions to (1) has the same parity whenever q is nondegenerate with respect to M.

6.7. *Note.* The assumption that M is nondegenerate cannot be dropped from the hypothesis of Theorem 6.2, as can be seen from the example below. Let

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then (1) is to solve

w_1	w_2	z_1	z_2	$= q$
1	0	0	-1	q_1
0	1	-1	-2	q_2

$$w_1, w_2, z_1, z_2 \geqslant 0, \qquad w_1 z_1 + w_2 z_2 = 0.$$

Only complementary cones with nonempty interiors are indicated in Fig. 2.

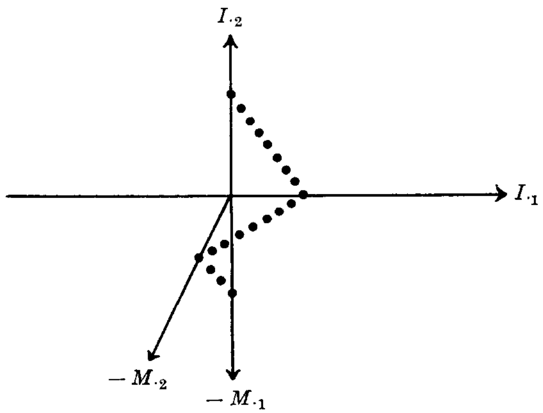

FIG. 2

$$q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

leads to one solution to (1) and

$$q = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

leads to two solutions to (1) even though both these are nondegenerate with respect to M here.

Here M is degenerate because the matrix $(I_{\cdot 2}, -M_{\cdot 1})$ is singular. The argument in Paragraph 6.4 fails.

6.8. *Note.* The assumption that q is nondegenerate with respect to M cannot be dropped from the hypothesis of Theorem 6.2, as can be seen from Fig. 3. Let

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

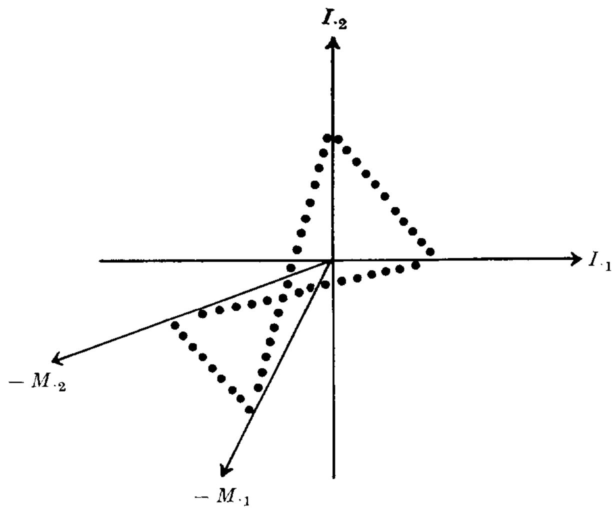

FIG. 3

When q is nondegenerate with respect to M the number of solutions to (1) is an odd number but, when $q = -M_{\cdot 2}$, (1) has exactly two distinct solutions.

6.9. COROLLARY. *If M is nondegenerate and not a Q-matrix, then the number of solutions to* (1) *is an even number for all q which are nondegenerate with respect to M.*

6.10. *Proof.* By Paragraph 2.10, the set of all q for which (1) has a solution is the union of the 2^n complementary cones. Each complementary cone is a closed set in R^n, and hence their union (being a union of a finite number of closed sets) is itself closed. The set of all q for which (1) has no solution is the complement of this union, and thus is an open set. Because M is not a Q-matrix, this open set is nonempty. Therefore the set of all q for which (1) has no solution is a nonempty open cone.

6.11. By Paragraph 2.16, the set of all q which are degenerate is the union of a finite number of subspaces of R^n, each of which has dimension $\leqslant (n-1)$. Hence the set of all q which are degenerate with respect to M has no interior.

6.12. By 6.10 and 6.11, we conclude that there must exist a q nondegenerate with respect to M, for which (1) has no solution, i.e., zero solutions. Now, by applying Theorem 6.2, we conclude that the number of solutions to (1) has the same parity as zero, i.e., even parity, whenever q is nondegenerate with respect to M.

6.13. *Note.* Corollary 6.9 is not necessarily true if M is degenerate, as seen from Example 6.7.

6.14. *Note.* The converse of Corollary 6.9 is not necessarily true unless $M \geqslant 0$. This is discussed in Note 8.17.

7. ON PROBLEMS WITH A CONSTANT NUMBER OF COMPLEMENTARY FEASIBLE SOLUTIONS

7.1. Here we show that, if the number of complementary feasible solutions is a constant for all nonzero $q \in R^n$, then that constant is equal to one and M is a P-matrix. We also show that, if the number of complementary feasible solutions is a constant for all $q \in R^n$ nondegenerate with respect to M, then that constant is equal to one and in this case all the complementary cones which have nonempty interiors form a partition of R^n.

7.2. THEOREM. *If the number of complementary feasible solutions is a constant for all $q \in R^n$, $q \neq 0$, then M is a P-matrix and that constant is equal to one.*

Proof.

7.3. Whatever M may be, (1) always has at least one solution for every $q \geqslant 0$ (the solution is $w = q$; $z = 0$). If M is not a Q-matrix, there exists a $q \neq 0$ for which (1) has no solution at all. Hence, if M is not a Q-matrix, the number of solutions to (1) cannot be a constant for all $q \neq 0$. Thus, under the hypothesis of Theorem 7.2, M must be a Q-matrix.

7.4. The number of complementary feasible solutions is finite whenever q is nondegenerate with respect to M. By Corollary 3.3, if M is not nondegenerate, there exists a $q \neq 0$ for which (1) has an infinite number of solutions.

Since the number of solutions to (1) is a constant for any $q \neq 0$, M must therefore be a nondegenerate matrix. Hence, all the principal submatrices of M are also nondegenerate. Also, every subcomplementary set of column vectors is linearly independent and every complementary cone has a nonempty interior.

7.5. Let $\{A_{\cdot 1}, \ldots, A_{\cdot n-1}\}$ be any subcomplementary set of column vectors. We now show that the hyperplane generated by this subcomplementary set of column vectors strictly separates the points representing the left-out complementary pair of column vectors $I_{\cdot n}$ and $-M_{\cdot n}$.

Suppose not. Then the interiors of the complementary cones $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}, I_{\cdot n}\}$ and $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}, -M_{\cdot n}\}$ have a nonempty intersection. Consider two hyperplanes (see Fig. 4) each of which is

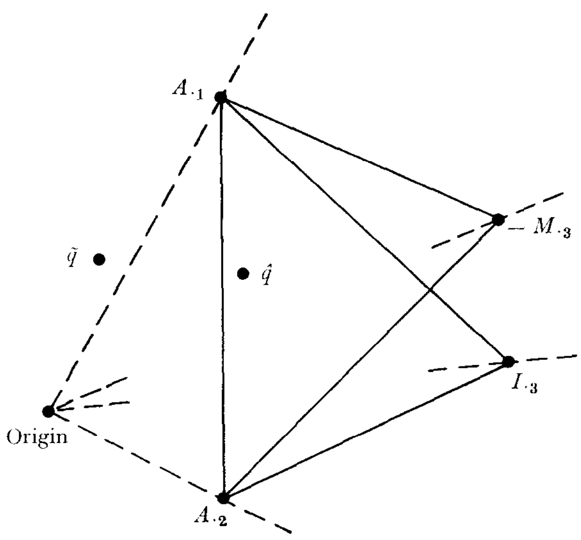

FIG. 4

generated by a linearly independent subset of $(n-1)$ column vectors of $(I, -M)$. If these two hyperplanes are distinct, then their intersection is a subspace of dimension $(n-2)$. The set $\{A_{\cdot 1}, \ldots, A_{\cdot n-1}\}$ is linearly independent, and there are only a finite number of subspaces generated by subsets of $(n-1)$ or less column vectors of $(I, -M)$. So we can pick a point

$$q^* = \sum_{j=1}^{n-1} \lambda_j A_{\cdot j},$$

where $\lambda_1, \ldots, \lambda_{n-1}$ are all > 0, such that q^* is not in any subspace generated by $(n-2)$ or less column vectors of $(I, -M)$ and also not in the intersection of $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}\}$ with a hyperplane which is distinct from the hyperplane through $A_{\cdot 1}, \ldots, A_{\cdot n-1}$ and is generated by a subset of $(n-1)$ linearly independent column vectors of $(I, -M)$.

So, if q^* is also on the hyperplane through some other subcomplementary set of column vectors, then that hyperplane must coincide with the hyperplane through $A_{\cdot 1}, \ldots, A_{\cdot n-1}$. Since both $I_{\cdot n}$ and $-M_{\cdot n}$ are not on the hyperplane through $A_{\cdot 1}, \ldots, A_{\cdot n-1}$, we conclude that, if q^* lies in the hyperplane through some other subcomplementary set of column vectors, then that set must be of the form $\{B_{\cdot 1}, \ldots, B_{\cdot n-1}\}$, where $B_{\cdot j} \in \{I_{\cdot j}, -M_{\cdot j}\}$ for $j = 1$ to $n-1$.

The intersection of the cones $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}, I_{\cdot n}\}$ and $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}, -M_{\cdot n}\}$ has a nonempty interior. Hence, by the choice of q^*, we can find an $\alpha > 0$ sufficiently small that, if we pick

$$\hat{q} = q^* + \alpha I_{\cdot n},$$

then $\hat{q}$ is nondegenerate with respect to M and it is in the intersection of both the cones. By the nondegeneracy of $\hat{q}$, $\hat{q} + \lambda_n I_{\cdot n}$ is also nondegenerate for all but a finite number of values of λ_n. So we could pick an $\alpha_0 > 0$ sufficiently small that

$$q = \sum_{j=1}^{n-1} \lambda_j A_{\cdot j} + \beta I_{\cdot n}$$

is nondegenerate with respect to M for all β satisfying $\beta \neq 0$, $-\alpha_0 < \beta < \alpha_0$. Hence, if we had chosen our original α so small that $0 < \alpha < \alpha_0$, and $\hat{q}$ is in the interior of both $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}, I_{\cdot n}\}$ and $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n-1}, -M_{\cdot n}\}$, then

$$\tilde{q} = \sum_{j=1}^{n-1} \lambda_j A_{\cdot j} - \alpha I_{\cdot n}$$

is outside both these complementary cones and is strictly separated from $\hat{q}$ by the hyperplane through $A_{\cdot 1}, \ldots, A_{\cdot n-1}$. Also q^* is the only point which is degenerate on the closed line segment joining the points $\hat{q}$ and $\tilde{q}$.

Thus, if $\tilde{q}$ lies in any complementary cone in which $\hat{q}$ does not lie, then q^* must lie on a face of that complementary cone. Hence by the choice of q^* we conclude that, if $\tilde{q}$ lies in any complementary cone in which $\hat{q}$ does not lie, then that cone must be of the form pos$\{B_{\cdot 1}, \ldots, B_{\cdot n}\}$, where $B_{\cdot j}$ is contained in $\{I_{\cdot j}, -M_{\cdot j}\}$ for $j = 1$ to n, and the hyperplane through $B_{\cdot 1}, \ldots, B_{\cdot n-1}$ must coincide with the hyperplane through $A_{\cdot 1}, \ldots, A_{\cdot n-1}$. But the hyperplane through $A_{\cdot 1}, \ldots, A_{\cdot n-1}$ separates $\tilde{q}$ from both $I_{\cdot n}$ and $-M_{\cdot n}$. So $\tilde{q}$ cannot lie in any complementary cone in which $\hat{q}$ does not lie.

Hence the number of solutions to (1) when $q = \tilde{q}$ is strictly less (at least by two) than the number when $q = \hat{q}$, leading to a contradiction.

By a similar argument, we verify that the hyperplane through any subcomplementary set of column vectors strictly separates the points representing the left-out complementary pair of column vectors.

7.6. We now show that the principal subproblem of (1) in the variables $(w_1, \ldots, w_{n-1}; z_1, \ldots, z_{n-1})$ satisfies a similar separation property. The column vector corresponding to w_j in this subproblem is the jth column vector of the unit matrix of order $(n-1)$, which we denote by $\mathscr{I}_{\cdot j}$, and the column vector corresponding to z_j in this subproblem is $-(m_{1j}, m_{2j}, \ldots, m_{n-1,j})^T$ which we denote by $-m_{\cdot j}$. We note that the column vectors in the subproblem are obtained by deleting the last component from the column vectors in the original problem.

Let $\{a_{\cdot 1}, \ldots, a_{\cdot i-1}, a_{\cdot i+1}, \ldots, a_{\cdot n-1}\}$ be any subcomplementary set of column vectors in the subproblem. We want to show that the hyperplane in R^{n-1} through these column vectors strictly separates $\mathscr{I}_{\cdot i}$ and $-m_{\cdot i}$.

Let $A_{\cdot r}$ be the column vector corresponding to $a_{\cdot r}$, $r = 1, \ldots, i-1, i+1, \ldots, n-1$, in the original problem. Then $\{A_{\cdot 1}, \ldots, A_{\cdot i-1}, A_{\cdot i+1}, \ldots, A_{\cdot n-1}, I_{\cdot n}\}$ is a subcomplementary set in the original problem. By Paragraph 7.5 the hyperplane in R^n through these column vectors strictly separates $I_{\cdot i}$ and $-M_{\cdot i}$. Suppose this hyperplane is

$$DX = 0,$$

where $D = (d_1, \ldots, d_n)$ and $X \in R^n$. Then

$$\begin{aligned} DA_{\cdot 1} &= 0, \\ &\vdots \\ DA_{\cdot i-1} &= 0, \end{aligned}$$

$$\begin{aligned} DI_{\cdot i} &> 0 \quad \text{and} \quad D(-M_{\cdot i}) < 0, \\ DA_{\cdot i+1} &= 0, \\ &\vdots \\ DA_{\cdot n-1} &= 0, \\ DI_{\cdot n} &= 0. \end{aligned}$$

Now $DI_{\cdot n} = d_n = 0$. Let $d = (d_1, \ldots, d_{n-1})$. Because $d_n = 0$, the preceding equations imply that

$$\begin{aligned} da_{\cdot 1} &= 0, \\ &\vdots \\ da_{\cdot i-1} &= 0, \\ d\mathscr{I}_{\cdot i} &> 0 \quad \text{and} \quad d(-m_{\cdot i}) < 0, \\ da_{\cdot i+1} &= 0, \\ &\vdots \\ da_{\cdot n-1} &= 0. \end{aligned}$$

Let $x = (x_1, \ldots, x_{n-1}) \in R^{n-1}$. Thus $dx = 0$ is the hyperplane in R^{n-1} through the subcomplementary set $\{a_{\cdot 1}, \ldots, a_{\cdot i-1}, a_{\cdot i+1}, \ldots, a_{\cdot n-1}\}$ of the subproblem and it strictly separates $\mathscr{I}_{\cdot i}$ and $-m_{\cdot i}$.

Hence the subproblem also satisfies a similar separation property. By a similar argument we can verify that every principal subproblem of (1) of order $(n-1)$ satisfies the separation property.

7.7. *Induction hypothesis.* For any complementarity problem of order $r \leqslant n-1$, with column vectors $(I, -N)$, if N is nondegenerate and if the hyperplane through every subcomplementary set of column vectors strictly separates the points representing the left-out complementary pair of column vectors in the problem, then N is a P-matrix.

7.8. The induction hypothesis is easily verified for the case $r = 1$. By nondegeneracy, $N = (m_{11}) \neq 0$. Since $r = 1$, the subcomplementary set is the null set and hence the hyperplane through the subcomplementary set is the singleton consisting of the origin itself. Since this separates the points on R^1 representing 1 and $-m_{11}$, we should have $-m_{11} < 0$. So $N = (m_{11}) > 0$ and hence is a P-matrix in this case.

7.9. Hence, by the induction hypothesis and 7.6, every principal submatrix of M or order $n-1$ is a P-matrix. Thus all principal subdeterminants of M of order $\leqslant n-1$ are strictly positive.

Since M is nondegenerate by 7.4, determinant of $M \neq 0$. So M^{-1} exists. Thus the constraints (1) can be written as

$$z - M^{-1}w = Q,$$

$$z \geqslant 0, \quad w \geqslant 0, \quad z^T w = 0, \tag{23}$$

where

$$Q = -M^{-1}q.$$

If (1) has a constant number of solutions for every $q \in R^n$, $q \neq 0$, then (23) has a constant number of solutions for each $Q \in R^n$, $Q \neq 0$.

Hence, by the arguments used previously, all principal subdeterminants of M^{-1} or order $(n-1)$ or less are strictly positive. Let α be the value of the principal subdeterminant of M^{-1} obtained by striking off the first row and column from M^{-1}. Then

$$\alpha = \frac{m_{11}}{\text{determinant of } M}\,. \tag{24}$$

But $\alpha > 0$, $m_{11} > 0$. Thus, by (24), the determinant of M is also strictly positive. So all principal subdeterminants of M are strictly positive. Hence M is a P-matrix.

So the induction hypothesis 7.7 holds when $r = n$ also. It has been verified for $r = 1$ in 7.8. Hence by induction it holds for all n.

Thus, by 7.5, Theorem 7.2 is true for all n.

7.10. Theorem. *If the number of complementary feasible solutions is a constant for all $q \in R^n$, which are nondegenerate with respect to M, then that constant is equal to one and, in this case, the set of complementary cones with nonempty interiors forms a partition for R^n.*

Proof.

7.11. Whatever M may be, (1) has a solution if $q \geqslant 0$ and we can always find a q nondegenerate with respect to M which is nonnegative.

If M is not a Q-matrix by 6.12, we can find a q nondegenerate with respect to M, for which (1) has no solution. Hence, if the number of solutions to (1) is a constant for all q nondegenerate with respect to M, M must be a Q-matrix.

7.12. Let $\{A_{\cdot 1}, \ldots, A_{\cdot i-1}, A_{\cdot i+1}, \ldots, A_{\cdot n}\}$ be any subcomplementary set of column vectors. If the two complementary cones $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot i-1}, I_{\cdot i}, A_{\cdot i+1}, \ldots, A_{\cdot n}\}$ and $\text{pos}\{A_{\cdot 1}, \ldots, A_{\cdot i-1}, -M_{\cdot i}, A_{\cdot i+1}, \ldots, A_{\cdot n}\}$ have nonempty interiors, then the intersection of their interiors must be empty. For, if their interiors have a nonempty intersection, by an argument similar to that in 7.5 we can find two points $\hat{q}$, $\tilde{q}$ both nondegenerate with respect to M, such that (1) has two more solutions when $q = \hat{q}$ than its number of solutions when $q = \tilde{q}$, leading to a contradiction of the hypothesis.

Thus, through every subcomplementary set of column vectors, there exists a hyperplane separating the points corresponding to the left-out complementary pair of column vectors. However, since we only know that the number of complementary feasible solutions is a constant for every q *nondegenerate* with respect to M, it is not possible to claim that this separation will be strict. Consequently we refer to this as the *weak separation property*.

7.13. As in 7.6, it can be shown that every principal subproblem of (1) satisfies a similar weak separation property. The separation may not be strict in the subproblems also.

7.14. Suppose the problem

$$u = \tilde{M}v + \tilde{q},$$

$$u \geqslant 0, \qquad v \geqslant 0, \qquad u^T v = 0, \tag{25}$$

has been obtained by performing a series of principal pivots, or a block principal pivot on the system (1). Then, by 2.21 and 2.22, we see that problem (25) also satisfies the weak separation property and consequently all the principal subproblems of (25) also satisfy a similar weak separation property.

7.15. *Induction hypothesis.* In any complementarity problem of order $r \leqslant n - 1$ satisfying the weak separation property (i.e., that through every subcomplementary set of column vectors there exists a hyperplane which separates the left-out complementary pair of column vectors), if the interiors of any pair of complementary cones have a nonempty intersection, then both complementary cones are equal to the nonnegative orthant.

7.16. The induction hypothesis is easily verified for the case $r = 1$. In this case, $M = (m_{11})$ and the weak separation property holds, if and only if $m_{11} \geqslant 0$, and hence the weak separation property in this case implies the stronger result that the interiors of no two distinct complementary cones have a nonempty intersection.

7.17. We now show that, if the interiors of two complementary cones in the original problem, both of which have the rays generated by two or more column vectors of the unit matrix in common, have a nonempty intersection, then the two cones are equal to the nonnegative orthant of R^n. Suppose the two cones are pos$\{I_{\cdot 1}, I_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, I_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$. Let $\mathscr{I}_{\cdot 2}$, $a_{\cdot r}$, $b_{\cdot r}$ for $r = 3$ to n represent the column vectors $I_{\cdot 2}$, $A_{\cdot r}$, $B_{\cdot r}$ with their first components removed. Then the relative interiors of pos$\{\mathscr{I}_{\cdot 2}, a_{\cdot 3}, \ldots, a_{\cdot n}\}$ and pos$\{\mathscr{I}_{\cdot 2}, b_{\cdot 3}, \ldots, b_{\cdot n}\}$ both subsets of R^{n-1}, and complementary cones of the principal subproblem in $(w_2, \ldots, w_n; z_2, \ldots, z_n)$ have a nonempty intersection. By 7.13 and the induction hypothesis 7.15, this implies that both pos$\{\mathscr{I}_{\cdot 2}, a_{\cdot 3}, \ldots, a_{\cdot n}\}$ and pos$\{\mathscr{I}_{\cdot 2}, b_{\cdot 3}, \ldots, b_{\cdot n}\}$ are equal to the nonnegative orthant of R^{n-1}. Hence the second to the nth rows of both the matrices $(I_{\cdot 1} \vdots I_{\cdot 2} \vdots A_{\cdot 3} \vdots \cdots \vdots A_{\cdot n})$ and $(I_{\cdot 1} \vdots I_{\cdot 2} \vdots B_{\cdot 3} \vdots \cdots \vdots B_{\cdot n})$ are positive multiples of row vectors of the unit matrix.

Now, considering similarly the principal subproblem in the variables $(w_1, w_3, \ldots, w_n; z_1, z_3, \ldots, z_n)$, we conclude that the first rows of both these matrices $(I_{\cdot 1} \vdots I_{\cdot 2} \vdots A_{\cdot 3} \vdots \cdots \vdots A_{\cdot n})$ and $(I_{\cdot 1} \vdots I_{\cdot 2} \vdots B_{\cdot 3} \vdots \cdots \vdots B_{\cdot n})$ are also positive multiples of the row vectors of the unit matrix. Hence the column vectors of both these matrices are positive multiples of the column vectors of permutation matrices and both the cones pos$\{I_{\cdot 1}, I_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, I_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$ are equal to the nonnegative orthant of R^n.

7.18. Now suppose that the interiors of two complementary cones, which have the rays generated by at least two column vectors in common, have a nonempty intersection. Suppose these cones are pos$\{A_{\cdot 1}, A_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ and pos$\{A_{\cdot 1}, A_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$.

Make a block principal pivot so that each of the variables associated with the column vectors $A_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}$ become associated with the unit vectors in the transformed problem. The transformed problem also satisfies the weak separation property by 7.14 and, by using 7.17, we conclude that under this block principal pivot each of the matrices $(A_{\cdot 1} \vdots A_{\cdot 2} \vdots \cdots \vdots A_{\cdot n})$ and $(A_{\cdot 1} \vdots A_{\cdot 2} \vdots B_{\cdot 3} \vdots \cdots \vdots B_{\cdot n})$ is transformed into matrices whose column vectors are positive multiples of column vectors of a permutation matrix. This is possible under a block principal pivot only if each of these matrices themselves has the same property which implies that both pos$\{A_{\cdot 1}, A_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ and pos$\{A_{\cdot 1}, A_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$ must be equal to the nonnegative orthant of R^n.

7.19. Now consider two complementary cones generated by complementary sets of column vectors which contain only one column vector in common, the interiors of which have a nonempty intersection. Suppose the common column vector is a column vector of the unit matrix. Let the complementary cones be pos$\{I_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, B_{\cdot 2}, \ldots, B_{\cdot n}\}$ where, for each $i = 2$ to n, $(A_{\cdot j}, B_{\cdot j})$ is a permutation of $(I_{\cdot j}, -M_{\cdot j})$. By 7.12 the hyperplane through $\{I_{\cdot 1}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$ separates the points represented by $A_{\cdot 2}$ and $B_{\cdot 2}$. If this separation is strict, then, since the interiors of pos$\{I_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, B_{\cdot 2}, \ldots, B_{\cdot n}\}$ have a nonempty intersection, the interiors of pos$\{I_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, A_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$ must have a nonempty intersection too. By 7.18, this implies that pos$\{I_{\cdot 1}, A_{\cdot 2}, A_{\cdot 3}, \ldots, A_{\cdot n}\}$ = pos$\{I_{\cdot 1}, A_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$, which contradicts the hypothesis that the interiors of pos$\{I_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, B_{\cdot 2}, \ldots, B_{\cdot n}\}$ have a nonempty intersection, for the interiors of pos$\{I_{\cdot 1}, A_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$ and pos$\{I_{\cdot 1}, B_{\cdot 2}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$ are disjoint, by the separation property. Thus $A_{\cdot n}$ must be on the hyperplane through $\{I_{\cdot 1}, B_{\cdot 3}, \ldots, B_{\cdot n}\}$. By a similar argument, this implies that $A_{\cdot r}$ lies on the hyperplane through $\{I_{\cdot 1}, B_{\cdot 2}, \ldots, B_{\cdot r-1}, B_{\cdot r+1}, \ldots, B_{\cdot n}\}$ and $B_{\cdot r}$ lies on the hyperplane through $\{I_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot r-1}, A_{\cdot r+1}, \ldots, A_{\cdot n}\}$ for $r = 2$ to n.

This implies that the two complementary cones pos$\{I_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$ and pos$\{I_{\cdot 1}, B_{\cdot 2}, \ldots, B_{\cdot n}\}$ are equal. Also, since the pair $(A_{\cdot j}, B_{\cdot j})$ is a permutation of $(I_{\cdot j}, -M_{\cdot j})$ for each $j = 2$ to n, we conclude that the rays generated by $I_{\cdot 2}, \ldots, I_{\cdot n}$ are all extreme rays of this cone. Hence both of these cones are equal to the nonnegative orthant of R^n.

In general, if the two complementary cones whose interiors have a nonempty intersection are generated by complementary sets of column vectors containing one column vector in common, which is not one of the $I_{\cdot j}$, we could use an argument based on block principal pivoting as in 7.18 and conclude that these two complementary cones must equal the nonnegative orthant of R^n.

7.20. The only remaining case is that of two complementary cones like $\mathrm{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n}\}$ and $\mathrm{pos}\{B_{\cdot 1}, \ldots, B_{\cdot n}\}$ where, for each $j = 1$ to n, the pair $(A_{\cdot j}, B_{\cdot j})$ is a permutation of $(I_{\cdot j}, -M_{\cdot j})$. Thus these two cones together contain all the $2n$ column vectors in the problem. Suppose the interiors of these two complementary cones have a nonempty intersection.

Consider the hyperplane through $\{A_{\cdot 2}, \ldots, A_{\cdot n}\}$, which is a face of one of these cones. By the weak separation property 7.12, this hyperplane separates $A_{\cdot 1}$ and $B_{\cdot 1}$. If the separation is strict, then, since the interiors of $\mathrm{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n}\}$ and $\mathrm{pos}\{B_{\cdot 1}, \ldots, B_{\cdot n}\}$ have a nonempty intersection, the interiors of the two cones $\mathrm{pos}\{B_{\cdot 1}, \ldots, B_{\cdot n}\}$ and $\mathrm{pos}\{B_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$ also have a nonempty intersection. By 7.18, this implies that $\mathrm{pos}\{B_{\cdot 1}, \ldots, B_{\cdot n}\} = \mathrm{pos}\{B_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$, and hence the interiors of $\mathrm{pos}\{B_{\cdot 1}, \ldots, B_{\cdot n}\}$ and $\mathrm{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n}\}$ are disjoint by the separation property, which is a contradiction.

Therefore $B_{\cdot 1}$ lies on the hyperplane through $\{A_{\cdot 2}, \ldots, A_{\cdot n}\}$. By a similar argument, we conclude that $B_{\cdot r}$ lies on the hyperplane through $\{A_{\cdot 1}, \ldots, A_{\cdot r-1}, A_{\cdot r+1}, \ldots, A_{\cdot n}\}$ and $A_{\cdot r}$ lies on the hyperplane through $\{B_{\cdot 1}, \ldots, B_{\cdot r-1}, B_{\cdot r+1}, \ldots, B_{\cdot n}\}$ for each $r = 1$ to n. Hence $\mathrm{pos}\{A_{\cdot 1}, \ldots, A_{\cdot n}\} = \mathrm{pos}\{B_{\cdot 1}, \ldots, B_{\cdot n}\}$ and, since the pair $(A_{\cdot j}, B_{\cdot j})$ is a permutation of $(I_{\cdot j}, -M_{\cdot j})$ for each $j = 1$ to n, the ray generated by $I_{\cdot j}$ is an extreme ray of this cone for each $j = 1$ to n. Thus both of these cones are equal to the nonnegative orthant of R^n.

7.21. Then, if the induction hypothesis holds for all $r \leqslant n - 1$, it also holds for $r = n$. By 7.15, we conclude that, if a complementarity problem satisfies the weak separation property 7.12; then, if the interiors of any two complementary cones have a nonempty intersection, both cones must be equal to the nonnegative orthant of R^n. Thus, if the problem has two or more complementary feasible solutions for some q, that q must lie in the nonnegative orthant of R^n.

But under the hypothesis of Theorem 7.10 the number of complementary feasible solutions is a constant for all q nondegenerate with respect to M. By 7.11, we therefore conclude that this constant must equal 1. So, whenever q is nondegenerate with respect to M, (1) has a unique solution. Thus the set of complementary cones which have nonempty interiors form a partition of R^n.

7.22. *Example.* A problem where the number of solutions to (1) is a constant ($= 1$) for all q which are nondegenerate with respect to M, but not for all $q \neq 0$, is obtained by taking $n = 2$ and

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The complementary cones corresponding to this problem which have

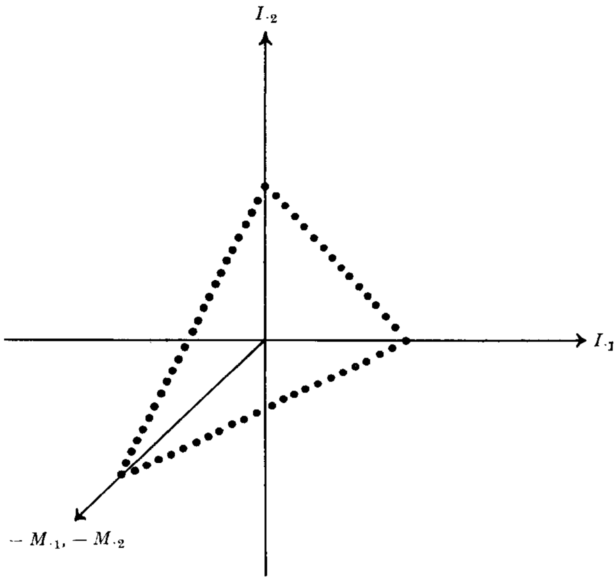

FIG. 5

nonempty interiors are depicted in Fig. 5. Here (1) has a unique solution whenever

$$Q \neq \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix}$$

for any $\alpha > 0$. When

$$q = \begin{bmatrix} -\alpha \\ -\alpha \end{bmatrix}, \qquad \alpha > 0,$$

(1) has an infinite number of solutions, namely,

$$w = 0, \qquad z = (\lambda\alpha, (1-\lambda)\alpha), \quad \text{for any} \quad 0 \leqslant \lambda \leqslant 1.$$

We also notice that only the weak separation property holds, since the hyperplane through $-M_{\cdot 1}$ separates $I_{\cdot 2}$ and $-M_{\cdot 2}$, but not strictly.

7.23. *Example.* A problem where the weak separation property 7.12 is satisfied, but in which the number of complementary feasible solutions is not a constant for all q nondegenerate with respect to M, is obtained by taking $n = 2$ and

$$M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

The complementary cones corresponding to this problem which have nonempty interiors are shown in Fig. 6.

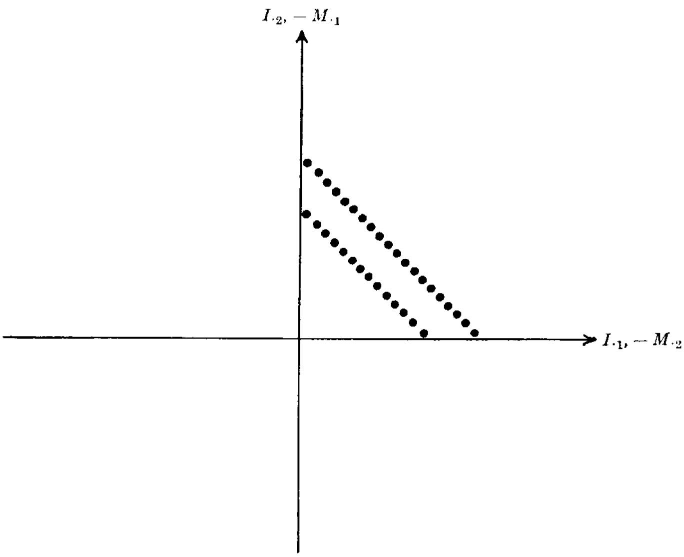

FIG. 6

The interiors of the two complementary cones $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}\}$ and $\text{pos}(-M_{\cdot 1}, -M_{\cdot 2})$ have a nonempty intersection, and they are both equal to the nonnegative orthant of R^n. M is not even a Q-matrix in this case.

7.24. *Example.* We noticed that, if (1) has a constant number of solutions for any nonzero right-hand constant vector, then every principal subproblem of (1) satisfies a similar property. However, if we are only given that (1) has a constant number of solutions for any q nondegenerate with respect to M, then principal subproblems of (1) may not satisfy a similar property. As an example, let $n = 2$,

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

In this case (1) has a unique solution whenever q is nondegenerate with respect to M. However, its principal subproblem in $(w_1; z_1)$,

$$w_1 = 0z_1 + q_1,$$

$$w_1 \geqslant 0, \qquad z_1 \geqslant 0, \qquad w_1 z_1 = 0,$$

has no solution when $q_1 < 0$ and has one solution if $q_1 \geqslant 0$.

7.25. COROLLARY. *If (1) has a constant number of solutions for all q which are nondegenerate with respect to M, then every principal subproblem of (1) has at most one solution for any of its right-hand constant vectors which is nondegenerate and not nonnegative.*

Proof. This follows from the fact that in this case every principal subproblem of (1) also satisfies the weak separation property 7.12 and hence the induction hypothesis 7.15 holds for it.

So, if the right-hand constant vector is nondegenerate, the principal subproblem can have two or more solutions only if the constant vector is nonnegative.

8. THE ODD NUMBER THEOREM FOR NONNEGATIVE Q-MATRICES

8.1. Here we show that, if M is a nonnegative Q-matrix, then the number of complementary feasible solutions is an odd number whenever q is nondegenerate with respect to M. This result may not hold if M is not nonnegative.

8.2. THEOREM. *If $M \geqslant 0$ and is a Q-matrix, then the number of complementary feasible solutions is an odd number for any q nondegenerate with respect to M.*

Proof. Proof is by induction on n.

8.3. If $n = 1$, then $M = (m_{11})$ and M is a Q-matrix if and only if $m_{11} > 0$ by Theorem 5.2. Here $q = (q_1)$ and for each $q \in R^1$ there is exactly one complementary feasible solution. Hence Theorem 8.2 is true when $n = 1$.

8.4. *Induction hypothesis.* Suppose Theorem 8.2 is true for all complementarity problems of order $(n - 1)$ or less. We now show that this implies that Theorem 8.2 also holds for problems of order n.

8.5. By Corollary 5.4, all principal submatrices of M are also Q-matrices. Consider the principal subproblem in $(w_2, \ldots, w_n; z_2, \ldots, z_n)$ with the right-hand constants $= \mathcal{Q}$. If $\bar{\mathcal{Q}}$ is nondegenerate in the subproblem, then it has an odd number of complementary feasible solutions when $\mathcal{Q} = \bar{\mathcal{Q}}$, by the induction hypothesis 8.4.

Let

$$\bar{q} = \begin{bmatrix} q_1 \\ \bar{\mathcal{Q}} \end{bmatrix},$$

where $q_1 > 0$, be nondegenerate with respect to M.

Since $q_1 > 0$ and $M \geqslant 0$, the variable z_1 must be equal to zero in any solution to (1) when $q = \bar{q}$. Thus, if $(\tilde{w}; \tilde{z})$ is a solution to (1) when $q = \bar{q}$, $\tilde{z}_1 = 0$, and hence $(\tilde{w}_2, \ldots, \tilde{w}_n; \tilde{z}_2, \ldots, \tilde{z}_n)$ is a complementary feasible solution to the subproblem when $\mathcal{Q} = \bar{\mathcal{Q}}$.

Also, if $(w_2^*, \ldots, w_n^*; z_2^*, \ldots, z_n^*)$ is any complementary feasible solution to the subproblem when $\mathcal{Q} = \bar{\mathcal{Q}}$, define

$$w_i^* = q_1 + \sum_{j=2}^{n} m_{1j} z_j^* > 0;$$

and then $(w_1^*, w_2^*, \ldots, w_n^*; 0, z_2^*, \ldots, z_n^*)$ is a complementary feasible solution to the original problem when $q = \bar{q}$.

Thus every complementary feasible solution of the original problem leads to a complementary feasible solution to the subproblem and vice versa. Hence both problems must have the same number of complementary feasible solutions. Hence, when $q = \bar{q}$, (1) has an odd number of solutions.

By a similar argument we conclude that the original problem has an odd number of complementary feasible solutions whenever q is nondegenerate with respect to M and at least one component in the vector q is positive.

It only remains to be shown that the same result holds even when $q < 0$.

8.6. We now show that, on every unbounded edge of $K(q)$ lying in the almost complementary set $C_1(q)$, both the variables w_1 and z_1 tend to $+\infty$, while $z_2, \ldots, z_n$ remain finite.

From 2.5, $K(q)$ is the set of all $(w; z)$ satisfying

$$w_i = M_{i\cdot}z + q_i, \qquad i = 1, \ldots, n,$$

$$z \geqslant 0, \qquad w \geqslant 0. \tag{26}$$

Since M is a Q-matrix, $m_{ii} > 0$ for all $i = 1, \ldots, n$ by Theorem 5.2. Consider any unbounded edge of $K(q)$. If all the variables $z_1, \ldots, z_n$ remain finite on this edge, then by (26) all the variables $w_1, \ldots, w_n$ also remain finite, and hence the edge cannot be an unbounded edge. Thus on every unbounded edge in $K(q)$ at least one of the variables $z_1, \ldots, z_n$ must tend to $+\infty$. If z_i tends to $+\infty$ on this edge, then from (26) and the facts that $M \geqslant 0$, $m_{ii} > 0$, and q_i is finite and fixed, w_i must also tend to $+\infty$ along this edge. Then, if any unbounded edge of $K(q)$ lies in the almost complementary set $C_1(q)$, the variables $z_2, \ldots, z_n$ should all remain bounded on that edge. Hence z_1 must tend to $+\infty$ on that edge and consequently w_1 also tends to $+\infty$ on that edge.

Thus on every unbounded edge in $C_1(q)$ the variable $w_1 \to +\infty$.

8.7. Suppose $\tilde{q}$ is nondegenerate with respect to M. Then there exists an $\alpha_0 > 0$ such that for all $\alpha > \alpha_0$ the point $\tilde{q} - \alpha I_{\cdot 1}$ is nondegenerate with respect to M. Hence the entire half-line

$$\{q\colon\ q = \tilde{q} - \alpha I_{\cdot 1}, \quad \alpha > \alpha_0\} \tag{27}$$

lies in the interior of a set of complementary cones. We now show that the number of complementary cones in which this half-line lies is precisely the number of unbounded edges in $C_1(\tilde{q})$. Let

$$F = \{(w; z)\colon\ (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2), \quad \theta \geqslant 0\}$$

be an unbounded edge in $C_1(\tilde{q})$. Then

$$(w_i^1 + \theta w_i^2)(z_i^1 + \theta z_i^2) = 0, \quad \text{for all} \quad i \neq 1, \quad \text{for all} \quad \theta \geqslant 0,$$

and $w_1^2 > 0$ by 8.6. Hence

$$(w; z) = (0, w_2^1 + \theta w_2^2, \ldots, w_n^1 + \theta w_n^2; z_1^1 + \theta z_1^2, \ldots, z_n^1 + \theta z_n^2)$$

is a complementary feasible solution for

$$q = \tilde{q} - (w_1^1 + \theta w_1^2)I_{\cdot 1} \quad \text{for all} \quad \theta \geqslant 0$$

and, since $w_1^2 > 0$, as θ varies from 0 to ∞

$$\{q\colon\ q = \tilde{q} - (w_1^1 + \theta w_1^2)I_{\cdot 1}, \quad \theta \geqslant 0\}$$

is eventually the same half-line as in (27).

Also, for any $\bar{\alpha} > \alpha_0$, $\tilde{q} - \bar{\alpha}I_{\cdot 1}$ cannot lie in any complementary cone which has $\text{pos}\{I_{\cdot 1}\}$ as a generator. For, if it does, there exists a subcomplementary set of columns $\{B_{\cdot 2}, \ldots, B_{\cdot n}\}$ such that

$$\tilde{q} - \bar{\alpha}I_{\cdot 1} = \lambda_1 I_{\cdot 1} + \sum_{i=2}^{n} \lambda_i B_{\cdot i}$$

for some $\lambda_1, \ldots, \lambda_n \geqslant 0$. Then $\tilde{q} - (\bar{\alpha} + \lambda_1)I_{\cdot 1}$ lies in the subspace through the subcomplementary set $\{B_{\cdot 2}, \ldots, B_{\cdot n}\}$, contradicting the assumption that $\tilde{q} - \alpha I_{\cdot 1}$ is nondegenerate with respect to M for all $\alpha > \alpha_0$.

Hence, if the half-line in (27) lies in some complementary cone, say $\text{pos}\{A_{\cdot 1}, A_{\cdot 2}, \ldots, A_{\cdot n}\}$, then $A_{\cdot 1}, \ldots, A_{\cdot n}$ must be linearly independent and $A_{\cdot 1} = -M_{\cdot 1}$. Now we can express this half-line as

$$\left\{q\colon\ q = \sum_{i=1}^{n} \beta_i^1 A_{\cdot i} + (\alpha - \alpha_0) \sum_{i=1}^{n} \beta_i^2 A_{\cdot i}, \quad \alpha > \alpha_0\right\}$$

for some $\beta^1, \beta^2 \geqslant 0$. Thus

$$\tilde{q} = \alpha I_{\cdot 1} + \sum_{i=1}^{n} \beta_i^1 A_{\cdot i} + (\alpha - \alpha_0) \sum_{i=1}^{n} \beta_i^2 A_{\cdot i} \tag{28}$$

for any $\alpha > \alpha_0$.

Suppose $(w^r; z^r)$ is obtained by setting $w_1^1 = \alpha_0$, $w_1^2 = 1$, and the variable associated with the column vector $A_{\cdot i}$ equal to β_i^r, $i = 1, \ldots, n$ and all the other variables in $(w; z)$ equal to zero, for $r = 1, 2$. Then (27) implies that

$$F = \{(w; z)\colon\ (w; z) = (w^1 + \theta w^2; z^1 + \theta z^2), \quad \theta \geqslant 0\}$$

is an unbounded edge in $C_1(\tilde{q})$.

Thus every unbounded edge in $C_1(\tilde{q})$ gives rise to a complementary cone in the interior of which the half-line in (27) lies and vice versa: Hence the number of unbounded edges in $C_1(\tilde{q})$ is equal to the number of complementary feasible solutions for $\tilde{q} - \alpha I_{\cdot 1}$, where α is a sufficiently large number.

8.8. Thus, for any q_1 such that $\hat{q} = (q_1, \tilde{q}_2, \ldots, \tilde{q}_n)^T$ is nondegenerate with respect to M, the number of unbounded edges in $C_1(\hat{q})$ is a constant. This number is equal to the number of complementary cones in which the half-line (27) eventually lies as α_0 is made large.

8.9. By the nondegeneracy of $\tilde{q}$ we know that there exists a β_0 such that, for all $\beta > \beta_0$, $(\beta, \tilde{q}_2, \ldots, \tilde{q}_n)^T$ is nondegenerate with respect to M. Hence we can always pick a $q_1^* > 0$ such that $q^* = (q_1^*, \tilde{q}_2, \ldots, \tilde{q}_n)^T$ is nondegenerate with respect to M. Since $q_1^* > 0$, the number of complementary feasible solutions when $q = q^*$ is an odd number. Therefore by **6.3**(iii) the number of unbounded edges in $C_1(q^*)$ is an odd number, and hence by 8.8 the number of unbounded edges in $C_1(\tilde{q})$ is an odd number. By **6.3**(iii) the number of complementary feasible solutions when $q = \tilde{q}$ is therefore an odd number.

Hence, under the induction hypothesis, Theorem 8.2 holds for the original problem of order n. By 8.3 and by induction, Theorem 8.2 is true for all n.

8.10. COROLLARY. *If M is a Q-matrix and if there exists a complementary set of column vectors $\{A_{\cdot 1}, \ldots, A_{\cdot n}\}$ which is linearly independent, such that each of the remaining vectors $B_{\cdot 1}, \ldots, B_{\cdot n}$ among the column vectors of $(I, -M)$ satisfies*

$$B_{\cdot j} \in \text{pos}\{-A_{\cdot 1}, \ldots, -A_{\cdot n}\} \quad \textit{for all} \quad j = 1, \ldots, n,$$

then the number of complementary feasible solutions is an odd number for all q which are nondegenerate with respect to M.

Proof. Transform the column vectors $A_{\cdot 1}, \ldots, A_{\cdot n}$ into the column vectors of the unit matrix by making the necessary principal pivots or a block principal pivot. Then Corollary 8.10 follows from Theorem 8.2 and Paragraphs 2.21 and 2.22.

8.11. In the special case when $n = 2$, the restriction that $M \geqslant 0$ can be removed from the hypothesis of Theorem 8.2. This is discussed below.

8.12. THEOREM. *If $n = 2$ and M is a Q-matrix then the number of complementary feasible solutions is an odd number whenever q is nondegenerate with respect to M.*

Proof.

8.13. *Case* 1. If pos$\{-M_{\cdot 1}, -M_{\cdot 2}\}$ is a subset of the nonpositive orthant of R^2, Theorem 8.12 follows from Theorem 8.2.

8.14. *Case* 2. *If* pos$\{-I_{\cdot 1}, -I_2\} \subset$ pos$\{-M_{\cdot 1}, -M_{\cdot 2}\}$, *the hypothesis that M is a Q-matrix implies that $-M_{\cdot 1}$ and $-M_{\cdot 2}$ are contained one each in* pos$\{I_{\cdot 1}, -I_{\cdot 2}\}$ *and* pos$\{-I_{\cdot 1}, I_{\cdot 2}\}$, *respectively.* (See Fig. 7.)

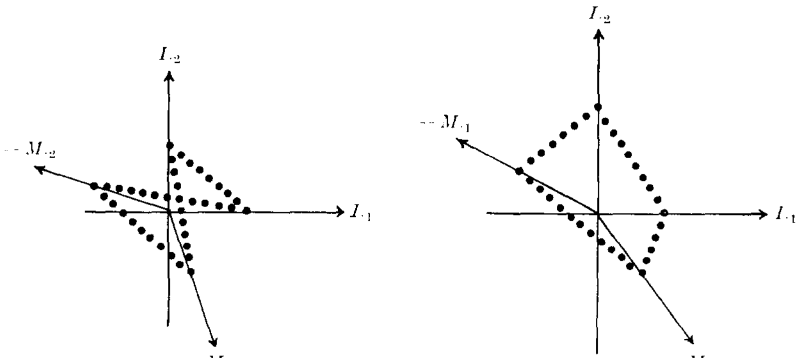

FIG. 7

We verify that in this case the number of complementary feasible solutions is 1 or 3 for every q nondegenerate with respect to M.

8.15. *Case* 3. Since M is a Q-matrix, the only other possibility is that exactly one of $-M_{\cdot 1}$ or $-M_{\cdot 2}$ is contained in the interior of

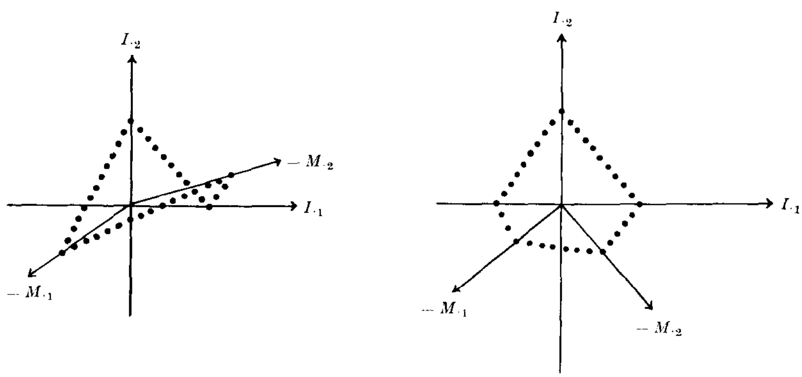

FIG. 8

$\text{pos}\{-I_{\cdot 1}, -I_{\cdot 2}\}$. (See Fig. 8.) Suppose it is $-M_{\cdot 1}$. Then the hypothesis that M is a Q-matrix implies that either $-M_{\cdot 2} \in \text{pos}\{I_{\cdot 1}, M_{\cdot 1}\}$ or $-M_{\cdot 2} \in \text{pos}\{I_{\cdot 1}, -I_{\cdot 2}\}$. In either case we verify that the number of complementary feasible solutions is either 1 or 3 for all q nondegenerate with respect to M.

8.16. COROLLARY. *If $n = 2$, there exists a q nondegenerate with respect to M for which the number of complementary feasible solutions is at most one.*

8.17. *Note.* When $n \geqslant 3$, Theorem 8.2 is not necessarily true if $M \not\geqq 0$, and Corollary 8.16 may not be true.

As an example, consider

$$M = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}.$$

It can be shown that this is a Q-matrix by verifying that the union of all the 8 complementary cones is R^3. Also, M is a nondegenerate matrix. We verify that $\tilde{q} = (1, 1, 1)^T$ is nondegenerate with respect to M. When $q = \tilde{q}$ there are four distinct complementary feasible solutions, because $\tilde{q}$ lies in each of the complementary cones $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}, I_{\cdot 3}\}$, $\text{pos}\{-M_{\cdot 1}, I_{\cdot 2}, I_{\cdot 3}\}$, $\text{pos}\{I_{\cdot 1}, -M_{\cdot 2}, I_{\cdot 3}\}$, and $\text{pos}\{I_{\cdot 1}, I_{\cdot 2}, -M_{\cdot 3}\}$ and in none of the others.

By Theorem 6.2, the number of complementary feasible solutions is an even number for all q nondegenerate with respect to M and, since M is a Q-matrix, this number must be $\geqslant 2$.

This shows that the converse of Corollary 6.9 is not necessarily true unless $M \geqslant 0$.

8.18. *Note.* When $n \geqslant 3$, the number of complementary feasible solutions can be $\geqslant 2$ for all $q \in R^n$. The example in 8.17 shows this. Thus, when $n \geqslant 3$, the complementary cones can span the whole space more than twice around.

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