Fractal and nonfractal behavior in Levy flights

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The d-dimentional space-continuous time-discrete Markovian random walk with a distribution of step lengths, which behaves like $x^{-(\alpha+d)}$ with $\alpha > 0$ for large x, is studied. By studying the density–density correlation function of these walks, it is determined under what conditions the walks are fractal and when they are nonfractal. An ensemble average of walks is considered and the lower entropy dimension D of the set of stopovers of the walks in this ensemble is calculated, and $D = \min\{2,\alpha,d\}$ is found. It is also found that the fractal nature of the walks is related to a finite value of the mean first passage time. The crossover of the correlation function from the fractal to nonfractal regimes is studied in detail. Finally, it is conjectured that these results for the lower entropy dimension apply to a wide class of symmetric Markov processes.

I. INTRODUCTION

The morphology of random fractals has recently become of considerable interest. One of the primary motivations for this interest has been the central role that these morphologies appear to play in a variety of kinetic growth processes. Among major questions to be understood in these processes are the questions of what conditions are necessary and sufficient for fractal growth to occur, and how the crossover to nonfractal growth regimes takes place. Unfortunately, even relatively simple, moderately realistic growth models are sufficiently complicated to render analytic progress toward understanding these questions difficult. Under these circumstances, it is therefore useful to study a much simpler process which exhibits both fractal and nonfractal growth and in which one can make analytic progress both in characterizing the nature of the fractal object generated in the fractal regime, and in studying the crossover between the fractal and nonfractal regions. To this end, we will study the process of Levy flights, which, in a certain sense, exhibit crossover from fractal to nonfractal growth as the step-length exponent of the walk is varied. Although the Hausdorff dimension of the stopovers of a Levy flight is always zero, the lower entropy dimension[1] (LED) for the process is nontrivial and corresponds to our intuitive motion of a "mass dimension." This dimension, defined for an ensemble average of walks (see below) will be used to distinguish between fractal and nonfractal regimes of the walk. Aside from their utility as analog growth processes, Levy flights are also of interest in their own right. Some work on the subject has been done by Mandelbrot,[2] and on the related subject of Weierstrassian random walks by Hughes, Montroll, and Shlesinger and Montroll and Shlesinger.[3] Furthermore, after the work reported in the present paper was completed, we became aware of the work of Hioe[4] in which a number of our results are obtained in the context of a lattice version of Levy flights.

The structure of the rest of this paper is as follows: First, we shall introduce some preliminary notions including a definition of the LED. Then we shall relate this dimension to the density–density corelation function, after which we shall calculate the asymptotic behavior of the density–density correlation function for the processes of interest. We shall end up with an expression for the LED of the stopovers of the Levy flight defined over a certain ensemble, as well as obtaining a relationship between the fractal nature of the Levy flight and the mean first passage time. We will also be able to study in detail the crossover between the fractal and nonfractal regions of the walk as we vary the step-length exponent. We will conclude with several comments and speculations.

The process we will study, a discrete-time continuous-space Levy flight, is a Markovian random walk process controlled by the probability function $P(n+1,\mathbf{x}|n,\mathbf{y})d\mathbf{x}\,d\mathbf{y}$ which is the conditional probability for the walker to be in the region $\mathbf{x}+d\mathbf{x}$ at time step $n+1$, if he was in the region $\mathbf{y}+d\mathbf{y}$ at time n. Here $\mathbf{x}$ and $\mathbf{y}$ are points in a continuous d-dimensional space, $d\mathbf{x}\equiv d^d\mathbf{x}$, $d\mathbf{y}\equiv d^d\mathbf{y}$, and n is an integer. We restrict ourselves to $P(n+1,\mathbf{x}|n,\mathbf{y}) = f(\mathbf{x}-\mathbf{y})$, and we will be particularly concerned with cases in which $f(\mathbf{x}-\mathbf{y})\sim|\mathbf{x}-\mathbf{y}|^{-(\alpha+d)}$ for large $|\mathbf{x}-\mathbf{y}|$. The Levy flight is thus a random walk with a variable step length whose size distribution is determined by $f(\mathbf{x}-\mathbf{y})$. To interpret the Levy flight as a "growth process," we imagine placing a particle at the end point of every step. Among the quantities we will discuss is the lower entropy dimension (LED), D, of the collection of these end points or stopovers defined by averaging over a suitable ensemble of walks. This D is a measure of how $N(L)$, the average number of particles contained in a nonempty region of linear dimension L, scales with L: i.e., $N(L)\sim L^{D(L)}$ and is thus consistent, for this process, with our intuitive notion of a mass dimension. If $D(L)$ is independent of L over some range then the system has a well-defined LED over that range.

Before proceeding with the calculation properly, it is useful to carefully define the quantities in which we shall be interested and to clearly state how averages are to be understood. Consider then the Levy flight defined by

$$P_n(\mathbf{x}) = \int d\mathbf{y}\, f(\mathbf{x}-\mathbf{y})P_{n-1}(\mathbf{y}), \tag{1}$$

where $P_n(\mathbf{x})$ is the probability density for the nth step to land on point x. We start our process at time $n=0$ at point $\mathbf{x}=0$, so that in terms of the conditional probability defined above,

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$$P_n(\mathbf{x}) \equiv P(n,\mathbf{x}|0,0). \quad (2)$$

Now, suppose we have generated a single sample of a Levy flight with a total of m steps. Let $\rho_m(\mathbf{x})d\mathbf{x}$ be the number of stopovers contained in the region $d\mathbf{x}$ about the point $\mathbf{x}$. The density–density correlation function is then

$$C'_m(\mathbf{r};\mathbf{x}) = \rho_m(\mathbf{x}+\mathbf{r})\rho_m(\mathbf{x}). \quad (3)$$

This quantity can be integrated over $\mathbf{r}$ to obtain

$$N'(L;\mathbf{x}) = \int_0^L d^d\mathbf{r}\, C'_m(\mathbf{r};\mathbf{x}), \quad (4)$$

which is the number of points contained in the region of linear dimension L weighted by $\rho_m(\mathbf{x})$, the number of particles at $\mathbf{x}$. Finally, we may average this quantity over a number of such m-step Levy flights and over all starting points $\mathbf{x}$ to obtain

$$\begin{aligned} N(L) &\equiv \langle N'(L;\mathbf{x})\rangle \\ &= \left\langle \int_0^L d^d\mathbf{r}\, C'_m(\mathbf{r};\mathbf{x}) \right\rangle \\ &= \left\langle \int_0^L d^d\mathbf{r}\, \rho_m(\mathbf{x}+\mathbf{r})\rho_m(\mathbf{x}) \right\rangle \\ &= \int_0^L d^d\mathbf{r} \langle \rho_m(\mathbf{x}+\mathbf{r})\rho_m(\mathbf{x})\rangle \\ &= \int_0^L d^d\mathbf{r} \langle C'_m(\mathbf{r};\mathbf{x})\rangle = \int_0^L d^d\mathbf{r}\, C_m(\mathbf{r}), \end{aligned} \quad (5)$$

where $\langle\ \rangle$ means averaging over the ensemble of samples. An explicit procedure for performing this average will be explained below. As we shall see, as a result of our averaging procedure, $N(L)$ and $C_m(\mathbf{r})$ will be independent of x. In any case, the x dependence for large m would be trivial since the process is translationally invariant. Therefore, $N(L)$, the average number of particles contained in a region of linear dimension L, having a behavior like $N(L) \sim L^D$ is equivalent to $C_m(\mathbf{r})$, the average density–density correlation function behaving like $C_m(\mathbf{r}) \sim r^{D-d}$.

II. THE AVERAGE DENSITY–DENSITY CORRELATION FUNCTION

We now want to calculate the average density–density correlation function for the processes in which we are interested. The result of this calcualtion will be an expression for the LED of the Levy flight averaged over a suitable ensemble. We will also be able to relate the fractal nature of the Levy flight to its mean first passage time, and we will be able to study in some detail the crossover from a fractal to nonfractal structure for the walk as we vary the step-length exponent. Unless explicitly stated otherwise in the sequel, when we refer to properties of the Levy flight, it should be understood that these statements refer to quantities averaged over the ensemble of sample flights, the construction of which we now explain.

To do this, we begin by defining a modified correlation function,

$$C_m(\mathbf{r}|j,\mathbf{x}) = \langle \rho_m(\mathbf{x}+\mathbf{r})\rho_m(\mathbf{x})\rangle_{(j,\mathbf{x})},$$

where $\langle\ \rangle_{(j,\mathbf{x})}$ means averaging over those systems in the ensemble in which the jth particle (i.e., the jth vertex of the given path) is between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$. Then

$$C_m(\mathbf{r}|j,\mathbf{x}) = \sum_{l=1}^{m}{}' P(l,\mathbf{r}+\mathbf{x}|j,\mathbf{x}), \quad (6)$$

where the prime on the sum means $l \neq j$. This is just the average particle density at the point $\mathbf{r}+\mathbf{x}$ if the jth particle is at the point $\mathbf{x}$. Averaging over $\mathbf{x}$, we have the correlation function averaged over an ensemble of samples in which the position of the jth particle is taken as one end point of the correlation function: i.e.,

$$C_m(\mathbf{r}|j) = \int d^d\mathbf{x}\, P_j(\mathbf{x}) C_m(\mathbf{r}|j,\mathbf{x}). \quad (7)$$

Using Eq. (1) it is clear that $P(l,\mathbf{x}|m,\mathbf{y}) = P_{l-m}(\mathbf{x}-\mathbf{y})$ for $l \geqslant m$, so that

$$C_m(\mathbf{r}|j) = \sum_{l=1}^{j-1} P_l(\mathbf{r}) + \sum_{l=1}^{m-j} P_l(\mathbf{r}). \quad (8)$$

Finally, if we randomly choose one particle in the object as the origin for calculating the correlation function, it is equally likely to be any of the particles, so that

$$C_m(\mathbf{r}) = \frac{1}{m}\sum_{j=0}^{m} C_m(\mathbf{r}|j) = 2\sum_{l=1}^{m}\left(1 - \frac{l-\frac{1}{2}}{m}\right)P_l(\mathbf{r}). \quad (9)$$

We now want to take $m \to \infty$ in this expression. First we show that $C_m(\mathbf{r})$ and $\Sigma_{l=1}^m P_l(\mathbf{r})$ diverge and converge together as $m \to \infty$. To see this, note that if $C_m(\mathbf{r})$ diverges as $m \to \infty$, then $\Sigma_{l=1}^m P_l(\mathbf{r})$ also diverges since, recalling that $P_l(\mathbf{r}) \geqslant 0$, it follows from Eq. (9) that $\Sigma_{l=1}^m P_l(\mathbf{r}) \geqslant \frac{1}{2} C_m(\mathbf{r})$. Furthermore, we can prove that if $\Sigma_{l=1}^m P_l(\mathbf{r})$ diverges as $m \to \infty$, then so does $C_m(\mathbf{r})$ as follows: If $\Sigma_{l=1}^m P_l(\mathbf{r}) \to \infty$, as $m \to \infty$, then for a given $\mathbf{r}$ there exists, for any L, an M such that $\Sigma_{l=1}^m P_l(\mathbf{r}) \geqslant L$. This means that for $m > 2M$,

$$\begin{aligned} C_m(\mathbf{r}) &> 2\sum_{l=1}^{M}\left(1 - \frac{l}{m}\right)P_l(\mathbf{r}) \\ &> 2\sum_{l=1}^{M}\left(1 - \frac{M}{m}\right)P_l(\mathbf{r}) > 2\sum_{l=1}^{M}\left(1 - \frac{M}{2M}\right)P_l(\mathbf{r}) \\ &> 2\frac{1}{2}L = L. \end{aligned}$$

Therefore, for large enough m, $C_m(\mathbf{r})$ is larger than any preassigned number L, and so diverges as $m \to \infty$.

Finally we note that if $C_m(\mathbf{r})$ converges we have

$$C(\mathbf{r}) = \lim_{m=0} C_m(\mathbf{r}) = 2\sum_{l=1}^{\infty} P_l(\mathbf{r}). \quad (10)$$

The right-hand side of Eq. (10) is twice the mean first passage time for this random walk.

Now we use a Fourier transform to rewrite Eq. (10) as

$$C(\mathbf{r}) = 2\frac{1}{(2\pi)^{d/2}}\int d\mathbf{k}\frac{\tilde{f}(\mathbf{k})}{1-\tilde{f}(\mathbf{k})}e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad (11)$$

where

$$\tilde{f}(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}}\int d\mathbf{r}\, f(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$$

is the d-dimensional Fourier transform of $f(\mathbf{r})$. We have used $\tilde{P}_l(\mathbf{k}) = \tilde{f}^l(\mathbf{k})$. If we consider only those processes which are independent of the angular variables, Eq. (11) is reduced to a form of Hankel transform,

$$C(r) = r^{-(d-1)/2} \int_0^\infty dk \frac{\tilde{f}(k)}{1-\tilde{f}(k)} \times k^{(d-1)/2} (kr)^{1/2} J_{(d-2)/2}(kr), \quad (11')$$

where $r = |\mathbf{r}|$, $k = |\mathbf{k}|$.

Let us now compute $C(r)$ and the LED for Levy flights. We consider walks for which the kernel in Eq. (1) has the form

$$f(r) \sim r^{-d} \sum_{i=0}^{n} b_i r^{-\alpha_i}, \quad \alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \equiv \alpha > 0, \quad b_0 \neq 0, \quad (12)$$

for large r and some integer $n > 0$ ($\alpha = \infty$ is included as a special case).

It is easy to show that (see Appendix A)

$$\tilde{f}(k) = 1 - \beta k^A + o(k^A) \quad \text{as } k \to 0, \quad (13)$$

where $A = \min\{2,\alpha\}$. Notice that $\tilde{f}(0) = 1$, otherwise the $P_n(\mathbf{x})$ cannot be interpreted as probabilities.

Using (13) in (11) it is not difficult to determine the necessary and sufficient conditions for the convergence of $C(r)$. We find that $C(r)$ converges (a) for $d \geqslant 3$ and any $\alpha > 0$, (b) for $d = 2$ and $\alpha < 2$, and (c) for $d = 1$ and $\alpha < 1$. Using (13) in (11), we see that for these values of d and α, $C(r) \sim r^{-(d-A)}$ as $r \to \infty$, and since $C(r) \sim r^{D-d}$, $D = A$ for these values of d and α. By Eq. (10), the mean first passage time is also finite for these vlaues of d and α.

For values of d and α for which $C(r)$ is divergent, we need to study $C_m(r)$ in the $m \to \infty$ limit a little more carefully. This is done in some detail in Appendix B. Here we report the results of this calculation. We find that for (d,α) such that $C(r)$ diverges, $\lim_{m\to\infty} C_m(0) \to \infty$, but $\lim_{m\to\infty} [C_m(0) - C_m(r)]$ is a finite function of r. Therefore, it is also possible to extract for this case a value of the LED by rescaling the correlation function by its value at the origin. Defining $C_m(r) = C_m(r)/C_m(0)$, we find $\lim_{m\to\infty} C_m(r) = 1$, and so the LED in this case is $D = d$. This is the case in which the LED of the trail of points left by a typical sample of the Levy flight passages has the naive dimension of space, and is, by Eq. (10), also the case in which the mean first passage time diverges. The value of the LED for all of these cases, for both divergent and convergent values of $C(r)$ can be summarized by the formula $D = \min\{2,\alpha,d\}$. Notice that we can mimic those cases in which $f(r)$ falls faster than a power as $r \to \infty$ by setting $\alpha = \infty$. We then find the usual Gaussian result for short range random walks, namely $D = 2$ for $d \geqslant 2$, and $D = 1$ for $d = 1$.

III. THE CROSSOVER REGIME BETWEEN FRACTAL AND NONFRACTAL

The structure of a typical sample of the Levy flight process, as we can infer from the results of an ensemble average, are markedly different in the fractal and nonfractal regimes. Since, to our knowledge, this is one of the only analytically tractable systems to exhibit this crossover, it is of considerable value to explicitly display the behavior of the correlation function in the crossover regime. This is done in Appendix C. Here we wish to point out some features of this crossover and comment on the qualitative differences in the behavior of a typical Levy flight in the fractal and nonfractal regimes. First, we want to make it clear that there are really three qualitatively different types of behavior possible for the Levy flight: (i) For $D < d \leqslant 2$ and for $D < 2$ and $d \geqslant 3$ the Levy flight is fractal-like and self-similar and the mean first passage time is finite. (ii) For $D = d \leqslant 2$ the Levy flight is nonfractal and space filling and the mean first passage time is infinite. (iii) For $D = 2$ and $d \geqslant 3$ the Levy flight is not space filling, but neither is it fractal. (This case also corresponds to the usual short-range finite step length random walk above two dimensions.) Because the walk is not space filling the mean first passage time is finite in this case, also.

The dynamics for case (i) differs markedly from the dynamics for cases (ii) and (iii). In cases (ii) and (iii) in which the step length distribution, $f(r)$, falls relatively rapidly, there will be no very large jumps and the stopovers will tend to congregate near the origin of the walk with the distribution of steps forming a Gaussian-like distribution which grows smoothly in width (and for $d \leqslant 2$, in height) at time goes by. For $d = 1$ and 2 the phase space is restricted enough so that these dynamics will cause $C_m(0)$ to diverge as $m \to \infty$ causing the mean first passage time to be infinite. For $d \geqslant 3$ there are enough random walk paths to prevent $C_m(0)$ from diverging as $m \to \infty$, and so the mean first passage time is finite. If, on the other hand, $f(r)$ does not fall rapidly enough, as is the situation in case (i), the dynamics is very different. In this case very large jumps will be possible, and the whole space will be sampled, although not densely. Indeed, in computer simulations of fractal Levy flights it is observed that the fractal structure is generated by the walker spending some time in a given region of space, then taking a single very large step to a far distant region, spending some time there, and repeating the process in a scale invariant way. This dynamics differs markedly from the smoothly spreading Gaussian distribution of cases (ii) and (iii). In terms of the density–density correlation function, we show in Appendix B that for $d = 1,2$, if we set $\alpha = d - \epsilon$, then for small positive ϵ, $C(r) \sim (1/\epsilon) r^{-\epsilon}$. Thus $C(r) \to \infty$ as $\epsilon \to 0^+$ and $[C(r) - C(0)] \sim \ln r$ for large r and $\epsilon = 0$, a behavior reminiscent of simple crossover effects in critical phenomena. This paradigm is worth keeping in mind as one studies more realistic and complex growth processes with fractal–nonfractal crossover.

IV. SUMMARY

In this paper we have analyzed the structure of Levy flights in the continuum. Using the lower entropy dimension as a criterion, we have found that the set of stopover points can exhibit both fractal and nonfractal behavior depending on the value of d, the number of dimensions in which the walk is embedded, and α, the power with which the jump distribution falls off asymptotically. We were also to exhibit in detail the behavior of an ensemble average Levy flights at the fractal–nonfractal crossover point. We showed furthermore that if the mean first passage time diverges, the LED is equal to d, and the typical Levy flight (understood as a representative of our ensemble) is not fractal-like. If the mean first passage time is finite, then the typical Levy flight will not be space filling and will generally be fractal unless $d \geqslant 3$ and $\alpha \geqslant 2$, in which case the dimension of the walk will be $D = 2$, just as for the ordinary random walk with fixed, finite step length.

We have analyzed the Levy flight for the specific step size distribution of Eq. (10). However, a careful examination of the derivation of our results clearly suggests an interesting generalization. We believe that the expression for the lower entropy dimension of the stopovers of this random walk, $D = \min\{2,\alpha,d\}$, will be correct for any symmetric distribution $f(r)$ where α is defined by

$$\alpha = \sup\left\{\alpha' \,\middle|\, \int |\mathbf{x}|^{\alpha'} f(|\mathbf{x}|) d^d\mathbf{x} < \infty\right\}.$$

The random Levy flight we have studied has a very rich structure, but, using the techniques of this paper, is amenable to considerable analysis. Such models should prove to be simple but useful archetypes in the study of fractal kinetic growth processes.

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APPENDIX A: LEADING BEHAVIOR OF $\tilde{f}(k)$ FOR SMALL k

In this Appendix we show that for

$$f(r) \sim r^{-d} \sum_{i=0}^{n} b_n r^{-\alpha_i}; \quad \alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \equiv \alpha > 0,$$

$b_0 \neq 0$ for large r, we have

$$\tilde{f}(k) = 1 - \beta k^A + o(k^A), \quad \text{as } k \to 0,$$

where $A = \min\{2,\alpha\}$.

Sketch of proof: Without loss of generality, let us consider the case $f(r) = r^{-(d+\alpha)}\sum_{i=0}^{\infty} c_i r^{-i}$ for $\alpha > 0$, r large. By using the integral representation of $J_\nu(x)$ for $d \geqslant 2$ we have

$$\tilde{f}(k) = c\int_0^\infty dr\, f(r) r^{d-1} \int_0^1 dy (1-y^2)^{(d-3)/2} \cos kyr, \tag{A1}$$

where c is a normalization constant. Equation (A1) can be rewritten as

$$\tilde{f}(k) = 1 - c\int_0^1 dy (1-y^2)^{(d-3)/2} \times \left\{\int_0^\infty dr\, f(r) r^{d-1}(1-\cos kyr)\right\}. \tag{A2}$$

Let us first concentrate on the integral,

$$\int_0^\infty dr\, f(r) r^{d-1}(1-\cos kyr)$$

in (A2). We divide the integral into two parts by some large number R above which the expansion of $f(r)$ around $r = \infty$ is valid, then expand the integrands properly, we have

$$\int_0^\infty dr\, f(r) r^{d-1}(1-\cos kyr) = \sum_{i=1}^{\infty} (ky)^{2i}\left[\int_0^R dr \frac{(-1)^i f(r)}{(2i)!} r^{2i+d-1}\right] + \sum_{i=0}^{\infty} c_i \int_R^\infty dr\, r^{-(1+\alpha+i)}(1-\cos rky).$$

Define

$$e_i(R) = \int_0^R dr \frac{(-1)^i f(r)}{(2i)!} r^{2i+d-1}, \tag{A3}$$

then it is easy to see $e_i(R)$'s are finite for any R for $\infty > R \geqslant 0$. Next we divide the integral in the second summation into two parts by $(1/ky)(>R)$, then expand $(1-\cos rky)$ in the first part and rescale the integral variable in the second part, and then we have

$$\begin{aligned}
&\int_R^\infty dr\, r^{-(1+\alpha+i)}(1-\cos rky) \\
&= -\int_{1/ky}^R dr\, r^{-(1+\alpha+i)}(1-\cos rky) + \int_{1/ky}^\infty dr\, r^{-(1+\alpha+i)}(1-\cos rky) \\
&= -\sum_{i=1}^{\infty} \frac{(-1)^{j+1}}{(2j)!}(ky)^{2j}\int_{1/ky}^R dr\, r^{-(1+\alpha+i)+2j} + (ky)^{-(\alpha+i)}\int_1^\infty dr\, r^{-(1+\alpha+i)}(1-\cos r) \\
&= -\sum_{j=1}^{\infty} (ky)^{2j}\frac{(-1)^{j+1}}{(2j)!}\left[\frac{R^{2j-(1+\alpha+i)+1}}{2j-(1+\alpha+i)+1} - \frac{(ky)^{-2j+(1+\alpha+i)-1}}{2j-(1+\alpha+i)+1}\right] \\
&\quad + (ky)^{-(1+\alpha+i)+1}\int_1^\infty dr\, r^{-(1+\alpha+i)}(1-\cos r) \\
&= -\sum_{j=1}^{\infty} (ky)^{2j} g_j(R,i) + (ky)^{-(\alpha+i)} h_i,
\end{aligned} \tag{A4}$$

where

$$g_j(R,i) = \frac{(-1)^{j+1}}{(2j)!} \frac{R^{2j-(1+\alpha+i)+1}}{2j-(1+\alpha+i)+1},$$

and

$$h_i = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j)!} \frac{1}{2j-(1+\alpha+i)+1} + \int_1^{\infty} dr\, r^{-(1+\alpha+i)}(1-\cos r).$$

There could be a $\ln(ky)$ term for α = integer in the above procedure, but it will not be the leading term, so it will not affect our derivation.

Now we go back to (A3), and we found

$$\int_0^{\infty} dr f(r) r^{d-1}(1-\cos rky) = \sum_{i=1}^{\infty} (ky)^{2i}\Big[e_i(R) - \sum_{j=0}^{\infty} g_i(R,j)c_j\Big] + \sum_{i=0}^{\infty} c_i h_i (ky)^{-(\alpha+i)}. \tag{A5}$$

Then we see

$$\tilde{f}(k) = 1 - c\sum_{i=1}^{\infty} k^{2i}\Big\{\Big[e_i(R) - \sum_{j=0}^{\infty} g_i(R,j)c_j\Big] \times \int_0^1 (1-y^2)^{(d-3)/2} y^{2i}\, dy\Big\} - c\sum_{i=0}^{\infty} k^{-(\alpha+i)}\Big[c_i h_i \int_0^1 (1-y^2)^{(d-3)/2} y^{2i}\, dy\Big]. \tag{A6}$$

Since $c_0 \neq 0$

$$h_0 = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2j)!} \frac{1}{2j-\alpha} \neq 0.$$

The leading term in the second summation is in order of $k^{-\alpha}$. The leading term in the first summation is k^{2i} for some integer $i > 0$. If $\alpha > 2$, from the probability theory we know that the second monent exists, therefore, $\tilde{f}(k) = 1 - \beta k^2 + o(k^2)$. From all the above procedures, we have shown for $d \geqslant 2$,

$$\tilde{f}(k) = 1 - \beta k^A + o(k^A) \text{ with } A = \min\{2,\alpha\} \text{ and } \beta \neq 0.$$

The proof for $d = 1$ is very similar (and also simpler).

APPENDIX B: FINITENESS OF $C(0) - C(r)$

In this Appendix we show that if $C(r) = \lim_{m\to\infty} C_m(r)$ diverges, then $\lim_{m\to\infty}[C_m(0) - C_m(r)]$ is a finite function of r so that $\lim_{m\to\infty} C_m(r) = 1$, where $C_m(r) = C_m(r)/C_m(0)$.

The Fourier transform of $C_m(r)$ may be written

$$\tilde{C}_m(k) = 2\sum_{l=1}^{m}\Big(1 - \frac{l}{m}\Big)[\tilde{f}(k)]^l = 2\Big[\tilde{f}\frac{H_m}{H_1} - \frac{\tilde{f}}{m}\frac{H_m}{H_1^2} + \frac{\tilde{f}^{m+1}}{H_1}\Big], \tag{B1}$$

where $H_m(k) \equiv 1 - [f(k)]^m$.

We need to examine the cases $d = 1$ and $d = 2$ separately.

(a) For $d = 1$,

$$C_m(0) - C_m(r) \sim \int_0^{\infty} dk\,[1-\cos(kr)] \times \Big\{\tilde{f}\frac{H_m}{H_1} - \frac{\tilde{f}}{m}\frac{H_m}{H_1^2} + \frac{\tilde{f}^{m+1}}{H_1}\Big\}. \tag{B2}$$

Recalling that $\tilde{f}(k) < 1$ for $k > 0$ and $\tilde{f}(0) = 1$, it is clear that for $m \to \infty$ only the first term in the curly brackets survives, so

$$C(0) - C(r) \sim \int_0^{\infty} dk\,(1-\cos(kr))\frac{\tilde{f}(k)}{H_1}. \tag{B3}$$

For $k \to 0$, the right-hand side of (B3) behaves like $\int_0 (K^2/K^A)dk$ and so is convergent. For $k \to \infty$ the right-hand side of (B3) is also convergent, having the behavior $\int^{\infty} \tilde{f}(k)(1-\cos kr)dk$. Therefore $C(0) - C(r)$ is a finite function of r.

(b) For $D = 2$; after integrating over the angular degrees of freedom,

$$C(0) - C(r) \sim \int_0^{\infty} k\, dk \frac{\tilde{f}(k)}{H_1} \times \Big[\int_0^1 (1-y^2)^{-1/2}(1-\cos(kry))dy\Big]. \tag{B4}$$

For $k \to 0$ the right-hand side of (B4) has the behavior $\int_0 (K^3/K^A)dk$, which is convergent, and for $k \to \infty$, the right-hand side of (B4) behaves like

$$\int^{\infty} k\, dk \tilde{f}(k)\Big[\int_0^1 (1-y)^{-1/2}(1-\cos(kry))dy\Big],$$

which is also convergent. Therefore $C(0) - C(r)$ is a finite function of r in this case also.

APPENDIX C: LEADING BEHAVIOR OF $C(r)$ IN CROSSOVER REGIME

In this Appendix we study the crossover between the fractal and nonfractal regimes by examining the leading behavior of $C(r)$ for large r and values of α close to the critical crossover value.

(i) $d = 1$. Here the critical value of α is $\alpha = 1$. Let $\alpha = 1 - \epsilon$.

(a) $\epsilon \leqslant 0$. In this case we know from the results of Appendix A that $\lim_{m\to\infty} C_m(r) = \infty$ and $C(r)/C(0) = 1$, which we interpret as implying nonfractal behavior with the LED $D = d = 1$.

(b) $\epsilon > 0$.

$$C(r) = \lim_{m\to\infty} C_m(r) = \int_0^{\infty} \frac{\tilde{f}(k)}{1-\tilde{f}(k)}\cos(kr)dk = r^{-\epsilon}\int_0^{\infty} \frac{(\tau/r)^{1-\epsilon}}{1-\tilde{f}(\tau/r)}\tilde{f}(\tau/r)\frac{\cos\tau}{\tau^{1-\epsilon}}d\tau. \tag{C1}$$

We now want to show that the leading behavior of the integrals as $r \to \infty$ is a constant proportional to $1/\epsilon$. To do this we note that

$$\frac{(\tau/r)^{1-\epsilon}}{1-\tilde{f}(\tau/r)}\tilde{f}\left(\frac{\tau}{r}\right)\equiv I\left(\frac{\tau}{r}\right) \tag{C2}$$

is bounded and that

$$\lim_{\tau\to\infty}\frac{\cos\tau}{\tau^{1-\epsilon}}=0. \tag{C3}$$

From this we can show that

$$\lim_{r\to\infty}\int_0^\infty I\left(\frac{\tau}{r}\right)\frac{\cos\tau}{\tau^{1-\epsilon}}d\tau=\int_0^\infty\left[\lim_{r\to\infty}I\left(\frac{\tau}{r}\right)\right]\frac{\cos\tau}{\tau^{1-\epsilon}}d\tau, \tag{C4}$$

and, since $I(0)$ is a finite constant, the integral in (C4) has the behavior

$$\int_0^\infty\left[\lim_{r\to\infty}I\left(\frac{\tau}{r}\right)\right]\frac{\cos\tau}{\tau^{1-\epsilon}}d\tau\sim\int_0^\infty\frac{\cos\tau}{\tau^{1-\epsilon}}d\tau\sim\frac{1}{\epsilon}.$$

The leading behavior of $C(r)$ for r large and $\epsilon>0$ is thus

$$C(r)\sim(1/\epsilon)r^{-\epsilon}.$$

Note that as $\epsilon\to0$ for large r,

$$C(r)\sim(1/\epsilon)[1-\epsilon\ln r]=(1/\epsilon)-\ln r.$$

Here we see explicitly that as $\epsilon\to0^+$, $C(r)$ consists of a divergent piece plus a finite function of r, which at the crossover point is proportional to $\ln r$.

(ii) $d=2$. The derivation of the behavior of $C(r)$ in this case is quite similar to the one-dimensional case. Defining $\alpha=2-\epsilon$, we have, as before, nonfractal behavior with the LED $D=d=2$ for $\epsilon<0$. For $\epsilon>0$ we can write

$$C(r)=\int_0^\infty k\,dk\frac{\tilde{f}(k)}{1-\tilde{f}(k)}\int_0^1(1-y^2)^{1/2}\cos(rky)dy$$

$$=r^{-\epsilon}\int_0^\infty d\tau\frac{(\tau/r)^{2-\epsilon}}{1-\tilde{f}(\tau/r)}\tilde{f}\left(\frac{\tau}{r}\right)\frac{J_0(\tau)}{\tau^{1-\epsilon}}. \tag{C5}$$

As before

$$\frac{(\tau/r)}{1-\tilde{f}(\tau/r)}\tilde{f}\left(\frac{\tau}{r}\right)$$

is bounded, and

$$\lim_{\tau\to\infty}\frac{J_0(\tau)}{\tau^{1-\epsilon}}=0,$$

so that

$$\lim_{r\to\infty}\int_0^\infty d\tau\frac{(\tau/r)^{2-\epsilon}}{1-\tilde{f}(\tau/r)}\tilde{f}\left(\frac{\tau}{r}\right)\frac{J_0(\tau)}{\tau^{1-\epsilon}}$$

$$=\int_0^\infty d\tau\left[\lim_{r\to\infty}\frac{(\tau/r)^{2-\epsilon}}{1-\tilde{f}(\tau/r)}\tilde{f}\left(\frac{\tau}{r}\right)\right]\frac{J_0(\tau)}{\tau^{1-\epsilon}}\sim\frac{1}{\epsilon}.$$

Therefore for small positive ϵ, the leading behavior of $C(r)$ for large r is

$$C(r)\sim(1/\epsilon)r^{-\epsilon}.$$

[1]B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1983), p. 359, and references therein.

[2]B. Mandelbrot, see Ref. 1, p. 288ff.

[3]B. D. Huges, E. W. Montroll, and M. F. Shlesinger, J. Stat. Phys. **28**, 111 (1982); E. W. Montroll and M. F. Shlesinger, in *Nonequilibrium Phenomena II: From Stochastics to Hydrodynamics*, edited by J. L. Lebowitz and E. W. Montroll (North-Holland, Amsterdam, 1984), p. 1.

[4]F. T. Hioe, in *Random Walks and Their Application in the Physical and Biological Sciences*, AIP Conf. Proc. No. 109, edited by M. F. Shlesinger and B. J. West (American Institute of Physics, New York, 1984), p. 85.