AQFT from n-functorial QFT

Urs Schreiber*

February 25, 2009

Abstract

There are essentially two different approaches to the axiomatization of quantum field theory (QFT): algebraic QFT, going back to Haag and Kastler, and functorial QFT, going back to Atiyah and Segal. More recently, based on ideas by Baez and Dolan, the latter is being refined to "extended" functorial QFT by Freed, Hopkins, Lurie and others. The first approach uses local nets of operator algebras which assign to each patch an algebra "of observables", the latter uses n-functors which assign to each patch a "propagator of states".

In this note we present an observation about how these two axiom systems are naturally related: we demonstrate under mild assumptions that every 2-dimensional extended Minkowskian QFT 2-functor ("parallel surface transport") naturally yields a local net, whose locality derives from the 2-categorical exchange law, and which is covariant if the 2-functor is equivariant. This is obtained by postcomposing the propagation 2-functor with an operation that mimics the passage from the *Schrödinger picture* to the *Heisenberg picture* in quantum mechanics. The argument has a straightforward generalization to general Lorentzian structure, bare lightcone structure and higher dimensions. It does not, however, by itself imply anything about the existence of a vacuum state or about positive energy representations.

*urs.schreiber@math.uni-hamburg.de

Contents

1 Introduction **3**

2 The situation for 1-dimensional QFT **7**

3 Nets of local monoids **10**

4 Extended 2-dimensional Minkowskian FQFT **12**

5 The main point: AQFT from extended FQFT **14**

6 Covariance/Equivariance **22**

7 Examples **27**

7.1 1-dimensional case . . . 27
7.2 Examples from parallel 2-transport . . . 28
7.3 2-Functors constant on one object . . . 30
7.4 Nets from wedge algebras . . . 30
7.5 Lattice models . . . 31
7.6 Boundary FQFT and boundary AQFT . . . 33
7.7 2-C^*-category codomains . . . 34
7.8 Hopf spin chain models . . . 35

8 Further issues **37**

A 2-Vector spaces and the canonical 2-representation **39**

1 Introduction

Out of the numerous tools and concepts that physicists have used for the description of quantum field theory few are well defined beyond simple toy examples. Still, in many cases they "work", often with dramatic success. Axiomatizations of QFT attempt to extract from the ill-defined symbols that appear in the physics literature those properties which are actually being used in structural proofs.

- While the path integral itself usually is ill-defined, all that often matters is the assumption that it satisfies the gluing law [58]. Taking this law as an axiom leads to the Atiyah-Segal formulation of functorial QFT.
- Similarly, while the products of physical field observables are usually ill-defined, all that often matters is the assumption that they satisfy the locality property [13]. Taking this as an axiom leads to the Haag-Kastler formulation of algebraic QFT.

The power of axiomatizations is that they lead to a more robust and clearer picture. The danger of axiomatizations is that they fail to capture important phenomena. Therefore it is especially important to understand how different axiomatizations of the same situation are related.

AQFT: nets of local algebras. Nets of local operator algebras have been introduced [25] (see [26] for a review) in order to formalize the concept of the algebra of *local observables* in quantum field theory. One way to think of such a net is as a co-presheaf on a sub-category of open subsets of a given Lorentzian manifold X with values in algebras. These co-presheaves are required to satisfy a couple of conditions (the first two mandatory, the third and fourth usually desired but sometimes dropped, the fifth crucial for real-world physical examples):

1. (**isotony**) all co-restriction morphisms are required to be inclusions of sub-algebras – this makes the co-presheaf a *net*;
2. (**locality/"microcausality"**) the inclusions of two algebras assigned to two spacelike separated open subsets into the algebra assigned to a joint superset are required to commute with each other.
3. (**covariance**) the net is covariant with respect to the action of a group G on X (for instance the Poincaré-group or the conformal group) if there is a family of algebra isomorphisms between the algebras assigned to any region and its image under the group action, compatible with the group product and the net structure.
4. (**time slice axiom**) the algebra of a subset is equal to that assigned to any neighbourhood of any of its Cauchy surfaces.
5. (**vacuum state and spectrum condition**) there is a *state* (a suitable linear functional) on the total algebra of the co-presheaf which behaves like a physical vaccum state (in that for instance it is translation invariant and induces a positive-energy representation of the translation group).

Out of the study of these structures a large subfield of mathematical physics has developed, which is equivalently addressed as *algebraic quantum field theory*, or as *axiomatic quantum field theory* or as *local quantum field theory*, but usually abbreviated as **AQFT**. For a review of physical applications see [20].

FQFT: n-functorial cobordism representation. Remarkably, all three of the terms – algebraic, axiomatic, local – would equally well describe what is probably the main alternative parallel development: the study of representations of cobordism categories, i.e. of functors from categories whose objects are $(d-1)$-dimensional manifolds and whose morphism are d-dimensional cobordisms between these to a category of vector spaces. An pedagogical introduction to this concept is in [4].

Such functors have been introduced to formalize the concept of the quantum propagator acting on the space of quantum states and imagined to arise from an integral kernel given by a path integral. While this functorial approach did not receive a canonical name so far, here we shall refer to it as *functorial quantum field theory* and abbreviate that as **FQFT**.

FQFT has most famously been studied in the context of *topological* QFT, from which Atiyah originally deduced his sewing axioms [2]. A review is [10]. While topological FQFT is by far the most tractable and hence the best understood one, FQFT is not restricted to the topological case: equipping the cobordisms for instance with conformal structure yields conformal QFT, an observation which is the basis of Segal's functorial axiomatization of QFT [53]. Restricting to 2-dimensional conformal cobordisms of genus 0 this yields the axioms of vertex operator algebras [28], see [35] for review and generalization. The result in [18] can be regarded as providing examples for Segal's CFT axioms (though in that work Atiyah's formulation of the functoriality axiom is being referred to).

Similarly, ordinary non-relativistic quantum mechanics ((1+0)-dimensional QFT) is about (monoidal) representations (i.e. functors to Vect) of the (monoidal) category of 1-dimensional *Riemannian* cobordisms [55]. Taking this point of view on ordinary quantum mechanics seriously leads to Abramsky-Coecke's *categorical semantics of quantum protocols* [1]. See [15] for an overview.

In this vein, here we shall be concerned with functors on cobordisms with *pseudo*-Riemannian structures, and with flat Lorentzian structure (Minkowski structure) in particular.

In [21, 22] it was suggested that the FQFT picture can and should be refined to an assignment of data of "order n" to codimension n spaces for all n, such that this assignment respects all possible gluings. Formally this should mean that for d-dimensional quantum field theory the 1-category of cobordisms is refined to a d-category of cobordisms [14, 57] whose k-morphisms are k-dimensional cobordisms between $(k-1)$-dimensional cobordisms, and that one considers d-functors from this d-category to a suitable codomain d-category. Baez and Dolan began to draw the grand picture emerging here in [7], which was recently picked up by Hopkins and Lurie [27].

This extended n-functorial description of d-dimensional QFT is only beginning to be explored. First concrete descriptions of Chern-Simons and Wess-Zumino-Witten theory in this context appeared in [21, 22, 55] and in various talks given by Freed and Hopkins, aspects of which have recently been made available as [23]. Much progress has been made with understanding the extended FQFT of finite group Chern-Simons theory (Dijkgraaf-Witten theory) [11]. The general idea (for smooth n-groups) is currently best understood not for quantum but for "classical" propagation, where it describes *parallel transport* in n-bundles ($\simeq (n-1)$-gerbes) with connection [47, 8, 49, 50, 51].

But there are numerous indications that the picture is correct, useful and compelling. In [19] we shall demonstrate that the formulation of 2-dimensional CFT and 3-dimensional TFT appearing in [18] (see [46] for a review) is secretly a 2- and 3-FQFT of this form.

The relation. An obvious question, which does not seem to have been addressed before, is: *What is the relation between the axioms of AQFT and FQFT?*

Intuitively it is clear that the locality of local nets captures the same physical aspect as the n-functoriality of n-FQFTs does: that assignments to larger patches are already determined by the assignment to their pieces. But the nature of the assignments are different. We shall demonstrate that every FQFT determines an AQFT in the sense of items 1 through 4 of the above list by postcomposing with the higher analog of the functor

$$\mathrm{End} : \mathrm{Vect}_{\mathrm{iso}} \to \mathrm{Algebras}$$

which sends each vector space to its algebra of endomorphisms and each isomorphism of vector space to the corresponding isomorphism of algebras.

The above functor is held in high esteem, if only implicitly so, in quantum mechanics, where it encodes the passage from what is called the *Schrödinger picture* to the *Heisenberg picture* of quantum mechanics: given a unitary morphism of Hilbert spaces of the form $E \xrightarrow{e^{itH}} E$ for H some self-adjoint operator, which sends each element $\psi \in E$ to the element $e^{itH}\psi$, its image under the above functor is the isomorphism of endomorphism algebras

$$\mathrm{End} : (\, E \xrightarrow{e^{itH}} E \,) \;\mapsto\; (\, \mathrm{End}(E) \xrightarrow{e^{itH}\circ(-)\circ e^{-itH}} \mathrm{End}(E) \,)$$

which sends any operator A on E to $e^{itH} A e^{-itH}$.

The situation is summarized in table 1.

Remark. The reader should beware that we do not consider or require in the present article structure related to item 5 of the above list of AQFT characteristics, involving existence and nature of vacuum states on our local nets. In this sense our notion of AQFT for the purpose of this article is considerably weaker than what is appropriate in the context of concrete physical applications, and in particular some of our examples in section 7 are formal examples in this sense, that will not extend to examples for AQFTs in a more strict physical sense that demands a suitable vacuum state. On the other hand, nothing in our discussion precludes the existence of a natural extra condition on FQFTs which would induce suitable vaccum structure on the corresponding AQFT. But discussion of this point shall not concern us here.

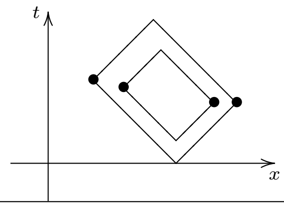
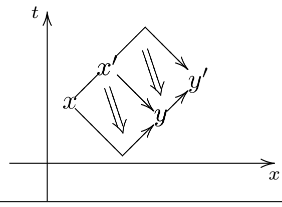
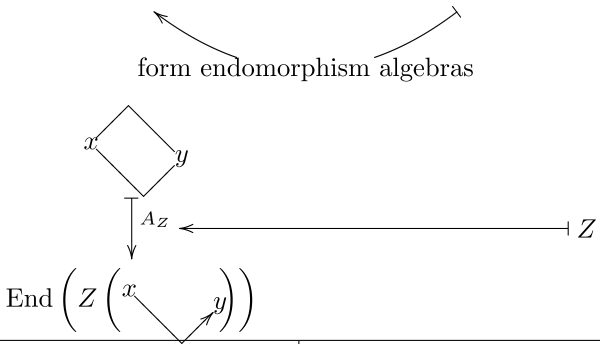
names	algebraic QFT (also: axiomatic QFT, local QFT)	functorial QFT
abbreviations	**AQFT**	**FQFT**
idea	assign algebras (of observables) to patches, compatible with inclusion	assign (time evolution) operators to patches, compatible with composition (gluing)
axioms due to	Haag, Kastler	Atiyah, Segal
aspect of QFT	Heisenberg picture	Schrödinger picture
formal structure	co-presheaf	transport n-functor
cartoon of domain structure		
relation		
main existing general theorems	spin-statistics theorem, PCT theorem	results about topological invariants
main existing nontrivial examples	chiral 2-d CFT	topological QFTs, full rational 2-d CFT

Table 1: **The two approaches** to the axiomatization of quantum field theory together with their interpretation and relation as discussed here. The rectangular diagrams are explained in sections 3 and 4. The construction of the AQFT $\mathcal{A}_Z$ from the extended FQFT Z is our main point, described in section 5.

Plan. We start in section 2 by discussing everything for the very simple case of 1-dimensional QFT (quantum mechanics), which should help to set the scene. Then in section 3 we quickly review those essentials of AQFT and in section 4 those of FQFT which we need later on. Here we restrict to $d = 2$ dimensions for ease of discussion. The generalization to higher dimensions is relatively obvious and straightforward, we briefly comment on that in section 8.

Our main definition is def. 9 in section 5, which gives the prescription for turning an FQFT 2-functor into a 2-dimensional local net of algebras. Our main result is theorem 1, which states that this definition works. Theorem 2 says that this construction extends to a 2-functor from the 2-category of FQFT 2-functors to the category of local nets, and, similarly, theorem 3 in section 6 says that the obvious notion of equivariance on FQFT induces the right notion of covariance in AQFT.

We close by discussing some examples in section 7 and some further issues in section 8.

2-categories. See [38] for the basics of 2-categories and 2-functors between them. For the time being we can and will entirely restrict attention to *strict* 2-categories and strict 2-functors between them. A review of all the basics of strict 2-categories that we need here can be found for instance in the appendix of [51]. After we have established our construction for strict 2-categories the generalization to arbitrary weak 2-categories is immediate.

Acknowledgement. I am grateful to David Corfield, Christoph Schweigert, Zoran Škoda, Jim Stasheff, Jamie Vicary and Konrad Waldorf for comments on earlier versions of this text, to Bruce Bartlett for discussion of aspects of some of the examples, to Maarten Bergvelt for discussion of relations between chiral nets and vertex operator algebras, to Jacques Distler for general discussion about AQFT and QFT, to Liang Kong for describing to me his work with Yi-Zhi Huang, to Peter Teichner for discussion of aspects at the beginning of section 2 and to Klaus Fredenhagen and an anonymous referee for pointing out the relevance of the example in section 7.4.

I had very useful discussion with Roberto Conti at an Oberwolfach CFT workshop in 2007, when I started thinking about the ideas presented here. Finally I heartily thank Pasquale Zito for a pleasant visit, for very useful discussion about his thesis and about Hopf spin chain models and for teaching me about asymptotic inclusion and pointing me to the relevant references.

This work was being completed while the author enjoyed a research fellowship at the *Hausdorff Center for Mathematics* in Bonn.

2 The situation for 1-dimensional QFT

To put the following construction into perspective, it is useful to indicate what the transition from FQFT to AQFT that we are after looks like for the simple case where we are dealing with 1-dimensional quantum field theory, also known as quantum mechanics.

Functorial quantum mechanics – Schrödinger picture. There are some slight variations on the theme of how to think of ordinary quantum mechanics – and in particular of possibly *time dependent* quantum mechanics – as a transport functor. These slight variations will have analogs also in higher dimensions, and hence are worth considering.

Let $X = \mathbb{R}$ be the real line, thought of as the *worldline* of a particle and in particular thought of as equipped with the obvious trivial Minkowski structure, which regards each vector as timelike. Let $P_1(X)$ be the category of homotopy classes of future-directed paths in X. Hence the objects of $P_1(\mathbb{R})$ are the points of $\mathbb{R}$ and there is a unique morphism from x to y whenever $x \leq y$. In other words, $P_1(X)$ happens to be nothing but $\mathbb{R}$ regarded as a poset.

There is the closely related category, $1\mathrm{Cob}_{\mathrm{Riem}}$, whose objects are disjoint unions of points and whose morphisms are abstract 1-dimensional cobordisms equipped with a Riemannian structure. If we forget the monoidal structure on $1\mathrm{Cob}_{\mathrm{Riem}}$ (which is important, but not for our purposes here) and restrict it to just a single point, then we find

$$1\mathrm{Cob}_{\mathrm{Riem}} \simeq \mathbf{B}\mathbb{R}_{0,+} = \left\{ \bullet \xrightarrow{t} \bullet \;\mid t \in [0,\infty) \right\} ,$$

where on the right we have the one-object category whose space of morphisms is the non-negative real half-line with composition given by addition of real numbers. There is a canonical projection functor

$$P_1(\mathbb{R}) \twoheadrightarrow 1\mathrm{Cob}_{\mathrm{Riem}}$$

which sends the path $x \longrightarrow y$ to the Riemannian cobordism $\bullet \xrightarrow{t=(y-x)} \bullet$ of the same lenght.

Now, ordinary time-independent quantum mechanics is a functor

$$Z : 1\mathrm{Cob}_{\mathrm{Riem}} \to \mathrm{Vect}_{\mathrm{isos}}$$

which sends the single object of $1\mathrm{Cob}_{\mathrm{Riem}}$ to the *space of states*, E, and sends the Riemannian cobordism of length t to an automorphism

$$Z : (\bullet \xrightarrow{t} \bullet) \mapsto (E \xrightarrow{\exp(itH)} E) ,$$

for H some endomorphism of the complex vector space E – the *Hamiltonian*. Here we take $\mathrm{Vect}_{\mathrm{isos}}$ to be the category whose objects are vector space and whose endomorphisms are linear *iso*morphisms.

By the above, we can understand this as a functor on paths on the worldline, $P_1(\mathbb{R})$, which happens to factor through $\mathbf{B}\mathbb{R}_{0,+}$:

$$\begin{array}{ccc} P_1(\mathbb{R}) & \longrightarrow & \mathrm{Vect}_{\mathrm{isos}} \\ \downarrow & & \uparrow Z \\ \mathbf{B}\mathbb{R}_{0,+} & \overset{\simeq}{\longrightarrow} & 1\mathrm{Cob}_{\mathrm{Riem}} \end{array} .$$

Using the interpretation of such functors as vector bundles with connection [49], we can think of this as a vector bundle on the real line obtained from an $\mathbb{R}_{0,+}$-equivariant vector bundle over the point.

A more general situation is obtained when one considers *time dependent* quantum mechanics. Here the space of states and the Hamiltonian is allowed to change. There is then a 1-parameter family $t \mapsto E_t$ of spaces of states and H is no longer necessarily constant. This, then, is the case of a general functor $P_1(\mathbb{R}^2) \to \mathrm{Vect}_{\mathrm{isos}}$:

$$(x \longrightarrow y) \mapsto (E_x \xrightarrow{P\exp(i\int_x^y H(t)\,dt)} E_y) ,$$

where the expression on the right denotes the path-ordered exponential, which is nothing but the parallel transport with respect to the connection 1-form $A = H\,dt$. (More on that in section 7.)

A slightly different but very similar concept plays an important role in [55], where quantum field theories *over* a space X are considered, as functors from a category of cobordisms that come equipped with maps to X: The category $1\mathrm{Cob}_{\mathrm{Riem}}(\mathbb{R})$ of cobordisms equipped with a (smooth, say) map to the real line is not quite the same as $P_1(\mathbb{R})$, but very similar. There is an obvious canonical functor

$$P_1(\mathbb{R}) \longrightarrow 1\mathrm{Cob}_{\mathrm{Riem}}(\mathbb{R})$$

which sends a path γ in $\mathbb{R}$ to the Riemannian cobordism of the same length equipped with the obvious map to $\mathbb{R}$ which coincides with γ.

This way, from every "1-dimensional QFT over $\mathbb{R}$" in the sense of [55]

$$F : \; 1\mathrm{Cob}_{\mathrm{Riem}}(\mathbb{R}) \to \mathrm{Vect}_{\mathrm{isos}}$$

one obtains an instance of ordinary time-dependent quantum mechanics by pulling back to $P_1(\mathbb{R})$:

$$\begin{array}{ccc} P_1(\mathbb{R}) & \xrightarrow{\quad Z \quad} & \mathrm{Vect}_{\mathrm{isos}} \\ & \searrow \quad \nearrow_{F} & \\ & 1\mathrm{Cob}_{\mathrm{Riem}}(\mathbb{R}) & \end{array} .$$

(In [55] Euclidean QFT is considered such that the morphisms assigned by Z are not in general invertible. While this is of no real relevance for the point of the above discussion, notice that later on, when we pass from FQFT to AQFT, we make crucial use of the fact that we assume FQFTs to assign invertible time propagators.)

Depending on the precise details, the functor Z is usually demanded to factor through vector spaces with suitable extra structure. Topological vector spaces and Hilbert spaces are common choices. For our current purposes all such extra structure does not add anything to the aspects that we are interested in here and will be ignored until we come to concrete examples in section 7.

Algebraic quantum mechanics – Heisenberg picture. Given such a functor Z, we can form for each point $x \in X$ the *endomorphism algebra* of the vector space, by sending

$$x \mapsto \mathrm{End}(Z(x)) .$$

In the case that there is extra structure on our vector spaces we would demand suitable endomorphisms. In the case of Hilbert spaces one usually demands all endomorphisms to be *bounded* operators.

The endomorphism algebras thus obtained is known often as the *algebra of observables*. In the present case, we would be tempted to associate this algebra at time x with the entire future of x.

So let $S(X)$ be the category whose objects are open sets $O_x := \{x' \in X | x' > x\}$ and whose morphisms are inclusions $O_x \subset O_y$ of open subsets. Of course, due to the simplicity of the present setup, $S(X)$ is canonically isomorphic to the opposite of $P_1(X)$ itself, hence is itself just the opposite catgeory of $\mathbb{R}$ regarded as a poset. But for the discussions to follow it is useful to think of $S(X)$ as a category of open subsets of X.

The crucial point now is that sending spaces of states to their algebras of endomorphisms sends the functor

$$Z : P_1(X) \to \mathrm{Vect}_{\mathrm{iso}}$$

to a functor $\mathcal{A}_Z$ defined by

$$\begin{array}{ccc} S(X) & \xrightarrow{\quad A_Z \quad} & \mathrm{Algebras} \\ & {}_{Z}\searrow \quad \nearrow_{\mathrm{End}} & \\ & \mathrm{Vect}_{\mathrm{iso}} & \end{array} .$$

The functor $\mathcal{A}_Z$ sends open subsets in $S(X)$ to the algebras of endomorphisms of the spaces of states sitting over their boundary, and it sends inclusions of open subsets to the inclusion of the algebras which is induced from using conjugation with the propagator that is assigned to the path connecting the respective boundaries. More precisely:

$$\mathcal{A}_Z : (O_y \subset O_x) \mapsto (\; \mathrm{End}(Z(y)) \xrightarrow{Z(x\to y)^{-1}\circ(-)\circ Z(x\to y)} \mathrm{End}(Z(x)) \;) .$$

Of course this means that all inclusions of algebras here are actually isomorphisms. But this is again just due to the simplicity of the one-dimensional example. In conclusion, since there is no content in the locality axiom in 1 dimension, this means that $\mathcal{A}_Z$ is indeed a net of local monoids.

It is this simple situation which we want to generalize from 1- to 2-dimensional QFT.

3 Nets of local monoids

We start by considering a simple version of the relevant axioms of nets of local algebras on Minkowski space. Compare with section 2.1 of [26]. Various refinements and generalizations are possible but add no further insight into the main point we want to make here. In particular, we shall ignore all extra structure that might be present on the algebras that appear below (such as them being C^*- or von-Neumann algebras) and even be content with regarding them just as *monoids* (i.e. forgetting their vector space structure). Our main point, that the inclusion and the locality axioms of local nets follow from taking endomorphisms on n-functors, is entirely independent of all such details. An interesting question is which extra structure on the n-functor will induce which extra structure on the local nets. While this shall not be our main concern here, the examples in section 7 give some indications.

So let $X = \mathbb{R}^2$ thought of as equipped with the standard Minkowski metric on $\mathbb{R}^2$ of which we will need only the induced lightcone structure on $\mathbb{R}^2$, hence only the conformal class of the standard Minkowski metric.

By a causal subset of X we shall mean as usual the interior of the intersection of the future of one point with the past of another.

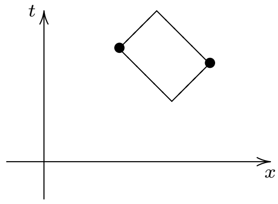
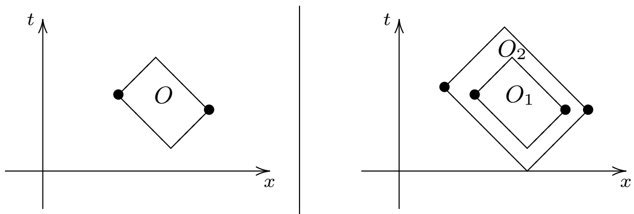


Figure 1: A "causal subset" of 2-dimensional Minkwoski space is the interior of a rectangle all whose sides are lightlike. Such subsets are entirely fixed in particular by their left and right corners.

Definition 1 *We denote by $S(X)$ the category whose objects are open causal subsets $V \subset X$ of X and whose morphisms are inclusions $V \subset V'$.*



an object O in $S(\mathbb{R}^2)$ | a morphism $O_1 \to O_2$ in $S(\mathbb{R}^2)$

Figure 2: The category $S(\mathbb{R}^2)$ of causal subsets of 2-dimensional Minkowski space. Objects are causal subsets, morphisms are inclusions of these.

In order to concentrate just on the properties crucial for our argument, we shall now talk about nets of local *monoids* (sets equipped with an associative and unital product). Write Monoids for the category of monoids and monoid homomorphisms and write $\mathrm{Monoids}_{\mathrm{incl}} \hookrightarrow \mathrm{Monoids}$ for the subcategory containing only injections (monomorphisms).

Definition 2 *Two objects O_1, O_2 in $S(X)$ are called spacelike separated if all pairs of points $(x_1, x_2) \in O_1 \times O_2$ are spacelike separated.*

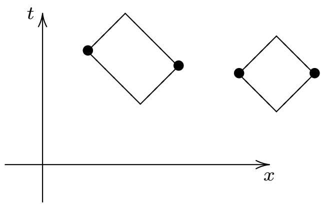


Figure 3: Two spacelike separated causal subsets of $\mathbb{R}^2$.

Definition 3 *A functor*

$$\mathcal{A} : S(\mathbb{R}^2) \to \text{Monoids}\,,$$

is a **net** *of monoids on 2-dimensional Minkwoski if it sends all morphisms in* $S(\mathbb{R}^2)$ *to injections (monomorphisms) of monoids, i.e. if it factors as*

$$\begin{array}{ccc} S(\mathbb{R}^2) & \xrightarrow{\mathcal{A}} & \text{Monoids} \\ & \searrow \quad \nearrow & \\ & \text{Monoids}_{\text{incl}} & \end{array} ,$$

This is a **net of local monoids** *if for all spacelike separated* $O_1, O_2 \subset O$ *the corresponding algebras commute with each other in* O*, i.e.*

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0$$

as an identity in $\mathcal{A}(O)$*.*

Notice that a monoid (possibly an algebra) A can be regarded as a one-object category $\mathbf{B}A := \left\{ \bullet \xrightarrow{a} \bullet \;|a \in A \right\}$ (possibly enriched over vector spaces). As such, these monoids naturally form the 2-category whose objects are monoids, whose morphisms are homomorphisms and whose 2-morphisms are intertwiners. See also appendix A.

Definition 4 *We write* $\mathbf{AQFT}(\mathbb{R}^2)$ *for the sub-2-category of the 2-functor 2-category* $\text{2Funct}(S(\mathbb{R}^2), \text{Cat})$ *whose objects are local nets A, regarded as functors*

$$S(\mathbb{R}^2) \xrightarrow{\mathcal{A}} \text{Monoids} \xhookrightarrow{\mathbf{B}(-)} \text{Cat}$$

taking values in one-object categories, whose morphisms are ordinary (as opposed to lax or pseudo) natural transformations between these, and whose 2- morphisms are modifications between those.

Monoidal categories of endomorphisms of local nets. From this it is immediate that for $\mathcal{A} \in \mathbf{AQFT}(\mathbb{R}^2)$ the endomorphisms $\text{End}_{\mathbf{AQFT}(\mathbb{R}^2)}(\mathcal{A})$ form a monoidal category (since it arises from a one-object 2-category). This is the monoidal catgegory defined in definitions 8.1 and 8.5 in [26] and proven there to be monoidal in proposition 8.30. The full subcategory

$$\Delta(\mathcal{A}) \subset \text{End}_{\mathbf{AQFT}(\mathbb{R}^2)}(\mathcal{A})$$

of *local* (meaning supported on some $O \in S(\mathbb{R}^2)$) and *transportable* (meaning independent of support region up to isomorphism) endomorphisms is the main entity of interest in, and maybe in AQFT in general. The famous Doplicher-Roberts reconstruction theorem was motivated by the study of $\Delta(\mathcal{A})$. This is discussed in great detail in [26].

Symmetries, covariance and equivariance. Let G be a group acting on $\mathbb{R}^2$ and preserving the causal set structure in that the action lifts to a functor

$$g : S(\mathbb{R}^2) \to S(\mathbb{R}^2)$$

for all $g \in G$. For $\mathcal{A}$ any local net we write

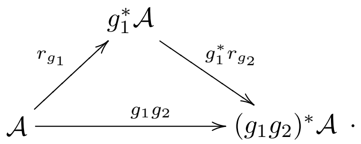
$$g^*\mathcal{A} : \; S_2(\mathbb{R}^2) \xrightarrow{g} S_2(\mathbb{R}^2) \xrightarrow{\mathcal{A}} \text{Monoids}$$

for the pullback of the net along the action of $g \in G$.

Definition 5 *An equivariant structure on a local net $\mathcal{A}$ is a choice of isomorphisms*

$$\mathcal{A} \xrightarrow{r_g} g^*\mathcal{A}$$

for all $g \in G$ such that for all $g_1, g_2 \in G$ we have



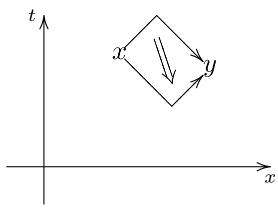
Remark. This is 1-categorical descent [56] along the nerve of the action groupoid $X//G$ of the category-valued presheaf Funct$(S(-), \text{Monoids})$.

Remark. In the AQFT literature this equivariant structure is often called a *covariant* structure (for instance assumption 3 on p. 14 of [26]) and is often expressed in terms of the total algebra $\text{colim}_{S(\mathbb{R}^2)}\mathcal{A}$ (compare fact 5.10 on p. 41 of [26]).

4 Extended 2-dimensional Minkowskian FQFT

Instead of regarding causal subsets as a category under inclusion of subsets, we can think of them as living in a 2-category under *composition* (gluing).

Definition 6 *Let $P_2(\mathbb{R}^2)$ be the 2-category whose objects are the points of $\mathbb{R}^2$, whose morphisms are piecewise lightlike right-moving paths in $\mathbb{R}^2$ and whose 2-morphisms are* generated *from the closure of causal bigons*



regarded as 2-morphisms as indicated, under gluing along pieces of joint boundary. Composition is by gluing along pieces of joint boundary, in the obvious way.

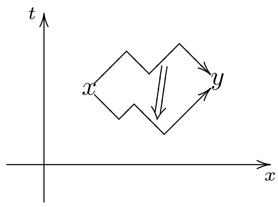


Figure 4: A typical 2-morphism in $P_2(\mathbb{R}^2)$

Remark. The restriction that 1-morphism have to go "right" and 2-morphisms "downwards" simplifies the discussion a bit but is otherwise of no real relevance. Various generalizations of $P_2(\mathbb{R}^2)$ can be considered without changing the substance of the following arguments.

Just as with local nets, there are many variations of definitions of extended quantum field theories on 2-dimensional Minkowski space which one could consider. We choose to take the following simple definition. (Compare with the notion of parallel surface transport [8, 50, 51]).

Definition 7 *For any 2-groupoid C, an extended FQFT on 2-dimensional Minkowski space is a 2-functor*

$$Z : P_2(\mathbb{R}^2) \to C\,.$$

We write $\mathbf{FQFT}(\mathbb{R}^2, C) := \mathrm{2Funct}(P_2(\mathbb{R}^2), C)$ *for the 2-functor 2-category and* $\mathbf{FQFT}_{\mathrm{isos}}(\mathbb{R}^2, C)$ *for the maximal strict 2-groupoid inside it.*

In concrete application C will usually be a 2-category of 2-vector spaces (which in general is not strict), as for instance those whose objects are (von Neumann) algebras, whose morphisms are bimodules over these, and whose 2-morphisms are bimodule homomorphisms [55]. We will see such an example in section 7 based on some constructions summarized in appendix A.

But for the moment we do not need to make any concrete choice concerning C. The only necessary requirement for the following is actually that the 2-morphisms in C all be invertible and that horizontal composition by the images of the 1-morphisms under Z is injective.

Equivariant structures. Let G be a group acting by diffeomorphisms on $\mathbb{R}^2$ which respects causal subsets in that the action extends to a functor

$$g : S_2(\mathbb{R}^2) \to S_2(\mathbb{R}^2)$$

with the induced 2-functor (denoted by the same symbol)

$$g : P_2(\mathbb{R}^2) \to P_2(\mathbb{R}^2)$$

for all $g \in G$. There is a canonical notion of what it means for a 2-functor $Z : P_2(\mathbb{R}^2) \to C$ to be *equivariant* with respect to this action [51, 47]: for $g \in G$ denote by

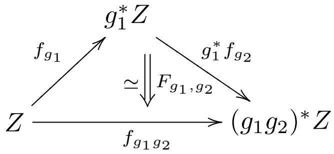
$$g^*Z : \; P_2(\mathbb{R}^2) \xrightarrow{g} P_2(\mathbb{R}^2) \xrightarrow{Z} C$$

the pullback of Z along the diffeomorphism G.

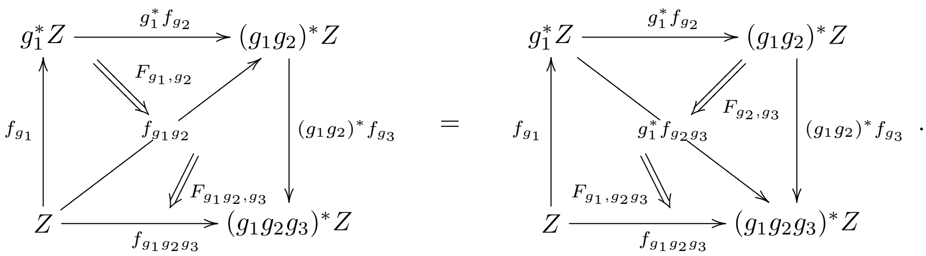
Definition 8 (equivariance of 2-functors) *A G-equivariant structure on Z is choice of isomorphisms f_g of 2-functors (i.e. strictly invertible pseudonatural transformations)*

$$Z \underset{\simeq}{\xrightarrow{f_g}} g^*Z$$

for all $g \in G$, and a choice for all $g_1, g_2 \in G$ of invertible 2-morphisms (i.e. modifications of pseudonatural transformations)



such that for all $g_1, g_2, g_3 \in G$ the tetrahedra 2-commute:



Remark. In the case that G acts freely, this is nothing but 2-categorical descent [56] along $Y := (\ X \longrightarrow\!\!\!\!\!\rightarrow X/G\ )$ with coefficients in the 2-category-valued presheaf $\mathrm{2Funct}(P_2(-), C)$ [47]. If G does not act freely it is descent with respect to the nerve of the action groupoid of G.

5 The main point: AQFT from extended FQFT

We define a map from FQFTs in the sense of definition 7 to AQFTs in the sense of definition 3 and demonstrate, theorem 1, that it indeed sends 2-functors Z to local nets of monoids $\mathcal{A}_Z$. Then we observe, theorem 2, that this construction extends to a 2-functor from FQFTs to AQFTs on $\mathbb{R}^2$. We end the section with a discussion of the properties of $\mathcal{A}_Z$ in light of the time slice axiom.

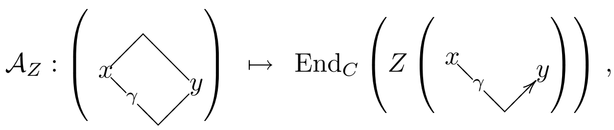
Definition 9 *Given any extended 2-dimensional FQFT, i.e. a 2-functor*

$$Z : P_2(\mathbb{R}^2) \to C$$

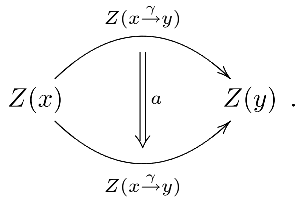
we define a functor

$$\mathcal{A}_Z : S(\mathbb{R}^2) \to \mathrm{Monoids}\,.$$

On objects it it acts as

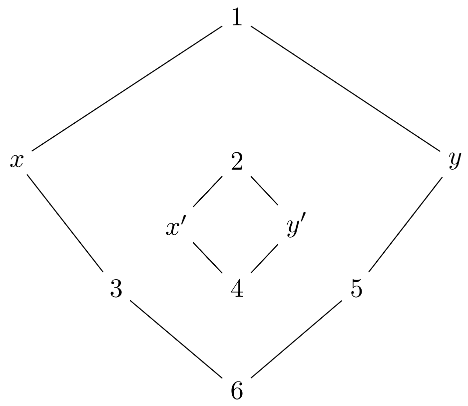


where on the right we form the monoid of 2-endomorphism a in C on the 1-morphism $Z(x \xrightarrow{\gamma} y)$ in C that is the past boundary of $O_{x,y}$,

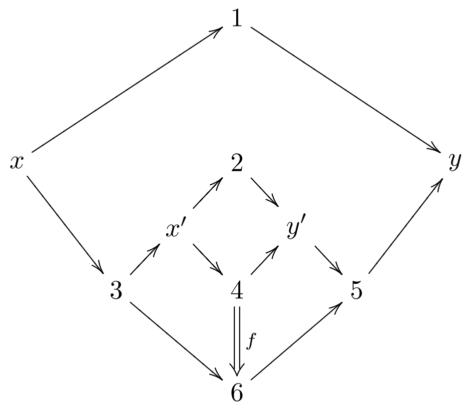


On morphisms $\mathcal{A}_Z$ is defined to act as follows.

For any inclusion $O_{x',y'} \subset O_{x,y} \in S(\mathbb{R}^2)$



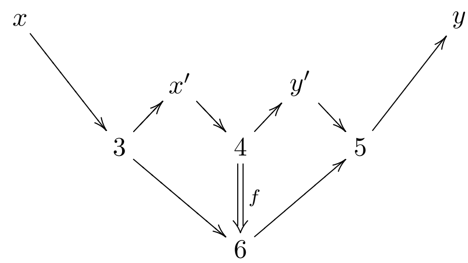
(the numbers here and in the following are just labels for various points in order to help us navigate these diagrams) we form the pasting diagram



in $P_2(\mathbb{R}^2)$. Here the obvious projections along light-like directions (for instance from x' onto $x \to 6$ yielding 3) is used. It is at this point that the light-cone structure crucially enters the construction.

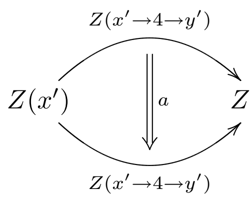
Let f' be the 2-morphism obtained by whiskering (= horizontal composition with identity 2-morphisms) the indicated 2-morphism f with the 1-morphisms $x \to 3$ and $5 \to y$.

$f' :=$



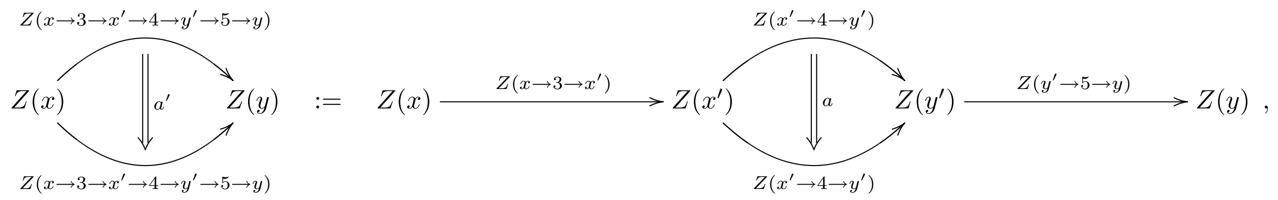
.

For any $a \in \mathrm{End}_C Z(x', 4, y')$,



,

let a' be the corresponding re-whiskering by $Z(x,3,x')$ from the left and by $Z(y',5,y)$ from the right:



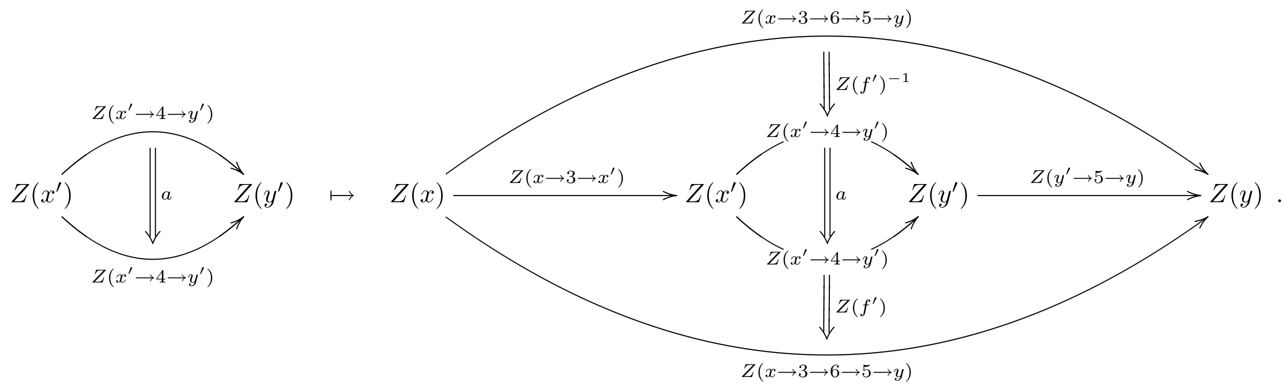
Then we obtain an injection

$$\mathrm{End}_C(Z(x',4,y')) \hookrightarrow \mathrm{End}_C(Z(x,3,6,5,y))$$

by setting

$$a \mapsto Z(f') \circ a' \circ Z(f')^{-1} \,,$$

i.e.



Remark. Notice that this prescription is essentially nothing but the one we described already for the 1-dimensional case in section 2: to open subsets we assign the endomorphism algebra of the space of states assigned to one part of their boundary. To an inclusion of open subsets we then assign the inclusion of such algebras obtained by *parallel transporting* the algebra of the inner set into the algebra of the outer set using conjugation with the propagators that the 2-functor assigns to 2-morphisms in $P_2(\mathbb{R}^2)$. The difference to the 1-dimensional case here is that this conjugation operation involves some (the obvious) re-whiskering. We will see that it is essentially this re-whiskering and the exchange law in 2-categories which lead to the locality of the net of monoids obtained this way.

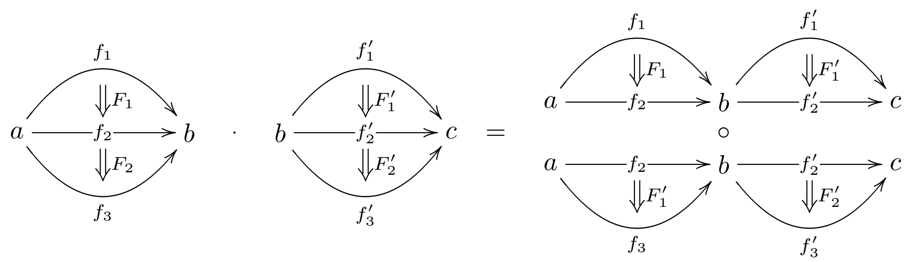


Figure 5: The exchange law in 2-categories, which is the functoriality of horizontal composition on the Hom-categories, says that the 2-dimensional order of composition of 2-morphisms is irrelevant.

Now we come to our main point.

Theorem 1 *The functor $\mathcal{A}_Z$ is a net of local monoids.*

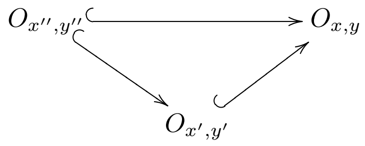
Proof. We need to demonstrate three things

1. that the above assignment is functorial;

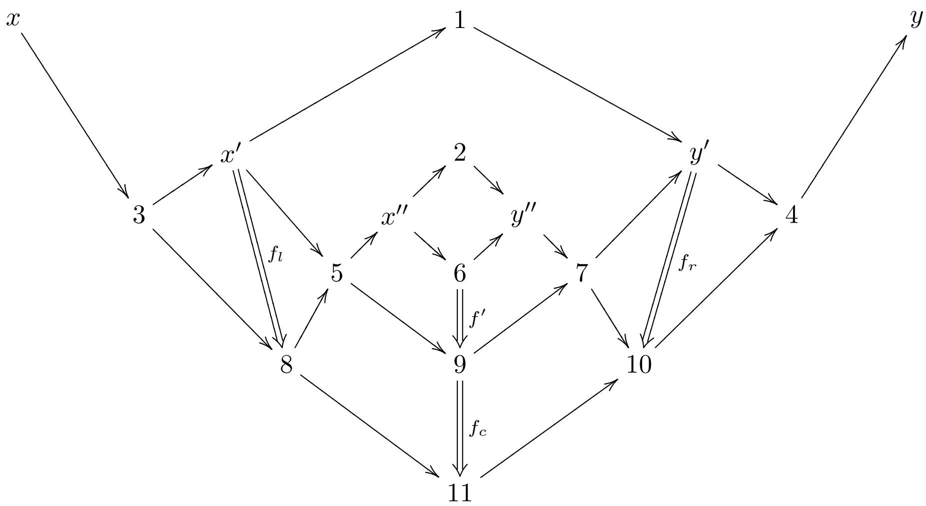
2. that the above assignment satisfies the locality axiom;

The first two properties turn out to be a direct consequence of 2-functoriality of Z and the exchange law in 2-categories.

To see functoriality, consider a chain of inclusions

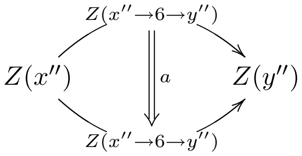


in $S(\mathbb{R}^2)$ and the corresponding pasting diagram

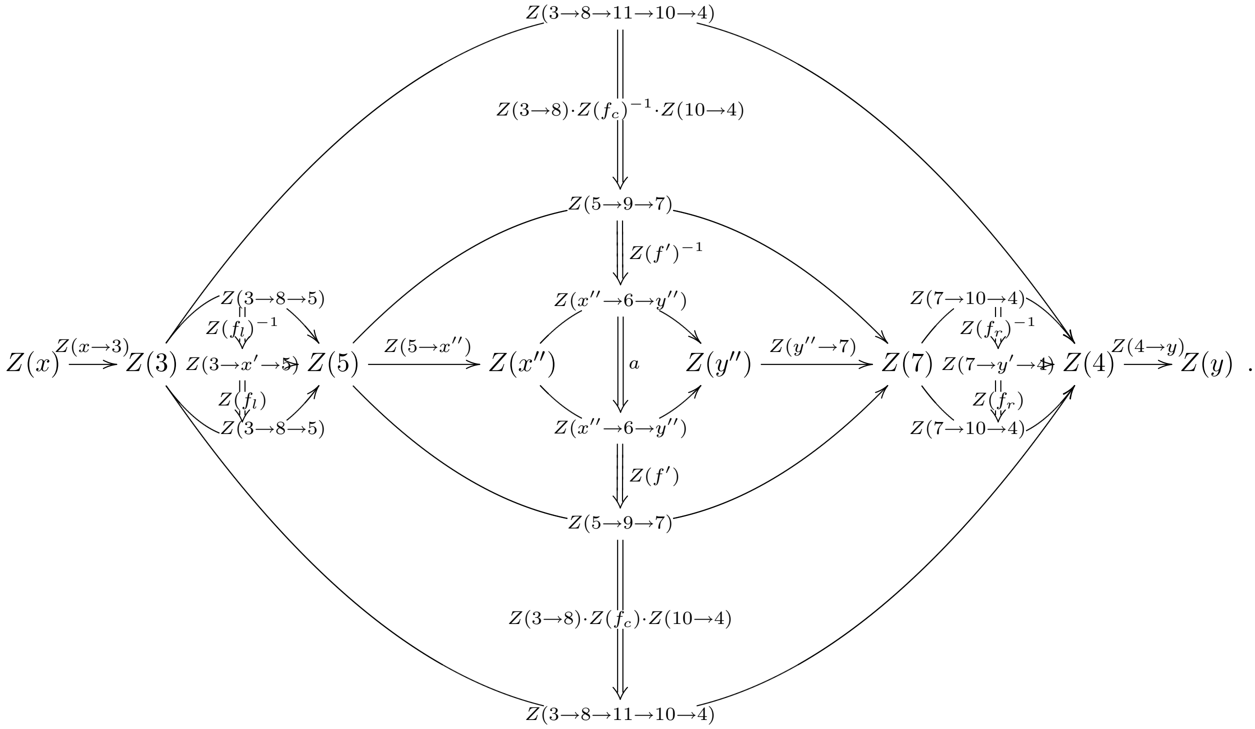


in $P_2(\mathbb{R}^2)$. The composite inclusion

$$\mathrm{End}_C(Z(x'' \to 6 \to y'')) \hookrightarrow \mathrm{End}_C(Z(x' \to 5 \to 9 \to 7 \to y')) \hookrightarrow \mathrm{End}_C(Z(x \to 3 \to 8 \to 11 \to 10 \to 4 \to y))$$

sends 

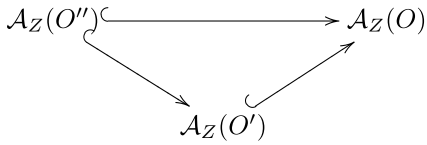
to



The contributions from f_l and f_r manifestly cancel and we are left with the pasting diagram for the direct inclusion

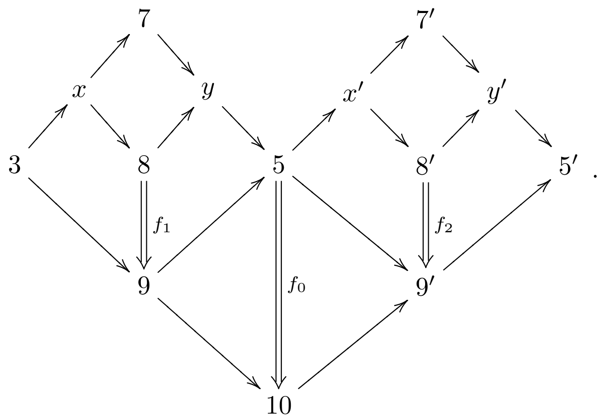
$$\mathrm{End}_C(Z(x'' \to 6 \to y'')) \hookrightarrow \mathrm{End}_C(Z(x \to 3 \to 8 \to 11 \to 10 \to 4 \to y))\,.$$

This shows that

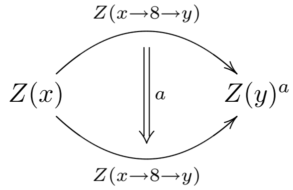


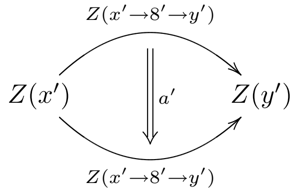
commutes, as desired.

To see locality, let $O_{x,y}$ and $O_{x',y'}$ be two spacelike separated causal subsets inside $O_{(3,5')}$. The relevant pasting diagram in $P_2(\mathbb{R}^2)$ is of the form

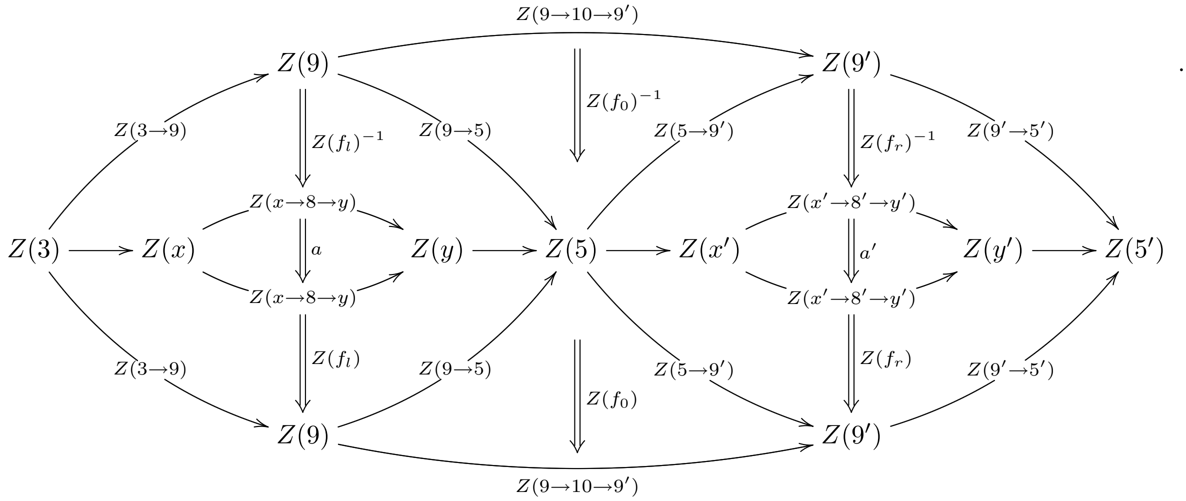


(We are displaying a very symmetric configuration only for notational convenience. The argument does not depend on that symmetry but just on the fact that $O_{x,y}$ does not intersect the past of $O_{x',y'}$ and vice versa.)

Now given any two endomorphisms 

and 

we can either first include a in $\mathrm{End}_C(Z(3\to 9\to 10\to 9'\to 5'))$ and then a', or the other way around. Either way, the total endomorphism in $\mathrm{End}_C(Z(3\to 9\to 10\to 9'\to 5'))$ is



This means that the inclusions of a and a' in $\mathrm{End}_C(Z(3\to 9\to 10\to 9'\to 5'))$ commute. $\square$

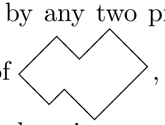
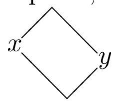
Theorem 2 *This construction extends to a 2-functor*

$$\mathbf{FQFT}_{\mathrm{isos}}(\mathbb{R}^2, C) \to \mathbf{AQFT}(\mathbb{R}^2)$$

faithful on 1-morphisms and trivial on 2-morphisms.

Proof. The proof is very analogous to the proof of theorem 3 in the next section, only slightly simpler. $\square$

Extension to Cauchy covers and the time slice axiom. The above construction restricts attention to causal subsets, while the principle underlying the construction is more general. We make some remarks on this generalization and its relation to the time slice axiom.

The category $P_2(X)$ from definition 6 – still for $X = \mathbb{R}^2$, for definiteness – contains (as its 2-morphisms) more subsets of $\mathbb{R}^2$ than the category $S(X)$ from definition 1 contains as objects: the former contains subsets bounded by any two piecewise lightlike rightbound paths with same source and target point, such as the interior of , while the latter contains only the causal double cones $O_{x,y} :=$ 

, which are the usual domains considered in AQFT.

Definition 10 (Cauchy neighbourhoods) *We write $S'(X)$ for the category whose objects of these more general open subsets bounded by piecewise lightlike rightbound paths (morphisms are still inclusions of open subsets).*

For the following paragraphs we shall refer to the objects of $S'(X)$ as *local convex causal Cauchy neighbourhoods*, or simply as *Cauchy neighbourhoods* for reasons to be discussed shortly.

Notice that we have an obvious inclusion $S_X \hookrightarrow S'(X)$ as well as a natural surjection $S'_X \twoheadrightarrow S(X)$ (obtained by sending an object U of $S'(X)$ to $O_{s(U),t(U)}$ where s and t are the source and target objects of U when regarded as a 2-morphism in $P_2(X)$) such that

$$S(X) \overset{\sigma}{\hookrightarrow} S'(X) \overset{p}{\twoheadrightarrow} S(X)\,, \qquad p\circ\sigma = \mathrm{Id}\,.$$

Definition 11 (local net on Cauchy neighbourhoods) *For $Z : P_2(X) \to C$ a 2-functor as before, let $\mathcal{A}'_Z : S'(X) \to \mathrm{Monoids}$ be the functor constructed verbatim as in definition 9 but with objects of $S(X)$ generalized everywhere to objects of $S'(X)$.*

Everything goes through exactly as before, and in fact our original construction of $\mathcal{A}$ is just the restriction of the construction of $\mathcal{A}'$ to causal subsets. Moreover, one notices that the endomorphism monoid assigned by $\mathcal{A}'$ to a Cauchy neighbourhood U is equal to the endomorphism monoid assigned by $\mathcal{A}$ to $O_{s(U),t(U)}$, since in the definition of the inclusion morphisms (definition 9) in the net there is in this case *no re-whiskering* involved in translating from the endomorphism monoid of the Cauchy neighbourhood to that of its double cone causal subset – compare the remark at the end of definition 9. We can summarize this by

Proposition 1 *$\mathcal{A}_Z$ equals the restriction of $\mathcal{A}'_Z$ along the inclusion σ and $\mathcal{A}'$ is naturally isomorphic to the pullback of $\mathcal{A}$ along the projection p:*

$$\begin{array}{ccc} S(X) & & \\ \downarrow & \searrow^{\mathcal{A}_Z} & \\ S'(X) & \xrightarrow{\mathcal{A}'_Z} & \mathrm{Monoids} \\ \downarrow^{p} & \Downarrow\simeq \nearrow_{\mathcal{A}_Z} & \\ S(X) & & \end{array}\,.$$

To interpret this physically, recall that a *Cauchy surface* in a globally hyperbolic Lorentzian manifold is a codimension 1 hypersurface such that every timelike curve intersects it precisely once. Cauchy surfaces are the supports of initial data for causal time evolution in globally hyperbolic Lorentzian manifolds. Noticing that the Cauchy surfaces of causal subsets $O_{x,y}$ are precisely the spacelike paths in $O_{x,y}$ connecting x and y we find that

Observation 1 *The objects in the fiber of* $S'(X) \xrightarrow{p} S(X)$ *over an object* $O_{x,y} \in S(X)$ *are precisely the convex open neighbourhoods* $U \subset O_{x,y}$ *of Cauchy surfaces in* $O_{x,y}$ *which arise as open covers* $U = \cup_i O_i$ *by causal subsets* $O_i \hookrightarrow O_{x,y}$*.*

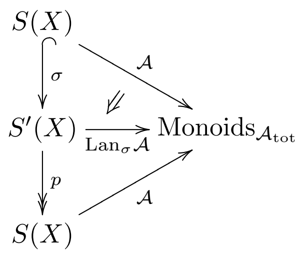
This justifies the term "Cauchy neighbourhoods" for the objects of $S'(X)$.

In light of this interpretation, proposition 1 asserts that the local net $\mathcal{A}_Z$ does (regarded as a co-presheaf on $S'(X)$) not actually depend on the full interior of any given causal subset, but just on that of any of the neighbourhoods of Cauchy surfaces in that causal subset.

There is one sense in which this statement is trivial: given *any* local net $\mathcal{A} : S(X) \to \mathrm{Monoids}$, we can always extend it to a net $\mathcal{A}'$ on Cauchy neightbourhoods simply by setting $\mathcal{A}' : \; S'(X) \xrightarrow{p} S(X) \xrightarrow{\mathcal{A}} S'(X)$, and this pair $(\mathcal{A}, \mathcal{A}')$ will form a commuting diagram as in proposition 1. But what our construction shows is that if $\mathcal{A}$ arises as the endomorphism co-presheaf of a 2-functor, then also this $\mathcal{A}'$ naturally has an interpretation as an endomorphism copresheaf.

For comparison, we state the usual *time slice axiom* in a form that exhibits its role in the context of the diagram appearing in proposition 1. Recall for that the notion of *Kan extensions* of functors along morphisms out of their domain, for instance from chapter 4 of [33]: the Kan extension is a *universal* solution to the problem of enlarging the domain of a functor, such as from $S(X)$ to $S'(X)$.

Definition 12 (time slice axiom for local nets of monoids) *A local net of monoids* $\mathcal{A} : S(X) \to \mathrm{Monoids}$*, regarded as a net of submonoids of* $\mathcal{A}_{\mathrm{tot}} = \mathrm{colim}_{S(X)}\mathcal{A}$*, (assumed to exist) to be written* $\mathcal{A} : S(X) \to \mathrm{Monoids}_{\mathcal{A}_{\mathrm{tot}}}$ *satisfies the* time slice axiom *if its left Kan extension* $\mathrm{Lan}_\sigma \mathcal{A}$ *along the inclusion of causal subsets into their Cauchy neighbourhoods coincides with their pullback along the projection from Cauchy neighbourhoods to causal subsets, i.e. if the lower triangle in*



commutes.

Proposition 2 *We have*

- $\mathrm{Lan}_\sigma \mathcal{A}$ *assigns to each* $U \in S'(X)$ *the monoid* $\vee_{\sigma(O) \subset U} \mathcal{A}(U)$ *which is the monoid* generated *from all submonoids* $\{\mathcal{A}(O)\}_{\sigma(O) \subset U}$ *which correspond to causal subsets inside* U*;*
- *the condition in definition 12 therefore demands that for* $\{O_i \in S(X)\}_i$ *any maximal cover of a Cauchy neighbourhood* $(U \in S'(X)) \subset \sigma(O)$ *of a causal subset* $O \in S(X)$ *by causal subsets* O_i *(i.e. by* all *causal subsets* O_i *with* $\sigma(O_i) \subset U$ *) we have*

$$\mathcal{A}(\sigma(U)) = \vee_i \mathcal{A}(O_i) \, .$$

Proof. To compute the left Kan extension $\mathrm{Lan}_\sigma \mathcal{A}$ notice that we can regard $S(X)$ and $S'(X)$, being posets, as categories enriched over the discrete monoidal category $\mathcal{V} = \{\emptyset, \{\bullet\}\}$ with product the cartesian product of sets (there is eigher an inclusion $O \hookrightarrow O'$ or not, so all Hom-sets are either empty or the singleton). Moreover, the category Monoids is *tensored* over $\mathcal{V}$ if we set the product of a monoid with the empty set to be the trivial monoid. In this case equation (4.24) in [33] applies which says that the left Kan extension is given by the coend

$$(\mathrm{Lan}_\sigma \mathcal{A})(U) = \int^{O_i \in S(X)} \mathrm{Hom}_S(\sigma(O_i), U) \cdot \mathcal{A}(O_i) \, ,$$

where $\mathrm{Hom}_S(\sigma(O_i), U)$ is either empty if O_i is not a subset of U, in which case the expression $\mathrm{Hom}_S(\sigma(O_i), U) \cdot \mathcal{A}(O_i)$ is the trivial monoid, or is the singleton if O_i is a subset of U, in which case the expression is just $\mathcal{A}(O_i)$ itself. This means that the coend reduces to the colimit over the $\mathcal{A}(O_i)$ for $O_i \subset U$

$$\cdots \simeq \int^{\sigma(O_i)\subset U} \mathcal{A}(O_i) = \mathrm{colim}_{\sigma(O_i)\subset U}\mathcal{A}(O_i) =: \vee_i \mathcal{A}(O_i)\,.$$

□

In summary we have

- the idea expressed by the time slice axiom is that a net on causal subsets extends to a net on Cauchy neighbourhoods and is then determined on double cones by its value on any of the double cone's Cauchy neighbourhoods;
- without further information the only reasonable extension of a net to Cauchy neighbourhoods is by $U \mapsto \vee_{O_i \subset U}\mathcal{A}(O_i)$, which we identified with the *universal* extension in the sense of Kan extensions of functors;
- but a net arising as the endomorphism co-presheaf of a 2-functor, as described here, has as such a (possibly different) natural extension obtained by applying the endomorphism construction to Cauchy neighbourhoods themselves.

A net arising as an endomorphism co-presheaf $\mathcal{A}_Z$ may fail the time slice axiom in its usual form in that $\mathcal{A}_Z(U)$ is not the same as $\vee_{O_i \subset U}\mathcal{A}_Z(O_i)$, still its value on any O, which is an endomorphism monoid associated to the boundary of O, is isomorphic to the corresponding endomorphism monoid of any Cauchy neighbourhood U inside O.

6 Covariance/Equivariance

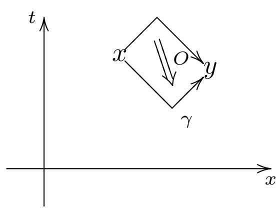
We had seen definitions for equivariance ("covariance") of local nets and of FQFT 2-functors. The following theorem says that these notions are compatible under our relation of the two.

Theorem 3 *Every G-equivariant structure, definition 8,*
on the FQFT $Z : P_2(\mathbb{R}^2) \to C$ induces a G-equivariant structure, definition 5, on the AQFT $\mathcal{A}_Z$ obtained from it according to definition 9.

Proof. For any $g \in G$ the component map of the pseudonatural transformation f_g is

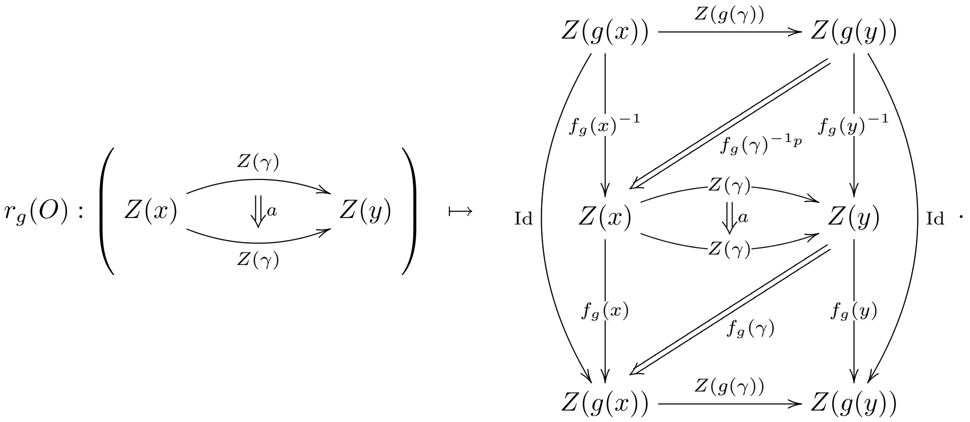
$$f_g : (\, x \xrightarrow{\gamma} y \,) \mapsto \begin{array}{ccc} Z(x) & \xrightarrow{Z(\gamma)} & Z(y) \\ {\scriptstyle f_g(x)}\big\downarrow & \swarrow_{f_g(\gamma)} & \big\downarrow{\scriptstyle f_g(y)} \\ Z(g(x)) & \xrightarrow{Z(g(\gamma))} & Z(g(y)) \end{array} \,.$$

For γ the target boundary of the causal subset O,



conjugating with the components on the right defines the monoid isomorphism

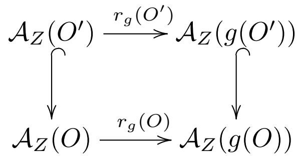
$$r_g(O) : \mathrm{End}_C(Z(\gamma)) \to \mathrm{End}_C(Z(g(\gamma)))$$



Here $f_g(\gamma)^{-1_p}$ denotes the inverse of the 2-cell $f_g(\gamma)$ with respect to vertical pasting (which is the ordinary inverse up to a re-whiskering).

We need to check that this construction

1. yields a morphism of nets in that it makes for all $O' \subset O$ the naturality squares

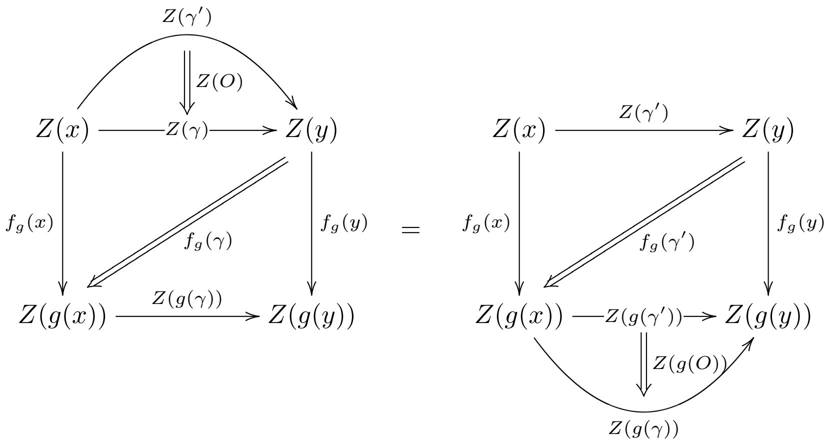


   commute;

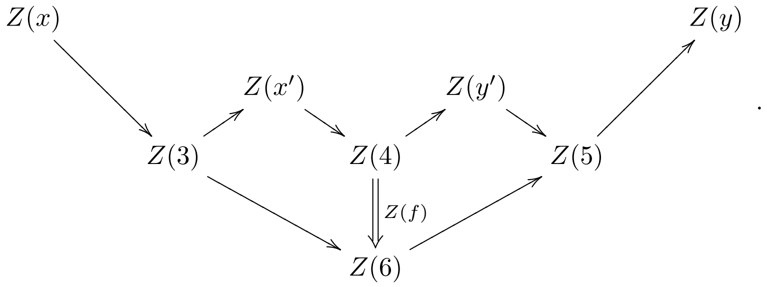
2. produces the commuting triangles in definition 5.

This can be seen as follows.

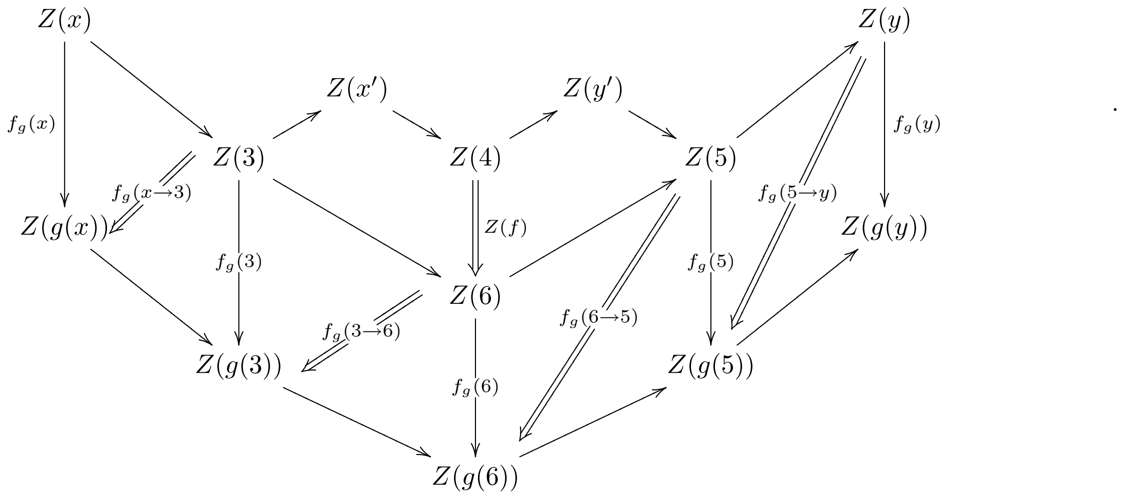
1. The pseudo-naturality condition on the components of f_g



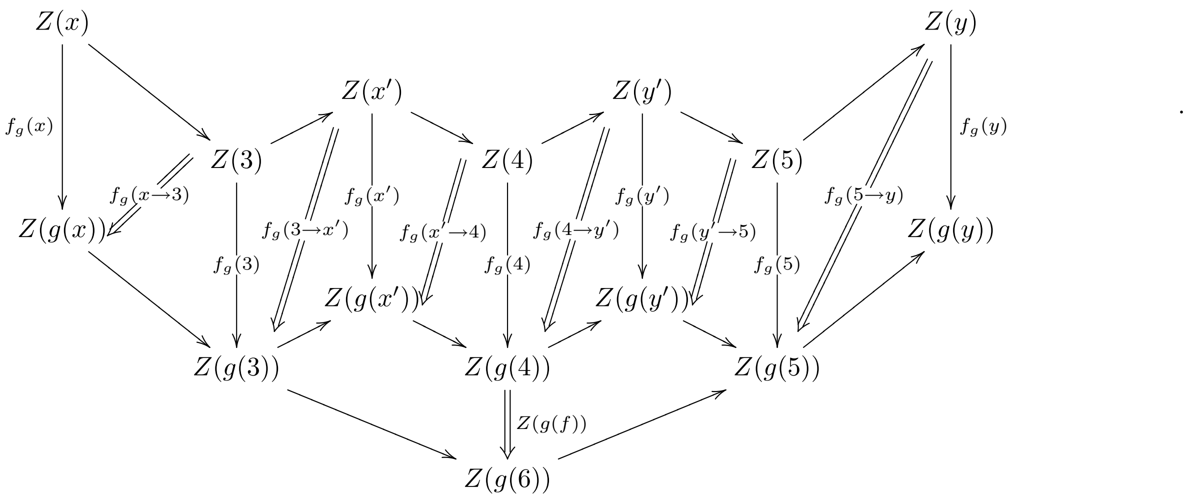
for all O implies precisely the condition $r_g(O)|_{A(O')} = r_g(O')$ when applied to our definition 9 of the inclusion map $A(O') \hookrightarrow A(O)$: that inclusion was obtained by conjugating with



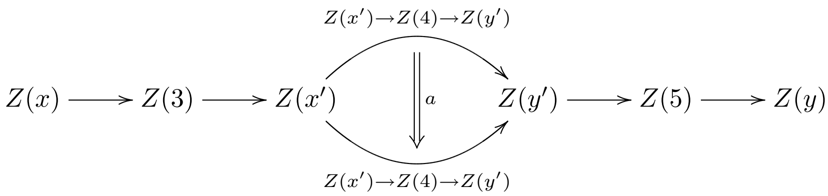
Following this by the action of $r_g(O)$ amounts to conjugating with



By pseudonaturality of f_g this equals conjuation with

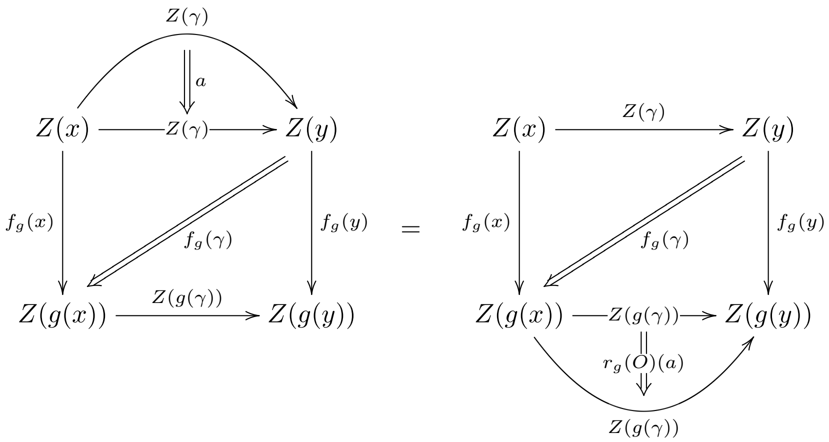


Since the endomorphism a to be conjugated is localized on $Z(x') \to Z(y')$

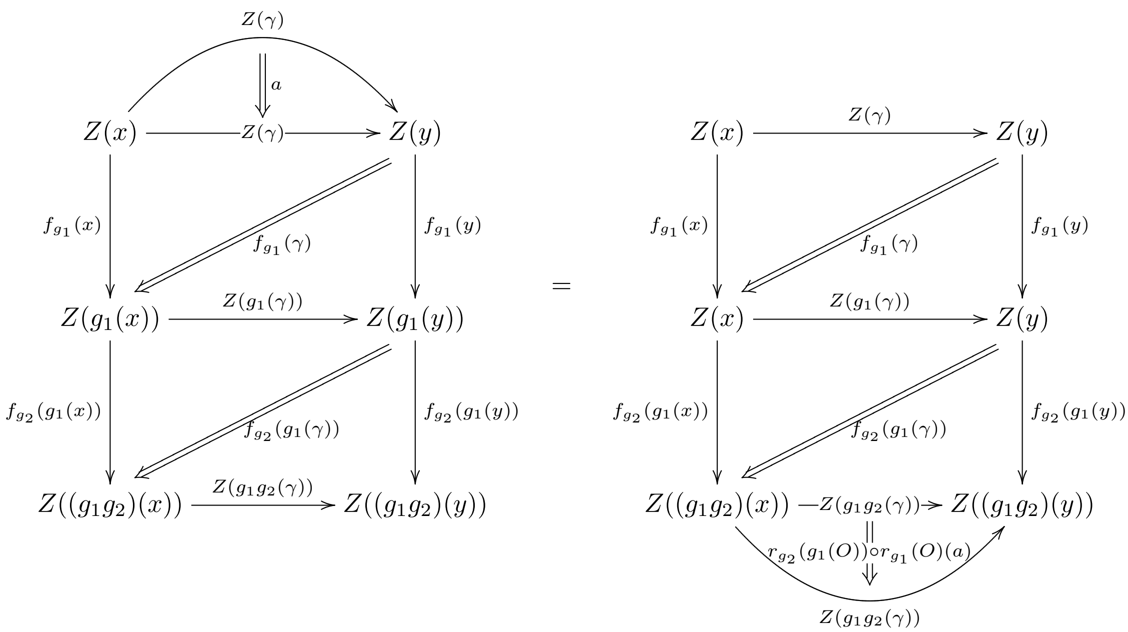


both $f_g(x \to 3 \to x')$ and $f_g(y' \to 5 \to y)$ drop out when conjugating and only conjugation with $f_g(x \to 4 \to y')$ acts nontrivially. But that precisely amounts to first applying $r_g(O')$ and then injecting into O.

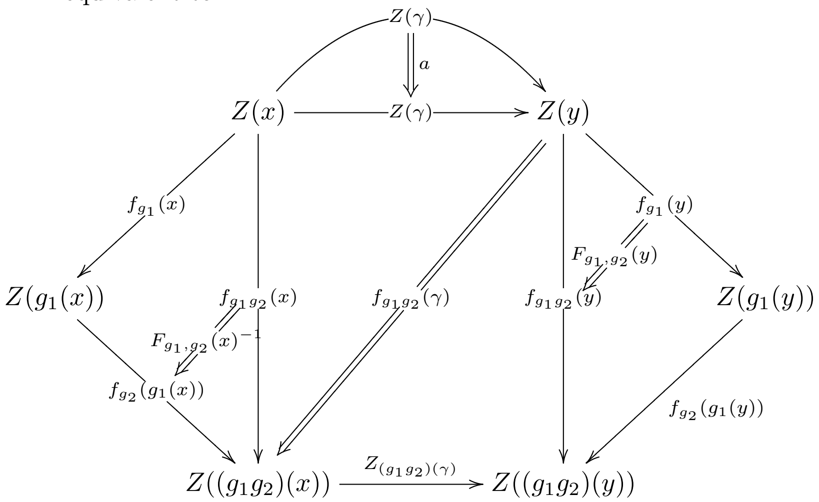
2. The equivariance triangle condition in definition 8 says precisely that $r_g(O)$ makes the required covariance triangle in definition 5 commute: To see this it is convenient to equivalently rewrite the previous equation for $r_g(O)$ as



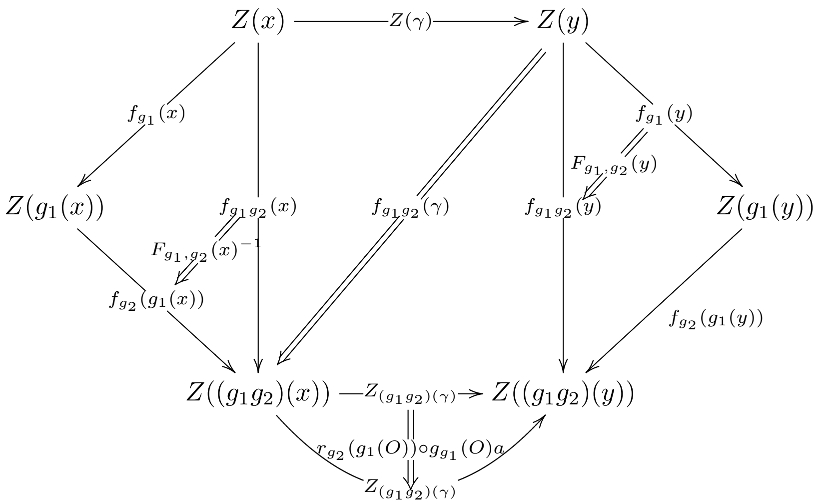
for all $a \in \mathrm{End}(Z(\gamma))$. Accordingly, we have for the composition of two transformations



for all $a \in \mathrm{End}(Z(\gamma))$. Using now the triangle of pseudonatural transformations in definition 8 this is equivalent to

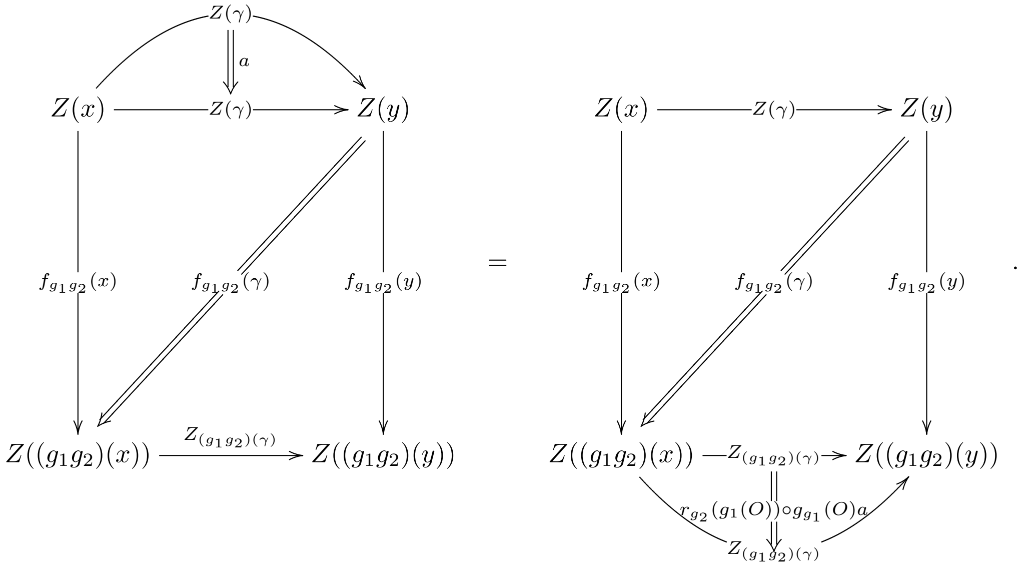


$=$



.

But in this equation we can cancel the $F_{\cdot,\cdot}$ on both sides to obtain



This shows that $r_{g_2}(g_1(O)) \circ r_{g_1}(O)(a) = r_{g_1 g_2}(O)(a)$.

□

7 Examples

7.1 1-dimensional case

Before looking at concrete examples for 2-FQFTs on Minkowski space it is again helpful to first recall some simple facts in the 1-dimensional case from our perspective.

We can regard ordinary quantum mechanics as given by an associated $U(E)$-bundle with connection on the real line (the "worldline") for E some Hilbert space. This bundle is necessarily trivializable. After picking a trivialization its globally defined Lie$(U(E))$-valued connection 1-form is

$$A = iHdt \in \Omega^1(\mathbb{R}^1, \mathfrak{u}(E))$$

with t the canonical coordinate and H a self-adjoint operator on E: the Hamilton operator. The quantum time evolution operator

$$Z : (\ t_0 \longrightarrow t_1\ ) \mapsto (\ E \xrightarrow{P\exp(\int_{[t_0,t_1]} A)} E\ )$$

is nothing but the parallel transport with respect to A (see for instance [49]).

In general H depends on t, in which case one speaks of *time dependent* quantum mechanics and the above formula, with its "path ordered exponential" on the right, is what is usually referred to as the *Dyson formula* in quantum mechanics textbooks. In that case there is no translational invariance on the worldline.

If however H is constant we have *time independent* quantum mechanics. In that case the quantum time evolution propagator reads

$$Z : (\ t_0 \longrightarrow t_1\ ) \mapsto (\ E \xrightarrow{P\exp(\int_{[t_0,t_1]} A)} E\ ) = (\ E \xrightarrow{\exp(i(t_1 - t_0)H)} E\ )\,.$$

In either case, there is a canonical equivariant structure, definition 8, on Z with respect to the action of $\mathbb{R}$ on $\mathbb{R}$ by translations: for $a \in \mathbb{R}$ the components of the natural transformation

$$Z \xrightarrow{f_t} a^* Z$$

are simply

$$f_a : x \mapsto (\; E_x \xrightarrow{Z(x \to x+a)} E_{x+a} \;) \, .$$

Naturality of f_t and commutativity of the equivariance coherence triangle both follow directly from the functoriality of Z. The equivariant structure on the net $\mathcal{A}_Z$ induced by this according to section 6 is that which acts on each local algebra $\mathcal{A}_Z(O_x)$ by the Heisenberg propagation rule $a \mapsto Z(x \to x+a) \circ a \circ Z(x \to x+a)^{-1}$.

7.2 Examples from parallel 2-transport

The above shows that the dynamics of quantum mechanics (1+0-dimensional QFT) can be entirely thought of as a vector bundle (or Hilbert bundle, rather) with connection on the "worldline" $\mathbb{R}$.

Similarly, 2-vector 2-bundles [9, 59] ($\simeq$ gerbes) with connection [8, 50, 51, 47] on the "worldsheet" $\mathbb{R}^2$ can be regarded as giving the dynamics of (1+1)-dimensional QFT. Indeed, every parallel transport 2-functor on $\mathbb{R}^2$ as in [8, 50, 51] gives an example of a 2-FQFT in the sense definition 7, simply by restricting it from all 2-paths in $\mathbb{R}^2$ to those contained in $P_2(\mathbb{R}^2)$. From each such 2-functor one obtains, by theorem 1, a local net of monoids. Whether this local net of monoids has any covariance depends, according to proposition 3, or whether or not the 2-functor has any equivariant structure. Whether the net of *monoids* obtained from the 2-functor is actually a net of algebras with certain extra structure (in particular C^*, von Neumann) depends on what precisely the 2-functor takes values in over 1-morphisms, because that determines what the endomorphism monoids are like.

While not every 2-bundle on 2-dimensional base space is necessarily trivializable, we here want to restrict attention to the case that the 2-bundle is trivializable. (If not, global effects such as described in [19] will play a role, too.) Then we can assume its parallel transport 2-functor to come from globally defined differential form data. If we require the 2-functor to be *strict* and to take values in a 2-groupoid with a single object, which we shall denote $\mathbf{B}G$, then theorem 2.20 in [50] says that it comes precisely from a pair consisting of a 1-form and a 2-form

$$A \in \Omega^1(\mathbb{R}^2, \mathfrak{g}), \; B \in \Omega^2(\mathbb{R}^2, \mathfrak{h})$$

with values in Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ which form a differential crossed module ($\mathfrak{h} \xrightarrow{t} \mathfrak{g} \xrightarrow{\alpha} \mathrm{der}(\mathfrak{g})$) such that

$$F_A + t_* \circ B = 0 \, ,$$

where $F_A \in \Omega^2(\mathbb{R}^2, \mathfrak{g})$ is the curvature 2-form of A. We write

$$Z_{(A,B)} : P_2(\mathbb{R}^2) \to \mathbf{B}G$$

for the 2-functor obtained this way. The local net $A_{Z_{(A,B)}}$ obtained from this by theorem 1 is a *local net of groups*.

We get proper nets of local *algebras* by passing instead to an *associated* parallel 2-transport functor [51], which is induced by a 2-representation of G on 2-vector space, i.e. a 2-functor

$$\rho : \mathbf{B}G \to \mathrm{2Vect} \, ,$$

where 2Vect denotes a 2-category of 2-vector spaces. In particular, [48], there are large classes of 2-representations which factor through the bicategory of bimodules

$$\begin{array}{ccc} \mathbf{B}G & \xrightarrow{\rho} & \mathrm{2Vect} \\ & \searrow \quad \nearrow & \\ & \mathrm{Bimod} & \end{array} \; ,$$

More details on this are summarized in appendix A and in [51].

The corresponding associated 2-FQFT functor

$$Z_{\rho(A,B)} : \ P_2(\mathbb{R}^2) \xrightarrow{Z_{(A,B)}} \mathbf{B}G \longrightarrow \mathrm{Bimod} \longrightarrow 2\mathrm{Vect}$$

sends each edge to a bimodule over some algebra. 2-Functors of this form and interpreted as 2-FQFTs have in particular been considered in [55].

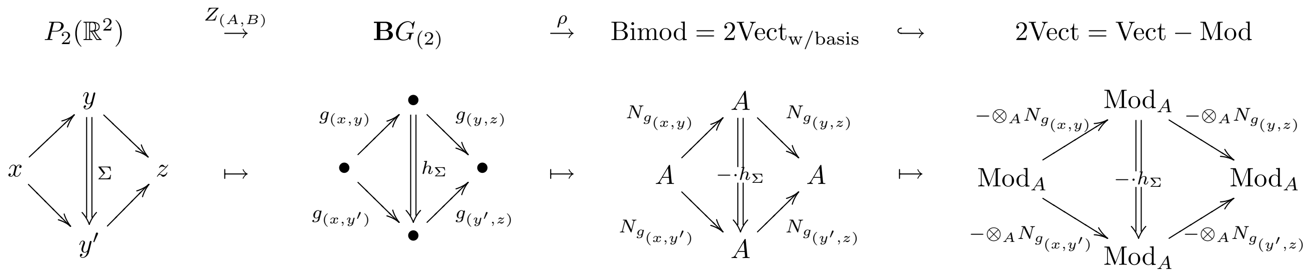


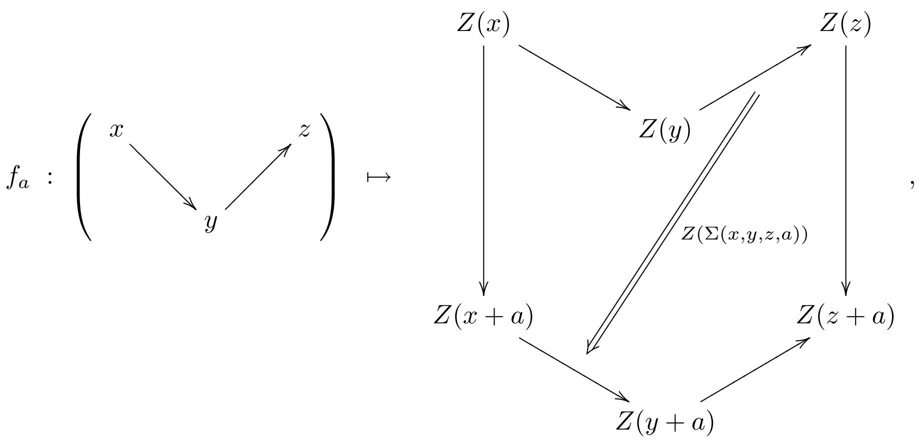
Figure 6: 2-Vector transport coming from a 2-connection $(A, B) \in \Omega^\bullet(\mathbb{R}^2, (\mathfrak{h} \to \mathfrak{g}))$ with values in the strict Lie 2-algebra $(\mathfrak{h} \to \mathfrak{g})$ and the canonical representation ρ of the corresponding strict Lie 2-group $G_{(2)}$ on 2-vector spaces. The 2-FQFT obtained this way assigns algebras to points, bimodules to paths and bimodule homomorphisms to surfaces. The corresponding local net $A_{Z_{(A,B)}}$ assigns algebras of bimodule endomorphisms.

Therefore the corresponding local net $A_{Z_{\rho(A,B)}}$ sends each $O \in S(\mathbb{R}^2)$ to an algebra of bimodule endomorphisms. This is reminiscent of various other constructions that have been considered in the context of AQFT. But a more detailed discussion will have to be given elsewhere.

As in the 1-dimensional case, we canonically have an equivariant structure on Z and on $\mathcal{A}_Z$ with respect to any 1-parameter group of translations which resepcts the light-cone structure. Let in particular $\mathbb{R}$ act by translation along the canonical time coordinare on $\mathbb{R}^2$. Then for $a \in \mathbb{R}$ the component of the pseudonatural transformation

$$Z \xrightarrow{f_a} a^* Z$$

is



where $\Sigma(x, y, z, a)$ denotes the surface swept out by the path $x \to y \to z$ when translating it continuously to $(x + a) \to (y + a) \to (z + a)$. This surface is not part of $P_2(\mathbb{R}^2)$ the way we have defined it, but is a more general 2-path in $\mathbb{R}^2$ on which we can evaluate our 2-functor Z, by assumption.

Pseudonaturality and coherence of the assignment f_a for all $a \in \mathbb{R}$ is a direct consequence of the 2-functoriality of Z, very similar to the 1-dimensional case. The induced equivariant structure on the net $\mathcal{A}_Z$ is the local Heisenberg picture time propagation.

7.3 2-Functors constant on one object

A simple class of examples worth looking at to get a feeling for the situation are those FQFT 2-functors Z on $P_2(\mathbb{R}^2)$ which assign a fixed object $V \in \mathrm{Obj}(C)$ to each point of $\mathbb{R}^2$, send all paths to the identity morphism on that object and all surfaces to the identity 2-morphism on this identity 1-morphism.

The local net $\mathcal{A}_Z$ obtained from such a 2-functor is constant. It assigns the same monoid to all causal subsets:

$$\mathcal{A}_Z : O \mapsto \mathrm{End}(\mathrm{Id}_V)\,.$$

For this to be a local net, it must be true that $\mathrm{End}(\mathrm{Id}_V)$ is a commutative monoid. And indeed it is: this is the Eckmann-Hilton argument which holds in general for 2-endomorphisms of identity 1-functors. The argument is entirely analogous (and that is of course no coincidence) to that which shows that the second homotopy group of any space is abelian.

In [24] the endomorphisms of the identity on an object V in a 2-category C is interpreted as the *trace* of the identity on V, which in turn is interpreted in [11] as the *dimension* of V:

$$A_Z(O) = \mathrm{End}(\mathrm{Id}_V) =: \mathrm{Tr}(\mathrm{Id}_V) =: \dim(V)\,.$$

For instance (see [11]) if $V = \mathrm{Rep}(H)$ is the category of representations of some group or groupoid H, regarded as a 2-vector space, then $\dim(V) = Z(\mathbb{C}(H))$ is the center of the group ring of H.

Another example, [24]: if C is the bicategory of bimodules, $C = \mathrm{Bimod}$, and V is any algebra, then $\dim(V)$ is the 0th Hochschild cohomology of V. Full Hochschild cohomology is obtained by taking the derived category of bimodules.

Of particular interest are objects V with a representation (meaning: 2-representation!) of the Poincaré group G in two dimensions, or some related group, on them. 2-Representations of the Poincaré group have been examined for instance in [16]. The constant FQFT 2-functor on such an object canonically carries a nontrivial G-equivariant structure in the sense of section 6, hence induces a covariant structure on the corresponding local net.

7.4 Nets from wedge algebras

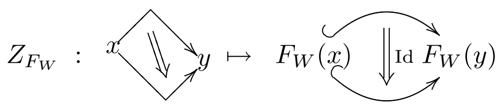
A special case of interest of the constructions in section 7.2 is the following:

Denote by $W := \{(x^0, x^1) \in \mathbb{R}^2 |\;\; x^1 \leq -|x^0|\} \subset \mathbb{R}^2$ the standard *left wedge* in 2-dimensional Minkowski space. Let $F_W : \mathbb{R}^2 \to \mathrm{Algebras} \downarrow A_W$ be a family of subalgebras of a fixed algebra A_W – to be called the *wedge algebra* – over 2-dimensional Minkowski space with the special property that every algebra in a wedge $y + W$ is a subalgebra of that at y:

$$\forall x, y \in \mathbb{R}^2 : (x - y \in W \;\Rightarrow\; A_x \hookrightarrow A_y)\,.$$

Wedge algebras of this general form are well known in algebraic quantum field theory: for instance, following [52], they play a major role in [37] (see definition 2.1.1, where of course the are equipped with more structure than considered for our purposes here).

Notice that the data of a wedge algebra naturally defines a 2-functor $Z_{F_W} : P_2(\mathbb{R}^2) \to \mathrm{Algebras}$ with values in the strict 2-category of algebras, algebra homomorphisms and intertwiners by the assignment



for all causal subsets between x and y as indicated. The reader may find it useful to think of this after the further inclusion Algebras $\to$ Bimod (see also appendix A) under which the right hand becomes

$$\cdots \mapsto \; F_W(x) \underset{F_W(y)}{\overset{F_W(y)}{\Downarrow \mathrm{Id}}} F_W(y)$$

with $F_W(y)$ regarded as an $F_W(x)$-$F_W(y)$-bimodule in the obvious way. In either case, the endomorphism algebra of the image under Z_{F_W} of any 1-morphism $(x \to y) \in P_2(\mathbb{R}^2)$ is seen to be a relative commutant in that

$$\mathrm{End}(Z_{F_W}(x \to y)) = (F_W(x))' \cap F_W(y)\,,$$

where $(F_W(y))' \subset A_W$ denotes the commutant of $F_W(y)$ in A_W, the algebra of elements of A_W that commute with all elements of $F_W(y)$. Hence the local net $\mathcal{A}_{Z_{F_W}}$ obtained from this 2-functor assigns

$$\mathcal{A}_{Z_{F_W}} \;:\; x \diamond_{O} y \mapsto (F_W(x))' \cap F_W(y)\,.$$

Local nets of this form obtained from wedge algebras are discussed for instance in [37] (there, again, equipped with more structure than considered here, see equation (2.18) in view of equation (2.1.3b)).

7.5 Lattice models

All our definitions and constructions make sense for $S(\mathbb{R}^2)$ and $P_2(\mathbb{R}^2)$ replaced by their restrictions $S(\mathbb{Z}^2)$ and $P_2(\mathbb{Z}^2)$ along that embedding $\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ which makes addition of $(1,0)$ a lightlike translation. This allows to see a class of important examples without the need to worry about weak 2-categories and issues in functional analysis.

Let

$$C := \mathbf{B}\mathrm{Vect} = \left\{ \bullet \underset{W}{\overset{V}{\Downarrow \phi}} \bullet \;|(V \xrightarrow{\phi} W) \in \mathrm{Vect} \right\}$$

be the strict 2-category obtained from the strict monoidal category of finite-dimensional vector spaces: it has a single object, its 1-morphisms are finite dimensional vector spaces with composition of morphisms being the tensor product of vector spaces, and 2-morphisms are linear maps $V \xrightarrow{\phi} W$ between vector spaces.

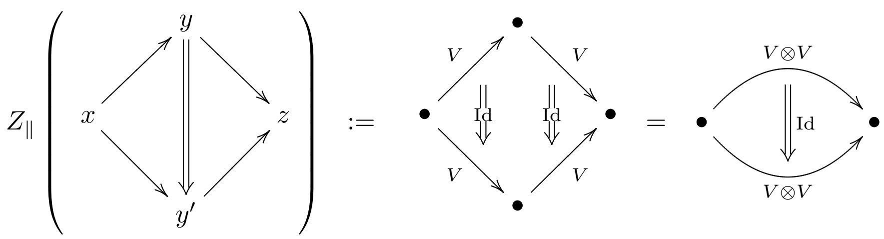
Pick a fixed finite dimensional vector space V and consider the two 2-FQFT 2-functors

$$Z_{\|} : P_2(\mathbb{Z}^2) \to \mathbf{B}\mathrm{Vect}$$

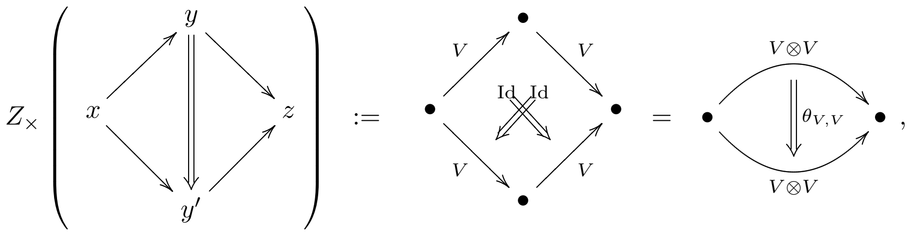
and

$$Z_{\times} : P_2(\mathbb{Z}^2) \to \mathbf{B}\mathrm{Vect}$$

which assign V to every elementary 1-morphism in $P_2(\mathbb{Z}^2)$ and which assign to every elementary square the linear map



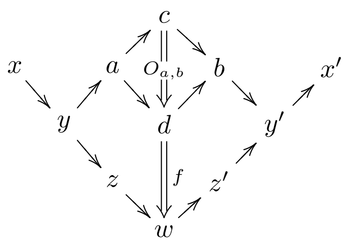
and



respectively, where $V \otimes W \xrightarrow{\theta_{V,W}} W \otimes V$ denotes the canonical symmetric braiding isomorphism in Vect.

The monoids assigned by the corresponding local nets $\mathcal{A}_{Z_\parallel}$ and $\mathcal{A}_{Z_\times}$ are algebras of the form $\mathrm{End}(V^{\otimes n})$, where n is the total number of elementary edges in the respective boundary of a region.

Given the inclusion of regions $O_{a,b} \subset O_{x,x'}$



we get, according to definition 9, inclusions

$$A_{Z_\parallel}, A_{Z_\times} : \mathrm{End}(V^{\otimes 2}) \hookrightarrow \mathrm{End}(V^{\otimes 6})$$

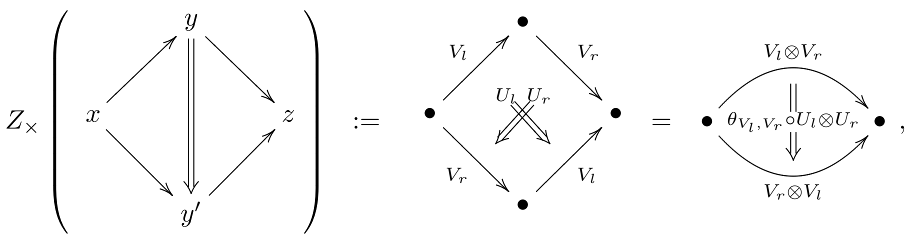
of endomorphism algebras given by

$$A_{Z_\parallel} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A & B & 0 & 0 \\ 0 & 0 & C & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \; A_{Z_\times} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & B & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & C & 0 & 0 & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where each entry in these matrices is an endomorphism of V.

The locality of the net $\mathcal{A}_{Z_\parallel}$ is manifest. The algebras assigned to two elementary regions clearly commute if and only if the two regions are spacelike separated. For $\mathcal{A}_{Z_\times}$ the algebras of course also commute if the regions are spacelike separated, but here they also commute if the two regions are *timelike* separated. Only if two elementary regions are lightlike separated do the inclusions of algebras due to $\mathcal{A}_{Z_\times}$ not commute.

There are various variations of this example. In particular for $Z_\times$ one would want to consider the case where two different vector spaces V_l and V_r and two nontrivial automorphisms $U_l : V_l \to V_l$ and $U_r : V_R \to V_r$ are assigned to elementary causal subsets as follows:



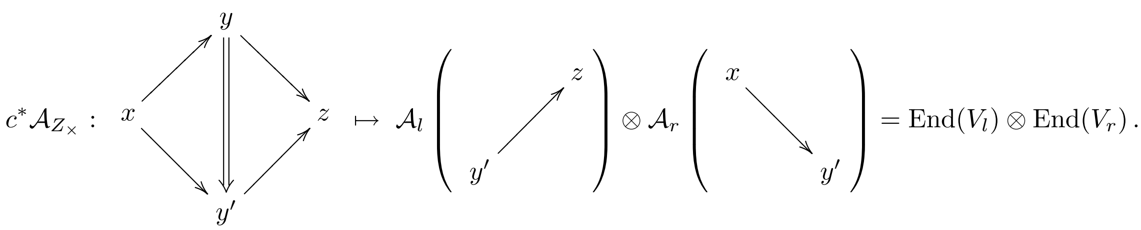
Denote by

$$c : \mathrm{End}(V_r) \otimes \mathrm{End}(V_l) \hookrightarrow \mathrm{End}(V_r \otimes V_l)$$

the canonical inclusion of algebras and by

$$c^*\mathcal{A}_{Z_\times} \hookrightarrow \mathcal{A}_{Z_\times}$$

the local sub-net of $\mathcal{A}_{Z_\times}$ obtained by restricting along c everywhere. Then $c^*\mathcal{A}_{Z_\times}$ is what is called a *chiral* AQFT. Its structure is encoded entirely in the two independent projections onto two orthogonal lightlike curves.



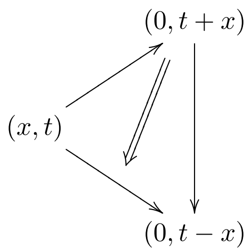
Restricting attention to just one of these and then "compactifying" that to a circle leads to the models [30, 32] of 2-dimensional (conformal) field theories as local nets on the circle.

This important example is further expanded on in section 7.6.

7.6 Boundary FQFT and boundary AQFT

AQFT on spaces with boundary has been introduced in [40] for the case of the Minkowski half-plane $X = \mathbb{R}^2_<$. Here we briefly indicate how boundary conditions are formulated for FQFT and how we recover the picture in [40] from this point of view.

We obtain the poset of causal subsets on the half plane, $S(\mathbb{R}^2_<)$, by starting with $S(\mathbb{R}^2)$ and intersecting everything with $\mathbb{R}^2_<$. We form $P_2(\mathbb{R}^2_<)$ by first restricting to 2-paths that run entirely within $\mathbb{R}^2_<$ and then throwing in new boundary generators for 1- and 2-morphisms of the form



From examples of classical parallel n-transport [47] and from the 2-functorial description of rational CFT [19] it is known that boundary conditions for n-functors Z correspond to choices of morphism from some trivial n-functor I into the restriction of the given one to the boundary:

$$I \longrightarrow Z|_{\partial X} \; .$$

We illustrate this in the context of the last example, $Z_\times : P_2(\mathbb{R}^2) \to \mathbf{B}\mathrm{Vect}$, from section 7, which lead to the discussion of chiral nets $i^*\mathcal{A}_{Z_\times} \subset \mathcal{A}_{Z_\times}$.

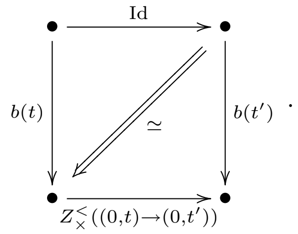
For that purpose, let I be the 2-functor $I : P_2(\mathbb{R}^2) \to \mathbf{B}\mathrm{Vect}$ which is constant on the single object of $\mathbf{B}\mathrm{Vect}$ and consider 2-functors $Z_\times^< : P_2(\mathbb{R}^2_<) \longrightarrow \mathbf{B}\mathrm{Vect}$ which coincide with our $Z_\times$ in the bulk. Then we have the simple but important

Proposition 3 *If a morphism*

$$b : I \to Z_\times^<|_{\partial\mathbb{R}^2_<}$$

exists and is time independent in that its component map is constant on objects (but not the 0 dimensional vector space), then $Z_\times^<$ assigns the identity to all boundary paths.

Proof. The components of the morphism, which is a pseudonatural transformation of 2-functors, are 2-cells in **B**Vect of the form

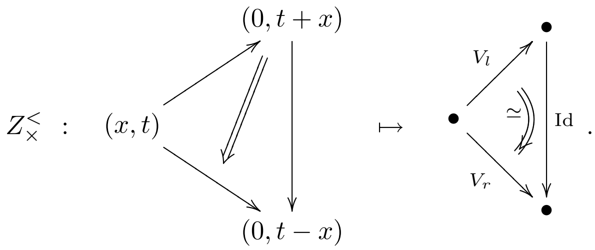


By assumption of time independence of the boundary condition we have $b(t) = b(t') = b(0)$. This means that $Z^<_\times((0,t) \to (0,t'))$ must be a vector space such that there exists an isomorphism of vector spaces

$$b(0) \otimes Z^<_\times((0,t) \to (0,t')) \simeq b(0)\,.$$

□

So in this case the 2-functor $Z^<_\times$ will specify identifications of the vector spaces V_l and V_r at the boundary



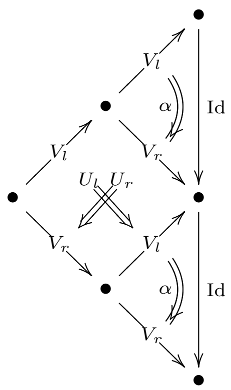


Figure 7: The image under the boundary FQFT 2-functor $Z^<_\times$ of a spacelike wedge on the left Minkowski half plane.

By taking endomorphisms this defines a net of algebras on the boundary, which entirely encodes the chiral part $c^*\mathcal{A}_{Z^<_\times}$ of $\mathcal{A}_{Z^<_\times}$. This way we arrive at the picture of boundary AQFT given in [40]. Further details should be discussed elsewhere.

7.7 2-C^*-category codomains

In most applications to physics one wants the algebras in a local net to be C^*-algebras. A natural type of 2-category in which endomorphism algebras of 1-morphisms are C^*-algebras is that of 2-C^*-categories: categories enriched in C^*-categories.

Definition 13 *A* C^*-category *(or* C^*-algebroid*: the many-object version of a C^*-algebra) is a category C enriched in complex Banach spaces (meaning that for all objects ρ, σ, τ of C we have that $C(\rho, \sigma)$ is a complex Banach space and that composition*

$$\circ_{\rho,\sigma,\tau} : C(\rho, \sigma) \times C(\sigma, \tau) \to C(\rho, \tau)$$

is a morphism of complex Banach spaces) which is equipped with an involutive antilinear functor

$$(\cdot)^* : C \to C^{\mathrm{op}}$$

that satisfies the C^*-condition

$$\forall \rho, \sigma \in \mathrm{Obj}(C) : \forall S \in C(a, b) : \begin{cases} S^* \circ S \text{ is positive in } C(\rho, \rho) \\ \|S^* \circ S\| = \|S\|^2 \end{cases} ,$$

where $\|\cdot\| : C(\rho, \sigma) \to \mathbb{C}$ is the Banach norm.

A C^*-algebra A is precisely the endomorphism algebra of an object ρ in a C^*-category, $A = C(\rho, \rho)$. We write $\mathbf{B}A$ for the one object C^*-category whose single endomorphism algebra is A.

C^*-categories form a strict monoidal 2-category $(C^*\mathrm{Cat}, \times)$ whose morphisms are Banach space functors (continuous on each Hom-space). Therefore one can enrich in C^*-categories themselves:

Definition 14 *A (strict)* 2-C^*-category *is a category enriched in* $C^*\mathrm{Cat}$.

A discussion of aspects of 2-C^*-categories can be found in [60].

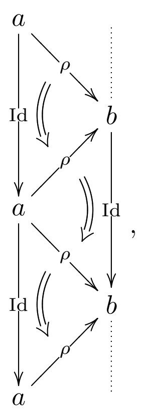
The canonical example of a strict 2-C^*-category is $\mathrm{Ampli}_{C^*} \subset \mathrm{Bimod}_{C^*}$, the 2-category whose objects are unital C^*-algebras, whose morphisms are amplimorphisms between these and whose 2-morphisms are intertwiners between those. Bimod_{C^*} is very similar, but is not strict. See [39] and section 2 of [60].

So we have

Observation 2 *For $Z : P_2(X) \to \mathcal{C}$ a transport 2-functor with values in a 2-C^*-category $\mathcal{C}$, the corresponding local net A_Z is a net of C^*-algebras.*

7.8 Hopf spin chain models

Recall the description of lattice models with boundary from section 7.6. Consider the extreme case where there is a left and right boundary which are separated only by a single lattice spacing:

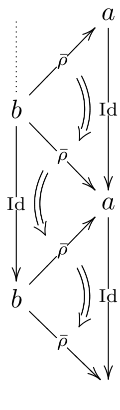


,

where for simplicity we are concentrating on the case that Z sends each edge to one and the same morphism $\rho : a \to b$ in $\mathcal{C}$.

Physically, we can think of this as a lattice model for an open string stretching from a brane of type a to a brane of type b. It s a crude lattice model, consisting of a single "string bit".

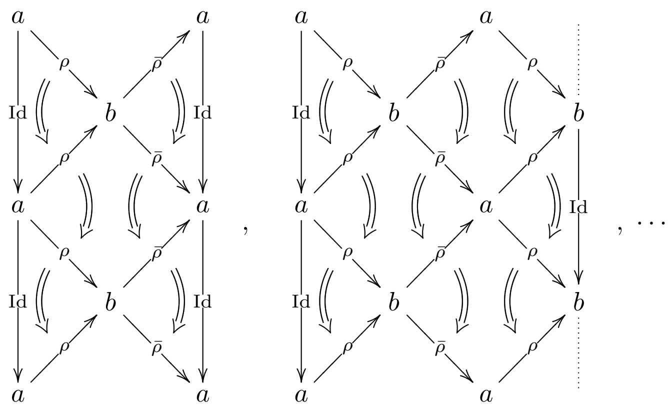
Consider another such strip, labeled by another morphism $\bar{\rho} : b \to a$



.

As the notation suggests, we want to think of $\bar{\rho}$ to be *conjugate* to ρ, meaning that ρ and $\bar{\rho}$ form an ambidextrous adjunction [36] between a and b such that the unit of the left-handed adjunction is the *-adjoint of the counit of the right-handed adjunction, and vice versa. (see p. 8 of [60]).

Then it makes sense to think of this as a lattice model for an open string, or rather a "string bit", as before, but now with that string taken to stretch from the b-type brane to the a-type brane. We can then consider lattice models built from the above building blocks by gluing the above strip-wise 2-functors horizontally:



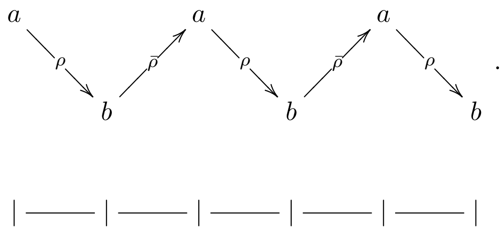
The algebras assigned by the corresponding net A_Z to the elementary causal bigon $O_{\rho,\bar{\rho}}$ and $O_{\bar{\rho},\rho}$ are

$$A_Z(O_{\rho,\bar{\rho}}) = \mathrm{End}_{\mathcal{C}}(\bar{\rho} \circ \rho)$$

and

$$A_Z(O_{\bar{\rho},\rho}) = \mathrm{End}_{\mathcal{C}}(\rho \circ \bar{\rho}) \,.$$

If $\mathcal{C}$ is a 2-C^*-category and ρ is an "irreducible 1-morphism generating a 2-C^*-category of depth two" as in section 4 of 7, then these are C^*-Hopf algebras H and $\hat{H}$ which are duals of each other [41, 60]. Due to the fact that the 2-morphisms in the above diagrams do not mix ρ and $\bar{\rho}$, we can understand the nature of the net A_Z obtained from the above 2-functor Z already by concentrating on the endomorphism algebras assigned to a horizontal zig-zag



If we to restrict to evaluating the net A_Z on zig-zags of even length, this gives rise to a net on the latticized real axis with the property that algebras $A_Z(I_1)$ and $A_Z(I_2)$ commute if the intervals I_1 and I_2 are not just disjoint but differ by at least one lattice spacing. Precisely these kind of 1-dimensional nets are considered in [42], where they are addressed as *Hopf spin chain models*.

8 Further issues

There are various immediate further questions to be addressed. We shall be content here with just briefly commenting on the following.

Continuum limits of lattice models and von Neumann algebra valued nets. We have shown that 2-functorial FQFTs very generally give rise to local nets of *monoids* and observed that 2-functors with values in 2-C^*-categories give rise to local nets of C^*-algebras. One would want to identify concretely those 2-functors which induce the celebrated local nets of *von Neumann algebra factors*. It is to be expected that many local nets of von Neumann algebras can be obtained from taking continuum limits of lattice models. By considering in this continuum limit the relation between lattice AQFT and lattice FQFT discussed in section 7, one should be able to construct examples of the desired 2-functors.

But while the idea of obtaining AQFT nets from continuum limits of lattice models seems to be straightforward and of considerable relevance, there exist to date apparently no published studies of this problem. A discussion of the problem of 2-functorial FQFT corresponding to local nets of von Neumann algebra factors will therefore have to be given elsewhere.

General Lorentzian structure AQFT was originally conceived entirely in its application to quantum field theories on Minkowski space, which is the case we have been concentrating on above. A generalization of Poincaré-covariant nets on causal subsets in Minkowski space to nets on globally hyperbolic Lorentzian spaces has later been proposed in [12].

The possibly most natural and immediate generalization to AQFT on a fixed general Lorentzian space was indicated in [44]: on a Lorentzian manifold X an AQFT net should be *locally local*: the locality axiom should hold after restriction of the net to any globally hyperbolic subspace of X. The same should be true for the time slice axiom.

No guesswork is required for generalizing the concept of Minkowskian FQFT 2-functors to general Lorentzian 2-functors: the concept of the 2-functor itself makes unambiguous sense for any choice of 2-path 2-category in X. So we can use our construction of local nets from 2-functors to *derive* locality properties of nets on Lorentzian spaces. Doing so confirms the idea of [44]:

Let (X, g) be any 2-dimensional oriented and time-oriented Lorentzian manifold.

In generalization of definition 1 consider

Definition 15 *A causal subset $O \subset X$ is a subset of a globally hyperbolic subset of X which is the interior of a non-empty intersection of the future of one point with the past of another. Write $S(X)$ for the category with such causal subsets as objects and inclusion of subsets as morphisms.*

In generalization of definition 6 consider

Definition 16 *Let $P_2(X)$ be the strict 2-category whose objects are the points in X, whose 1-morphisms are piecewise lightlike and right-moving paths (with respect to the chosen orientation and time-orientation of X) and whose 2-morphisms are generated under gluing along common boundaries from closures of causal subsets.*

Our construction in definition 9 immediately generalizes to a construction of a net $\mathcal{A}_Z : S(X) \to \text{Monoids}$ from a 2-functor $Z : P_2(X) \to C$. All the arguments need to be done within globally hyperbolic subsets of X, where they go through literally as before. We can *read off* from the result of this construction the locality properties of $\mathcal{A}_Z$:

Proposition 4 *The net $\mathcal{A}_Z : S(X) \to$ Monoids obtained from any 2-functor $Z : P_2(X) \to C$ is* locally local *and satisfies the* local time slice axiom*: for any inclusion*

$$i : Y \hookrightarrow X$$

with Y globally hyperbolic we have that $i^*\mathcal{A}_Z$ is a local net satisfying the time slice axiom.

This concept of local locality is compatible with [12] but does not presuppose any covariance condition on the net.

Higher dimensional QFT. We had considered, for ease of discussion, in definition 4 the 2-category $P_2(X)$ whose 2-morphisms are generated from gluing the closures of 2-dimensional causal subsets along common boundaries. But nothing in our constructions crucially depends on gluing of causal subsets, and in fact gluing of causal subsets becomes less natural in higher dimensions. As the examples we presented in section 7, where we obtained FQFT 2-functors by *restricting* 2-functors on a larger 2-category of 2-paths to $P_2(X)$, clearly indicate, the 2-category $P_2(X)$ can be replaced by any 2-category of 2-paths in X which is large enough that every causal subset in X can be regarded as a 2-morphisms in there, so that every FQFT 2-functor can be evaluated on causal subsets. And this statement then immediately generalizes to higher dimensions.

For X a d-dimensional Lorentzian manifold, we should take the category $S(X)$ to be that whose objects are causal subsets in X, which are those subsets that arise within any globally hyperbolic subset of X as the interior of the future of one point with the past of another point. Morphisms are inclusions.

The d-category $P_d(X)$ used to described Lorentzian FQFT on X can be any sub-d-groupoid of the path d-groupoid [47] which is large enough so that every causal subset in X comes from a d-morphism in $P_d(X)$ and such that the obvious higher dimensional generalizations of the diagrams in section 5 exist in $P_d(X)$. In particular, one can always use the *full* path d-groupoid.

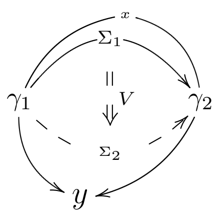


Figure 8: A 3-morphism in a 3-path 3-category: a volume V, cobounding two surfaces Σ_1 and Σ_2, which each cobound two paths γ_1 and γ_2 which each cobound two points x any y.

With such a setup, all our constructions here should have essentially straightforward generalizations to higher dimensions, leading to a construction of local nets on X from any FQFT d-functor on X. In such a context the spatial separation of two causal subsets would manifest itself not in the position of endomorphisms in a 1-dimensional string of products, but in their position in a higher dimensional algebra.

A 2-Vector spaces and the canonical 2-representation

In section 7 we obtained examples of FQFT 2-functors from differential form data and a choice of 2-representation. Here we briefly indicate a bit of background concerning these 2-representations.

For our purposes here a 2-vector space is an abelian module category, i.e an abelian category equipped with an action by a monoidal category. Notice that the category of k-vector spaces is the category of k-modules

$$\mathrm{Vect}_k = k - \mathrm{Mod}\,.$$

Accordingly we write

$$2\mathrm{Vect} = \mathrm{Vect}_{\mathrm{Vect}} = \mathrm{Vect} - \mathrm{Mod}$$

for the 2-category of abelian categories equipped with a (left, say) $(\mathrm{Vect}, \otimes)$-action. Since Vect is symmetric monoidal, one can keep going this way and in principle define recursively the n-category

$$n\mathrm{Vect} = (n-1)\mathrm{Vect} - \mathrm{Mod}\,.$$

Notice in particular that then $0\mathrm{Vect} = k$.

There are other monoidal categories over which one may want to consider 2-vector spaces. For instance if we denote by $\mathrm{Disc}(k)$ the discrete category over the ground field (the ground field as its objects and only identity morphisms), then

$$\mathrm{Disc}(k) - \mathrm{Mod} \simeq \mathrm{Cat}(\mathrm{Vect})$$

is the 2-category of categories internal to vector spaces, which in turn is equivalent to chain complexes concentrated in degree 0 and 1. These are the 2-vector spaces considered in [5]. $\mathrm{Disc}(k)$-modules are the "right" notion for 2-vector space for higher Lie theory, but probably not [3] as models for fibers of interesting 2-vector bundles.

The entirety of the 2-category of all Vect-modules is quite untractable. What is more accessible and more useful is the 2-category of 2-vector space that "have a basis". Noticing that an ordinary vector space V has a basis if there is a set S such that $V \simeq \mathrm{Hom}_{\mathrm{Set}}(S, k)$, we should define a basis for a 2-vector space V to be a category S such that $V \simeq \mathrm{Hom}(S, \mathrm{Vect})$. If S is itself Vect-enriched this says that V is a category of algebroid modules. We shall restrict attention to S having a single object, in which case we are left with modules for ordinary algebras.

This way we find the bicategory Bimod of algebras, bimodules and bimodule homomorphisms sitting inside 2Vect as a sub-2-category of 2-vector spaces with basis:

$$\mathrm{Bimod} \hookrightarrow 2\mathrm{Vect}$$

$$\left(A \underset{N'}{\overset{N}{\Downarrow \phi}} B \right) \mapsto \left(\mathrm{Mod}_A \underset{-\otimes_A N'}{\overset{-\otimes_A N}{\Downarrow -\otimes_A \phi}} \mathrm{Mod}_B \right)\,.$$

Notice how Mod_A is a *category of modules* which is itself a *module category* over Vect. The 2-category of Kapranov-Voevodsky 2-vector spaces [29] is the full sub 2-category of Bimod on all algebras of the form $k^{\oplus n}$ for $n \in \mathbb{N}$.

$$\mathrm{KV2Vect} \hookrightarrow \mathrm{Bimod}\,.$$

While Bimod is not a strict 2-category, it is a *framed bicategory* in the sense of [54]: there is the strict 2-category Algebras of algebras, algebra homomorphisms and intertwiners (the obvious 2-category for algebras regarded as one-object Vect-enriched categories), and the obvious inclusion

$$\mathrm{Algebras} \hookrightarrow \mathrm{Bimod}$$

is full and faithful on all Hom-categories. Noticing that similarly groups, when regarded as one-object groupoids, live in the 2-category Groups of groups, group homomorphisms and inner automorphisms, we get a strict 2-functor

$$\mathrm{Groups} \longrightarrow \mathrm{Algebras}$$

induced from forming for each group its group algebra. For each group H there is the 2-group $\mathrm{AUT}(H) := \mathrm{Aut}_{\mathrm{Groups}}(H)$ and the canonical inclusion

$$\mathbf{B}\mathrm{AUT}(H) \hookrightarrow \mathrm{Groups}$$

induces, combined with the above discussion, the canonical 2-representation of $\mathrm{AUT}(H)$ given by

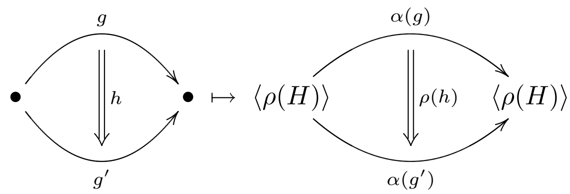
$$\rho_{\mathrm{can}} : \mathbf{B}\mathrm{AUT}(H) \longrightarrow \mathrm{Groups} \longrightarrow \mathrm{Algebras} \longrightarrow \mathrm{Bimod} \longrightarrow 2\mathrm{Vect} .$$

The logic of this construction generalizes to arbitrary strict 2-groups $G_{(2)}$ coming from crossed modules of groups $(H \stackrel{t}{\to} G \stackrel{\alpha}{\to} \mathrm{Aut}(G))$ (see for instance [50] for a review) and algebras obtained from a representation of H:

Proposition 5 *For* $\rho : \mathbf{B}H \to \mathrm{Vect}$ *a representation of* H *such that the action of* G *on* H *extends to algebra automorphisms of the representation algebra* $\langle \rho(H) \rangle$*, the assignment*

$$\tilde{\rho} : \mathbf{B}(H \to G) \to \mathrm{Algebras}$$

given by



is a strict 2-functor.

Accordingly we obtain a 2-representation

$$\mathbf{B}(H \to G) \xrightarrow{\tilde{\rho}} \mathrm{Algebras} \longrightarrow \mathrm{Bimod} \longrightarrow 2\mathrm{Vect} .$$

All this should go through when the vector spaces here are equipped with more structure. In particular, for G a compact, simple and simply connected group, for $\rho : \mathbf{B}\hat{\Omega}G \to \mathrm{Hilb}$ a positive-energy representation of the weight 1 central extension of its loop group and for vNBimod the bicategory of vonNeumann algebras and their bimodules composed under Connes-fusion, [55] the above should extend to a 2-representation

$$\mathbf{B}\mathrm{String}(G) \to \mathrm{vNBimod}$$

of the strict String 2-group [6].

References

[1] Samson Abramsky and Bob Coecke, *A categorical semantics of quantum protocols*, Proceedings of the 19th IEEE conference on Logic in Computer Science (LiCS'04). IEEE Computer Science Press (2004) [arXiv:quant-ph/0402130]

[2] Michael Atiyah, *Topological quantum field theory*, Publications Mathématiques de l'IHÉS, 68 (1988), p. 175-186

[3] Nils Baas, Marcel Bökstedt andTore Kro, *Two-Categorical Bundles and Their Classifying Spaces* [arXiv:math/0612549]

[4] John Baez, *Quantum Quandaries: a Category-Theoretic Perspective*, Structural Foundations of Quantum Gravity, eds. Steven French, Dean Rickles and Juha Saatsi, Oxford U. Press, Oxford, 2006, pp. 240-265. [arXiv:quant-ph/0404040]

[5] John Baez and Alissa Crans, *Lie 2-algebras*, Theory and Applications of Categories 12 (2004), 492-528 [arXiv:math/0307263]

[6] John Baez, Alissa Crans, Urs Schreiber, Danny Stevenson, *From loop groups to 2-groups*, Homology, Homotopy and Applications, Vol. 9 (2007), No. 2, pp.101-135 [arXiv:math/0504123]

[7] John Baez and Jim Dolan, *Higher dimensional algebra and topological quantum field theory*, [arXiv:q-alg/9503002]

[8] John Baez and Urs Schreiber, *Higher gauge theory*, in Contemporary Mathematics, 431, Categories in Algebra, Geometry and Mathematical Physics, [arXiv:math/0511710]

[9] Toby Bartels, *2-Bundles* PhD Thesis, UC Riverside, 2006 [arXiv:math/0410328]

[10] Bruce Bartlett, *Categorical aspects of Topological Quantum Field Theories*, M.Sc Thesis, Utrecht University, 2005 [arXiv:math/0512103]

[11] Bruce Bartlett, *On unitary 2-representations of finite groups and TQFT*, PdD Thesis, University of Sheffield, 2008

[12] Romeo Brunetti, Klaus Fredenhagen, Rainer Verch, *The generally covariant locality principle – A new paradigm for local quantum physics*, Commun.Math.Phys. 237 (2003) 31-68 [arXiv:math-ph/0112041]

[13] Detlev Buchholz, Rudolf Haag, *The Quest for Understanding in Relativistic Quantum Physics*, J.Math.Phys. 41 (2000) 3674-3697, [arXiv:hep-th/9910243]

[14] Eugenia Cheng and Nick Gurski, *Towards an n-category of cobordisms*, Theory and Applications of Categories, Vol 18, 2007, No. 10, pp. 274-302

[15] Bob Coecke, *Kindergarten Quantum Mechanics*, talk at Quantum Information, Computation and Logic (Perimeter Institute, 2005) [arXiv:quant-ph/0510032]

[16] Josep Elgueta, *Representation theory of 2-groups on Kapranov and Voevodsky's 2-vector spaces*, Advances in Mathematics Volume 213, Issue 1, 1 August 2007, Pages 53-92 [arXiv:math/0408120]

[17] D. E. Evans and Y. Kawahigashi, *Quantum symmetries on operator algebras*, Oxford University Press, Oxford, 1998

[18] Jens Fjelstad, Jürgen Fuchs, Ingo Runkel and Christoph Schweigert, *Uniqueness of open/closed rational CFT with given algebra of open states*, Adv. Theor. Math. Phys. Volume 12, Number 6 (2008), 1283-1375 [arXiv:hep-th/0612306]

[19] Jens Fjelstad and Urs Schreiber, *Rational CFT is parallel transport*, in preparation [http://www.math.uni-hamburg.de/home/schreiber/cc.pdf]

[20] Klaus Fredenhagen, Karl-Henning Rehren and Erhard Seiler, *Quantum Field Theory: Where We Are*, Lecture Notes in Physics 721 (2007) 61-87 [arXiv:hep-th/0603155]

[21] D. S. Freed, *Higher algebraic structures and quantization*, Commun. Math. Phys. **159** (1994) 343-398, [arXiv:hep-th/9212115].

[22] D. S. Freed, *Quantum groups from path integrals*, proceedings of Particles and Fields (Banff, 1994), 63107, CRM Ser. Math. Phys., Springer, New York, 1999, [arXiv:q-alg/9501025].

[23] D. S. Freed, *Remarks on Chern-Simons Theory*, AMS Bulletin, 0273-0979(09)01243-9 [arXiv:0808.2507]

[24] Nora Ganter, Mikhail Kapranov, *Representation and character theory in 2-categories*, Advances in Mathematics, Volume 213, Issue 1, 1 August 2007, Pages 53-92 [arXiv:math/0602510]

[25] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Springer 1992, 1996.

[26] Hans Halvorson and Michael Mueger, *Algebraic Quantum Field Theory*, in J. Butterfield and J. Earman (eds.) Handbook of the Philosophy of Physics (pp. 731-922), Amsterdam: North-Holland (2007) [arXiv:math-ph/0602036]

[27] Jacob Lurie, *On the Classi*
cation of Topological Field Theories, [http://www-math.mit.edu/∼lurie/papers/cobordism.pdf]

[28] Yi-Zhi Huang, *Geometric interpretation of vertex operator algebras*, Proc. Natl. Acad. Sci. USA, Vol 88, (1991) pp. 9964-9968

[29] Kapranov Voevodsky, *2-categories and Zamolodchikov tetrahedra equations* in Proc. Symp. Pure Math. 56, Part 2, 177-260. American Mathematical Society, 1994

[30] Yasuyuki Kawahigashi, *Classification of operator algebraic conformal field theories in dimensions one and two*, proceedings of XIV International Congress on Mathematical Physics at Lisbon (2003) [arXiv:math-ph/0308029]

[31] Yasuyuki Kawahigashi, *Conformal Field Theory and Operator Algebras*, Proceedings of ICMP, Rio de Janeiro (2006) [arXiv:0704.0097]

[32] Yasuyuki Kawahigashi and Roberto Longo, *Classification of two-dimensional local conformal nets with $c < 1$ and 2-cohomology vanishing for tensor categories*, CMP, Volume 244, Number 1, January 2004 , pp. 63-97(35) [arXiv:math-ph/0304022]

[33] G. M. Kelly, *Basic concepts of enriched category theory*, Cambridge University Press, Lecture Notes in Mathematics 64, 1982; Reprints in Theory and Applications of Categories, No. 10 (2005) pp. 1-136

[34] Yasuyuki Kawahigashi, Roberto Longo, *Local conformal nets arising from framed vertex operator algebras*, Advances in Mathematics Volume 206, Issue 2, 2006, 729-751 [arXiv:math/0407263]

[35] Liang Kong, *Open-closed field algebras*, Commun. Math. Phys. 280, 207–261 (2008), [arXiv:math/0610293]

[36] Aaron Lauda, *Frobenius algebras and ambidextrous adjunctions*, Theory and Applications of Categories, Vol. 16, 2006, No. 4, 84-122 [arXiv:math/0502550]

[37] G. Lechner, *On the construction of quantum field theories with factorizing S-Matrices*, PhD thesis, Göttingen (2006)

[38] Tom Leinster, *Basic Bicategories*, [arXiv:math/9810017]

[39] Roberto Longo, Karl-Henning Rehren, *Nets of subfactors*, Rev. Math. Phys., 7 (1995)

[40] Roberto Longo, Karl-Henning Rehren, *Local fields in boundary conformal QFT*, Rev.Math.Phys. 16 (2004) 909 [arXiv:math-ph/0405067]

[41] Michael Müger, *From Subfactors to Categories and Topology I. Frobenius algebras in and Morita equivalence of tensor categories*, J, Pure Appl. Alg. **180** (2003), 81-157, [arXiv:math/0111204]

[42] Florian Nill and Kornél Szlachány, *Quantum chains of Hopf algebras with order-disorder fields and quantum double symmetry*, CMP volume 187 number 1, 1997 [arXiv:hep-th/9509100]

[43] A. Ocneanu, *Quantized group, string algebras and Galois theory for algebras*, in *Operator algebras and applications*, Vol. 2, p. 119, eds. D. E. Evans and M. Takesaki, Cambridge University Press, 1988

[44] Karl-Henning Rehren, weblog comment, [http://golem.ph.utexas.edu/ distler/blog/archives/000987.html#c005396]

[45] John Roberts and Roberto Longo, *A theory of dimension*, K-Theory, Vol. 11, Number 2, 1997 ,p. 103-159

[46] Ingo Runkel, Jens Fjelstad, Jürgen Fuchs and Christoph Schweigert, *Topological and conformal field theory as Frobenius algebras*, Contemp.Math. 431:225-248, 2007 [arXiv:math/0512076]

[47] H. Sati, U. Schreiber, Z. Škoda, D. Stevenson, *Twisted nonabelian differential cohomology*, in preparation, [http://www.math.uni-hamburg.de/home/schreiber/nactwist.pdf]

[48] Urs Schreiber, *The canonical 2-representation*, [http://www.math.uni-hamburg.de/home/schreiber/canrep.pdf]

[49] Urs Schreiber and Konrad Waldorf, *Parallel transport and functors*, to appear in Journal of Homotopy and Related Structures [arXiv:0705.0452].

[50] Urs Schreiber and Konrad Waldorf, *Smooth functors vs. differential forms*, [arXiv:0802.0663].

[51] Urs Schreiber and Konrad Waldorf, *Connections on non-abelian Gerbes and their Holonomy*, [arXiv:0808.1923]

[52] Bert Schroer, *Modular Wedge Localization and the $d = 1 + 1$ Formfactor Program*, Annals Phys. 275 (1999) 190-223 [arXiv:hep-th/9712124]

[53] Graeme Segal, *The definition of conformal field theory*, Topology, geometry and quantum field theory, London Math. Soc. LNS 308, edited by U. Tillmann, Cambridge Univ. Press 2004, 247-343.

[54] Michael Shulman, *Framed Bicategories and Monoidal Fibrations*, Theory and Applications of Categories, Vol. 20, 2008, No. 18, pp 650-738 [arXiv:0706.1286]

[55] Stephan Stolz and Peter Teichner, *What is an elliptic object?*, Topology, geometry and quantum field theory, London Math. Soc. LNS 308, Cambridge Univ. Press 2004, 247-343. [http://math.berkeley.edu/∼teichner/Papers/Oxford.pdf]

[56] Ross Street, *Categorical and combinatorial aspects of descent theory*, Applied Categorical Structures, Vol. 12, Num. 5-6, 2004 [arXiv:math/0303175]

[57] Dominic Verity, *Cobordisms and weak complicial sets*, Australian Category Theory Seminar, February 13, 2008, Macquarie University

[58] Kevin Walker, *TQFTs*, [http://canyon23.net/math/tc.pdf]

[59] Christoph Wockel, *A global perspective to gerbes and their gauge stacks*, [arXiv:0803.3692]

[60] Pasquale Zito, *2-C^*-categories non-simple units*, PhD thesis, [arXiv:math/0509266]