The Feynman Propagator and Cauchy's Theorem

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The aim of these notes is to show how to derive the momentum space form of the Feynman propagator which is $\Delta(p) = i/(p^2 - m^2 + i\epsilon)$. For most of this course and for most work in QFT, "propagator" refers to the Feynman propagator[2].

Wightman function

The **Wightman function** $D(x-y)$ is a useful mathematical construction and while it contains physical information, it does not itself have a natural physical interpretation. It is a useful algebraic tool in some circumstances. In terms of operators it is defined as the vacuum expectation value of fields in a fixed order

$$D(x-y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|\,0\rangle \,. \tag{1}$$

In principle this should be written as a function of two variables, $D(x,y)$, but we will assume our theories are space-time translation (and Lorentz invariant) which implies this can only be a function of $x-y$.

The field operator for free fields[3] is given by

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{-ip\cdot x} + \hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{ip\cdot x}\right) \ , \qquad \omega_p = |\sqrt{\boldsymbol{p}^2+m^2}| \tag{2}$$

So then the Wightman function is simply

$$D(x-y) \quad = \quad \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}\mathrm{e}^{-ip(x-y)} \,. \tag{3}$$

Cauchy's theorem

We now want to use a result from complex analysis. Suppose an analytic function $f(z)$ has simple poles at $z = z_i$ where $i = 1, \ldots, n$. This means that near $z = z_i$ the function diverges as

$$f(z) = \frac{R_i}{z - z_i} + \ldots \tag{4}$$

where the remaining terms are finite as $z \to z_i$ and R_i is known as the residue at $z = z_i$. For simple poles like this, the R_i is simply the part of the function $f(z)$ without the pole but evaluated at the pole z_i i.e. you find

$$R_i = \lim_{z \to z_i}(z - z_i)f(z) \,. \tag{5}$$

This means the residue is in principle different for every pole z_i.

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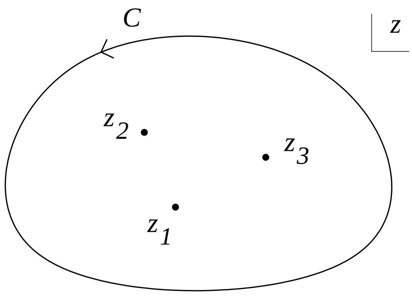
[1]Extended and adapted from notes by Prof.Waldram.

[2]There are other types of propagator such as the retarded propagator which we have seen as the Green function for the Klein-Gordon equation with retarded boundary conditions. This is linked to vacuum expectation values of pairs of field operators ordered in a different way from other types of propagator.

[3]This is the full field in the Heisenberg picture for a free field theory or, for interacting scalar fields, this is the field operator in the interaction picture.

Cauchy's theorem states



$$\int_C f(z)\mathrm{d}z = 2\pi\mathrm{i}\sum_i R_i \tag{6}$$

where the sum is over those points $z = z_i$ enclosed by the closed curve C.

Feynman Propagator

One definition of the **Feynman propagator**[4] Δ, is as the vacuum expectation value of the time-ordered expectation value two free fields of the form (2),

$$\Delta(x-y) = \langle 0|\mathrm{T}\hat{\phi}(x)\hat{\phi}(y)|\,0\rangle\,. \tag{7}$$

The **time-ordering operator** T is defined so that when acting on two fields we have

$$\mathrm{T}\hat{\phi}(x)\hat{\phi}(y) = \theta(x^0-y^0)\hat{\phi}(x)\hat{\phi}(y) + \theta(y^0-x^0)\hat{\phi}(y)\hat{\phi}(x)\,, \tag{8}$$

where the **Heaviside function** or, as I will often call it informally, the **theta function** $\theta(t)$ is given by[5]

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}\,. \tag{9}$$

We will ignore the $t = 0$ case for now.

From the definition (7) of the vacuum expectation value of time-ordered products, or from the expression (11), we have the simple result that

$$\Delta(x-y) = \theta(x^0-y^0)D(x-y) + \theta(y^0-x^0)D(y-x)\,. \tag{10}$$

Then using the expression for the Wightman function of free fields, (1), we have that

$$\Delta(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[\theta(x^0)\mathrm{e}^{-ip\cdot x} + \theta(-x^0)\mathrm{e}^{ip\cdot x}\right]\,, \tag{11}$$

where $p^\mu = (\omega_p, \boldsymbol{p})$ and $p \cdot x = p_\mu x^\mu = \omega_p t - \boldsymbol{p}.\boldsymbol{x}$ in the exponentials.

In energy-momentum space, the Feynman propagator is $\Delta(p)$ where

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)}\frac{i}{p^2-m^2+i\epsilon}\,. \tag{12}$$

[4] There are two other ways to define this which we will encounter in this course. First it is a Green function of the Klein-Gordon equation with appropriate boundary conditions, known as Feynman boundary conditions. These boundary conditions are not well known outside the context of relativistic QFT. The second is via contractions, $\hat{\phi}(x)\hat{\phi}(y)$, encountered in the discussion of perturbation theory and Wick's theorem.

[5] More generally the time-ordering operator T orders operators so that every operator in the expression, say $\widehat{A}(t)$, has only earlier (later) operators $\widehat{B}(t')$ to the right (left), i.e. $t > t'$ ($t < t'$). See below for comments about equal time case.

Here ϵ is an infinitesimal positive real number and the integrations are along the real axes.

The aim is to prove that this momentum space form $\Delta(p)$ of (12) is completely consistent with the space-time coordinate form.

As we are aiming for expressions involving three-momentum integrals through the Wightman function D of (3), it is useful to rewrite the four-momentum integrals of (12) as

$$\Delta(x-y) \quad = \quad \int \frac{d^3p}{(2\pi)^3}\, e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} \frac{1}{2\pi} \int_{C_0} dz\, e^{-izt} \frac{i}{z^2-\omega_p^2+i\epsilon}\,. \tag{13}$$

where C_0 is the curve running from $-\infty$ to $+\infty$ along the real energy axis. Now we identify that

$$(z-\omega_p+i\epsilon')(z+\omega_p-i\epsilon') \quad = \quad z^2-(\omega_p-i\epsilon')^2 = z^2-\omega_p^2+2i\epsilon'\omega_p+\epsilon'^2 \tag{14}$$

As ϵ is infinitesimal we can ignore the ϵ'^2 as compared to the $2i\epsilon'\omega_p$ term (provided $m>0$). As $\omega_p>0$ (again assuming $m>0$) the $2\epsilon'\omega_p$ acts as a positive infinitesimal, and we can call this $\epsilon \equiv 2\epsilon'\omega_p$. So we see that

$$z^2-\omega_p^2+i\epsilon \equiv (z-\omega_p+i\epsilon')(z+\omega_p-i\epsilon') \tag{15}$$

That is the integrand in (13) has one pole at $z=+\omega_p-i\epsilon'$ in the lower-half plane and second pole at $z=-\omega_p+i\epsilon'$ in the upper-half plane as shown in figure 1.

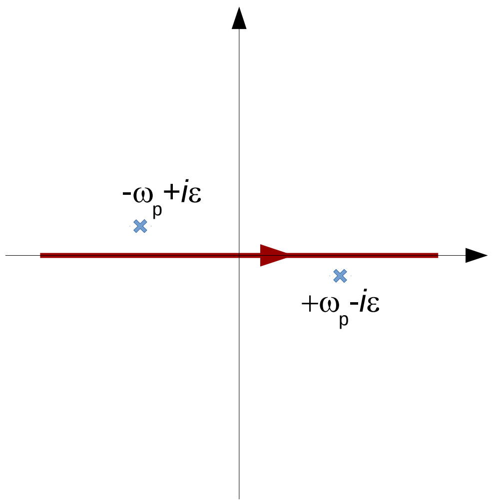


Figure 1: Energy integration curve C_0 (in red) and poles (blue crosses) for the integrals (12) and (13). Integration shown in the complex p_0 plane with $\Re(p_0)$ ($\Im(p_0)$) plotted along the horizontal (vertical) axis.

The expression for Δ in terms of the Wightman function $D(x,y)$ (10) involves two terms for different time-orderings. So consider first $t=x^0-y_0>0$. In this case we can complete the energy integration in (13) along a semi-circle at infinity in the lower half-plane of the complex energy variable z where $\Im(z)<0$. The integral along this lower semi-circle (C_-) gives zero as $\exp(-izt)=\exp(-i.-i(\infty)t)\exp(-i.\Re(z)t)\to 0$ so it can be added on. So using the closed contour C_0+C_- as shown in figure 2 gives us for $t=x^0-y_0>0$

$$\theta(x^0-y_0)\Delta(x-y) = \theta(x^0-y_0)\int \frac{d^3p}{(2\pi)^3}\, e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} \frac{1}{2\pi}\left(-2\pi i.\ e^{-izt}\frac{i}{(z+\omega_p-i\epsilon')}\bigg|_{z=+\omega_p}\right). \tag{16}$$

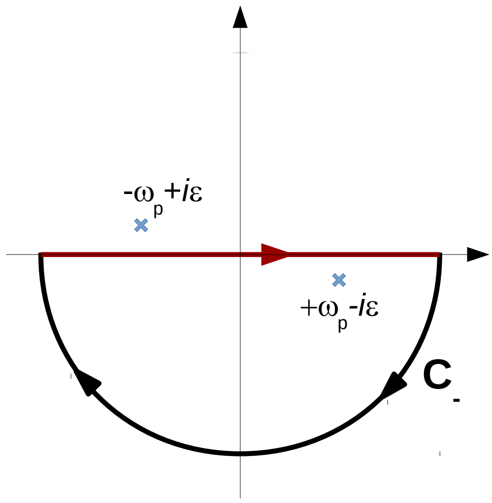


Figure 2: Closed energy integration curve $C_0 + C_-$ used for positive time case of the integrals (12) and (13). Integration shown in the complex p_0 plane with $\Re(p_0)$ ($\Im(p_0)$) plotted along the horizontal (vertical) axis.

where we note that the closed curve is running in a negative sense so we get a factor of $-2\pi i$ times the residue at the pole enclosed by the contour. Tidying this up gives

$$\theta(x^0 - y_0)\Delta(x - y) = \theta(x^0 - y_0) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega_p} e^{-i\omega_p(x_0 - y_0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \theta(x^0 - y_0) D(x, y) \quad (17)$$

by comparing with (3).

The second case where $t = (x^0 - y_0) < 0$ works in a similar way. In this case we complete the energy integration in (13) along a semi-circle at infinity in the upper half-plane of the complex energy variable z where $\Im(z) > 0$, see figure 3. The integral along this upper semi-circle (C_+) gives zero as $\exp(-izt) = \exp(-i. + i(\infty)t) \exp(-i.\Re(z)t) \to 0$ so it can be added on to the integration curve. So with the closed contour $C_0 + C_+$ of figure 3 we find

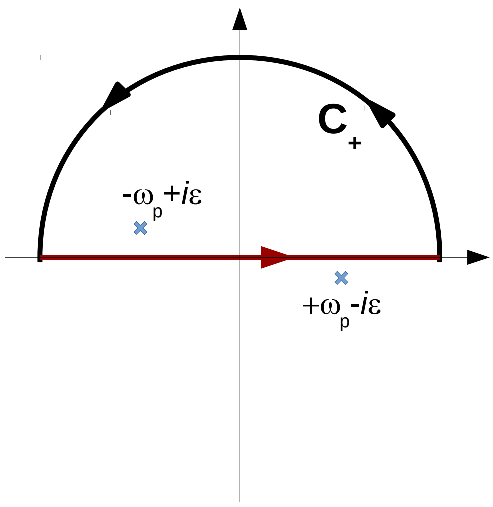


Figure 3: Closed energy integration curve $C_0 + C_+$ used for negative time case of the integrals (12) and (13). Integration shown in the complex p_0 plane with $\Re(p_0)$ ($\Im(p_0)$) plotted along the horizontal (vertical) axis.

for $t = (x^0 - y_0) > 0$ that

$$\theta(y^0 - x_0)\Delta(x-y) = \theta(y^0 - x_0) \int \frac{d^3p}{(2\pi)^3}\, e^{+i\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})} \frac{1}{2\pi} \left(+2\pi i.\ e^{-izt} \frac{i}{(z - \omega_p + i\epsilon')} \bigg|_{z=-\omega_p} \right) . \tag{18}$$

where we note that the closed curve is running in a negative sense so we get a factor of $-2\pi i$ times the residue at the pole enclosed by the contour. Tidying this up gives

$$\begin{aligned} \theta(y^0 - x_0)\Delta(x-y) &= \theta(y^0 - x_0) \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega_p}\, e^{+i\omega_p(x_0 - y_0) + i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} && (19) \\ &= \theta(y^0 - x_0) \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3} \frac{1}{2\omega_{p'}}\, e^{+i\omega_{p'}(x_0 - y_0) - i\boldsymbol{p}'\cdot(\boldsymbol{x}-\boldsymbol{y})} && (20) \\ &= \theta(y^0 - x_0) D(y,x) && (21) \end{aligned}$$

by comparing with (3). Note that we can change integration variable from $\boldsymbol{p}$ to $\boldsymbol{p}' = -\boldsymbol{p}$ and find the integrand is invariant, in part since the dispersion relation ω_p is a function of $|\boldsymbol{p}|$ which is invariant under this change.

Putting our two parts (17) and (21) together we have that

$$\Delta(x-y) = \theta(x^0 - y_0) D(x,y) + \theta(y^0 - x_0) D(y,x) \tag{22}$$

as required from (10).

Definition of Time-Ordering for Equal Times

One problem with our definition of time ordering is that theta functions are not 'proper' functions in a strict mathematical sense. For two operators $\widehat{A}$ and $\widehat{B}$ our typical definition is

$$\mathrm{T}\widehat{A}(t_1)\widehat{B}(t_2) = \theta(t_1 - t_2)\widehat{A}(t_1)\widehat{B}(t_2) + \theta(t_2 - t_1)\widehat{B}(t_2)\widehat{A}(t_1) \tag{23}$$

which clearly depends on how the "Heaviside step function"[6] is defined around zero argument. There are ways of dealing with this properly and mathematically. For instance one may use the theory of distributions rather than ordinary functions, but we would need to be much more careful with our mathematical definitions all the way through this course.

However, this definition of time ordering T need not be inconsistent. The key requirement here is that we get the same answer which ever way we approach the $t_1 = t_2$ limit. That is we want

$$\begin{aligned} \lim_{(t_1 - t_2) \to 0^+} \mathrm{T}\widehat{A}(t_1)\widehat{B}(t_2) &= \lim_{(t_1 - t_2) \to 0^-} \mathrm{T}\widehat{A}(t_1)\widehat{B}(t_2) && (24) \\ \Rightarrow \quad \widehat{A}(t_1)\widehat{B}(t_1) &= \widehat{B}(t_1)\widehat{A}(t_1) && (25) \end{aligned}$$

That is there is no inconsistency if the two operators commute at equal times $[A(t), \widehat{B}(t)] = 0$.

As we are interested in time ordered products of fields, the equal time commutators for fields guarantee time ordering is well defined for products of fields and their hermitian conjugates as used in perturbation theory. The ETCR are true for free fields in a free theory or for fully interacting fields. So fields and their Hermitian conjugates all commute at equal times in the interaction picture; check by hand if you like.

[6] Like the Dirac delta function, the "Heaviside step function" or "theta function" is not strictly a function.