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De Broglie-Bohm Theory: A Hidden Variables Approach to Quantum Mechanics.

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Abstract

Despite its great predictive power the orthodox formalism of Non-relativistic Quantum Mechanics has many conceptual problems. De Broglie-Bohm theory shows not only can a hidden variables theory reproduce the orthodox quantum formalisms predictions, but also give a deeper insight into the mechanisms behind the statistical results. Here we introduce both de Broglie's (1927) Pilot-Wave Theory and Bohm's (1952) Bohmian Mechanics and show how they address some of the problems of orthodox Quantum theory using 'hidden variables'. We then look at the dynamical origin of the Born probability rule $\rho = |\psi|^2$ in the de Broglie-Bohm pilot-wave formulation and look at how the nodes of the wavefunction drive the mixing of trajectories and relaxation to $|\psi|^2$.

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1 Introduction

First discovered by Louis de Broglie in 1927 [1] and later rediscovered by David Bohm in 1952 [2], de Broglie-Bohm Theory is a description of non-relativistic quantum mechanics that introduces particle positions as hidden variables.

The wavefunction of a system of particles, with its time evolution given by the Schrödinger equation, does not give the complete description of the system like in orthodox quantum theory. The complete description of the system is given by both its wavefunction and the particle positions. The dynamics of the particles are determined by the guiding equation which expresses the particle velocities in terms of the wavefunction.

In de Broglie-Bohm theory the particles exist in definite positions objectively with no need for observers or measurement. This means that many of the problems of Orthodox Non-Relativistic Quantum Mechanics are avoided. By Orthodox Quantum theory we mean any theory where the wavefunction gives a complete description of a system, and so the theory is fundamentally statistical in nature.

Orthodox Quantum theory has had great predictive success but there are many conceptual difficulties. A major point of contention is that the dynamics that the wavefunction obeys differ depending on whether the system is being observed. The instantaneous non-local nature of the collapse dynamics are at complete odds to the Schrödinger evolution. For a fundamental theory this seems highly undesirable.

Other conceptual difficulties arise on the quantum-classical scales. The fact that there is no definite distinction of where the system-apparatus divide should be or whether there should be one is another area where Orthodox Quantum Theory fails to give a satisfying answer.

J.S. Bell [16], suggested that there were two ways out from the conceptual frailties of the orthodox quantum theory. The first possibility was that something could be added to theory. By this he meant the wavefunction

was not a complete description and that there must be some 'hidden variables' that complete the description. The second way out is by modifying the schrödinger equation so that it can encompass the random collapse dynamics, for example GRW spontaneous collapse models.

Reasons to consider hidden variable theories in general arise from deficiencies in the orthodox quantum formalism. De Broglie-Bohm theory says that the statistical nature arises from underlying deterministic processes much like in classical statistical mechanics. Not only does it smooth out/eliminate the conceptual difficulties of orthodox quantum mechanics, it also gives us a different interpretation about individual quantum events, interpretations of experiments and what it actually means to make a measurement.

De Broglie-Bohm theory puts quantum mechanics on a sturdy mathematical foundation since all the theory requires as postulates are the Schrödinger equation and the guiding equation. All else follows as secondary implications and can be derived from the dynamics. Non-locality is also explicit in the de Broglie-Bohm formulation of Quantum Mechanics.

There are two main formulations of de Broglie-Bohm theory. The first, Pilot-Wave theory sometimes called the minimalist interpretation, which will be discussed in chapter 1. Bell was one of the main proponents of the theory since de Broglie originally formulated it in 1927. Bell referred to the actual particles as beables, unlike observables in orthodox quantum theory.

The second formulation is called Bohmian Mechanics which reformulates de Broglie's pilot-wave theory in line with the classical regime of second-order equations of motion. It uses a modified form of the Hamilton-Jacobi equation in its formulation. We discuss this in chapter 2.

In chapter 3 we study how in de Broglie Bohm Theory $\rho = |\psi|^2$ is not a postulate but is derived from the dynamics in terms of a relaxation argument analogous to thermal equilibrium in classical statistical mechanics. We shall further study the mixing role played by nodes in the relaxation process.

2 De Broglie Pilot-wave Dynamics

We will now introduce the original formulation of pilot wave theory looking at some simple applications, how the theory deals with measurement, and how it reproduces the empirical results of non-relativistic quantum mechanics. Pilot-Wave Theory was formulated between 1923-1927 by Louis de Broglie. He was motivated by the need of an explanation for experimental results that were incompatible with the predictions of classical mechanics, namely the diffraction and interference of single photons. De Broglie sought to reconcile the wave and particle characteristics of the results of the experiments by trying to unify the classical variational principles of Maupertuis, $\delta \int m \mathbf{v} \cdot d\mathbf{x} = 0$, and Fermat, $\delta \int dS = 0$, which apply to particles and waves respectively [3].

2.1 Pilot Wave Theory Equations of Motion

For non-relativistic pilot-wave theory of a many-body closed system¹ of N spinless particles, a state is defined by:

1. $\Psi(q, t)$ The systems wavefunction, a complex-valued field on $3N$ dimensional configuration space, where $q = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$ is the point in configuration space labelled by N particles in \mathbb{R}^3 , which evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \sum_{i=1}^N -\frac{\hbar^2}{2m_i} \nabla_i^2 \Psi + V \Psi. \quad (2.1)$$

Where m_i is the mass of the i -th particle, $\hbar = h/2\pi$ with h being Plank's constant, and V is a potential energy function and independent of time.

¹where a 'system' may in principle include; atoms, particles, fields, recording devices, experimenters, the environment, even the whole universe! and where closed means nothing outside the system can effect it's dynamics

2. The actual trajectories of the N particles $\mathbf{x}_i(t)$ which evolve according to the *guiding equation*

$$\mathbf{v}_i = \frac{d\mathbf{x}_i}{dt} = -\frac{i\hbar}{2m_i|\Psi|^2}(\Psi^*\nabla_i\Psi - \Psi\nabla_i\Psi^*) = \frac{\hbar}{m_i}\text{Im}\frac{\Psi^*\nabla_i\Psi}{|\Psi|^2} \quad (2.2)$$

where we have used $y = -i(z - z^*)/2$ to show the second equality ($z = x + iy$)².

Because of the wavefunctions role in the guiding equation (2.2) it is also referred to as the *pilot wave* or *guiding wave*. We also note the dynamical law for the particle beables (2.2) is first-order, which is of course in stark contrast to Newton's second-order equation of motion $F = m\ddot{\mathbf{x}}$.

If initial conditions $\Psi(q, 0)$ and $\mathbf{x}_i(0)$ are specified, the wavefunction $\Psi(q, t)$ and the trajectories of the N particles $\mathbf{x}_i(t)$ are determined for all t . We say that the theory is deterministic as the system is in a definite state for all t . This is opposed to orthodox quantum theory which is probabilistic, with a state defined by probabilities of outcomes.

In standard quantum theory the probability density³ of measuring a system with configuration q at time t is given by $|\Psi(q, t)|^2 d\Omega$, where Ω is a volume in configuration space. For pilot-wave theory we say $|\Psi(q, t)|^2 d\Omega$ is the probability density of the system actually having configuration q (regardless of measurement) with every particle \mathbf{x}_i having definite position. the probability of a system being in configuration q in a volume on configuration space Ω is given by

$$P = \int_{\Omega} |\Psi|^2 d\Omega. \quad (2.3)$$

Differentiating both sides of (2.3) with respect to time and assuming the volume to be constant with get

$$\frac{dP}{dt} = \int_{\Omega} \frac{\partial|\Psi|^2}{\partial t} d\Omega = \int_{\Omega} \left(\frac{\partial\Psi^*}{\partial t}\Psi + \Psi^*\frac{\partial\Psi}{\partial t} \right) d\Omega. \quad (2.4)$$

²note for the case we are studying, of a spinless particle, Ψ^* may be cancelled in the last part of (2.2), but we leave it in it's most general form where spinor valued wavefunctions may be introduced.

³we are assuming all wavefunctions are normalized, $|\Psi|^2 = 1$.

Now using the Schrödinger equation (2.1) to find $\frac{\partial \Psi}{\partial t}$ and its conjugate and substituting into (2.4) and cancelling the potential terms we have

$$\frac{dP}{dt} = \int_{\Omega} \sum_{i=1}^N \frac{i\hbar}{2m_i} (\Psi^* \nabla_i^2 \Psi - \Psi \nabla_i^2 \Psi^*) d\Omega, \quad (2.5)$$

and using the product rule for the divergence operator

$\sum_{i=1}^N (\Psi^* \nabla_i^2 \Psi - \Psi \nabla_i^2 \Psi^*) = \sum_{i=1}^N \nabla_i \cdot (\Psi^* \nabla_i \Psi - \Psi \nabla_i \Psi^*)$ we arrive at

$$\frac{dP}{dt} = \int_{\Omega} \sum_{i=1}^N \frac{i\hbar}{2m_i} \nabla_i \cdot (\Psi^* \nabla_i \Psi - \Psi \nabla_i \Psi^*) d\Omega. \quad (2.6)$$

Finally substituting the second part of (2.4) for the left hand side of (2.6) we arrive at

$$\int_{\Omega} \left(\frac{\partial |\Psi|^2}{\partial t} + \sum_{i=1}^N \nabla_i \cdot \mathbf{j}_i \right) d\Omega = 0, \quad (2.7)$$

which holds for all Ω , and the integrand must vanish everywhere so,

$$\left(\frac{\partial |\Psi|^2}{\partial t} + \sum_{i=1}^N \nabla_i \cdot \mathbf{j}_i \right) = 0. \quad (2.8)$$

where $\mathbf{j}_i = \frac{i\hbar}{2m_i} (\Psi^* \nabla_i \Psi - \Psi \nabla_i \Psi^*) = \mathbf{v}_i(t) |\Psi|^2$ is the 3-vector probability current along the trajectory of the i -th particle. (2.8) is the continuity equation and ensures conservation of probability for the system. The above derivation of the continuity equation assumes the initial distribution of particles ρ is equal to the quantum equilibrium probability distribution $|\Psi|^2$ we will look later at the case where $\rho \neq |\Psi|^2$.

Using our expression for the probability current have another way to express the guidance equation

$$\dot{\mathbf{x}}_i(t) = \frac{\hbar}{m_i} \text{Im} \frac{\nabla_i \Psi}{\Psi} = \frac{\mathbf{j}_i}{|\Psi|^2}, \quad (2.9)$$

and we shall also point out that if the wavefunction is written in polar form, $\Psi = |\Psi| e^{iS/\hbar}$ where $S(q, t)$ is the phase, we can write the *guidance equation* in the form

$$\dot{\mathbf{x}}_i(t) = \frac{\nabla S}{m_i}. \quad (2.10)$$

We shall leave a deeper discussion for later, mainly in chapter 3. For each particle, ψ is called the *guiding wave* or pilot-wave which generates a velocity field throughout space and at each moment in time tells the particle how to move.

We shall now look at some simple applications of Pilot-Wave Theory.

2.2 Some Simple Examples[5]

In this chapter will will be finding the pilot-wave theory solutions of some simple one particle systems and discussing how they differ and relate to orthodox quantum theory. we will be using the Schrödinger and guidance equations for one particle i.e. equations (2.1) and (2.2) with $N = 1$. make sure to cite val in bib un published or in print like in his bibs

2.2.1 Plane Wave Solution

For the case or a free particle of mass m we want to find the solution of (2.1) with $N = 1$ and $V = 0$, and then use this solution to find the particle beable's trajectory using (2.2). The solution of the one particle time-dependent free Schrödinger equation in 3-dimensions may be found by noting that both sides of the Schrödinger equation are equal to the constant total energy E and solving each side separately,

$$i\hbar \frac{\partial \Psi}{\partial t} = E\Psi$$

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi = E\Psi,$$

and noting the total energy $E = \mathbf{p}^2/2m$ is just the kinetic energy which we have used to solve for the position solution, giving $T(t) \propto e^{-iEt/\hbar}$ and $X(x) \propto e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$ with the full solution given by their product

$$\Psi(\mathbf{x}, t) = Ae^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar} \quad (2.11)$$

where A is a constant and \mathbf{p} is the momentum vecotr conjugate to \mathbf{x} .

We can now use (2.10) to solve for the particle position and find the

trajectory. the velocity field is given by

$$\dot{\mathbf{x}}(t) = \frac{\nabla S}{m} = \frac{\mathbf{p}}{m}$$

and the trajectory is

$$\mathbf{x}(t) = \mathbf{x}_0 + \frac{\mathbf{p}}{m}t.$$

So we see the particle travels with uniform velocity in a straight line as a classical particle. The trajectories move orthogonal to surfaces of constant phase as shown in figure 2.1.

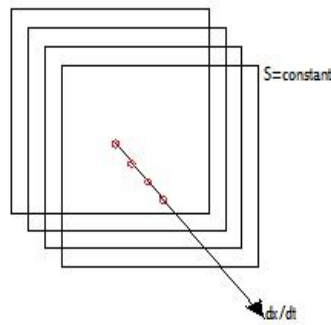


Figure 2.1: plane wave propagation, trajectories travel perpendicular to surfaces of constant phase

2.2.2 1-Dimensional Free Gaussian

For a Free particle of mass m in 1-dimension a Gaussian wave packet solution to the Schrödinger equation can be obtained by superposing plane wave solutions. A Gaussian wave packet has initial wavefunction given by

$$\Psi_0(x) = \frac{1}{(2\pi\Delta_0^2)^{1/4}} e^{-x^2/4\Delta_0^2} \quad (2.12)$$

where Δ_0 is the initial width of the wave packet. After time t under free Schrödinger evolution this evolves to

$$\Psi(x, t) = \frac{1}{(2\pi(\Delta_0 + i\hbar t/2m\Delta_0)^2)^{1/4}} e^{-x^2/4\Delta_0^2(1+i\hbar t/2m\Delta_0^2)} \quad (2.13)$$

with the width of the wave packet $\Delta(t) = \Delta_0\sqrt{1 + t^2/\tau^2}$, where $\tau = 2m\Delta_0^2/\hbar$ is the timescale.

The velocity of a particle with position x is

$$\frac{dx}{dt} = \frac{\hbar}{m} \text{Im} \left(\frac{1}{\Psi} \frac{\partial \Psi}{\partial x} \right) = \frac{xt}{t^2 + \tau^2} \quad (2.14)$$

where $\frac{1}{\Psi} \frac{\partial \Psi}{\partial x}$ is simply

$$\frac{-x}{2\Delta_0^2(1 + it/\tau)}$$

with the imaginary part easily found by Taylor expanding $(1 + it/\tau)^{-1}$. Taking a factor of t/τ out of the expansion and cancelling the minus sign leaves a geometric series in $-t^2/\tau^2$, which after some tidying gives

$$\frac{dx}{dt} = \frac{xt}{\tau^2} \left(\frac{1}{1 - (-t^2/\tau^2)} \right) = \quad (2.15)$$

where we have used the formula for a geometric series

$$\sum_{n=0}^{\infty} \left(-\frac{t^2}{\tau^2} \right)^n = \frac{1}{1 - (-t^2/\tau^2)}$$

Integrating (2.14) gives us the trajectory of the particle

$$x(t) = x_0\sqrt{1 + t^2/\tau^2} \quad (2.16)$$

and so we see the particle for initial position $x_0 > 0$ the particle travels to the right and for $x_0 < 0$ it travels to the left. Only if $x_0 = 0$ does the particle remain stationary. We can also see for $t \gg \tau$ the particle travels with uniform velocity. further from the origin (or center of the packet) the greater the acceleration of the particle moving forward in time [6]

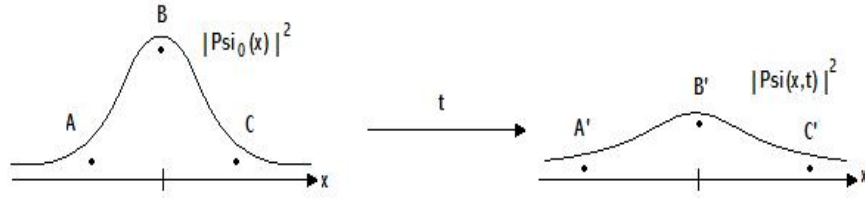


Figure 2.2: Spreading Gaussian Wavepacket

2.2.3 Stationary States[5]

For stationary states, that is energy eigenfunctions such as

$$\hat{H}\phi_n(x) = E_n\phi_n(x) \quad (2.17)$$

where $\psi(x,t) = \phi_n(x)e^{iE_nt/\hbar}$, then if the energy eigenfunction is real then $S = -E_nt$ and so does not depend on position. This means that the velocity field is zero, $\nabla S/m = 0$ and so the particle is at rest, which is opposed to what orthodox quantum theory tells us. This is also the case for any spherically symmetric potential, where the phase will not depend on position. An example of this is that in de Broglie-Bohm theory the ground state a Hydrogen atom is stationary before any sort of interaction including measurements.

In the case of the harmonic oscillator case the wavefunction of every state ψ_n has the gradient of it's phase equal to zero and so the actual particle is before measurement motionless. a measurement gives the particle a momentum and so we argue should we really call this a measurement in the sense of measuring a pre-existing quality of the system and leads us to look at what it means to make a 'measurement' in Pilot-Wave theory which we shall look at in section 4 of this chapter.

2.3 Quantum Measurement in de Broglie-Bohm Theory

As mentioned earlier in the chapter, de Broglie didn't create theory to solve the measurement problem or to provide a realist or deterministic theory, it was solely created to explain experiment. Although it was not de Broglie's motivation or intention to solve the measurement problem, pilot-wave theory manages to give a full description of a 'quantum measurement process' in completely consistently in terms of one single set of dynamical rules.

The measurement problem arises in the orthodox quantum theory due to the assumption that the measurement process may be described using analogous ideas to that of how we make measurements in classical physics. In classical physics, if we want to measure an attribute ω of a system using a measuring device with the output being some pointer described by position y that can be either 0 or 1 say, Hamilton's equations say 'switch on' an interaction Hamiltonian. This means that we add an interaction term, that couples the two systems, to the total Hamiltonian i.e.

$$H = a\omega p_y \quad (2.18)$$

where a is a coupling constant and p_y is the momentum conjugate to y . Evolving this in time gives

$$\omega(t) = \omega_0, \quad y(t) = y_0 + a\omega_0 t \quad (2.19)$$

so we see that the pointer is displaced by $a\omega_0 t$ from y_0 and so we can infer the value of ω_0 . The result of our measurement being the pre-existing quantity attributed to the system ω_0 .

This is fine for measurements in classical physics, where we have used the theory to tell us how to make a measurement, but for a new theory which explains different or wider ranging phenomena than classical mechanics a new theory of measurement must be found constructed from the new theory. In orthodox quantum theory this has not happened and the old classical regime of measurement has been used as a guide on how to make correct quantum measurements. To measure a property ω of a quantum system orthodox quantum mechanics takes (2.18) and quantizes this procedure via

the mapping

$$H = a\omega p_y \rightarrow \hat{H} = a\hat{\omega}\hat{p}_y \quad (2.20)$$

where we have just quantised the classical Hamiltonian. From this follows the same procedure of measurement as the classical theory, where in the orthodox quantum theory a correct measurement of a property ω of a system is obtained if the pointer y indicates the eigenvalue ω_n ($n = 1, 2$ in this example) where the pointer gives a correct measurement of the property ω and the experimenter would say the "the system has property ω with value ω_n ".

2.3.1 Orthodox vs. de Broglie-Bohm Measurements[2]

Measurement in Orthodox Quantum Theory

As a concrete example of a measurement in the orthodox quantum theory take a particle at $t = 0$ with a wavefunction in a superposition of two eigenfunctions of some operator ω

$$\psi_0(x) = c_1\phi_1(x) + c_2\phi_2(x) \quad (2.21)$$

where $|c_1|^2 + |c_2|^2 = 1$ and

$$\hat{\omega}\phi_1(x) = \omega_1\phi_1(x), \quad \hat{\omega}\phi_2(x) = \omega_2\phi_2(x)$$

Now to make a "measurement" of observable ω related to $\hat{\omega}$ a second system is introduced, which we will call the apparatus and refer to the system we want to measure the "system". The apparatus may be a "pointer" described by the particle position y with wavefunction $g_0(y)$ which is highly localised around $y = 0$ (so that the value of the pointer position y when the wavefunction is "collapsed" into an eigenstate by observation it can be concluded y "points" to a definite value 0 or 1). We now wish to couple the system and apparatus and evolve this coupled wavefunction using a Von

Neumann type interaction ⁴ Hamiltonian

$$\hat{H}_I = a\hat{\omega}\hat{p}_y, \quad (2.22)$$

where now in the quantised scheme $\hat{p}_y = -i\hbar\frac{\partial}{\partial y}$ is the momentum operator conjugate to y .

The combined system of apparatus and system being measured is initially

$$\Psi_0(x, y) = (c_1\phi_1(x) + c_2\phi_2(x))g_0(y) \quad (2.23)$$

For large enough coupling a and over a short enough timescale the total Hamiltonian can be approximated by the interaction Hamiltonian, $\hat{H}_{Tot} \approx \hat{H}_I$ for $a \gg 1$ and $t \rightarrow \epsilon$. The Schrödinger equation is then

$$i\hbar\frac{\partial\Psi(x, y, t)}{\partial t} = -a\hat{\omega}i\hbar\frac{\partial\Psi(x, y, t)}{\partial y} \quad (2.24)$$

with solution

$$\Psi(x, y, t) = c_1\phi_1(x)g_0(y - a\omega_1t) + c_2\phi_2(x)g_0(y - a\omega_2t) \quad (2.25)$$

where the wavefunction branches into two different non-overlapping eigenstates. This is due to the non-overlapping pointer packets 2.3 associated with each eigenstate of $\hat{\omega}$. If a y value within the localised packet $g_0(y - a\omega_i t)$ ($i = 1, 2$) is observed then we conclude the wavefunction has "collapsed" into the $\phi_i(x)$ (discarding all other eigenfunctions) eigenstate and we have the value ω_i for the measurement of the property ω (relating to operator $\hat{\omega}$) of the system. The probability of y being in wavepacket $g_0(y - a\omega_i t)$ i.e. "measuring" ω_i is given by $|c_i|^2$ [?].

Pilot-Wave Theory Measurements

For pilot-wave theory we add a configuration $(x(t), y(t))$ to the quantum state. We now have the same wavefunction as in the orthodox case as the *guiding wave* that lives in configuration space (x, y) .

⁴A Von Neumann interaction is one where the measuring apparatus is also treated as a quantum object and is assigned some wavefunction $|g\rangle$. The coupled wavefunction of the entire system being some product of the the wavefunctions of apparatus and system being measured i.e. $|g\rangle|\psi\rangle$.

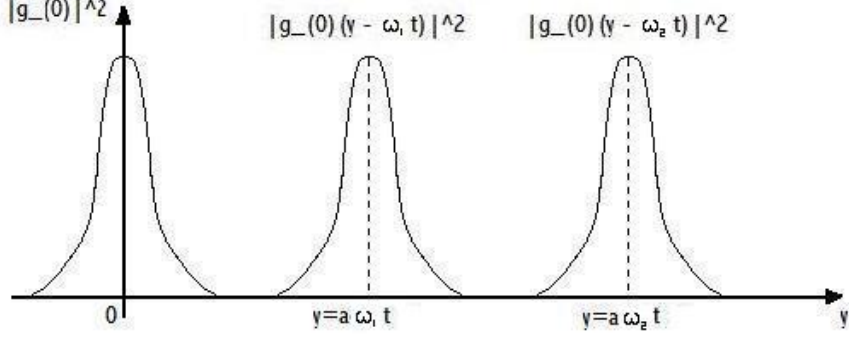


Figure 2.3: Non-overlapping pointer wavepackets $|g_0(y - a\omega_1 t)|^2$ and $|g_0(y - a\omega_2 t)|^2$, and initial pointer wavepacket $g_0(y)$. [5]

If the initial distribution of the ensemble is $\rho_0(x, y) = |\Psi_0(x, y)|^2$ then because the continuity equation for the probability distribution of the ensemble and the quantum continuity equation are the same we have $\rho(x, y, t) = |\Psi(x, y, t)|^2$. The ensemble then has probability distribution

$$\rho(x, y, t) = |c_1|^2 |\phi_1(x)|^2 |g_0(y - a\omega_1 t)|^2 + |c_2|^2 |\phi_2(x)|^2 |g_0(y - a\omega_2 t)|^2, \quad (2.26)$$

where there are no cross terms due to the non-overlapping pointer packets. This means that the distribution $\rho(x, y, t)$ of our ensemble separates or branches in configuration space with the actual trajectory of particle beables being guided into **one** of these non-overlapping regions as seen in 2.4.

In the pilot-wave view of measurement there is no need for observers to "collapse" the wavefunction and the problem of the remaining eigenstate's ontological status is avoided. These other eigenstates exist and still have ontological status but have no influence on the actual configuration $(x(t), y(t))$, i.e. the particle beables that are recorded in measurements. The configuration is now guided by just one of the initial eigenfunctions. This is pilot-wave explanation of the "collapse" process.

Unlike orthodox quantum theory, in pilot-wave theory the above process is not necessarily seen as a measurement of some pre-existing property of the system, this is has been called "naive eigenvalue realism" since in pilot-

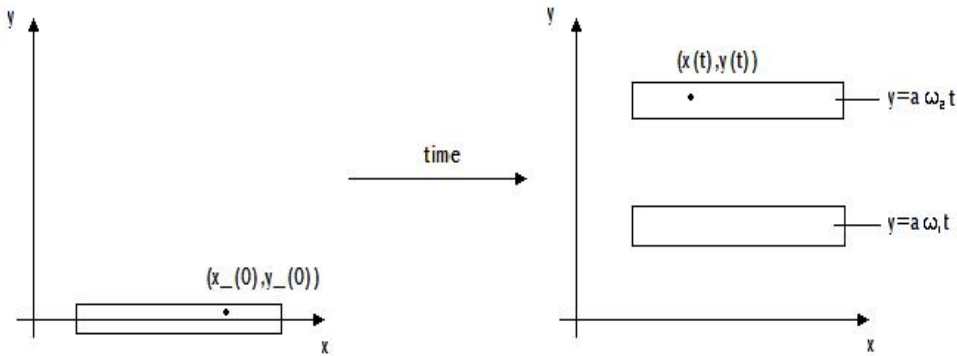


Figure 2.4: Separation of the probability distribution of an ensemble of particles ρ with identical wavefunction, with actual configuration (x, y) guided into one of the regions[5].

wave theory the eigenvalues obtained from measurements in the orthodox quantum theory are not always **real** properties of the system. The trajectory $x(t)$ is said to be guided by the reduced wavefunction $\phi_i(x)$ but this does not necessarily mean the system has or had some actual attribute ω with value ω_i (the eigenvalue of the i -th eigenstate). We shall see how this works in some examples shortly. There is one exception where eigenvalues correspond to **real** properties of the system and that is for position measurements where $\hat{\omega} = \hat{x}$.

In conclusion we have

Table 2.1: orthodox quantum theory vs. pilot-wave theory.

Orthodox Quantum Theory	Pilot-Wave Theory
"Observable $\hat{\omega}$ has value ω_i "	" Trajectory $x(t)$ is guided by $\phi_i(x)$ "

2.3.2 Recovery of Orthodox Quantum Mechanics

For pilot-wave theory to a suitable candidate for a real physical theory it must be capable of at least reproducing all the empirical results of the orthodox formulation that has been experimentally verified. This can easily

be done.

If we look at (2.26), the probability distribution of the ensemble, we would like to reproduce the orthodox theories probabilities $|c_i|^2$ of measuring a particle. If the probability of the configuration being in support i is given by $|c_i|^2$, then that has the same consequence of saying it has that probability of being measured there, thus reproducing the statistics of the orthodox formulation. So the probabilities of the configuration lying in either support are given by

$$\int_{all} dx \int_{support\ of\ |g_0(y-a\omega_1t)|^2} dy |c_1|^2 |\phi_1(x)|^2 |g_0(y-a\omega_1t)|^2 = |c_1|^2,$$

$$\int_{all} dx \int_{support\ of\ |g_0(y-a\omega_2t)|^2} dy |c_2|^2 |\phi_2(x)|^2 |g_0(y-a\omega_2t)|^2 = |c_2|^2,$$

which are the same probabilities as for the likelihood of measurement in the orthodox quantum theory. Note, we have assumed initial distribution $\rho_0 = |\Psi_0|^2$.

in de Broglie-Bohm probabilities are a subsidiary condition of a casual theory of individuals.

We either have to accept the non local collapse postulate and have two incompatible dynamics to describe a quantum systems evolution or say the spots on the screen are proof of hidden variables and have all encompassing dynamics and have explanation for what happens during what is seen as a 'collapse' of the wavefunction in orthodox quantum theory. and so no distinction in divide between system and apparatus is needed.

non locality is clouded by the born distribution and so information can't be sent faster than c .

2.3.3 An Example of a Quantum Measurement

As an example of a measurement in pilot-wave theory we shall look at a particle in the ground state of an infinite square well potential in 1-dimension, and show how this leads to misplaced trust in eigenvalue realism. That is that the what are said to be the results measurements in orthodox quantum theory are in general not linked to what we would define as a measurement, i.e. finding some pre-existing value of a property of a system.

The solution of the time independent Schödinger equation for the infinite

square well potential in one dimension is

$$\psi_0(x) = Ae^{ipx} + e^{-ipx} \quad (2.27)$$

the superposition of two momentum eigenstates. The velocity of the particle in pilot-wave theory is $\frac{\nabla S}{m} = 0$ since the phase is $S = 0$. This means the momentum of the particle before measurement is guided by ψ_0 and is $m\dot{x}_0 = 0$.

After measurement of momentum of the type described above, the wave function is either seen to "collapse" or branch in configuration space and the final *guiding wave* or collapsed wavefunction is $\phi_1(x) = e^{\mathcal{S}px}$ or $\phi_2(x) = e^{-\mathcal{S}px}$ and so the particle beables final momentum is $m\dot{x} = p$ or $-p$. What the "measurement" interaction has done is disturb the system and give the particle some momentum. To say we have measured this momentum would be incorrect.

2.4 Features of Pilot-Wave Dynamics and New Physics

Firstly an important point to make is that all measurements are reduced to position measurements in pilot-wave theory. This makes sense because even when measuring some attribute of a system that isn't its position, what it boils down to is reading of positions of instruments. Only position eigenvalues are real measurements.

Another point to make is that the theory is highly non Newtonian and naturally is written in terms of a first-order dynamics, which greatly contrasts with the second order dynamics of Newton. There is a one way force from the wavefunction acting on the particle(s), but the particle(s) do not exert a force on wavefunction, meaning violation of Newton's 3rd law.

Subjectivity of observer eliminated hidden variables have objective existence, and exist regardless of measurement interaction.

Two particles are entangled in de Broglie-Bohm theory if their wavefunctions are not factorable. This means for particles with positions x_1 and x_2 their joint wavefunction is not separable in x_1 and x_2 so that $\psi(x_1, x_2, t) \neq \psi_1(x_1, t)\psi_2(x_2, t)$. With regards to the guiding equation this would mean that their phases are not sums of individual phases for x_1 and x_2 such

that $S(x_1, x_2, t) \neq S(x_1, t) + S(x_2, t)$, and therefore the velocity field at x_1 depends on x_2 and vice versa.

Another triumph of pilot-wave theory is that we can keep classical logic and don't have to resort to quantum logic.

De Broglie's pilot-wave theory is not an alternative formulation of quantum mechanics. instead it tells us quantum theory is a special case of a much wider non-equilibrium physics.

3 Bohmian Mechanics

In 1952 David Bohm's two seminal papers [?] were published. Over the two papers he set out how quantum mechanics could be reconciled with a casual interpretation of some underlying "hidden" variables. Bohm was not aware of De Broglie's work on pilot-wave theory until some of his work on Bohmian mechanics had been published.

Unlike de Broglie's first order dynamics, Bohm formulated his theory in terms of a second order dynamics. Although more complicated and not strictly necessary [5], Bohm's formulation does have it's advantages. For de Broglie's dynamics it is hard to see how a classical limit can be achieved given that classical mechanical dynamics are described by second order equations. Bohm's formulation, although essentially the same a de Broglie's, recasts de Broglie's dynamics into a second order form using a modified version of the Hamilton-Jacobi equation. This gives an appropriate language that links classical and quantum mechanics. The main key feature of Bohm's formulation is the Quantum Potential, but before coming to this we shall look at classical Hamilton-Jacobi theory and see how it links the wave and ray features of optics.

3.1 Classical Hamilton-Jacobi Theory

The Hamilton-Jacobi Equation is is a reformulation of classical mechanics like the Lagrangian and Hamiltonian formulations. If we want to find the equations of motion of a classical system with n degrees of freedom using Hamilton's equations we must solve the $2n$ first-order differential equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i = \frac{\partial H}{\partial p_i} \quad i = 1, \dots, n. \quad (3.1)$$

Where H is the hamiltonian of the system and q_i and p_i are generalised coordinates and momenta. If the Hamilton-jacobi method is used the problem

of solving the $2n$ ordinary differential equations can be reduced to finding a single integral of a partial differential equation in $(n + 1)$ variables.

To do this we first want to find a canonical transformation of the generalised coordinates and momenta (q_i and p_i) so that the new Hamiltonian is zero ensuring the new coordinates are constant in time i.e.

$$(q, p) \longrightarrow (Q, P) \quad \text{where} \quad \dot{Q} = \dot{P} = 0. \quad (3.2)$$

A canonical transformation is one which preserves the form of Hamilton's equations (3.1). If K is our new Hamiltonian then for a canonical transformation we require

$$\dot{P}_i = -\frac{\partial K}{\partial Q_i} \quad \text{and} \quad \dot{Q}_i = -\frac{\partial K}{\partial P_i} \quad i = 1, \dots, n. \quad (3.3)$$

where $K = K(Q, P, t)$. The equations (3.1) are may be derived from varying the action integral

$$I = \int L dt = \int \left(\sum_{i=0}^n p_i \dot{q}_i - H \right) dt \quad (3.4)$$

with respect to the $2n$ independent variables q_i and p_i , where have used the Legendre transform to get to the final expression.

$$0 = \delta I \quad (3.5)$$

$$= \int \sum_{i=0}^n (\delta(p_i \dot{q}_i) - \delta H) dt \quad (3.6)$$

$$= \int \sum_{i=0}^n \left[(\dot{q}_i \delta p_i - \dot{p}_i \delta q_i) - \left(\frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i \right) \right] dt \quad (3.7)$$

where, once the common factors δq_i and δp_i are taken out we are left with two terms that must vanish for the integral to be zero, and so setting these terms to zero gives Hamilton's equations.

The same process can again be followed to find the equations of motion in terms of the transformed coordinates (Q, P)

$$I = \int \left(\sum_{i=0}^n P_i \dot{Q}_i - K \right) dt. \quad (3.8)$$

The integrands in (3.4) and (3.9) then differ by a total time derivative

$$\sum_{i=0}^n p_i \dot{q}_i - H - \frac{dS}{dt} = \sum_{i=0}^n P_i \dot{Q}_i - K. \quad (3.9)$$

The function S is called the *generating function* of the canonical transform and in general is an arbitrary of $2n$ of q, p, Q, P , and also depends upon t . This is because $Q = Q(q, p, t)$ and $P = P(q, p, t)$ means each of the two new generalised coordinates depend on two of the old generalised coordinates i.e. only $2n$ of these coordinates are independent. For the case of $S = S(q, Q, t)$ (which is nice due to the time derivatives factoring), plugging this into (3.9) collecting terms with like time derivatives and equating the separate terms on either side we find

$$p_i = \frac{\partial S}{\partial q_i} \quad (3.10)$$

$$P_i = -\frac{\partial S}{\partial Q_i} \quad (3.11)$$

$$K = H + \frac{\partial S}{\partial t}. \quad (3.12)$$

These relations allow us to construct the canonical transform by solving (3.10) for Q and then (3.11) for P , thus S generates the canonical transform.

Now we have a way of generating a canonical transform let us go back to what we had set out to do in (3.2) using (3.10), (3.11) and (3.12). What we want then is to find some canonical coordinates such that

$$\dot{Q}_i(q, p, t) = \dot{P}_i(q, p, t) = 0. \quad (3.13)$$

From (3.3) it is obvious that $K = 0$ is our requirement so that we have (3.13). With $K = 0$ (3.12) becomes

$$\frac{\partial S(q, Q, t)}{\partial t} + H(q, \frac{\partial S(q, Q, t)}{\partial q}, t) = 0, \quad (3.14)$$

where we have used (3.10) to substitute for the momentum in the hamiltonian. This is the *Hamilton-Jacobi equation*, it is a partial differential equation of $n + 1$ variables q_1, q_2, \dots, q_n, t . the function S^1 in this equation is called *Hamilton's principle function*. Since S is not explicit in the equation

¹It may also be shown that S is in fact equal to the action[7]

the solution must have n nonadditive constants $\alpha_1, \dots, \alpha_n$ as arguments of S i.e. $S = S(q, \alpha, t)$. These constants may be any function of the Q_i coordinates for example $\alpha_i = Q_i$. If we have some other form for our generating function such as $S(p, P, t)$ the α_i would be functions of P_i .

We have the Hamilton-Jacobi equation and would now like to solve it to find our laws of motion. The constant transformed coordinates Q_i may be written as n independent functions of the integration constants α_i so that we have $Q_i = \gamma_i(\alpha_1, \dots, \alpha_n)$ and the constant transformed momenta P_i is found using (3.11)

$$P_i = \frac{\partial S(q, \gamma, t)}{\partial \gamma_i} = \beta_i, \quad (3.15)$$

which is called the Jacobi law of motion. It can be solved for q_i in terms of the constants α_i and β_i assuming that the determinant of the Jacobian of the transformation is not singular ($\det(\partial^2 S / \partial q_i \partial \gamma_j) \neq 0$)². The original canonical momenta is found using (3.10). Once q_1 and p_1 have been found all that is left is to express the $2n$ constants α_i and β_i in terms of the actual initial conditions, q_{0i} and p_{0i} , of the problem. This may be done by evaluating (3.10) **and** (3.15) at t_0 to find respectively, p_0 which relates q_0 , p_0 and γ , **and** q_0 which relates β and γ . Once this is done we finally have expression for $q(t)$ and $p(t)$ in terms of their initial values.

3.1.1 Solving the 1-D Simple Harmonic Oscillator Using the Hamilton-Jacobi Equation

We will use the example of the 1-D Simple Harmonic Oscillator to illustrate how the Hamilton-Jacobi equation can be used to solve dynamical problems.

The Hamiltonian for the harmonic oscillator is

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 \quad (3.16)$$

substituting this into (3.14) and using $\frac{\partial S}{\partial q}$ in place of the momentum we have the Hamilton-Jacobi equation for the SHO

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + \frac{1}{2}kq^2 = 0 \quad (3.17)$$

guessing at a solution of (3.17) of the form $S(q, t) = S_1(t) + S_2(q)$ so that

²If it is singular we cannot solve for q_i 's in terms of the γ_i 's

we can use separation of variables and turn our PDE above into a set of ODE's. We find,

$$-\frac{dS_1}{dt} = \frac{1}{2m} \left(\frac{dS_2}{dq} \right)^2 + \frac{1}{2}kq^2. \quad (3.18)$$

Since S_1 on the left of (3.18) depends only on t and S_2 on the right only on q both sides must equal the same constant, we will call E . This gives a set of 2 PDE's

$$\frac{dS_1}{dt} = -E \quad \text{and} \quad \frac{1}{2m} \left(\frac{dS_2}{dq} \right)^2 + \frac{1}{2}kq^2 = E. \quad (3.19)$$

The time differential equation is easily solved by integrating to find $S_1 = -Et + C$ where C is the constant of integration. The integral in q is

$$S_2 = \int \sqrt{m(2E - kq^2)} dq. \quad (3.20)$$

To solve this we first bring it to the form

$$S_2 = \sqrt{mk} \int \sqrt{\frac{2E}{k} - q^2} dq \quad (3.21)$$

then make the substitution $q = \sqrt{2E/k} \sin t$, where $dq = \sqrt{2E/k} \cos t dt$. Factorizing the $2E/k$ term and using $1 - \sin^2 t = \cos^2 t$ we arrive at

$$S_2 = \sqrt{mk} \frac{2E}{k} \int \cos^2 t dt. \quad (3.22)$$

We then use the trigonometric half-angle identity $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$. After integrating we have

$$S_2 = \sqrt{mk} \frac{2E}{k} \frac{1}{2} \left[t + \frac{1}{2} \sin 2t \right] + c. \quad (3.23)$$

($c =$ constant of integration) then we use the double angle identity $\sin 2t = 2 \sin t \cos t$ and substitute q back in for t , where $\sqrt{k/2E}q = \sin t$ and $\cos t = \sqrt{1 - kq^2/2E}$ and so we finally arrive at our solution

$$S_2 = \sqrt{mk} \frac{1}{2} \left[\frac{2E}{k} \arcsin \left(\sqrt{\frac{kq^2}{2E}} \right) + q \sqrt{\frac{2E}{k} - q^2} \right] + c. \quad (3.24)$$

Now we have solved both PDE's we can write down the complete integral

$$S(q, t) = -Et + S_2(q, E) + \text{const.} \quad (3.25)$$

apart from the constant of integration, there is a new non-additive constant E in our solution. In this problem there is one degree of freedom $n = 1$ and so as we would expect we have only $\alpha_1 = E$ as our non-additive constant. Using (3.15) we can find the constant β_1 , by putting $t = 0$ and q_0 into (3.26), which we will relabel t_0 .

$$\frac{\partial S(q, \gamma, t)}{\partial E} = \beta_1 = t_0 = -t + \sqrt{\frac{m}{k}} \arcsin \sqrt{\frac{kq^2}{2E}}. \quad (3.26)$$

So we can find $q(t)$ in terms of a function involving constants $\alpha_1 = E$ and $\beta_1 = t_0$

$$q(E, t) = \sqrt{\frac{2E}{k}} \sin \left[\sqrt{\frac{k}{m}} (t_0 + t) \right]. \quad (3.27)$$

Now we have the desired canonical position, we can find the canonical momentum of the system using (3.10) giving

$$p = \frac{\partial S}{\partial q} = \sqrt{m(2E - kq^2)}. \quad (3.28)$$

where $p(t)$ is found by substituting (3.27) into (3.28).

In the above example we have used the form of Hamilton's principle function depending on the transformed momenta $S = S(q, P, t)$ and so we first find our constant E then find our initial coordinate which is the time t_0 . (The example above is from [20]).

Other important features we should mention that are important for taking H-J equations to the quantum theory. multi-valued S so different functional forms of S may give the same momentum for some initial conditions but not all. S describes an ensemble of trajectories found by holding α and varying β .

Remarks

In the Hamilton-Jacobi Method for solving a mechanical system, Hamilton's principle function S is connected to not just the solution for the trajectory that it has been solved for, but to an infinite set of trajectories, much like

the idea of an ensemble in pilot-wave theory, see [8].

3.2 Bohm's Hamilton-Jacobi Pilot-Wave Theory

Bohm's version of de Broglie-Bohm theory is based around a second-order dynamics much like classical mechanics. It is essentially a reformulation of de Broglie pilot-wave theory, although Bohm arrived at his formulation independently of de Broglie's work. Bohm started from writing the wavefunction in its polar form $\Psi(\mathbf{x}, t) = R(\mathbf{x}, t)e^{iS(\mathbf{x}, t)/\hbar}$ and substituting this into Schrödinger's wave equation. Working with the one particle Schrödinger equation we have

$$i\hbar \frac{\partial}{\partial t} R(\mathbf{x}, t)e^{iS(\mathbf{x}, t)/\hbar} = - \left(\frac{\hbar^2}{2m} \nabla^2 + V \right) R(\mathbf{x}, t)e^{iS(\mathbf{x}, t)/\hbar}. \quad (3.29)$$

First working with the left hand side of (3.29), taking the time partial differential using the product rule we find,

$$i\hbar \frac{\partial}{\partial t} R e^{iS/\hbar} = i\hbar \left(\frac{\partial R}{\partial t} + R \frac{i}{\hbar} \frac{\partial S}{\partial t} \right) e^{iS/\hbar}. \quad (3.30)$$

Now the right hand side, where $\nabla^2 = \nabla \cdot \nabla$, is

$$- \left(\frac{\hbar^2}{2m} \nabla^2 + V \right) R e^{iS/\hbar} = \left(\frac{\hbar^2}{2m} \left[\nabla^2 R + \frac{i}{\hbar} 2\nabla R \cdot \nabla S + \frac{i}{\hbar} R \left(\nabla^2 S + \frac{i}{\hbar} \nabla S \right) \right] + V R \right) e^{iS/\hbar}. \quad (3.31)$$

Equating (3.30) and (3.31) and cancelling the exponential terms, we can now find separate equations for $\frac{\partial R}{\partial t}$ and $\frac{\partial S}{\partial t}$ by separating the real and imaginary parts respectively into two equations (3.32) and (3.33).

$$\frac{\partial R}{\partial t} = -\frac{1}{2m} [R \nabla^2 S + 2\nabla R \cdot \nabla S], \quad (3.32)$$

$$\frac{\partial S}{\partial t} = - \left[\frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \right]. \quad (3.33)$$

We can rewrite (3.32) by calling $P^{\frac{1}{2}} = R$ so using $\frac{\partial R}{\partial t} = \frac{1}{2} P^{-1/2}$ and reversing the product rule (3.32) becomes

$$\frac{\partial P}{\partial t} + \nabla \cdot \left(P \frac{\nabla S}{m} \right) = 0, \quad (3.34)$$

and we see it is just the continuity equation for the ensemble with probability density $P(\mathbf{x}, t)$.

Equation (3.33) is similar in form to the Hamilton-Jacobi equation where S is Hamilton's principle function and $\nabla S/m$ is again the particle velocity. (3.33) is called the modified Hamilton-Jacobi equation and the term

$$-\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = -\frac{\hbar^2}{4m} \left[\frac{\nabla^2 P}{P} - \frac{1}{2} \frac{(\nabla P)^2}{P^2} \right] = Q. \quad (3.35)$$

is called the quantum potential. This shows that even a classically free particle may be acted upon by forces arising from the quantum potential.

Putting the above modified Hamilton-Jacobi equation into a second-order framework shows that it is consistent to regard Q as a potential acting on the particle like V . This is done by bringing (3.33) to the form

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} = -(V + Q). \quad (3.36)$$

now applying ∇ to (3.36) and using $d/dt = \partial/\partial t + \dot{\mathbf{x}} \cdot \nabla$ we have

$$\frac{\partial \nabla S}{\partial t} + \nabla \cdot \frac{(\nabla S)^2}{2m} = \left(\frac{\partial}{\partial t} + \frac{(\nabla S)}{m} \cdot \nabla \right) \nabla S = -\nabla(V + Q) \quad (3.37)$$

$$\frac{d}{dt}(m\dot{\mathbf{x}}) = -\nabla(V + Q) \quad (3.38)$$

giving us the particle equation of motion with the form of Newton's second law, where the particle is subject to a quantum force $-\nabla Q$ as well as any classical force.

Remarks

Using Bohm's formulation may not be necessary but it does have some useful features, for example classical mechanics is a limiting case of de Broglie-Bohm theory but Bohm's formulation makes this more apparent by showing that the quantum effects that effect the particle through the quantum potential decrease as $\hbar \rightarrow 0$.

in Bohm's view, $p = \nabla S$ is not a law of motion but a condition on the initial momentum that is preserved in time by his second order dynamics. relaxation of $p = \nabla S$ means a spread of momentum values at each position. If $p = \nabla S$ is the initial condition then it is preserved by time evolution.

An important prediction of the Bohm theory, made in Bohm's original 1952 paper, is that the electron in the ground state of a hydrogen atom is at rest since s states have spherical symmetry and thus have constant phase, it can be shown that the quantum force introduced by Bohm balances the classical electromagnetic potential.

4 Dynamical Origin of Quantum Probabilities/Born rule

Up until now we have only mentioned in passing that $\rho = |\psi|^2$ may be derived from the underlying dynamics in de Broglie-Bohm theory and does not need to be a postulate¹ as the Born rule is in orthodox quantum theory. In de Broglie-Bohm theory what we mean by $\rho = |\Psi|^2$ is that the initial distribution of an ensemble of particles, guided by Ψ , is to be distributed according to the Born rule. By ensemble we mean a collection of many identical configurations all with the same wavefunction, with the same potential and boundary conditions, but differing initial conditions i.e. a set of many different individual evolutions/trials of the same system with different initial conditions for each evolution/trial. This is how we arrive at probability distributions, but in fact one trial or experiment only has one actual configuration with one set of initial conditions (only one member of the (statistical) ensemble actually exists).

Bohm made suggestions in [?] that the born rule may be seen as having an analogous status to thermal equilibrium in classical statistical mechanics. Pauli also pointed this out [15]. In reply Bohm showed that for the case of a hydrogen molecule, excited to a doubly degenerate energy level E_0^2 , with initial non-equilibrium distribution $\rho \neq |\Psi|^2$ that random external collisions would cause the initial distribution to relax to $\rho = |\Psi|^2$ [17]. Due to difficulties in forming a general relaxation argument Bohm later introduced stochastic elements to his theory[18].

Valentini and Westman's paper [12] gave a general argument that initial non-equilibrium distributions would relax to $|\Psi|^2$ without the need to introduce stochastic 'fluid fluctuations'. [12] used the coarse-graining H-theorem to show that an initial non equilibrium distribution would make a

¹Most papers on de Broglie-Bohm theory assume $\rho = |\Psi|^2$ as a postulate.

²i.e. a system with two eigenstates both with degenerate eigenvalue E_0

significant approach to equilibrium, within a certain finite timescale τ and coarse-graining length ϵ (not proving that equilibrium is actually reached). Numerical simulations were performed for the case of ensembles, in a 2-D box with $V = 0$ for $x, y = 0.. \pi$ and $V = \infty$ everywhere else, with initial *guiding wave*³

$$\psi(x, y, 0) = \sum_{m,n=1}^4 \frac{1}{4} \left[\frac{2}{\pi} \sin(mx) \sin(ny) \right] e^{i\theta_{mn}} \quad (4.1)$$

at $t = 0$ and with energy eigenvalues $E_{mn} = \frac{1}{2}(m^2 + n^2)$, $m, n = 1, 2, 3, 4$. The statistical ensemble's initial distribution was

$$\rho(x, y, 0) = \frac{4}{\pi^2} \sin^2 x \sin^2 y. \quad (4.2)$$

The numerical simulations showed $\rho(x, y, 0)$ to approach equilibrium $|\psi|^2$ on a coarse grained level significantly after just one period of 4π i.e. that the H-function decreases in time.

The results showed that a large number of degrees of freedom are not necessary for efficient relaxation. Even for a single particle with a wavefunction that is a superposition of a small number of energy eigenfunctions equilibrium would be approached in finite time. Generally similar simulations would show any initial non-equilibrium state, obeying certain conditions such as absence of fine-grained microstructure, would evolve towards $|\psi|^2$. This went beyond previous results [19] which showed relaxation would occur for a many-body system with many D.O.F. Where it was then argued that a particle extracted from the relaxed equilibrium distribution and prepared with wavefunction ψ would have a distribution $|\psi|^2$.

Arguments as to why all quantum systems probed so far all have the quantum equilibrium distribution are given in [12] and [?]. The general idea is that all quantum systems we experiment on today have had a violent astrophysical history which will have caused relaxation to occur to well beyond accuracies that we measure the systems to. These arguments leave the likely option that particles in the early universe may not have $\rho = |\psi|^2$.

What we shall now look further at comments made in [12] about how the velocity field acts as a stirrer mixing two "fluids" i.e. two distributions ρ

³The values for the phases $\theta_{m,n}$ were taken as exactly the four decimal place values from Valentini and Westman's paper (see footnote on page 5 of [12]).

and $|\psi|^2$. It is highlighted that 'quasi-nodes' in $|\psi|^2$, where $|\psi|^2 < \epsilon$ where the velocity field varies rapidly, are where the most efficient mixing occurs. It has been mentioned, by Frisk [13], that trajectories close to nodes exhibit chaotic behaviour. Relaxation is an effect of relative dispersion of different trajectories in an ensemble, and so it seems plausible that if close proximity to nodes causes trajectories to diverge then that relaxation is driven by nodes in the *guiding-wave*.

4.1 Nodes, Divergent Trajectories and Relaxation

In this section we shall look at two trajectories with initially close positions and the same wavefunction. We shall study the case of a single particle in a 2-D box as in [12], using the same *guiding wave*/wavefunction (4.1). At time t under Schrödinger evolution (4.1) becomes

$$\psi(x, y, t) = \sum_{m,n=1}^4 \frac{1}{4} \left[\frac{2}{\pi} \sin(mx) \sin(ny) \right] e^{i(\theta_{mn} - E_{mn}t)} \quad (4.3)$$

which is plotted below in figures 4.1 and 4.2 for $t = 0, 2\pi$, and 4π . Where

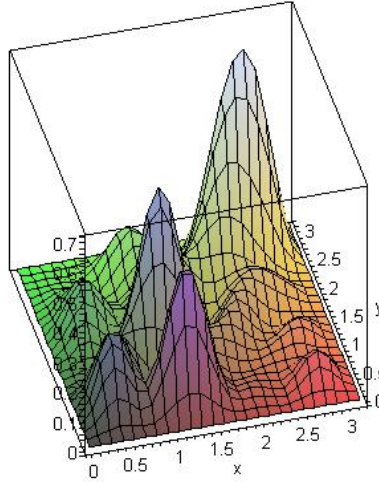


Figure 4.1: A plot of $\psi(x, y, 0) = \psi(x, y, 4\pi)$ in a 2-D box

ψ is periodic in 4π . This is the *guiding wave* for our system.

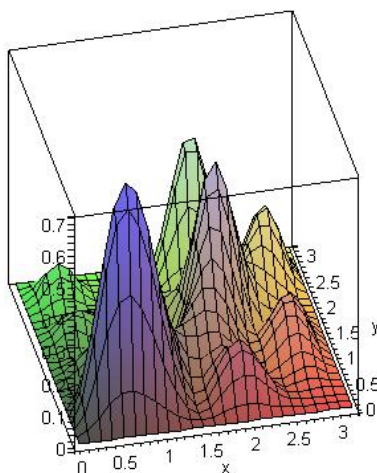


Figure 4.2: A plot of $\psi(x, y, 2\pi)$ in a 2-D box

Using the dsolve tool in Maple to numerically integrate the velocity fields⁴, in the two independent directions, we can plot the particle trajectories 4.3.

$$\frac{dx}{dt} = \frac{i}{2|\psi|^2} \left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right) \quad (4.4)$$

$$\frac{dy}{dt} = \frac{i}{2|\psi|^2} \left(\psi \frac{\partial \psi^*}{\partial y} - \psi^* \frac{\partial \psi}{\partial y} \right) \quad (4.5)$$

with initial conditions (4.2) and $x_{0,1} = \frac{\pi}{2}$, $y_{0,1} = \frac{\pi}{2}$ and $x_{0,2} = \frac{\pi}{2} + \epsilon$, $y_{0,2} = \frac{\pi}{2} + \epsilon$. The initial positions of the two trajectories $(x_{0,1}, y_{0,1})$ and $(x_{0,2}, y_{0,2})$ are separated by $\sqrt{2}\epsilon$, where $\epsilon = 0.001$. The trajectories are very complicated but clearly diverge at some point as the end points are clearly far apart and have separated. On close inspection it can also be seen that initially the trajectories stay quite close together and there is no great deviation from one another for an initial period.

To make this clearer a plot 4.4 of the Euclidean distance (4.6) between the two trajectories is given for $t = 0$ to 4π .

$$X_{ED}(t) = \sqrt{(x_1(t) - x_2(t))^2 + (y_1(t) - y_2(t))^2} \quad (4.6)$$

⁴Maple uses a Runge-Kutta 45 method

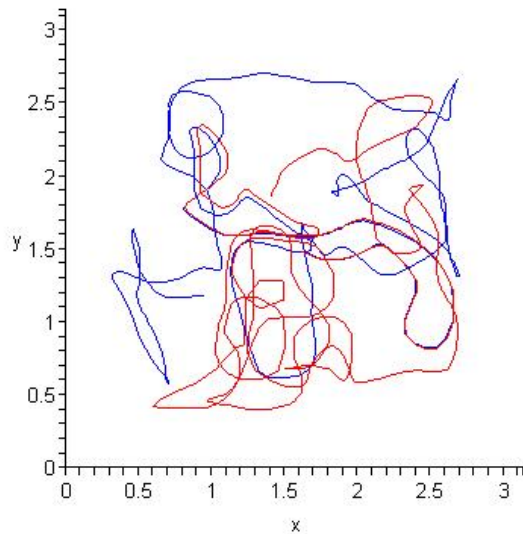


Figure 4.3: Trajectories of Two Particles with initial separation $\sqrt{2}\epsilon$

We now wish to see if significant divergences occur in the particle trajectories when either of the trajectories passes close to a "quasi-nodal region" (i.e. $|\psi|^2$ is less than some small value making a nodal likely to be close by). By significant divergences we mean divergences that are much larger than the normal rate of separation. To see if divergences occur when either trajectory passes close to a "quasi-nodal" region we make a plot of $|\psi(x_1(t), y_1(t), t)|^2$ where we have evaluated ψ along the trajectory $(x_1(t), y_1(t))$, see figure 4.5. Figure 4.6 is a joint plot of 4.4 (blue), 4.5 (red) and a plot of $|\psi(x_2(t), y_2(t), t)|^2$ (green).

From this we hope to see a correlation between large increases (i.e. a steep slope) in the blue plot when either the green or red plots are below some small value, during some initial period where the trajectories are still close. It should also be noted when, after the trajectories have initially diverged and are far apart, the trajectories suddenly come close together again and again there is a steep slope in the curve of $X_{ED}(t)$. either of these situations would show that "quasi-nodal" regions (that likely contain a node) cause the trajectories to become chaotic, suddenly and sharply

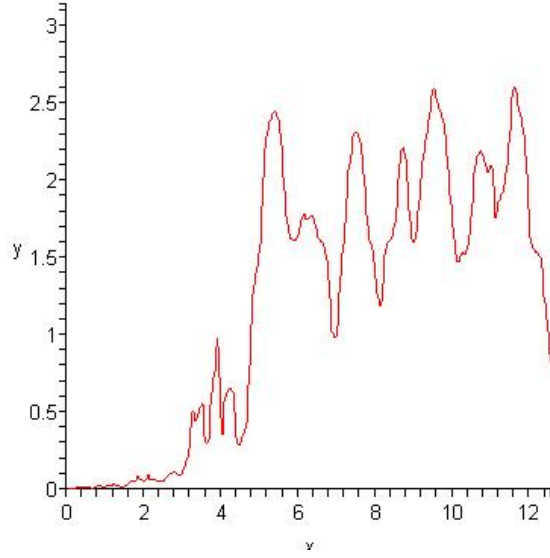


Figure 4.4: The distance between two initially close trajectories at time t

diverging, meaning memory of initial conditions is lost even with very small errors in the numerical analysis.

Such a divergence arising from two initially close trajectories passing by a "quasi-nodal" region would hint again at the idea that the velocity field acts like a stirrer mixing the statistical ensemble distribution into the quantum equilibrium distribution.

Remarks

Looking at a close up of the initial short time period of figure 4.6 in figure 4.7 the author can see no definite correlation between small values of one of the $|\psi(x_i(t), y_i(t), t)|^2$, $i = 1, 2$, curves and steep slopes in the $X_{ED}(t)$ curve. A clearer and more conclusive analysis may be carried out by plotting the modulus of the gradient $|\frac{dX_{ED}}{dt}|$ against the sum $|\psi_1(t)|^2 + |\psi_2(t)|^2$ for a finite number of points in t .

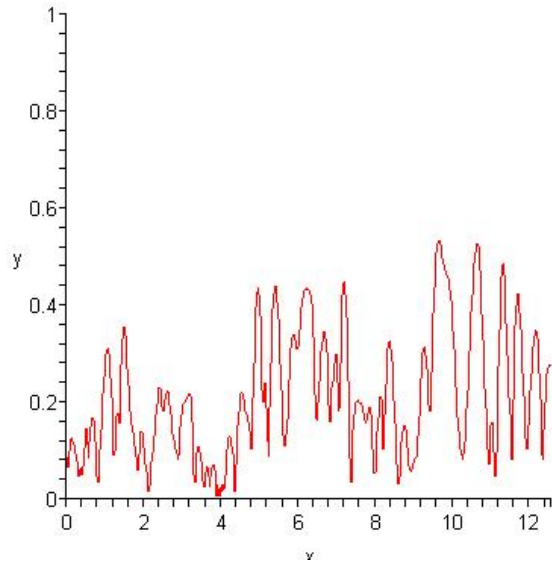


Figure 4.5: A plot of $|\psi(x_1, y_1, t)|^2$, the modulus squared of the systems wavefunction evaluated along the trajectory $(x_1(t), y_1(t))$

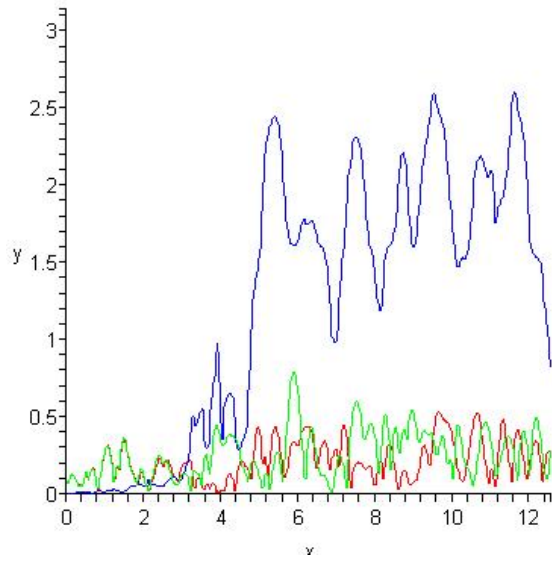


Figure 4.6: Plots of $|\psi|^2$ evaluated along $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ and also $X_{ED}(t)$

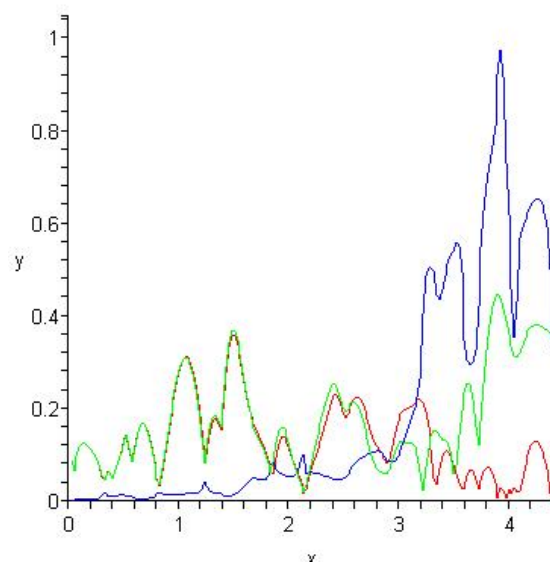


Figure 4.7: A close-up of figure 4.6

5 Conclusion

The orthodox quantum formalism is not the only theory that can reproduce all experimental results that it is meant to. De Broglie-Bohm Theory can explain quantum phenomena in an objective way using deterministic dynamics. Due to Bell's inequalities we know any such hidden variables theory must be non local, which is a desired feature of a theory that describes nature.

The de Broglie-Bohm formalism gives us different perspectives and ways of treating topics such as measurement. It is also a healthy activity to explore other lines of thought, especially if they are consistent with experiment.

It suggests many experiments, that would be worthwhile even for sceptics of the theory. One of these may be testing for non-equilibrium particles from the early universe that decoupled before they had time to relax to the quantum equilibrium distribution. For such a particle a double slit experiment would give a different diffraction pattern to what we are used to.

non-local communication may be achievable with non equilibrium particles, and exact properties could be shown to be reality with very weakly coupled system and apparatus set ups.

The analysis of chapter 3 gave inconclusive results and more would need to be done to show, with this analysis, that nodes are key to an efficient relaxation process.

Bibliography

- [1] Solvay Congress (1927), 1928, *Electrons et Photons: Rapports et Discussions du Cinquième Conseil de Physique tenu à Bruxelles du 24 au 29 Octobre 1927 sous les Auspices de l'Institut International de Physique Solvay*, Paris: Gauthier-Villars.
- [2] Bohm, D., 1952, "A suggested interpretation of the Quantum Theory in terms of 'Hidden' Variables, I and II," *Physical Review* 85: 166-193.
- [3] Valentini, A., 2009, "De Broglie-Bohm Pilot-Wave Theory: Many Worlds in Denial?" To appear in: *Everett and his critics*, eds. S. W. Saunders et al (Oxford University Press, 2009).
- [4] Valentini, A. (2001) Hidden Variables, Statistical Mechanics and The Early Universe. In *Chance in Physics: Foundations and Perspectives*, eds. J. Bircmont *et al.*. Berlin: Springer-Verlag, pp. 165-81
quant-ph/0104067
- [5] Valentini, A. *Introduction to Hidden Variables and Pilot-Wave Theory*, Lecture Course.
- [6] P.R. Holland, *The Quantum Theory of Motion*(1993), pages 158-165. Cambridge University Press.
- [7] P.R. Holland, *The Quantum Theory of Motion*(1993), pages 33-34. Cambridge University Press.
- [8] P.R. Holland, *The Quantum Theory of Motion*(1993), pages 27-33. Cambridge University Press.
- [9] Struyve, W, Phd. thesis, *The de Broglie-Bohm Pilot-Wave Interpretation of Quantum Theory*(2004). University of Ghent.

- [10] Towler, M. Pilot-Wave Theory, Bohmian Metaphysics, and The Foundations of Quantum Mechanics. Graduate Lecture Course.
- [11] Goldstein, S. (2001) Bohmian Mechanics, Stanford Encyclopedia of Philosophy. <http://plato.stanford.edu/entries/qm-bohm/>
- [12] Valentini, A. and Westerman, H. (2005). Dynamical Origin of Quantum Probabilities. *Proceedings of the Royal Society London A*, **461**, 253-272.
- [13] Frisk, H. (1997) Properties of the trajectories in Bohmian mechanics. *Phys. Lett. A*, vol. 227, issues 3-4, pages 139-142.
- [14] J.B. Keller. *phys. rev.* **89**, 1040, (1953)
- [15] W. Pauli, in: *Louis de Broglie: Physicien et Penseur* (Albin Michel, Paris, 1953)
- [16] J.S. Bell, *Speakable and Unsayable in Quantum Mechanics* (Cambridge University Press, Cambridge 1987).
- [17] Bohm, D. (1953) Proof That Probability Density Approaches $|\psi|^2$ in Causal Interpretation of the Quantum Theory. *Phys. Rev.* **89**, pages 458-466.
- [18] D. Bohm and J.P. Vigier, *Phys. Rev.* **96**, 208 (1954).
- [19] A. Valentini, *Phys Lett. A* **156**, 5 (1991).
- [20] Classical Mechanics by N.C.Rana and P.S. Joag pages 283-284