REVIEW ARTICLE

Thermodynamic Gravity And The Emergence of Space With Geometry

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Abstract

Thermodynamic aspects of black holes and Rindler horizons are reviewed along with quantum field theory in the corresponding curved spacetimes. The implication that gravity might be thermodynamic in origin is considered. Macroscopic space is seen to emerge at a course-grained limit via the application of holography in the context of horizons. Both Newtonian and Einsteinian gravity are seen (seperately) to be implied by an entropy maximisation principle and the suggestion that gravity is not fundamental is discussed and criticised. The fact that the thermodynamic perspective applies to Lanczos-Lovelock models, a more general (arbitrary dimensions) class of theories than Einstein's $D = 4$ gravity, is also explained.

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Contents

1 Introduction

The discovery that black holes are intrinsically thermodynamic systems (Bekenstein 1973, Hawking 1975-6) [2, 26, 12] and the striking similarity between the laws of black hole mechanics and those of thermodynamics systems [5], has provoked significant investigation. Together with the work that has been done on quantum field theory in the Rindler spacetime (Fulling 1973, Davies 1975, Unruh 1976)[9, 8, 29] these findings have led to the idea that gravity itself is actually a macroscopic thermodynamic phenomenon (Jacobson 1995)[14], perhaps similar to elasticity (Padmanabhan, Verlinde 2010)[30, 21, 23]. There is now a persuasive argument that gravity is an emergent, rather than fundamental force and that its origin lies in the statistical tendency toward maximal entropy. That is literally to suggest that the very nature of electromagnetism and the nuclear forces sets them entirely apart from gravitation.

One attractive feature of this perspective is that it would explain why the canonical quantisation of the microscopic forces have led to renormalisable quantum field theories in a relatively straightforward manner, while quantum gravity has proved so problematic. One can equivalently state the distinction by noting the dimensions of Newton's constant G in contrast to the coupling constants for the other forces, or that the equivalence principle holds for gravitational and inertial mass but not for the other charges. It would also explain the dominance of gravity at classical and cosmological scales along with its relative weakness in comparison to the short range forces; like other thermodynamic phenomena such as temperature or pressure, gravity might only arise with statistical aggregation in multi-particle systems. The analogy is drawn in [14] with sound waves in air (for which canonical quantisation is uncontrovertially inappropriate).

It is well understood that classical physics emerges naturally from quantum mechanics with course-graining, as interference terms are averaged out [11]. In light of that fact, it would not be so surprising if gravity (which is naturally associated with classical scales) also turned out to be statistically emergent (c.f. Ehrenfest theorem and Newton's second law). Furthermore, the generally covariant nature from which general relativity derives its name, seems to sit at odds with results in quantum theory such as the Kochen-Specker theorem [**?**] and the experimental violation of Bell's inequalties, which seem to imply an unavoidably contextual nature of quantum observation, contrary to Einstein's realist intuitions. In other words, GR requires some objective information (e.g. Riemann tensor) and quantum mechanics seems to imply that there is none. There are thus conceptual differences to be overcome in unifying the two, in addition to the technical hurdles concerning renormalisation[1]. It is at least an attractive idea then, to consider the possibilty of reclassifying gravity as emergent, perhaps in the non-contextual limit of some microscopic quantum theory, where $\hbar$ becomes negligible (viz. classical deterministic mechanics).

In this review, the relevant background theory of black hole mechanics [5], the thermodynamic properties of Rindler and Schwarzchild horizons [29, 2, 26] and the associated application [22] of the holographic principle [27] are first examined. The thermodynamic aspects of gravity [14] are then made clear in light of these findings and the most relevant and recent literature [30, 21, 23, 19, 20, 25] concerning a formal construction of the laws of gravity as an entropic force are reviewed. It will be clear from the discussion of these theories that a top-down approach is being taken, in that gravity is not recovered from some already well-understood underlying quantum theory, but rather is shown to follow from purely macroscopic considerations which could, in principle, apply to any microscopic theory which produced the required form of the entropy functional for spacetime and matter. The metric signature is $(-+++)$ except where otherwise specified and Greek indices range over all dimensions of spacetime.

1.1 The Notion of an Entropic Force

The application of statistical mechanics can, in some cases, be seen to lead to forces whose effects are the result of statistical tendancy of a system's configuration space to increase (exponentially). The fundamental nature of statistical methods is that one neglects, to some extent, the precise dynamics of the system at the fundamental, usually microscopic scale. Definitions of the thermodynamic quantities, such as temperature, pressure and entropy, all involve this kind of course-grained analysis, in which some details about the system have been averaged out, in favour of macroscopic variables. In this sense, it is obvious that a force defined in terms of entropy changes is not, unlike electromagnetism and the nuclear forces, a feature of physics at its most fundamental level, but merely a phenomenon observable at sufficiently large scales in systems with sufficiently many degrees of freedom.

The thermodynamic aspects of black holes have lead to the idea [30], [23] that gravitational effects should be analysed as entropic. The interesting question is what light, if any, the possibility of this treatment sheds on the traditional theories of gravity, most importantly, Einstein's geometrical theory. To

[1]Possibly not a coincedence, the technical problems could be symptoms of underlying inconsistent assumptions.

make clear the claim being made in the case of gravitation, it will be instructive to briefly review an analogy proposed in [30] and [21], namely elasticity.

1.2 Elasticity

Crudely, one might compare macroscopic spacetime to a stretched-out rubber band or polymer in a heat bath. Such a system has, by definition, the statistical tendency to coil up and reduce its extension x, as there are more ways it can be coiled up than extended. A basic discussion of the polymer in a heat bath is given in [30] and several good textbooks (e.g. [18]) on statistical and thermal physics include a discussion. Thermal "kicks" from particles in the heat bath encourage state transitions of the elastic material, such that an external force is required if one wishes to keep the extension from tending toward a minimum (by sheer chance). The external force acts in opposition to the *entropic* force arising due to the material trying to coil up, which is thermal/statistical rather than mechanical[2]. In that sense, it is not fundamental, but statistically emergent in the appropriate limit. The work done in extending a polymer in a thermal bath with temperature T, with the polymer/bath system in equilibrium, is the differential of the Helmholtz free energy:

$$dA = dE - TdS - SdT \tag{1.1}$$

and dE is given by, for an external force F,

$$dE = TdS - pdV + Fdx. \tag{1.2}$$

Since the volume change is cubic in the associated small extension it may be neglected and from (1.1) we have

$$dA = -SdT + Fdx. \tag{1.3}$$

If follows straightforwardly that

$$F = \left(\frac{\partial A}{\partial x}\right)_T \tag{1.4}$$

where the subscript T denotes that temperature remains constant. From (1.1) we have, with $dT = 0$,

$$F = \left(\frac{\partial E}{\partial x}\right)_T - T\left(\frac{\partial S}{\partial x}\right)_T. \tag{1.5}$$

[2]Although there are electrostatic forces between the molecules in the polymer.

One finds experimentally [18], that the internal energy is effectively not dependent on x for constant temperature, so we take the first term in (1.5) to be a vanishing contribution and arrive at

$$F = -T\left(\frac{\partial S}{\partial x}\right)_T . \tag{1.6}$$

Assuming that the external and entropic forces are balanced yields (1.6) as the expression for the entropic force.

The reasons for thinking that spacetime itself has microstructure and corresponding entropy (defined holographically on horizons) are discussed in this review. It is argued [21] that the appropriate entropy functional is such that gravitation is a reflection of the tendancy of large systems and the classical spacetime in which they live to maximise their combined entropy. The proposal of Verlinde [30] is that the ultimate nature of gravity is seen in an equation of the form

$$F_{gravity} = -T\left(\frac{\partial S}{\partial x}\right) . \tag{1.7}$$

Alternatively stated, pulling a mass away from a black hole is less like pulling an electron away from the nucleus of an atom, and more like stretching out the unsecured end of a polymer in a heat bath, with the other end secured inside the bath.

2 Black Holes And Rindler Space

The thermodynamic interpretation of gravity is best understood with reference to black holes. The reason for this is that black holes, which are an intrinsic feature of general relativity, turn out to behave just like thermodynamic systems. This is initially striking, and constitutes some of the most popular evidence that gravity is a thermodynamic phenomenon, although it will be argued in later sections of this review that one must be very careful about precisely what conclusions are to be drawn from these findings.

2.1 Physics in Accelerating Frames

By the equivalence principle, the physics experienced by observers in a gravitational field is equivalent to the physics experienced by accelerating observers in flat spacetime. It will therefore be instructive to consider the co-ordinate systems corresponding to such observers. Many of the arguments concerning the thermodynamic nature of black hole horizons can have analogies for another kind of horizon, existing in what is known as the Rindler spacetime. Let us, for simplicity, consider two-dimensional Minkowski spacetime from the point of view of an observer in an accelerating frame, using a treatment similar to chapter 9.5 of [4]. In inertial co-ordinates we have simply

$$ds^2 = -dt^2 + dx^2. \tag{2.1}$$

We want to find co-ordinates corresponding to an observer with uniform acceleration

$$a^\mu = \frac{d^2x^\mu}{d\tau^2} \tag{2.2}$$

where τ is the proper time for the observer. A natural solution for acceleration in the x-direction is given by

$$a^t = a\sinh(a\tau) \tag{2.3}$$

$$a^x = a\cosh(a\tau) \tag{2.4}$$

$$t(\tau) = \frac{1}{a}\sinh(a\tau) \tag{2.5}$$

$$x(\tau) = \frac{1}{a}\cosh(a\tau) \tag{2.6}$$

where $x^\mu(\tau)$ is the trajectory of the accelerating observer in the inertial co-ordinates and a is the magnitude of the acceleration. It is simple to confirm that with Minkowski signature, the negative value of g_{00} means that the requirement

$$\begin{aligned}\sqrt{a_\mu a^\mu} &= \sqrt{-a^2\sinh^2(a\tau) + a^2\cosh^2(a\tau)}\\ &= \sqrt{a^2[\cosh^2(a\tau) - \sinh^2(a\tau)]}\\ &= a\end{aligned} \tag{2.7}$$

is satisfied for the hyperbolic functions. The apparent curvature witnessed by an observer in the Rindler frame, is just described by the Minkowski metric in Rindler coordinates,

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2). \tag{2.8}$$

The scenario is depicted in Fig.(2.1) in which it is easy to see that the Rindler observer is confined to region outside the hyperbola, which is asymptotic to the null lines labeled H^+ and H^-. The observer, or indeed any matter in

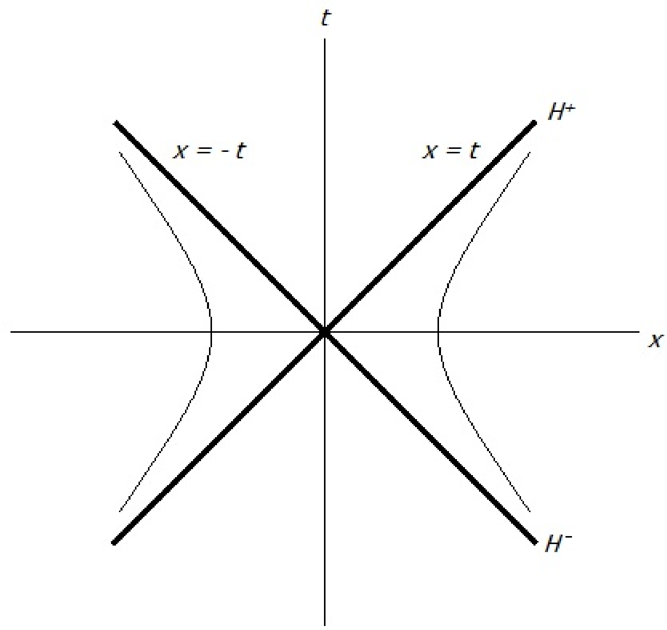

Figure 2.1: Rindler trajectories with inertial axes. The 45° lines form the past and future light cones for an inertial observer at (0,0).

the neighbourhood of the observer, cannot access the region bounded by the null lines since it takes an infinite amount of inertial coordinate time to reach the null lines themselves. For this reason, H^+ and H^- are known as Rindler horizons.

2.2 Similarity of Rindler and Schwarzchild Spacetimes

It is important to note at this stage, the close resemblance between the Rindler spacetime (really just Minkowski in alternative coordinates) and the spherically symmetric vacuum solution to Einstein's equations, namely the Schwarzchild spacetime described by[3]:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{(1 - 2M/r)} + r^2 d\Omega^2. \tag{2.9}$$

The singularity at $r = 2M$ implies the existence of an event horizon similar to the horizon experienced by the Rindler observer. In fact the causal structure of the spacetime near a Schwarzchild black hole closely resembles that of the

[3]with units such that G=1

Rindler spacetime. Indeed, in Kruskal coordinates

$$T = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh(\frac{t}{4M}) \tag{2.10}$$

$$R = \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh(\frac{t}{4M}) \tag{2.11}$$

the metric (2.9) becomes

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dT^2 + dR^2) + r^2 d\Omega^2 \tag{2.12}$$

in close analogy to (2.8). The Schwarzchild spacetime for static[4] observers is basically what Minkowski looks like to Rindler observers. This is unsurprising, of course, given the equivalence principle. The important difference is that black holes cannot be transformed away, they are an intrinsic properties of the manifold; the singularity in (2.9) has been moved to $r = 0$ in (2.12) but cannot be eliminated.

2.3 The Four Laws

In order to gain a full appreciation of the thermodynamic nature of black holes, it is necessary to consider the laws of black hole mechanics which apply, in addition to the simpler Schwarzchild case, to the Kerr, Reisser-Nordstrom, and Kerr-Newman families of black holes (which have non-zero angular velocity, charge and both respectively). The discussion presented here is similar, in parts, to that given in [24], although the proof of the First law here is formally somewhat different, and borrows a construction used in [14]. A detailed discussion of [14] is presented in section 5.3 in which the Einstein equations are seen to arise from thermodynamic considerations. In what follows, we will concentrate on the nature and origins of the laws themselves and leave the comparison with thermodynamics to the following section.

Zeroth Law

Surface gravity is constant over the entire horizon for a stationary black hole

The horizon is the union of all points along all of the null generators. To prove the law, we therefore need to show that the surface gravity is the same along each of the generators and across the set of generators, such that the respective derivatives are everywhere vanishing:

[4] Fixed r

$$(1)\ \kappa_{,\alpha}\, \xi^\alpha = 0 \quad \text{and} \quad (2)\ \kappa_{,\alpha}\, e^\alpha_A = 0.$$

where $\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right) = t^\alpha + \Omega \phi^\alpha$ and $e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)$ are the tangent vectors along and transverse to (respectively) the geodesics generating the horizon. The coordinate v is a spacetime coordinate[5] on the geodesics (the advanced time) and θ^A coordinatise the space orthogonal to the geodesics. The surface gravity κ may be defined by (see Appendix A.3) $\xi^\alpha{}_{;\beta}\, \xi^\beta = \kappa \xi^\alpha$ and we have:

$$\kappa^2 = -\frac{1}{2} \xi^{\alpha;\beta} \xi_{\alpha;\beta}. \tag{2.13}$$

Differentiating and using $\xi_{\alpha;\mu\nu} = R_{\alpha\mu\nu\beta} \xi^\beta$ (Appendix A.2) yields

$$2\kappa\kappa_{,\alpha} = -\xi^{\mu;\nu} R_{\mu\nu\alpha\beta} \xi^\beta. \tag{2.14}$$

Multiplying from the right by ξ^α we obtain condition (1) above. Alternatively, we can multiply by e^α_A in which case (2.14) becomes

$$2\kappa\kappa_{,\alpha}\, e^\alpha_A = -\xi^{\mu;\nu} R_{\mu\nu\alpha\beta} e^\alpha_A \xi^\beta \tag{2.15}$$

Since ξ^μ vanishes on the bifurcation two-sphere[6] (if the hole has one) and by condition (1) we can drag the derivative along the null generators without it varying, condition (2) must hold for the entire horizon. To see that the law also applies to black holes without a birurcation two sphere, consider two black holes, identical for $v > 0$ but differing by the fact that only one of them has a bifurcation horizon. Condition (2) is satisfied for the hole with the bifurcation two-sphere on all surfaces defined by constant v. Since the hole without the bifurcation two-sphere is identical for $v > 0$, it must also satisfy condition (2) since it has just all positive v surfaces of the other hole.

First Law

For a black hole of mass M, surface area A and angular momentum J,

$$\boxed{\delta M = \frac{\kappa}{8\pi} \delta A + \Omega \delta J} \tag{2.16}$$

[5] $v = t + r^*$ where t and r are from the Schwarzchild metric

[6] The sphere separating the past and future horizons, where the killing vector vanishes. See [24], chapter 5.1.10

Consider the general perturbations such that

$$\delta M = -\int_{\mathcal{H}} T_{\alpha\beta} t^{\beta} d\Sigma^{\alpha} \tag{2.17}$$

$$\delta J = \int_{\mathcal{H}} T_{\alpha\beta} \phi^{\beta} d\Sigma^{\alpha}. \tag{2.18}$$

Defining a tangent vector k^{α} such that $\xi^{\alpha} = -\kappa\lambda k^{\alpha}$, we may express the Horizon surface element as $d\Sigma^{\alpha} = k^{\alpha} d\lambda dA$, so that

$$\delta M = -\int_{\mathcal{H}} T_{\alpha\beta} t^{\beta} k^{\alpha} d\lambda dA \tag{2.19}$$

$$\delta J = \int_{\mathcal{H}} T_{\alpha\beta} \phi^{\beta} k^{\alpha} d\lambda dA, \tag{2.20}$$

$$\delta M - \Omega\delta J = -\int_{\mathcal{H}} T_{\alpha\beta}(t^{\beta} + \Omega\phi^{\beta}) k^{\alpha} d\lambda dA \tag{2.21}$$

$$= \int_{\mathcal{H}} T_{\alpha\beta} \xi^{\beta} k^{\alpha} d\lambda dA. \tag{2.22}$$

Now making the replacement for ξ^{β} we have

$$\delta M - \Omega\delta J = \kappa \int_{\mathcal{H}} \lambda T_{\alpha\beta} k^{\beta} k^{\alpha} d\lambda dA \tag{2.23}$$

and using the Einstein equation this can be written in terms of the Ricci tensor as

$$\delta M - \Omega\delta J = -\frac{\kappa}{8\pi} \int_{\mathcal{H}} \lambda R_{\alpha\beta} k^{\beta} k^{\alpha} d\lambda dA. \tag{2.24}$$

Considering the Raychaudhuri equation (see Appendix B.4) and neglecting second order terms in θ and σ, we have

$$\frac{d\theta}{d\lambda} = -R_{\alpha\beta} k^{\alpha} k^{\beta}, \tag{2.25}$$

which implies $\theta = -\lambda R_{\alpha\beta} k^{\alpha} k^{\beta}$ up to a constant shift, which can be scaled to zero. (2.24) then becomes,

$$\delta M = \frac{\kappa}{8\pi} \int_{\mathcal{H}} \theta d\lambda dA + \Omega\delta J. \tag{2.26}$$

Finally, recognising that $\int_{\mathcal{H}} \theta d\lambda dA$ is just the variation in the area of horizon in terms of the expansion of the generators, we have, as required,

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega\delta J. \tag{2.27}$$

Second Law

If the null energy condition is satisfied,

The surface area of a black hole cannot decrease

$$\boxed{\delta A \geq 0.} \tag{2.28}$$

The law can be equivalently stated by saying that the expansion of the null generators is everywhere non-negative on the horizon, $\theta \geq 0$. If $\theta < 0$ for any non-empty subset of the generators, then according to the focussing theorem (Appendix B.5), these geodesics must converge at a caustic point, where the expansion is negative infinity. This contradicts an observation made by Roger Penrose that null geodesics generating an horizon have no future end-points. Assuming that holds, we have $\theta \geq 0$ everywhere on the horizon.

Classically, the second law can also be understood via that fact that matter cannot leave the black hole but only enter it. Since the radius of the event horizon increases with the mass of the black hole, one expects that the surface area could not decrease. As we will see in section 3.3, this does not hold when quantum effects are considered. The second law also implies that two black holes may collide to form a larger hole, but one black hole cannot split to form two smaller black holes.

Third Law

It is impossible to reduce the surface gravity of a black hole to zero via a finite sequence of operations.

The third law prevents the creation of naked singularities [5], which would violate the cosmic censorship hypothesis. It was noted in [5] that if matter is thrown in to a black hole to increase the angular momentum and thus lower the surface gravity, one finds that the decrease in κ per particle thrown in diminishes as angular momentum J tends to mass M as κ tends to zero. Any process for which this does not happen takes an infinite amount of time. It was shown by Israel [13] in 1986 that this law can be stated and proved precisely. The proof is very involved and including a explanation here would amount to digressing too far from the main purpose of our discussion to be worthwhile.

Law	Thermodynamics	Black Hole Mechanics
Zeroth	Temperature is constant throughout system in equilibrium	Surface gravity is constant across entire horizon of a stationary black hole
First	$dE = TdS - pdV$	$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J$
Second	$\delta S \geq 0$	$\delta A \geq 0$
Third	Cannot reach absolute zero ($T = 0$)	Cannot achieve $\kappa = 0$

Table 1: Laws of thermodynamics and black holes

2.4 Comparison With Thermodynamics

The striking similarity with thermodynamics is now very clear. At first glance, it seems appropriate to draw the analogies $E \sim M$, $T \sim \kappa$ and $A \sim S$. The first of these relations needs little justification, while the other two are slightly more profound. The association of temperature with surface gravity will become clear when we derive the Hawking temperature in section 3.3. The area-entropy relationship was originally postulated by Bekenstein in 1971, who noted the similarity of the second laws, and an exact formula $S = \frac{Ac^3}{4G\hbar}$ was found by Stephen Hawking in 1974. It is certainly natural to at least suggest that the secret to quantum gravity might lie in a thermodynamic analysis, since the archetypal quantum gravitational object (the black hole) accidentally turned out to work just like a thermodynamic system. However, when one considers the nature of the method applied to black hole mechanics, the similarity with thermodynamics is, in fact, not as surprising as it might initially seem. Essentially the approach is to admit ignorance of the physics behind the horizon and define the thermodynamic properties *on* the horizon. In this sense, the black hole is methodologically treated as a thermodynamic system to begin with, so one might well expect the form of the laws to resemble one another.

As such, the laws of thermodynamics are just one instance of some more general statements about energy and probabilities in complex systems, which may have myriad applications. An obvious example is the second law, which is just a fact about statistics. With that considered, the initially striking similarity between black holes and thermodynamics might not be the statement that there is something intrinsically thermodynamical about black holes and gravity, but rather an expression of the more trivial fact that similar theoretical methodology has been applied in the two contexts. For example, the correspondence of area and entropy might just be that they have played the same role in a particular mathematical strategy, applied in two very different situations.

There is however, further evidence that black holes are truly thermodynamic. To understand this, let us consider quantum field theory in the Rindler and Swarzchild spacetimes.

3 Thermodynamics of Horizons

The purpose of the following discussion is two-fold: (1) elucidate and explain the origin of the thermodynamic properties of horizons and (2) calculate general expressions for such quantities, for use in the discussion of a thermodynamic interpretation of gravity in section 5.

3.1 Bogoliubov Transformations

The following discussion is similar, in parts, to that given in chapter 9.4 of [4]. Consider the Klein-Gordon equation for a free scalar field:

$$(\Box - m^2)\phi = 0 \tag{3.1}$$

where for spacetimes of arbitrary curvature the d'Alembertian is given by

$$\Box = \nabla_\mu \nabla^\mu = g_{\mu\nu} \nabla^\mu \nabla^\nu . \tag{3.2}$$

As for QFT in Minkowski spacetime, there exist complete sets $\{f_i(x^\mu)\}$ and $\{f_i^*(x^\mu)\}$ of solutions to (3.1), which are the field excitation modes and conjugate modes respectively. In the flat case, however, the theory enjoys a Lorentz invariant vacuum state and Fock-Hilbert space, despite the choice of a preferred quantisation frame (determined by selecting a time coordinate). The difference here, in the general (not necessarily flat) case when $\eta_{\mu\nu} \to g_{\mu\nu}$ as in (3.2), is that we lose this invariance. Inertial observers can now have different vacuum and multiparticle states; the particle number operator will have different eigenvalues in each frame. The formal demonstration of this involves what are known as Bogoliubov transformations, which relate alternative sets of basis modes. Suppose

$$\phi = \sum_i (\hat{a}_i f_i + \hat{a}_i^\dagger f_i^*) \tag{3.3}$$

$$\phi = \sum_i (\hat{a}_i g_i + \hat{a}_i^\dagger g_i^*) \tag{3.4}$$

define two non-identical sets of orthonormal basis modes with vacuum states $|0_f\rangle$ and $|0_g\rangle$ satisfying

$$\hat{a}_i|0_f\rangle = 0 \qquad \text{and} \qquad \hat{b}_i|0_g\rangle = 0$$

respectively. Orthonormality is defined by the vanishing of the Klein-Gordon inner product:

$$(\phi_i, \phi_j) = -i \int_{\Sigma_t} (\phi_i \partial_t \phi_j^* - \phi_j^* \partial_t \phi_i) d^{n-1}x \tag{3.5}$$

where Σ_t is a hypersurface of constant t. Let multiparticle states $|n_i\rangle$ be defined by successive application of the creation operators:

$$|n_i\rangle = \frac{1}{\sqrt{n_i!}} \left(\hat{a}_i^\dagger\right)^{n_i} |0_f\rangle \tag{3.6}$$

$$|n_i\rangle = \frac{1}{\sqrt{n_i!}} \left(\hat{b}_i^\dagger\right)^{n_i} |0_g\rangle. \tag{3.7}$$

The creation and annihilation operators satisfy the usual Clifford algebra

$$\begin{array}{ll} [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} & \qquad [\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij} \\ [\hat{a}_i, \hat{a}_j] = 0 & \qquad [\hat{b}_i, \hat{b}_j] = 0 \\ [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 & \qquad [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0, \end{array}$$

such that if the operators

$$\hat{N}_{f_i} = \hat{a}_i^\dagger \hat{a}_i \qquad \text{and} \qquad \hat{N}_{g_i} = \hat{b}_i^\dagger \hat{b}_i$$

act on the multiparticle states $|n_i\rangle$, their eigenvalues are just the number of particles n_i. This can be seen by dragging the annihilation operator to the right successively through the n_i creation operators and applying the Clifford algebra each time. Now we wish to to relate f_i's and the g_i's so that we may consider transformations between the two bases. This can be achieved via the expressions

$$f_i = (\alpha_{ji}^* g_j - \beta_{ji} g_j^*) \tag{3.8}$$

$$g_i = (\alpha_{ij} f_j + \beta_{ij} f_j^*) \tag{3.9}$$

where the Einstein convention is in place over the repeated j indices. The matrix components α_{ij} and β_{ij} are known as the Bogoliubov coefficients. Similar relations hold for the distinct sets of operators:

$$\hat{a}_i = (\alpha_{ji}\hat{b}_j + \beta^*_{ji}\hat{b}^\dagger_j) \tag{3.10}$$
$$\hat{b}_i = (\alpha^*_{ji}\hat{a}_j - \beta^*_{ji}\hat{a}^\dagger_j). \tag{3.11}$$

The Bogoliubov coefficients are given by

$$\alpha_{ij} = (g_i, f_j) \tag{3.12}$$
$$\beta_{ij} = -(g_i, f^*_j). \tag{3.13}$$

One can now see that something very strange is going to happen, since the expression for each of the annihilation operators have non-zero contributions from terms depending on the *creation* operators in the alternative quantisation, which follows from the fact that the f-modes are not orthonormal to the g-modes. This already suggests that one might count a different number of particles depending on which set of operators have been chosen. We can now consider the effect of the basis mode transformation on the Hilbert space. In particular let us consider the number of g-basis particles we would expect to see in the vacuum state associated with the f-modes. The expectation value of the g-number operator for the state $|0_f\rangle$ is found to be

$$\begin{aligned}\langle 0_f|b^\dagger_i b_i|0_f\rangle =& \langle 0_f|(\alpha_{ij}\hat{a}^\dagger_j - \beta_{ij}\hat{a}_k)(\alpha^*_{ik}\hat{a}_k - \beta^*_{ik}\hat{a}^\dagger_k)|0_f\rangle \\ =& \beta_{ij}\beta^*_{ik}\langle 0_f|\hat{a}_j\hat{a}^\dagger_k|0_f\rangle \\ =& \beta_{ij}\beta^*_{ik}\langle 0_f|\hat{a}^\dagger_k\hat{a}_j + \delta_{jk}|0_f\rangle \\ =& \beta_{ij}\beta^*_{ij} \\ =& |\beta_{ij}|^2 \end{aligned} \tag{3.14}$$

where the Einstein convention once again applies and has been used in the summation over k. In curved spacetime $|\beta_{ij}|^2$ can be non-zero (see (3.13)) and so we have seen the important consequence that the particle content of a given region of a curved spacetime is observer dependent. Since we are ultimately interested in the thermodynamics of horizons, the importance of the above result in our discussion obviously has to do with the question of what the Minkowski vacuum looks like to a Rindler observer. As one might expect from the equivalence principle, it turns out that this question is more or less physically equivalent to considering quantum field theory near a Schwarzchild black hole. What we will find is that black holes are not literally black, and empty spacetime for a Minkowski observer is in fact a thermal bath of particles in the Rindler frame!

3.2 The Unruh Temperature

One vitally important quantity pertaining to objective (2) above is the Unruh temperature, the temperature associated with the Rindler horizon. It can be shown [29] that a non-inertial observer in Minkowski spacetime should experience a non-zero temperature. This temperature is associated with a particle content expected in the Minkowski vacuum when quantisation is carried out in Rindler coordinates. In light of our discussion of the Bogoliubov transformations, one might want to draw the distinction between "Rindler particles" and "Minkowski particles". These would be defined as (multi)particle states corresponding to the two distinct sets of excitation modes, which are the solutions to the field equations of motion in the respective coordinate systems. One can then make the statement that the Minkowski vacuum contains Rindler particles, which can be viewed as a thermal system at finite temperature.

One way to calculate the temperature is by considering the power spectrum of plane waves which are subject to a (time-dependent) redshift in the frame of an accelerating observer. Such an approach is taken, among others, by Alsing in [1] and Padmanabhan in [23]. An appropriately Doppler shifted propagating wave is shown to have a Planckian power spectrum with temperature $T = a/2\pi$ in natural units. The calculation presented below, however, is closer to the method originally used by Unruh [29] and a similar approach is taken in chapter 9.5 of [4] and 4.5 of [3]. Consider the equation of motion for a scalar field (Klein-Gordon) in two dimensions of flat space-time, expressed in the inertial (Minkowski) coordinates:

$$(\partial_\mu \partial^\mu - m^2)\phi = (-\partial_t + \partial_x - m^2)\phi = 0. \tag{3.15}$$

where it will be convenient for our purposes to set $m = 0$. (3.15) can then be expressed in Rindler coordinates using (2.8),

$$\begin{aligned} 0 &= (\partial_\mu \partial^\mu)\phi & (3.16)\\ &= (g_{\mu\nu}\partial^\mu \partial^\nu)\phi & (3.17)\\ &= e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi. & (3.18) \end{aligned}$$

Let us define Minkowski modes f_k as solutions to (3.15) by

$$\phi(t,x) = \int dk \Big(\hat{a}_k f_k(t,x) + \hat{a}_k^\dagger f_k^*(t,x) \Big) \tag{3.19}$$

and Rindler modes $g_k^{(1)}$ as solutions to (3.16) existing in the $x > 0$ right wedge in figure(2.1) by

$$\phi(\eta,\xi) = \int dk \Big(\hat{b}_k^{(1)} g_k^{(1)}(\eta,\xi) + \hat{b}_k^{(1)\dagger} g_k^{(1)*}(\eta,\xi) \Big) \tag{3.20}$$

and $g_k^{(2)}$ in the $x < 0$ left wedge,

$$\phi(\eta, \xi) = \int dk \Big(\hat{b}_k^{(2)} g_k^{(2)}(\eta, \xi) + \hat{b}_k^{(2)\dagger} g_k^{(2)*}(\eta, \xi) \Big). \tag{3.21}$$

The respective vacuum states are determined by

$$\hat{a}_k |0_M\rangle = 0 \tag{3.22}$$

$$\hat{b}_k^{(2)} |0_R\rangle = \hat{b}_k^{(2)} |0_R\rangle = 0 \tag{3.23}$$

It was noted above that the discrepancy concerning the vacuum states in two distinct second quantisation pictures arises due to the mixing of creation and annihilation operators in transformations such as (3.10) and (3.11). So we expect $|0_M\rangle \neq |0_R\rangle$.The mixing effect can be equivalently stated by saying that the g-modes contain contributions from both positive and negative frequency Minkowski modes. Unruh took the approach of finding a new set of Rindler modes which can be expressed purely in terms of positive-frequency Minkowski modes. These will come with a set of operators which are unmixed in the Minkowski operators, such that they share the same vacuum state. In order to achieve this, combine the $g_k^{(1)}$ modes with the conjugate modes from the left wedge with negative k and vice-versa,

$$h_k^{(1)} = Z(e^{\pi\omega/2a} g_k^{(1)} + e^{-\pi\omega/2a} g_{-k}^{(2)}) \tag{3.24}$$

$$h_k^{(2)} = Z(e^{\pi\omega/2a} g_k^{(2)} + e^{-\pi\omega/2a} g_{-k}^{(1)}) \tag{3.25}$$

Where Z is a normalisation constant. Using the orthonormality condition $\big(g_{k_1}, g_{k_2}\big) = \delta(k_1 - k_2)$ for of the g-modes and $\big(g_{k_1}^*, g_{k_2}^*\big) = -\delta(k_1 - k_2)$, for the conjugate modes, we have

$$\big(h_{k_1}^{(1)}, h_{k_2}^{(1)}\big) = Z^2(e^{\pi\omega/a} - e^{-\pi\omega/a})\delta(k_1 - k_2) \tag{3.26}$$

$$= Z^2[2\sinh(\pi\omega/a)]\delta(k_1 - k_2) \tag{3.27}$$

and since Z must be fixed such that $\big(h_{k_1}^{(1)}, h_{k_2}^{(1)}\big) = \delta(k_1 - k_2)$, one obtains

$$Z = \frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}}. \tag{3.28}$$

Unruh found the Bogoliubov transformations:

$$\hat{b}_k^{(1)} = \frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} (e^{\pi\omega/2a} \hat{c}_k^{(1)} + e^{-\pi\omega/2a} \hat{c}_{-k}^{(2)\dagger}) \tag{3.29}$$

$$\hat{b}_k^{(2)} = \frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} (e^{\pi\omega/2a} \hat{c}_k^{(2)} + e^{-\pi\omega/2a} \hat{c}_{-k}^{(2)\dagger}) \tag{3.30}$$

with precisely this normalisation. The number of Rindler particles in the Minkowski vacuum is given by $\langle 0_M|\hat{N}_R|0_M\rangle$, which we can now express in terms of the $\hat{c}$ and $\hat{c}^\dagger$ operators. This gives

$$\begin{aligned}\langle 0_M|\hat{b}_k^{(1)\dagger}\hat{b}_k^{(1)}|0_M\rangle &= \frac{e^{-\pi\omega/2a}}{2\sinh(\pi\omega/a)}\langle 0_M|\hat{c}_k^{(1)}\hat{c}_k^{(1)\dagger}|0_M\rangle \\ &= \frac{e^{-\pi\omega/2a}}{e^{(\pi\omega/a)}-e^{-(\pi\omega/a)}}\langle k_M|k_M\rangle \\ &= \frac{1}{e^{(2\pi\omega/a)}-1}\delta(0) \end{aligned} \tag{3.31}$$

(we could have chosen a normalisation of the one particle states so that the delta function would not appear). For a Planck distribution, the average number of particles per mode with wave number k is given by the Planck distribution function

$$\bar{N}(k) = \frac{1}{e^{(\hbar\omega(k)/k_BT)}-1}. \tag{3.32}$$

Comparing this with (3.31) and using dimensional considerations to restore the hidden factor of c, one obtains

$$\boxed{k_BT = \frac{\hbar a}{2\pi c}} \tag{3.33}$$

where T is the Unruh temperature, which in natural units is clearly just $a/2\pi$.

3.3 The Hawking Temperature

Given the similarity between the Schwarzchild and Rindler spacetimes, one might expect an analogue of the Unruh effect in the vicinity of a Schwarzchild Horizon. Indeed, a temperature analogous to (3.33) arises quite naturally when one considers quantum field theory at the $r = 2M$ surface. In fact one would expect a temperature anyway given that the black hole has an entropy:

$$\frac{1}{T} = \frac{\partial S}{\partial E}. \tag{3.34}$$

In the classical analysis, energy leaving the black hole in the form of radiation is forbidden, but in can be shown [26] that quantum field theory provides a mechanism for the black hole to radiate. Pair creation at the event horizon leads to matter effectively escaping from the hole, with the corresponding black hole mass decrease resulting from negative energy falling permanently inside. A calculation of the corresponding temperature was originally carried

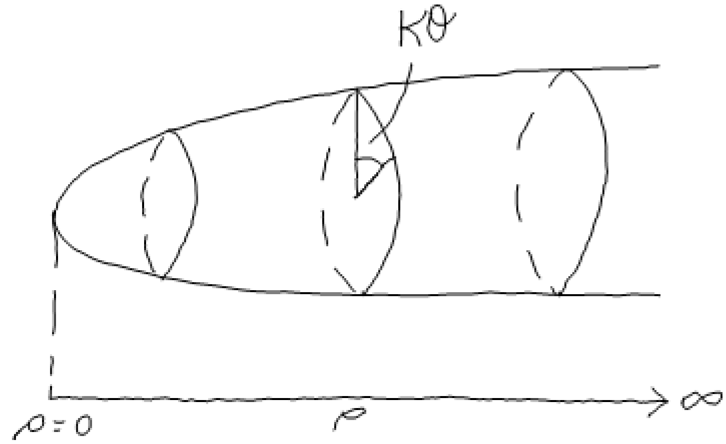

Figure 3.1: Euclidean Schwarzchild spacetime. The geometry looks like a cigar, tending to a cylinder infinitely far from the horizon.

out by Hawking for Minkowski spacetime, but it has since been shown that the same result may be obtained more easily by switching to a Euclidean metric signature (+,+,+,+). This can be achieved by setting $t = i\tau$ so that $t^2 = -\tau^2$ and the Schwarzchild metric is now given by

$$ds_E^2 = (1 - \frac{2M}{r})d\tau^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2 \tag{3.35}$$

with the new time coordinate. In plane polars this can be written[7]

$$ds_E^2 = \rho^2 \kappa^2 d\theta^2 + d\rho^2 \tag{3.36}$$

where the coordinate $\kappa\theta$ has inherited a periodicity in the Euclidean regime such that $\kappa\theta + 2\pi \sim \kappa\theta$ and so $\theta \sim \theta + \frac{2\pi}{\kappa}$. The quantum partition function will be given by first taking the path integral over fields which are periodic in Euclidean time (the field returns to an initial configuration ψ after one time-loop round the cigar in Fig.(3.1)) and then integrating over all possible initial field configurations ψ. So if $\Phi(0) = \psi = \Phi(2\pi/\kappa)$ then

$$Z = \int d\psi \int [D\Phi] e^{-S_E}, \tag{3.37}$$

where S_E is the Euclidean action. Writing this in terms of the evolution operator we have

$$Z = \int d\psi \langle\psi| e^{-(2\pi/\kappa\hbar)\hat{H}} |\psi\rangle = \mathrm{Tr} e^{-(2\pi/\kappa\hbar)\hat{H}} \tag{3.38}$$

[7] Surface gravity for the Swarzchild black hole is $\kappa = \frac{1}{4M}$

Then comparing this with the thermal partition function

$$Z = \mathrm{Tr} e^{-\beta \hat{H}} \tag{3.39}$$

where $\beta = (k_B T)^{-1}$, we now make the identification

$$\beta = \frac{2\pi}{\kappa \hbar}. \tag{3.40}$$

This determines the Hawking temperature to be

$$\boxed{T_H = \frac{\hbar \kappa}{2\pi k_B}} \tag{3.41}$$

or $T_H = \kappa/2\pi$ in natural units, c.f. the Unruh temperature. For alternative descriptions of a similar approach see [7] or [28]

3.4 Significance of The Thermodynamic Properties of Horizons

We have seen that, when both general relativity and quantum field theory are taken into account, entropy and temperature are actually intrinsic features of horizons. In light of these discoveries, the parallels between black hole mechanics and thermodynamics look less like interesting coincidences and more like exact identities. Black holes are not just *like* thermodynamic systems, they literally *are* thermodynamic systems. Furthermore, we are now pushed to the important and surprising conclusion that spacetime has microstructure, to wit that there are fundamental degrees of freedom giving rise to the temperature and entropy of horizons. It is open to debate what these degrees of freedom are, but it seems almost certain that, as macroscopic observers, our knowledge of them relies on the subject of the next section, namely the holographic principle.

4 The Holographic Principle

4.1 Information Loss Paradox

Most of the recent work that has been done so far on emergent models of gravity relies on the holographic principle. The principle, proposed in [27], allows information in a bounded region V of finite volume to be contained entirely on the boundary of that region ∂V. The suggestion that a fermion field lattice theory in three spatial dimensions might be projected, without loss of information, onto a 2-surface is discussed in [16]. So when we have horizons, the microscopic degrees of freedom inside the horizon are "projected" onto the horizon itself. In this sense, the entropy of the horizon can be thought of as measuring the lack of knowledge an observer has about microscopic degrees of freedom in a given region, due to the presence of an horizon causally isolating them from its interior. The analogy is drawn in [27] with a two-dimensional hologram image, which encodes all the information in a region of three dimensional space. There is necessarily some "blurring" of the image, limited by the wavelength of the radiation used to create the image, which corresponds with the fuzziness of quantum uncertainty, itself lower bounded by Planck's constant.

While there are numerous known applications of the Holographic principle, not least the AdS/CFT correspondence, the idea is offered in [27] as a solution to the black hole information loss paradox, which is the most relevant application to the discussion here. The reason is that we would like extend the notion of holography on the horizon to explain the emergence of classical space and geometry (explained in detail in section 5). Suppose that an observer sees some matter flow across an horizon. If we require that all observers can do physics with the degrees of freedom that are physically accessible to them, entropy has disappeared from the universe in the frame of this observer. A solution to this problem comes in the form of holography, and we must attribute a change in horizon entropy

$$\delta S = \delta E/T \tag{4.1}$$

when an energy δE flows across an horizon of temperature T. In this way, the information "lost into oblivion" is actually stored on the boundary of the causally inaccessible region.

4.2 Holography and The Einstein-Hilbert Action

It is particularly interesting to study the significance of holography in the context of an action formulation of Einsteinian gravity. A detailed discussion of this, as well as similar considerations relevant to the more general class of Lanczos-Lovelock actions, is given in the aptly named paper [22]. The Einstein-Hilbert action is, in appropriate units,

$$A_{EH} = \int_V d^D x \sqrt{-g} L_{EH} = \int_V d^D x \sqrt{-g} R \tag{4.2}$$

and so L_{EH} can be written

$$L_{EH} = \frac{1}{2}(\delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu) R^{\mu\nu}_{\rho\sigma} = \frac{1}{2}(\delta^\rho_\mu g^{\nu\sigma} - \delta^\sigma_\mu g^{\nu\rho}) R^\mu_{\nu\rho\sigma}. \tag{4.3}$$

It is straightforward to decompose L_{EH} into bulk (quadratic in derivatives of the metric) and surface (total derivative) term [22]:

$$L_{EH} = \underbrace{2\partial_\rho(\sqrt{-g}Q_\mu{}^{\nu\rho\sigma}\Gamma^\mu_{\nu\sigma})}_{L_{surface}} + \underbrace{2\sqrt{-g}Q_\mu{}^{\nu\rho\sigma}\Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho}}_{L_{bulk}} \tag{4.4}$$

where $Q_\mu{}^{\nu\rho\sigma} = \frac{1}{2}(\delta^\rho_\mu g^{\nu\sigma} - \delta^\sigma_\mu g^{\nu\rho})$. Varying the action

$$\int_V d^4x\sqrt{-g}(L_{EH} + L_{matter}) \tag{4.5}$$

with respect to the metric one obtains the usual Einstein field equations. An interesting fact pointed out by Padmanabhan, is that the relation

$$L_{surface} = -\partial_i(g_{ab}\frac{\partial L_{bulk}}{\partial(\partial_i g_{ab})}) \tag{4.6}$$

exposes an holographic relationship manifest in the EH action itself. With this holographic relation, all the information can be encoded in either the bulk or surface term alone.

5 Emergent Space and Entropic Gravity

5.1 Emergence of Space and The Newtonian Concepts

If the idea that gravity is an emergent phenomenon is to be taken seriously, an immediate question one must ask is whether even simpler notions, such as force, inertia or even space itself might be given an emergent description. This is precisely the view offered in [30] and the approach, unsurprisingly, involves holographic screens. For an arbitrary screen bounding an arbitrary region, one might refer to the interior of the region as *unemerged* space and the exterior of the region as *emerged* space. Consider, for example, a particle behind a screen moving deeper into the unemerged space along an axis perpendicular to the screen. The radial co-ordinate corresponding to this axis does not have a meaningful empirical interpretation as a physical spatial co-ordinate, since it can only be understood in terms of the holographic image of the particle on the screen which is spatially fixed. In that sense, dropping the screen radially inward corresponds to the emergence of physical space, as only *outside* the screen do we have the usual physical (rather than holographic) interpretation of spatial co-ordinates. Macroscopic space can then be thought of as foliated into two dimensional surfaces, with the radial dimension emergent.

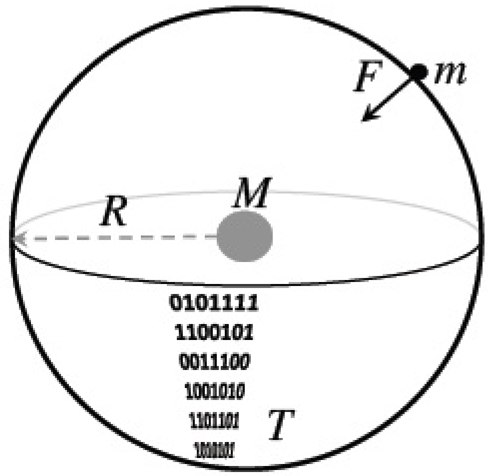

Figure 5.1: Image taken from [30]. A particle contributes entropy to the holographic screen.

5.1.1 The Principle of Inertia

Following Verlinde, let us consider moving a particle in the emerged part of space towards an holographic screen, but let us assume that the screen is spherical to begin with. As Bekenstein argued, when the particle is a distance of one Compton wavelength $\Delta x = \frac{\hbar}{mc}$ from the horizon of a black hole, it contributes some entropy to the Horizon. Let us take this quantity to be $\Delta S = 2\pi k_B$. Assuming a similar line of reasoning holds for Rindler horizons, one may use the Unruh temperature to introduce an acceleration a:

$$k_B T = \frac{\hbar a}{2\pi c}. \tag{5.1}$$

Verlinde's idea is to consider this as the temperature *required to produce* the acceleration a. In that scenario we can define the entropic force by

$$F = T\frac{\Delta S}{\Delta x} \tag{5.2}$$

and substituting for T, Δx and ΔS we have

$$F = \frac{1}{k_B}\frac{\hbar a}{2\pi c}\frac{2\pi k_B mc}{\hbar} = ma \tag{5.3}$$

as one would hope. Equations (5.2) and (5.3) together should be understood as the definition of inertia in the thermodynamic paradigm.

5.1.2 Gravitation and The Inverse Square Law

Let us now proceed in making some additional assumptions. First, it is natural to assume, given our knowledge of black holes and holography, that the information stored on an horizon is proportional to the area of the horizon (the basic idea behind the Bekenstein black hole entropy relation). This is equivalent to saying that the increase in area of an horizon due to matter flowing across it is linearly related to the mass of the matter. Let us postulate then that for an horizon storing N bits of information, $N \propto A$, i.e. $N = \alpha A$ if α is a constant. It will be convenient for the following derivation, to define now another constant $G = c^3/\alpha\hbar$ so that

$$N = \frac{Ac^3}{G\hbar}. \tag{5.4}$$

Second, let us assume an equipartition of energy on the horizon such that the total energy is given by

$$E = \frac{1}{2} N k_B T \tag{5.5}$$

with the purpose of this assumption being the definition of the temperature in terms of the information stored on the horizon. Now we can simply write

$$F_{gravity} = T\frac{\Delta S}{\Delta x} \tag{5.6}$$

and making the appropriate substitutions yields

$$\begin{aligned} F_{gravity} &= \left(\frac{2E}{Nk_B}\right)\left(\frac{2\pi k_B mc}{\hbar}\right) \\ &= \frac{4\pi G m}{A}\frac{E}{c^2} \\ &= \frac{GmM}{R^2} \end{aligned} \tag{5.7}$$

where in the last line the identifications $A = 4\pi R^2$ and $E = Mc^2$ have been made. The energy E should thus be considered as the holographic projection of a mass M at the centre of the spherical screen, evenly distributed over its surface. As is pointed out in [30], Newtonian principles have played a part in the realisation of the laws of black hole mechanics and holography anyway, so what is surprising and interesting here is not that one can arrive at Newton's gravity from black holes and holography, but rather that this particular approach treats gravity as an entropic force.

5.1.3 The Newton Potential

It is also suggested in [30] that (5.3) be reformulated in terms of the newton potential by expressing the acceleration as the gradient $a = -\nabla\Phi$ such that

$$\frac{\Delta S}{n} = -k_B \frac{\Delta\Phi}{2c^2} \tag{5.8}$$

This relation is crucial in understanding the interplay between emergent space, thermodynamics and gravity. What (5.8) basically suggests is that gravity arises due to the loss of information incurred with course graining (parameterised by Φ). A negative change in the Newton potential reflects a corresponding positive change in the entropy, so the gravitational field strength is directly associated with the information available to an observer in the emerged part of space! As we course grain, we are (virtually) displacing the screen inwards, more space is considered to have emerged and the area of the holographic screen has decreased along with the configuration space of the bounded region. That such a relationship can be shown to lead to the laws of gravitation is remarkable and profoundly elegant. Although the thermodynamic interpretation of gravity has been known since Jacobson 1995 and clues to that effect even earlier (Hawking, Bekenstein, Unruh etc.), the full implications of (5.8) are truly striking and if correct, provide a revolutionary understanding of the nature of gravity.

It is important to note that Verlinde does not consider this process to be dynamical [31] although others [25], [17] have suggested it might be. One might compare the process of emergence described above with the dynamical process of a black hole consuming more and more of the space around it as its area increases with its entropy. In such a scenario, space might be considered as dynamically un-emerging.

5.1.4 Loop Quantum Gravity as a Candidate Microscopic Theory

It should be noted that all of the results above have been arrived at from large scale phenomenological considerations; the holographic principle has enabled such work to be done without knowledge of a fundamental microscopic theory. For example, it was assumed that N bits of information are stored on an holographic screen, but no explanation has been given as to how this information is stored or what the fundamental horizon degrees of freedom actually are. Answers to these questions have been offered by Smolin [25]. The basic idea is to consider the holographic boundary as a punctured two-surface and define a Hilbert space on the boundary by assuming that each puncture can be related to the edge of a spin network inside the bounded region by holography. It is assumed that each spin network is associated with a $j = 1/2$ (two-dimensional) representation of SU(2), and it is thus possible to define $\mathcal{H}^{boundary}$ as the direct sum of N two-dimensional Hilbert spaces. So the N spin states (each one $|\uparrow\rangle$ or $|\downarrow\rangle$) associated with N punctures store N bits of information on the boundary. We can thus write the entropy of the boundary

$$S = \ln \dim \mathcal{H}^{boundary} = N \ln 2 \tag{5.9}$$

5.2 Objections and Replies Concerning The Newtonian Limit

There have already been some notable objections [10, 6] raised against the model put forward by Verlinde. One line of attack [10] has been to criticise the causal relationship between Δx and ΔS. Verlinde postulates a change in horizon entropy resulting from a change in the radial distance of the particle from the screen, but then uses that relationship to argue that the change in entropy results in an attractive force on the particle. The accusation made by Gao is that there is a contradiction in [30], in that the logic starts out as $\Delta x \to \Delta S$, then becomes $\Delta S \to \Delta x$ with gravity as entropic. Furthermore, the causal chain $\Delta x \to \Delta S \to F$ would seem to imply that when $\Delta x = 0$ we have $\Delta S = 0$ and therefore $F = 0$, which is false by observation because it would imply a zero force between objects at rest to one another.

The first of these objections is not, however, wholly convincing. There is no inherent contradiction in [30], since $\Delta S \leftrightarrow \Delta x$ is simply not inconsistent: this is precisely the scenario one has for the polymer in a heat bath where an increase in the polymer's entropy results in a decrease in its extension and pulling the polymer out straight results in a decrease in the entropy[8]. There is a kind of logical equivalence between the two statements. Verlinde has just calculated the relationship between ΔS and Δx by considering *one* of the causal directions (again similar to the case of the elastic polymer where one considers the work done in extending the polymer by a given amount and the corresponding change in entropy, and then expresses the force in terms of those quantities).

The second objection does point out a more legitimate problem, since it seems circular to suggest that gravity arises as a result of a change in entropy, itself a result of particles and objects moving closer to one another. Provided, however, that there is ultimately some explanation from an underlying micro-

[8]Although the entropy of the universe as a whole still increases.

scopic theory of why spacetime spontaneously increases its entropy[9], then the problem is not quite fatal.

Another valid point [10], is that in the case of the polymer, there are electrostatic interactions between the molecules - the entropic force is not magical! While an expression can be found in terms of change in entropy and extension, there still needs to be some underlying fundamental force at play. In the case of gravity it is unclear what this would be, but again one might hope an answer would come from the theory of spacetime microstructure. Clearly if gravity is not fundamental, then it must ultimately be reduced to another fundamental force.

Others have noted [15, 6] that Verlinde's assumption of a linear dependance of ΔS on Δx needs justification for two reasons. First, it is not clear that one can extend Bekenstein's argument to particles far away from the screen. Second, one might expect the entropy to vary with area A (quadratic in x) according to the Bekenstein-Hawking formula, rather than x. Lee actually answers this objection by suggesting that Verlinde's holographic screen be identified with Rindler horizons[10]. The Rindler observer measures $\Delta x = c^2/a$ for a particle on the horizon, and so using $\Delta E = T\Delta S$ one has:

$$mc^2 = \Delta E = T_U \Delta S = \frac{\hbar a}{2\pi c k_B}\Delta S \tag{5.10}$$

and thus

$$\Delta S = \frac{2\pi c k_B m \Delta x}{\hbar}. \tag{5.11}$$

as claimed by Verlinde. Further details, and other lines of argument to the same conclusion are given in [15].

5.3 Einsteinian Gravity From Thermodynamics

It had in fact been shown [14] prior to the work discussed above concerning the Newtonian limit, that the Einstein equations fit naturally into a thermodynamic interpretation, namely as an equation of state analogous to $p = T(\partial S/\partial V)$. The derivation considers Rindler horizons while assuming the area-entropy dependence $dS \propto \delta A$, the equation $\delta Q = TdS$ from thermodynamics and the Unruh temperature (3.33). As explained already in this review, it is a fundamental feature of any thermodynamic analysis, that heat is energy concerned with microscopic degrees of freedom ignored by the macroscopic

[9]This is what is missing in comparison with the polymer whose entropy increases due to thermal collisions with other particles in the bath.

[10]It seems clear from [31] that Verlinde already has this idea in mind. Nevertheless, the explanation in [15] details why this solves the problem.

observer (Boltzmann principle). In the following analysis, Rindler horizons (null hypersurfaces) play the role of obscuring fundamental degrees of freedom in the unemerged region of space (the role usually played by some diathermic barrier). The "heat" δQ is the energy flowing across the horizon. For some essential background concerning the structure of geodesic congruences, see Appendix B.

Consider now the existence of Rindler horizons at every point in spacetime. One can, of course, in principle, always find a local Rindler frames (LRFs) corresponding to any acceleration vector in this scenario simply by considering an observer with the appropriate acceleration. We would also like to consider the tangent space of each point on the manifold as a local Minkowski plane, which we may do by considering a free falling frame (FFF) in the usual way. If we consider the point $\mathcal{P}$ to be the stationary point on the hyperbolic trajectory of the Rindler observer as viewed in the local Minkowski FFF, we can assume that the expansion θ and shear σ of a past direced null congruence (see Appendix B) vanish at $\mathcal{P}$[11]. Now we have a situation which is familiar from our discussion of the Unruh effect and we may attribute a temperature to the Horizon given by (3.33). Notice also that, since the Minkowski plane has Lorentz symmetry, there exist boost generating Killing vectors (see Appendix). Let χ^μ be the Killing vector field generating the boost responsible for the existence of the Rindler horizon. Then the kinetic energy δQ associated with a Rindler observer is just $T_{\mu\nu}\chi^\mu$ integrated over the Horizon where $T_{\mu\nu}$ is the energy-momentum tensor for the "Rindler particles" in the Minkowski vacuum state. We have[12]

$$\delta Q = TdS = (\hbar\kappa/2\pi)dS = \int_{\mathcal{H}} T_{\mu\nu}\chi^\mu d\Sigma^\nu \tag{5.12}$$

where $d\Sigma^\mu$ are elements of the Horizon generators. Let us now define a tangent vector k^μ such that $\chi^\mu = -\kappa\lambda k^\mu$. Then we may express $d\Sigma^\mu$ in terms of this vector and the Horizon area element dA as $d\Sigma^\mu = k^\mu d\lambda dA$. Replacing these quantities in (5.12) yields

$$dS = -\frac{2\pi}{\hbar}\int_{\mathcal{H}} \lambda T_{\mu\nu}k^\mu k^\nu d\lambda dA. \tag{5.13}$$

Now from the proportionality of the horizon area and entropy we can write $dS = \eta\delta A$ for some unknown constant η so that (5.13) becomes

$$\delta A = -\frac{2\pi}{\eta\hbar}\int_{\mathcal{H}} \lambda T_{\mu\nu}k^\mu k^\nu d\lambda dA. \tag{5.14}$$

[11]Jacobson describes this as "local equilibrium" in analogy to traditional thermodynamics.
[12]with units such that $k_B = c = 1$.

Now, let us rewrite the area variation in terms of the expansion of the horizon generators,

$$\delta A = \int_{\mathcal{H}} \theta d\lambda dA \tag{5.15}$$

such that (5.14) becomes

$$\int_{\mathcal{H}} \theta d\lambda dA = -\frac{2\pi}{\eta\hbar} \int_{\mathcal{H}} \lambda T_{\mu\nu} k^\mu k^\nu d\lambda dA. \tag{5.16}$$

Since the objective is to recover Einstein's equations, the next step is to introduce the Ricci tensor. In other words, we want to see (in equilibrium) how spacetime needs to curve so that the horizon area varies with energy flowing across it, such that the two are proportional. Jacobson considered the Raychaudhuri equation (Appendix B.4) near enough to $\mathcal{P}$ that the second order terms in θ and σ can be neglected. This gives

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu} k^\mu k^\nu, \tag{5.17}$$

which implies $\theta = -\lambda R_{\mu\nu} k^\mu k^\nu$ up to a constant shift, which can be scaled to zero. Substituting into (5.16) yields

$$R_{\mu\nu} k^\mu k^\nu = \frac{2\pi}{\eta\hbar} T_{\mu\nu} k^\mu k^\nu. \tag{5.18}$$

This is required to hold for all null vectors k^μ, so we have, for some function f

$$\frac{2\pi}{\eta\hbar} T_{\mu\nu} = R_{\mu\nu} + f g_{\mu\nu} \tag{5.19}$$

where one recovers (5.18) by multiplying by $k^\mu k^\nu$ from the right and using the fact that null vectors satisfy $g_{\mu\nu} k^\mu k^\nu = 0$, by definition. By conservation of energy and momentum, taking the divergence of (5.19) in any index must give zero, so $\nabla_\mu (f g^{\mu\nu}) = \nabla_\mu R^{\mu\nu}$ and similarly for $\mu \leftrightarrow \nu$. Now by the Bianchi identity for the Ricci tensor and the fact the $\nabla_\mu g^{\mu\nu} = 0$, we have

$$\nabla_\mu f = 0. \tag{5.20}$$

The solution for f must be diffeomorphism invariant and since f is clearly scalar, we construct a solution in terms of $R = R_\mu{}^\mu$,

$$f = -R/2 + \Lambda \tag{5.21}$$

where Λ is an arbitrary integration constant, whose origin is essentially mysterious. Equation (5.19) now reads

$$R_{\mu\nu} - 1/2 R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{2\pi}{\eta\hbar} T_{\mu\nu} \tag{5.22}$$

which is Einstein's equation with $\eta = (4G\hbar)^{-1}$, the inverse of the square of twice the Planck length.

We have seen that the celebrated geometrical theory of gravity provided by general relativity in fact follows from thermodynamic considerations alone, once quantum field theory is taken into account in the Rindler coordinates. This remarkable fact has provided the basis for an understanding of Einsteinian gravity in terms of the foliated emergent space model [30] discussed in the preceding section. In that case, the reasoning is applied to timelike, rather than null screens.

5.4 Gravitational Entropy from Horizon Thermodynamics

Cconsider a virtual displacement of an horizon such that it engulfs some matter in the surrounding region. The change in entropy is $\delta S = \delta E/T$ and it natural to express dE (the energy flux across the horizon) as proportional to the energy momentum tensor $T_{\mu\nu}$, and a unit vector normal to the horizon n^μ. Specifically, in the LRF we have:

$$dE = \sqrt{-g} T_{\mu\nu} n^\mu n^\nu d^3x \tag{5.23}$$

For an explicit demonstration of Padmanabhan's approach, the most immediate requirement is a definition of the entropy of a given region of spacetime. Since we wish to avoid explicit reference to a theory of spacetime microstructure, let us follow Padmanabhan in the top-down approach and make use of our knowledge of horizon thermodynamics. In line with traditional thermodynamics, specifically what is required is an entropy functional. By considering the virtual displacement of an horizon and the resulting change in entropy, we can construct such a functional in some parameter n^μ (and derivatives $\nabla_\nu n^\mu$) such that the diffeomorphism $x^\mu \to x^\mu + n^\mu$ might be thought of as the analog of elastic deformation of a material. So n^μ measures the (virtual) displacement normal to a null surface (our horizon). We will find the gravitational field equations by the extremisation of the entropy functional. The basic idea is to assume $S = S_{matter} + S_{grav}$ and postulate the form of each term separately. If we define $\beta \equiv T^{-1} = 2\pi/\kappa$ then from (4.1) we have

$$\delta S_{matter} = \beta \delta E = \int_0^\beta dt \delta E$$

and using (5.23) this can be written

$$\int dS_{matter} = \int_0^\beta dt \int d^3x \sqrt{-g} T_{\mu\nu} n^\mu n^\nu \tag{5.24}$$

which generalises in spacetime dimension $D \geq 4$ to:

$$S_{matter} = \int_V d^D x \sqrt{-g} T_{\mu\nu} n^\mu n^\nu \tag{5.25}$$

for some spacetime volume V. To determine S_{grav} we follow the approach used in [23] and [21] by first postulating that the general form is quadratic in derivatives of n^μ:

$$S_{grav} = -4 \int_V d^D x \sqrt{-g} P_{\mu\nu}{}^{\rho\sigma} \nabla_\rho n^\mu \nabla_\sigma n^\nu \tag{5.26}$$

and then impose constraints to determine $P_{\mu\nu}{}^{\rho\sigma}$. Phadmanabhan points out the unusual situation that the extremum principle is required to hold for all vectors n^μ even though the entropy will be a functional precisely *in* the vector n^μ, i.e. $S = S[n^\mu]$. We are actually looking for a condition on the background metric, rather than n^μ. To this end, we will require that $P_{\mu\nu\rho\sigma}$ has all the symmetries of the Riemann tensor and that it is divergence free in all indices i.e. that $\nabla_\mu P^{\mu\nu\rho\sigma} = 0$ holds for interchange of any index with μ. The most general tensor satisfying these constraints can be expressed as a power series:

$$P^{\mu\nu\rho\sigma}(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}) = c_1 \overset{(1)}{P}{}^{\mu\nu\rho\sigma}(g_{\alpha\beta}) + c_2 \overset{(2)}{P}{}^{\mu\nu\rho\sigma}(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}) + ... \tag{5.27}$$

where the third term would be quadratic in $R_{\alpha\beta\gamma\delta}$, the fourth would be cubic and so on. The mth order term is given by:

$$\overset{(m)}{P}{}^{\rho\sigma}_{\mu\nu} \propto \delta^{\rho\sigma\mu_3...\mu_{2m}}_{\mu\nu\nu_3...\nu_{2m}} R^{\nu_3\nu_4}_{\mu_3\mu_4} ... R^{\nu_{2m-1}\nu_{2m}}_{\mu_{2m-1}\mu_{2m}} \tag{5.28}$$

and comparing this expression with the mth order Lanczos-Lovelock Lagrangian:

$$L^{(m)} = \frac{1}{64\pi} 2^{-m} \delta^{\mu_1\mu_2...\mu_{2m}}_{\nu_1\nu_2...\nu_{2m}} R^{\nu_1\nu_2}_{\mu_1\mu_2} R^{\nu_{2m-1}\nu_{2m}}_{\mu_{2m-1}\mu_{2m}} \tag{5.29}$$

we can write

$$\overset{(m)}{P}{}^{\rho\sigma}_{\mu\nu} = \frac{\partial L^{(m)}}{\partial R^{\mu\nu}_{\rho\sigma}}. \tag{5.30}$$

We are now ready to make sense of an expression for the total entropy. Using (5.26) and (5.25) we can write:

$$S[n^\mu] = S_{grav} + S_{matter} = -\int_V d^D x \sqrt{-g} (4 P_{\mu\nu}{}^{\rho\sigma} \nabla_\rho n^\mu \nabla_\sigma n^\nu - T_{\mu\nu} n^\mu n^\nu). \tag{5.31}$$

Einstein's gravity appears in D=4, where we take $P^{\mu\nu\rho\sigma}$ as dependent on the metric alone - i.e. the first order term in (5.27). Setting $c_1 = 1$ and using (5.30) and (5.29) to evaluate $\overset{(1)}{P}{}_{\mu\nu}^{\rho\sigma}$, we can write the entropy functional for Einstein's theory explicitly as:

$$S[n^{\mu}] = -\int_V d^4x\sqrt{-g}\left(\frac{1}{8\pi}(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \delta^{\rho}_{\nu}\delta^{\sigma}_{\mu})\nabla_{\rho}n^{\mu}\nabla_{\sigma}n^{\nu} - T_{\mu\nu}n^{\mu}n^{\nu}\right) \tag{5.32}$$

It is worth noting at this stage that ultimately one would hope to determine $P^{\mu\nu\rho\sigma}$ directly from some underlying microscopic theory. The construction here is essentially a toy model and this is a limitation which seems to apply to much of the work on emergent gravity to date. On the other hand, it is easy to imagine how these ideas could be incorporated into a fundamental quantum theory (although practically implementing this could be non-trivial). For example, one could try to derive (5.31) or equivalent from such a theory (above we have merely postulated a sensible form which will give the gravitational field equations when we impose entropy maximisation $\delta S = 0$) . It would be a very important finding if (5.31) or (5.32) was seen to follow from something like Loop Quantum Gravity or Causal Set theory for example.

5.5 Derivation of the Gravitational Field Equations

We wish now to impose that the entropy is extremal and that this holds for all null vector fields, so we require that $\delta S = 0$ with the additional constraint that $n_{\mu}\delta n^{\mu} = 0$ is satisfied. To achieve this we use a Lagrange multiplier method. The variation in (5.31) is thus:

$$\delta S = -2\int_V d^Dx\sqrt{-g}(4P_{\mu\nu}{}^{\rho\sigma}\nabla_{\rho}n^{\mu}(\nabla_{\sigma}\delta n^{\nu}) - T_{\mu\nu}n^{\mu}\delta n^{\nu} - \lambda(x)g_{\mu\nu}n^{\mu}\delta n^{\nu}). \tag{5.33}$$

Integrating by parts to move the derivative off the variation in n^{ν} we obtain (ignoring a vanishing boundary term as usual):

$$\delta S = 2\int_V d^Dx\sqrt{-g}\Big[4\nabla_{\sigma}P_{\mu\nu}{}^{\rho\sigma}\nabla_{\rho}n^{\mu} + 4P_{\mu\nu}{}^{\rho\sigma}\nabla_{\sigma}\nabla_{\rho}n^{\mu} + (T_{\mu\nu} + \lambda(x)g_{\mu\nu})n^{\mu}\Big]\delta n^{\nu}$$

and since $\nabla_{\sigma}P_{\mu\nu}{}^{\rho\sigma} = 0$, the first term vanishes leaving:

$$\delta S = 2\int_V d^Dx\sqrt{-g}\Big[4P_{\mu\nu}{}^{\rho\sigma}\nabla_{\sigma}\nabla_{\rho}n^{\mu} + (T_{\mu\nu} + \lambda(x)g_{\mu\nu})n^{\mu}\Big]\delta n^{\nu}.$$

Requiring that δS vanish for arbitrary δn^{ν} yields:

$$0 = 4P_{\mu\nu}{}^{\rho\sigma}\nabla_\sigma\nabla_\rho n^\mu + (T_{\mu\nu} + \lambda(x)g_{\mu\nu})n^\mu$$

$$= 2P_{\mu\nu}{}^{\rho\sigma}(\nabla_\rho\nabla_\sigma - \nabla_\sigma\nabla_\rho)n^\mu - (T_{\mu\nu} + \lambda(x)g_{\mu\nu})n^\mu$$

Since $P_{\mu\nu}{}^{\rho\sigma}$ is antisymmetric in ρ and σ. We can now recognise the commutators of the covariant derivatives and rewrite the equation in terms of the Riemann tensor using $[\nabla_\sigma, \nabla_\nu]u_\mu = R^\rho{}_{\mu\sigma\nu}u_\rho$, reducing the last line above to:

$$\left(2P_\nu{}^{\alpha\beta\gamma}R^\mu{}_{\alpha\beta\gamma} - T^\mu_\nu + \lambda\delta^\mu_\nu\right)n_\mu = 0 \tag{5.34}$$

By Lorentz invariance, this holds for all n^μ, so

$$2P_\nu{}^{\alpha\beta\gamma}R^\mu{}_{\alpha\beta\gamma} - T^\mu_\nu + \lambda\delta^\mu_\nu = 0. \tag{5.35}$$

It can be shown [23] that in the general case one obtains the Lanczos-Lovelock field equations:

$$\mathcal{R}^\mu_\nu - \frac{1}{2}L = \frac{1}{2}T^\mu_\nu + \Lambda\delta^\mu\nu \tag{5.36}$$

where $\mathcal{R}^\mu_\nu = P_\nu{}^{\alpha\beta\gamma}R^\mu{}_{\alpha\beta\gamma}$. Let us see for ourselves how this works out in the case of Einstein's equations. We want the first order Lanczos-Lovelock Lagrangian and so $P_\nu{}^{\alpha\beta\gamma} = \frac{1}{32\pi}(\delta_\nu{}^\beta\delta^{\alpha\gamma} - \delta_\nu{}^\gamma\delta^{\alpha\beta})$. Then (5.35) becomes

$$\begin{aligned} 0 &= \frac{1}{16\pi}(\delta_\nu{}^\beta\delta^{\alpha\gamma} - \delta_\nu{}^\gamma\delta^{\alpha\beta})R^\mu{}_{\alpha\beta\gamma} - T^\mu_\nu + \lambda\delta^\mu_\nu \\ &= \frac{1}{8\pi}R^\mu_\nu - T^\mu_\nu + \lambda\delta^\mu_\nu \\ &= R_{\eta\nu} - 8\pi T_{\eta\nu} + 8\pi g_{\eta\nu}\lambda \end{aligned} \tag{5.37}$$

where an index has been lowered using $g_{\eta\mu}$. Now if we introduce a constant[13] $\Lambda = \frac{1}{2}R + 8\pi\lambda$, (5.37) becomes

$$R_{\eta\nu} - 8\pi T_{\eta\nu} + g_{\eta\nu}(\Lambda - \frac{1}{2}R) = 0 \tag{5.38}$$

$$R_{\eta\nu} - g_{\eta\nu}\frac{1}{2}R + g_{\eta\nu}\Lambda = 8\pi T_{\eta\nu} \tag{5.39}$$

as required. So we have seen that Einstein's geometrical theory of gravity follows from extremisation of the entropy of spacetime and matter. Further, it is clear that this is not some strange coincidence, but must hold in general for any theory of gravity that can be formulated by means of an action constructed in arbitrary orders of derivatives of the metric and in arbitrary dimensions (Lanczos-Lovelock class of theories).

[13] That Λ is constant follows from the fact that the Einstein and stress energy tensors are both divergence free.

6 Summary and Conclusions

The mathematical arguments presented in this review show that a thermodynamic (entropic) description of gravity at least works on a formal level. There is a deep, a priori connection between the maximisation of entropy and the fact the matter curves spacetime. Additionally, there is a Newtonian limit in which one recovers the appropriate equations. Whether we see this as "entropy change causes gravitation" or "gravitation causes entropy change" remains debatable [10]. As explained in section 5.2 there is also the possibilty that gravitation simply *is* a change in the spacetime-matter entropy functional, i.e. that each causes the other. There have also been concerns [10] about the role of the Unruh temperature in the derviations by Jacobson and Verlinde, particularly concerning its use in the non-relativistic limit. The current situation may be summarised as follows:

- It is unclear what the microscopic degrees of freedom giving rise to the entropy and temperature of horizons are. The "smearing out" of the holographic image serves to obscure the nature of the interior of a black hole and the argument for the existence of spacetime microstructure is top-down.
- Most important is an understanding of what the temperature of spacetime really is; the energy inducing the state transitions of, say, a polymer in a heat bath, is the heat of the bath. If gravity is entropic, a good (better) explanation needs to be given as to what is driving the force.
- According to Verlinde's reasoning, gravity is related to the information (not)available to observers with a course-grained picture. This is equivalent to stating that the force arises as a result of an entropy gradient, itself the result of a gradient in the number of degrees of freedom available to the observer.
- Regardless of whether it is correct to consider gravity as an entropic force, the holographic mechanism for the emergence of space provided in [30] is promising. It seems likely that the harmony between quantum field theory and gravity comes in the form of holograpy.
- In [30], it is explained that the emergent model can be given a string-theoretic interpretation, with D-branes as the holographic screens, open strings living on the inside (unemerged region) and closed strings and gravity on the outside (emerged region). It has been suggested [25] that

> LQG could provide the microscopic theory and it has been shown that one can run Verlinde's argument in that scenario. The model is also compatible, in principle, with causal set theory, CDT's and pretty much any background independent theory that deals with a spacetime microstructure. Verlinde's holographic space scenario would allow such a theory to explain why we live in a classical spacetime, while they fundamentally deal with something quite different [25].

There is some work to be done then, before the emergent paradigm can be fitted into a complete theory of everything. Its main strength is that it would also make sense of the apparent thermodynamic nature of horizons. There are some very intriguing mathematical results, but precisely what they mean for the nature of gravity is still contentious and not fully understood.

A Appendix A

A.1 Flows

For an arbitrary vector field V, there exist **integral curves** $p_i(\lambda)$, defined by the condition that their tangent $\frac{d}{d\lambda}|_p$ at any point p on the manifold is simply V_p, the vector field evaluated at p. In a coordinate chart such that the curve can be written $x^\mu(\lambda)$ it follows that

$$\frac{dx^\mu(\lambda)}{d\lambda} = V^\mu(x(\lambda)) \tag{A.1}$$

where V^μ are the components of the vector field V. Solving for x^μ with initial value $x^\mu(0) = x_0^\mu$ defines a **flow** by

$$x^\mu(\lambda) = \sigma_V^\mu(\lambda, x_0) \tag{A.2}$$

such that λ parameterises a map along an integral curve defined by the vector field V. For a given fixed value of lambda this defines a diffeomorphism σ_V mapping points on the integral curve to points a fixed distance "further along" the integral curve.

A.2 Killing Vectors

If the diffeomorphisms generated by a vector field ξ in the manner described above are symmetries of the metric (isometries), then ξ is known as a **Killing vector field**. Formally, a Killing vector field ξ may be defined by the condition that the Lie derivative with respect that vector field vanishes acting on the metric:

$$\pounds_\xi g_{\alpha\beta} \equiv \lim_{\epsilon\to 0}\left(\frac{\sigma(\epsilon)^* g_{\alpha\beta}|_{p'} - g_{\alpha\beta}|_p}{\epsilon}\right) = 0 \tag{A.3}$$

where $\sigma(\epsilon)^* g_{\alpha\beta}|_{p'}$ is the pullback of the metric two form to the tangent space at p (a diffeomorphism generated by a flow on ξ).
A useful corollary is

$$\xi_{\alpha;\mu\nu} + \xi_{\alpha;\nu\mu} = 0. \tag{A.4}$$

Together with the fact that $[\nabla_\mu, \nabla_\nu]\xi_\alpha = R^\alpha{}_{\mu\nu\sigma}\xi_\alpha\xi^\sigma$ (A.4) implies

$$\begin{aligned} 2\xi_{\alpha;\mu\nu} &= R^\alpha_{\mu\nu\sigma}\xi_\alpha\xi^\sigma \\ \xi^\alpha\xi_{\alpha;\mu\nu} &= \xi^\alpha R\alpha\mu\nu\sigma\xi^\sigma \\ \xi_{\alpha;\mu\nu} &= R_{\alpha\mu\nu\sigma}\xi^\sigma \end{aligned}$$

for any Killing vector ξ.

Another important fact is that if the metric is independent of a coordinate in a spacetime chart, then there necessarily exists a corresponding Killing vector field defined over that chart. For example, the Schwarzchild spacetime has a Killing vector field whose components are given by $t^\mu = \frac{\partial x^\mu}{\partial t}$ and which satisfies $g_{\mu\nu} t^\mu t^\nu = 1 - \frac{2M}{r}$, i.e. it is timelike in the $r > 2M$ region, null at $r = 2M$ and spacelike inside the black hole where $r < 2M$. More generally, if the black hole is rotating (Kerr) we have a Killing vector with components $\xi^\mu = t^\mu + \Omega \phi^\mu$ where $\phi^\mu = \frac{\partial x^\mu}{\partial \phi}$ and ϕ is the usual polar coordinate.

A.3 Surface Gravity

For a Killing vector ξ we may define a quantity κ by:

$$\xi^\mu{}_{;\nu}\, \xi^\nu = \kappa \xi^\mu. \tag{A.5}$$

It follows from Frobenius's theorem [24] that this implies the explicit form of the surface gravity is

$$\kappa^2 = -\frac{1}{2} \xi^{\mu;\nu} \xi_{\alpha;\beta}. \tag{A.6}$$

Physically, the surface gravity of a black hole is the force required at infinity to keep a particle exactly on the horizon, divided by the mass of the particle. The name comes from the fact that this is precisely the acceleration due to gravity experienced at the horizon (or surface of, say, a planet). For the Schwarzchild metric these equations hold for the timelike killing vector t.

B Appendix B

The following discussion is similar, in parts, to both [4] and [24].

B.1 Geodesic Congruence

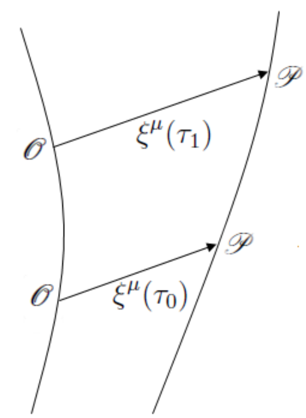

Figure B.1: Two neighbouring geodesics whose separation evolves with τ.

A set of non-intersecting geodesic curves, such that any point in a given region of spacetime lies on exactly one the curves, is known as a **geodesic congruence**. Consider two members of such a congruence as depicted in Fig.(B.1). Let us assume, for now, that we are working with timelike geodesics, parameterised by the proper time τ. $\mathscr{O}$ and $\mathscr{P}$ are separated by the displacement vector with components ξ^μ at points where $\tau = \tau_0$ and $\tau = \tau_1$. The value of ξ^μ depends, therefore, on τ such that

$$\xi^\mu(\tau_1) = \xi^\mu(\tau_0) + \Delta\xi^\mu(\tau_0). \tag{B.1}$$

We may define a tensor $B^\mu{}_\nu$ satisfying

$$\Delta\xi^\mu = B^\mu{}_\nu(\tau_0)\xi^\nu(\tau_0)\Delta\tau + \mathcal{O}(\Delta\tau^2) \tag{B.2}$$

and

$$\frac{d\xi^\mu}{d\tau} = B^\mu{}_\nu(\tau)\xi^\nu + \mathcal{O}(\xi^2) \tag{B.3}$$

if ξ^μ is sufficiently small. Let us now define a (timelike) tangent vector field u^μ such that $u^\mu = dx^\mu/d\tau$ everywhere along the curve. ξ^μ will not, in general, be parallel transported along the congruence for arbitrary evolution of the geodesics. The purpose of $B^\mu{}_\nu$ then, is to capture the nature of the evolution, via

$$\xi^\mu{}_{;\nu}u^\nu = B^\mu{}_\nu\xi^\nu \tag{B.4}$$

where the failure of ξ^μ to be parallel transported is expressed by the non-triviality of $B^\mu{}_\nu$, i.e. that it is non-vanishing in general.

Let us now consider the decomposition of $B^\mu{}_\nu$ into a trace $\theta = B^\mu{}_\mu$, traceless-symmetric and antisymmetric parts respectively:

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{B.5}$$

where $h_{\mu\nu}$ is the transverse part of the metric, satisfying $h_{\mu\nu}k^\nu = 0$ (just the spatial parts if the geodesics are timelike). The quantity θ, and matrices with components $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ are known as the **expansion, shear** and **rotation** respectively.

B.2 Expansion

One can gain a more intuitive understanding by considering, geometrically, a two-dimensional cross-section of the congruence (an horizontal slice in Fig.(B.1)). For simplicity, take the space to be flat so that $h_{\mu\nu} = \delta_{\mu\nu}$ and the shear and rotation to be zero. Then

$$B_{\mu\nu} = \begin{pmatrix} \frac{1}{2}\theta & 0 \\ 0 & \frac{1}{2}\theta \end{pmatrix}. \tag{B.6}$$

Now imagine turning the displacement vector ξ through 2π about $\mathscr{O}$, to trace out a circle of radius $r_0 = \sqrt{\xi^\mu \xi_\mu}$, defined by $\xi^\mu = r_0(\cos\phi, \sin\phi)$, in the plane orthogonal to the geodesic. Combining (B.2) and (B.6) we see that for a perturbation of ξ,

$$\Delta\xi^\mu = \frac{1}{2}\theta r_0 \Delta\tau(\cos\phi, \sin\phi) \tag{B.7}$$

so that the resulting change in area, to first order in $\Delta\tau$ is

$$\begin{aligned} \Delta A &= \pi(r_1^2 - r_0^2) \\ &= \pi\Big((r_0 + \frac{1}{2}\theta r_0 \Delta\tau)^2 - r_0^2\Big) \\ &= \pi\theta r_0^2 \Delta\tau \\ &= \theta A_0 \Delta\tau. \end{aligned} \tag{B.8}$$

From this we understand the name *expansion*, since θ measures the proper rate of change in area,

$$\theta = \frac{1}{A_0}\frac{\Delta A}{\Delta\tau}. \tag{B.9}$$

This can be extended naturally from our heuristic to the case of three spatial dimensions (one of the spatial dimensions in the diagram has been used to represent time). In the case of four-dimensional spacetime, the expansion is given by

$$\theta = \frac{1}{V_0}\frac{\Delta V}{\Delta \tau}, \tag{B.10}$$

where V is just the analogous volume.

B.3 Shear

In the case that the expansion and rotation are vanishing,

$$B_{\mu\nu} = \sigma_{\mu\nu} = \begin{pmatrix} \sigma_+ & \sigma_\times \\ \sigma_\times & -\sigma_+ \end{pmatrix} \tag{B.11}$$

such that $\mathrm{Tr}(B) = 0$ and $B_{\mu\nu} = B_{\nu\mu}$, in accordance with (B.5). This time, combining with (B.2) we have

$$\Delta\xi^\mu = r_0 \Delta\tau(\sigma_+ \cos\phi + \sigma_\times \sin\phi, -\sigma_\times + \sin\phi + \sigma\cos\phi). \tag{B.12}$$

After the perturbation,

$$r_1(\phi) = r_0(1 + \sigma_+ \Delta\tau \cos 2\phi + \sigma_\times \Delta\tau \sin 2\phi), \tag{B.13}$$

which is an ellipse, with its major axis at angle ϕ on fig(B.2).

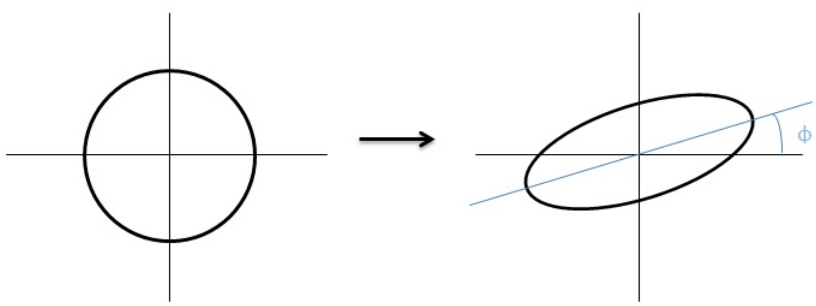

Figure B.2: The effect of shear

B.4 Raychaudhuri's Equation

We now derive an important result, the differential equation describing the evolution of the expansion. From (B.4), it follows that $B^\mu{}_\nu = \nabla_\nu u^\mu$ and thus the evolution equation for $B_{\mu\nu}$ is given by

$$\frac{D(B_{\mu\nu})}{d\tau} = u^\sigma \nabla_\sigma B_{\mu\nu} = u^\sigma \nabla_\sigma \nabla_\nu u_\mu. \tag{B.14}$$

Now using the relation of the commutator of the covariant derivatives to the Riemann tensor we have

$$
\begin{aligned}
u^\sigma \nabla_\sigma \nabla_\nu u_\mu &= u^\sigma (\nabla_\nu \nabla_\sigma u_\mu + R^\rho_{\mu\nu\sigma} u_\rho) \\
&= \nabla_\nu (u^\sigma \nabla_\sigma u_\mu) - (\nabla_\nu u^\sigma)(\nabla_\sigma u_\mu) - R_{\rho\mu\nu\sigma} u^\sigma u^\rho \\
\frac{D(B_{\mu\nu})}{d\tau} &= -B^\sigma{}_\nu B_{\mu\sigma} - R_{\rho\mu\nu\sigma} u^\sigma u^\rho. \qquad \text{(B.15)}
\end{aligned}
$$

The trace of this equation is

$$\frac{d\theta}{d\tau} = -B^{\sigma\nu} B_{\nu\sigma} - R_{\sigma\nu} u^\sigma u^\nu \qquad \text{(B.16)}$$

Recalling that the transverse metric is purely spatial for timelike geodesics, $h_{\mu\nu} h^{\mu\nu} = 3$, so after canceling several terms, one obtains $B^\sigma{}_\nu B_{\nu\sigma} = \frac{1}{3}\theta^2 + \sigma^{\nu\sigma}\sigma_{\nu\sigma} - \omega^{\nu\sigma}\omega_{\nu\sigma}$ where the signs have been fixed by the respective symmetry or antisymmetry in the indices. With this substitution, (B.16) becomes Raychaudhuri's equation for timelike goedesics:

$$\boxed{\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{\nu\sigma}\sigma_{\nu\sigma} + \omega^{\nu\sigma}\omega_{\nu\sigma} - R_{\mu\nu} u^\mu u^\nu} \qquad \text{(B.17)}$$

A similar line of reasoning yields the Raychaudhuri equation for a congruence of null geodesics. The calculation is practically slightly more complicated, in that it is less trivial to define the transverse spacetime (in the above case it was simply the spatial components), but the logic of the derivation and the result are very similar:

$$\boxed{\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^{\nu\sigma}\sigma_{\nu\sigma} + \omega^{\nu\sigma}\omega_{\nu\sigma} - R_{\mu\nu} u^\mu u^\nu} \qquad \text{(B.18)}$$

The only differences are the coefficient in the first term on the RHS due to the fact that the transverse space is two-dimensional and the alternative parameterisation such that the LHS is a derivative with respect to an affine parameter λ rather than the proper time.

B.5 Focusing Theorem

Let us consider an important implication of the Raychauduri equations. For a null[timelike] congruence which is hypersurface orthogonal (vanishing rotation), given the null[strong] energy condition, we have that $R_{\mu\nu} u^\mu u^\nu = 8\pi G (T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}) u^\mu u^\nu \geq 0$ and hence that $\frac{d\theta}{d\lambda} \leq 0$ [$\frac{d\theta}{d\tau} \leq 0$] so the expansion is decreasing with a corresponding increase in the affine parameter. This implies an attractive nature of gravity, manifest in the *focusing* of geodesics. We find from (B.18) that, in the null case, $\frac{d\theta}{d\lambda} \leq -\frac{1}{2}\theta^2$ and from (B.17), in the timelike case, $\frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2$. These imply $\theta^{-1} = \theta_0 + \lambda/2$ and $\theta^{-1} = \theta_0 + \tau/3$ respectively. If the congruence is initially converging, then in finite proper time, the geodesics will necessarily cross at some point, known as a **caustic**.

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