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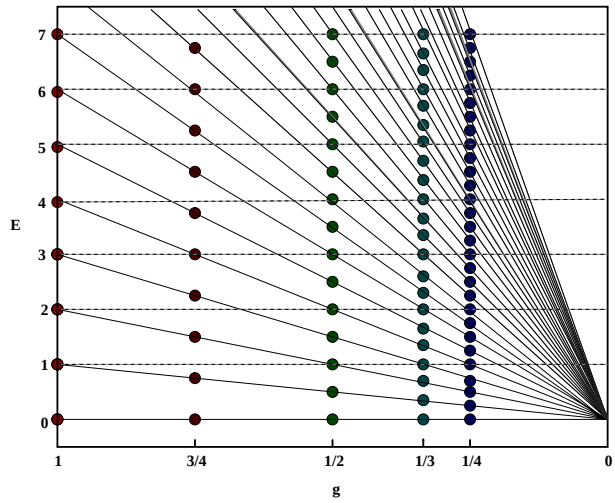
Exclusion Statistics: From Pauli to Haldane

M V N Murthy and R Shankar

The Institute of Mathematical Sciences
Chennai 600 113, India



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Chapter 1

Overview

Quantum statistics provides a way of understanding the statistical mechanics of particles whose dynamical evolution is inherently governed by quantum mechanics. The statistics of all observed particles are covered by the two well known realisations of quantum statistics, namely, the Bose-Einstein (BE) statistics and the Fermi-Dirac (FD) statistics. Particles that obey these statistics are called bosons and fermions respectively. While this is true for all elementary particles which can have arbitrarily large momenta and exist in asymptotically free states, in recent years there has been much interest in the physics of "quasi-particles" obeying *fractional statistics*. Such quasi-particles may correspond to an elementary excitations that can only exist in the interior of a many body interacting system.

A way of characteristing the statistics of identical particles is through their properties under exchange as in the case of ideal fermions and bosons. In arbitrary space dimensions, we characterise the particles as bosons or fermions if under the exchange of any two particles, the wave function of the system remains the same or changes sign. Topological considerations allow us to generalise this definition in two space dimensions. In 1977, Leinaas and Myrheim[1] showed that in two space dimensions it is possible to have particles obeying intermediate statistics different from the well known BE and FD statistics. Later Goldin, Menikoff and Sharp[2] obtained similar results following an algebraic approach using the commutators of particle and current density operators. Wilczek[3, 4, 5] later coined the name *anyons* for particles obeying these peculiar statistics and elaborated on the connection between anyons and charged vortices. The concept of statistical transmutation where anyons may be regarded as interacting bosons (or fermions) also took root through Wilczek's contributions. In the last two decades a lot of work has been done in this field. Many of these developments have been brought up to date and nicely summarized in the books by Lerda[6] and Khare[7] as also in the collection of articles on anyon superconductivity[8] and reviews[9, 10].

Anyons as a research area is now nearly three decades old. These systems have emerged as being interesting in their own right from the point of view of mathematical physics at both the classical and the quantum level. These systems also constitute an example of inequivalent quantizations due to non trivial fundamental group of the configuration space [11]. It is well known that the existence of fractional statistics is intimately connected to having multivalued wavefunctions, in the Schroedinger picture, which naturally occur in quantum mechanics on multiply connected spaces [12, 13]. The classification of multivalued wave functions is provided by one dimensional representations of the fundamental group of the multiply connected configuration space. The configuration space Q^N is the d-dimensional Euclidean space given by R_2^N with all the diagonal points, Δ, removed. The fundamental group of Q^N turns out to be

$$\pi_1(Q^N) = S_N \quad if \quad d \geq 3$$

and

$$\pi_1(Q^N) = B_N \quad if \quad d = 2,$$

where B_N is the braid group of N objects which contains the permutation group S_N as a subgroup. This immediately brings out the difference between two and three dimensions as far as the statistics of particles is concerned. However, most of the studies on anyons in two dimensions have been done in the context of many body quantum mechanics[14, 15, 16, 17, 18, 19, 20, 21] using both numerical[22, 23, 24, 25] and perturbative method [26, 27, 28, 29]. Though there have been few attempts at studying the statistical properties of the system[30, 31, 32, 33, 34, 35, 36, 37], a general formulation of quantum statistical mechanics of ideal gas of anyons is still lacking.

Anyons are particles whose many particle wave functions pick up a phase $e^{i\alpha\pi}$ under the exchange of any two particles. Standard bosons and fermions correspond to the case when $\alpha = 0, 1$. However arbitrary values of α are allowed by the configuration topology of the many particle system in one and two spatial dimensions[1]. The parameter α is called the statistics of the particle. Here after we refer to α as the *fractional exchange statistics* parameter. It is also well known that $\alpha = 1$, implies that the wave functions are antisymmetric under exchange which inturn implies the Pauli exclusion principle. For $\alpha = 0$, the wave function is symmetric and no restriction operates. The existence of such a principle has nontrivial consequences for the counting of states in a many body system. The question may then be raised as to whether this correspondence between statistics and exclusion principle be exploited to define fractional statistics through a generalised exclusion principle? Indeed the answer is yes. As we shall demonstrate in the following pages, a general formulation of the statistical mechanics of ideal gas that interpolates between bosons and fermions also emerges in the process.

In a seminal paper[38], Haldane proposed an alternate definition of statistics based on a generalization of the Pauli exclusion principle. Haldane's definition was motivated by physical examples such as quasi-particles in fractional quantum Hall systems and spinon excitations in one-dimensional antiferromagnetic chains. We follow closely the original paper where the idea was first introduced. As originally conceived, the definition applies to particles with finite dimensional Hilbert spaces.

Consider the wave function of a N identical particle system. Let the wave function be $\psi(r_1, r_2, ..., r_N)$ in arbitrary space dimension. Let us fix the coordinates of the $N-1$ particles and expand the many body wave function in a basis of of the wave function of the i-th particle, that is,

$$\psi(r_1, r_2, \cdots, r_i, \cdots, r_N) = \sum_\nu A_\nu(\{r_j; j \neq i\})\phi_\nu(r_i; \{r_j; j \neq i\}), \tag{1.1}$$

where ϕ_ν span a one dimensional Hilbert space whose dimension is taken to be d_N. Since we have an identical particle system, this dimension is independent of label i but will depend on the boundary conditions and the size of the system in which it is confined. Thus d_N is extensive and may be finite. It will change depending on the change in the number of particles, in general, for any interacting system with all the other conditions remaining the same.

To clarify the definition, consider particles on a lattice (or a set of discrete quantum states) of dimension d which is kept fixed. If the system contains N identical fermions, then a single fermion can occupie any of the $d-(N-1)$ available single particle state due to Pauli blocking. Thus $d_N^F = d-(N-1)$ denotes the dimension of available fermionic single particle state. On the other hand if the particles are boson, then $d_N^B = d$ since there is no Pauli blocking. We may then consider a generalised Pauli blocking by defining a differential

relation

$$\Delta d_N = -g\Delta N, \tag{1.2}$$

where ΔN is the allowed change in the particle number and g is a parameter. Thus the generalised Pauli principle implies

$$d_N^g = d - g(N-1), \tag{1.3}$$

where d_N^g is the number of states accessible to a particle in the presence of $N-1$ other particles. In effect g states are blocked. The crucial point here is that the number of single particle states given by d_N^g is not a constant as given by d. It depends on the number of particles in the system N. This is expected when localised particle states are non-orthogonal. For conventional bosons $g = 0$ and fermions $g = 1$ but in general g may be arbitrary. *Thus the relation (1.2) may be considered as a generalisation of the Pauli principle.* For Thermodynamic limit to exist, g must be independent of the number of particles. While interactions in general cause changes in the single particle level structure, only some very special type of interactions leave g independent of the number of particles and such interactions are necessarily *statistical interactions*. In all such cases we may regard g as a statistical parameter.

Even before the exclusion statistics was introduced by Haldane, the germ of the concept of exclusion statistics was already evident in systems of particles interacting by a pair potential of the form $g(g-1)/r^2$ in one dimension. Here r is the relative distance between particles. Since the wave function vanishes as $\psi \to r^g$ at short distances, we have $g = 0$ for bosons and $g = 1$ for fermions. Interestingly, the spectrum of the interacting case for $g = 2$ could be obtained by requiring that the particles neither occupy the same orbital nor neighbouring orbital in momentum space[39, 40, 41]. This in a sense is generalised Pauli principle, more exclusive than the exclusion principle for fermions. Infact the one dimensional inverse square model is a template for an exact realisation of FES.

Notice that we never had an occasion to refer to the dimension of the space in the foregoing analysis. Thus the generalised Pauli principle, as formulated by Haldane, is valid in arbitrary space dimensions unlike the fractional exchange statistics of anyons which is realised only in two dimensions. We refer to the quantum statistics arising from a generalised exclusion principle as *Fractional Exclusion Statistics*(FES) to distinguish it from the exchange statistics formalism.

We now consider the space of N- identical particles based on the generalised exclusion principle. For fermions the size of the full Hilbert space, that is the number of quantum states of N identical particles occupying d states, is given by

$$D_N^F = \frac{d!}{N!(d-N)!} = \frac{(d_N^F + N - 1)!}{N!(d_N^F - 1)!}. \tag{1.4}$$

Similarly for bosons we have

$$D_N^B = \frac{(d+N-1)!}{N!(d-1)!} = \frac{(d_N^B + N - 1)!}{N!(d_N^B - 1)!}. \tag{1.5}$$

A straightforward generalisation for a fixed N of the system invoking FES is

$$D_N(g) = \frac{(d_N^g + N - 1)!}{N!(d_N^g - 1)!} = \frac{(d + (1-g)(N-1))!}{N!(d - gN - (1-g))!} \tag{1.6}$$

which reduces to (1.4) for $g = 1$ (fermions) and (1.5) for bosons as it should. The relations (1.2) and (1.6) are the central relations that provide the basis for developing the statistical

mechanics of an ideal gas obeying the FES or Haldane statistics. In related development, Ramanathan [42] independently proposed a generalised interpolative quantum statistics based on a certain reordering of phase space achieved through Bose-counting strategy. Though the aim was to predict the existence of an infinite quantum Boltzmann-Gibbs statistics akin to the infinite statistics of Greenberg[43], its essential content is similar to the Haldane proposal. Thus the exclusion statistics had its origins in the proposal by Haldane[38] and, rather less known but equally important proposal by Ramanathan[42].

To demonstrate how such statistics could arise in an interacting model [44] we discuss the properties of a confined system. Let us, for simplicity, consider particles confined in an external harmonic potential in one space dimension. The energy of the N-particle system is then given by

$$E = \hbar\omega \left[\sum_i n_i + \frac{N}{2} \right], \tag{1.7}$$

where ω is the oscillator frequency. The occupation numbers n_i are ordered set of non-negative integers. For fermions these integers are atleast one unit apart. Suppose the interactions modify the spectrum in such a way the new energy is given by

$$E = \hbar\omega \left[\sum_i n_i + g\frac{N(N-1)}{2} + \frac{N}{2} \right]. \tag{1.8}$$

Indeed such a body shift of the energy does happen in a class of exactly solvable models in one dimension which will be considered in some depth later. For now it is sufficient to note that the total energy can be rewritten as

$$E = \hbar\omega \left[\sum_i \bar{n}_i + \frac{N}{2} \right], \tag{1.9}$$

where $\bar{n}_i$ are the modified excitation numbers given by

$$\bar{n}_i = n_i + g(i-1) \tag{1.10}$$

We note that the equation (1.9) is identical to the non-interacting energy of N particles but where excitation quanta are given by $\bar{n}_i$. Further the distance between excitation quanta is given by $\bar{n}_{i+1} - \bar{n}_i \geq g$, that is the modified excitation quanta are atleast a distance g apart resembling the distance between single particle levels as given in eq.(1.3).

We may therefore regard a model which has a spectrum given by eq.(1.8) as a system of interacting bosons (or fermions) or equivalently, as a system of non-interacting quasi-particles whose excitation quanta are modified such that they are a distance g apart. Thus at zero temperature when all levels are occupied up to some Fermi energy, each cell of width $\hbar\omega$ has an occupancy of g quasi-particles. Interestingly, removing a particle from one of these cells, leaves g holes. Thus the particle hole symmetry acquires a new meaning in the sense that the symmetry also requires changing g to $1/g$ which is often referred to as duality property in the context of exclusion statistics. We will discuss this in more detail later.

We should mention that there is a rich body of work on FES and its implications. Fractional Quantum Hall Systems (FQHE) are of course an example that is well known. Haldane's idea was first tested numerically for the Fractional Quantum Hall systems(FQHE) by Johnson and Canright[45]. The normal state of the cuprate superconductors is also believed to be a possible system where FES may occur. However the theory of these two dimensional systems is as yet incomplete and inconclusive. The theory of interacting electron systems in one dimension is unambigious. Fermi liquid theory does break down and

such quasiparticles do exist. The concurrence of exclusion statistics and non-Fermi liquid phases has been stated by Anderson[46] in the context of the normal state of the cuprate superconductors. Thus we expect FES to occur in quantum wires, if the correlations play an important role in their physics. The isotropic quantum antiferromagnetic chain is another physically realisable model. This model can be mapped on to the Luttinger model by the Jordan-Wigner transformation. The low energy physics is then described by a Luttinger liquid with radius parameter equal to $1/\sqrt{2}$. Later on we show that the system therefore would have quasiparticles with the exclusion statistics parameter $g = 1/2$, that is the quasi particles are in fact spinons. The overscreened Kondo impurity systems are another class of models where such quasiparticles are expected to exist. The solution of these models using conformal field theory (CFT)[47] predict nontrivial quantum numbers for quasiparticles. In particular, for the spin $s = 1/2$ which may have a physical realisation[48], where quasiparticles with $g = 1/2$ are predicted. Experimentally there is also evidence from neutron inelastic scattering experiment[49] on the compound $KCuF_3$, which is regarded as a one dimensional Heisenberg antiferromagnet above the temperature 40 K. The observed inelastic scattering is best fitted by spinon excitations in a spin chain whose pairwise interaction falls off as the inverse square of the lattice spacing [50]. The dynamic correlation function for such a system has been calculated by Haldane and Zirnbauer[51]. The many body states and operator algebra for exclusion statistics was dealt in detail in [52, 53].

In the following pages we develop the quantum statistics as viewed from the generalised Pauli principle of Haldane. The central idea of Haldane that such a principle be applied to elementary excitations that can only exist in the interior of a region of matter is developed further by deriving the thermodynamic properties of an "ideal gas" consisting of such quasi-particles. We discuss an earlier work by us[54] in which we had generalized Haldane's definition to the case where the Hilbert space of the particles is infinite dimensional. We showed that the statistics parameter g is determined by the high temperature limit of the second virial coefficient. This was then applied to the case of exchange anyons in two dimensions and to the case of quasiparticles in the Luttinger model. We derive the distribution function of such quasi-particles based on important contributions from Ramanathan[42], Wu[55] and Isakov[56] who independently derived the distribution function obeyed by particles with FES. Earlier in the specific case of particles occupying the lowest Landau level, the same distribution has been derived by Ouvry[57]. It is now known that the anyons occupying the lowest Landau level obey ideal FES in the sense it was defined by Haldane.

In most of the cases cited above, the existence of fractional exclusion statistics was assumed and consequences worked out. We look at the microscopic origins of FES and its realisation in some exactly solvable models. It turns out that in one-dimension at least there exist a class of models of interacting fermions which can be looked upon as a model of noninteracting anyons obeying FES. These are the well known Calogero-Sutherland model (CSM) where the particles interact via an inverse square interaction in some confining potential[39, 40, 41] and quasiparticles in Luttinger Model[58, 59]. That the particles of CSM obey FES has been shown by several authors recently[56, 60, 61, 62]. Furthermore we have argued that the quasiparticles of the Luttinger model also obey FES[54]. Infact just as the Hamiltonian of the particle-flux composite describes the anyons in two dimensions, the Hamiltonians in these one dimensional models provide the framework for fractional statistics particles in one dimension.

In any interacting many body system, the interaction causes the shift in single particles levels. However, a crucial property of exclusion statistical interactions is that they should cause shifts in single particle energies at all scales[54]. This property is realized by a large class of one dimensional models of interacting fermions where Fermi liquid theory breaks down [46, 63]. In fact it has been shown exactly that quasiparticles with nontrival exclusion

statistics exist in a class of models that are solved by the Bethe ansatz [56, 60]. In particular the quasiparticles of the Calogero-Sutherland model (CSM) behave like ideal exclusion statistics system [56, 60, 61, 62].

A feature of the exclusion statistics as gleaned from the analysis of various models is that the exclusion acts across a set of levels unlike in the case of Fermi or Bose statistics where the exclusion principle is stated with a single level in mind. It is this crucial difference that results in the occurance of the so called negative probabilities[64, 65]. A microscopic interpretation of exclusion statistics systems has been discussed by Chaturvedi and Srinivasan[66], where they show how this problem of negative probabilities may be solved for semions, $g = 1/2$. They have also indicated how their method may be generalised to other values of g. We do precisely that next and show that the particles obeying fractional exclusion statistics can be characterised by constraints on the sets of occupation numbers[67]. There are no negative probabilities if these constraints are obeyed. If these constraints are relaxed then the negative weights arise in order to compensate for the resulting over counting. Indeed this is the way we encounter negative probabilities in other systems in physics- for example in gauge theories, they arise in the ghost sectors. Ghosts come from the Jacobian associated with nonlinear gauges which essentially ensure the correct counting of states. Another example is that of the Wigner distribution function in quantum mechanics which is not positive definite precisely because some constraints are relaxed. A formulation based on the variable number of single particle states, which depends on the total number of particles in the system, has been discussed by Isakov[68] as a way to avoid the problem of negative weights.

The discussions in the following pages include many aspects of FES mentioned above and more. The central ideal of FES, that it provides a frame-work to describe in a rather simple way a certain class of strongly interacting systems is developed further. There may be many important contributions which may not be covered in these pages. What is included is partly based on our understanding of what is important in this field and partly due to our own work forming the basis of this presentation. Obviously it is almost impossible to cover all the of work in FES that has followed Haldane's seminal work. If some work of importance in the field is left out it is partly due to our ignorance or lack of understanding.

Chapter 2

Statistical mechanics of FES particles

The original definition of g is given in eq.(1.2) interms of the effective single particle dimensions embedded in a finite dimensional many particle Hilbert space. While this is a useful definition for systems in which the many particle spectrum admits an interpretation in terms of an effective single particle level spectrum, in general it may not be a very useful definition. Furthermore for a given single particle space d and number of particles N, the dimension of the many particle space is an integer only for some rational values rational of g. In this chapter we first generalise the definition given by Haldane to infinite dimensional spaces. In doing so we show that the statistics parameter g can be related to the second virial coefficient in the equation of state under certain conditions. The distribution function of an ideal gas obeying FES and its thermodynamic properties are derived in a manner that we recover the properties of ideal Bose and Fermi gas at $g = 0$ and $g = 1$ respectively. Thus the average occupancy of a state in the thermodynamic limit can be defined in terms of the statistical parameter $g > 0$.

2.1 Partition function in large d-limit

As noted in the introduction, the number of quantum states of an N identical particle system occupying d states is given by

$$D_N(g) = \frac{(d + (1-g)(N-1))!}{N!(d-1-g(N-1))!}. \tag{2.1}$$

The FES parameter g ensures that there is smooth and simple interpolation between bosonic limit $D_N^B = D_N(g = 0)$ and the fermionic limit $D_N^F = D_N(g = 1)$. In principle g may be greater than 1. We note that the above generalisation is a matter of definition for arbitrary values of g and is consistent with Haldane formulation of FES. Furthermore, the combinatorial expression introduced above is only a way of realising the original definition of g given by eq.(1.2) which admits more than one interpretation in terms of the dimension of the many particle states[65, 69, 70, 71]. An alternative proposal is discussed in Appendix A.

We now generalise Haldane's definition of FES as given above to the case where the Hilbert space of the particles is infinite dimensional[54]. We are therefore interested in the limit when the single particle dimension $d \to \infty$, and the number of particles N is finite such that $N/d \to 0$. Often this limit may be confused as the zero density limit. To clarify what this limit implies, assume that the particles are spread on a lattice of size L (per dimension) with a spacing a. Then $d = L/a$ and the limit $d \to \infty$ would imply $L \to \infty$ (infinite volume) or the lattice spacing $a \to 0$. Since the density is $\rho = N/L^\eta$, where η is the space

dimension, it is clear that $\rho \to 0$ when $L \to \infty$ or ρ is finite when $a \to 0$ (continuum limit). In what follows we are interested in the continuum limit when the density is finite and seek a definition of the exclusion statistics in this limit.

Expanding the many particle dimension in powers of single particle dimension d, we have,

$$D_N(g) = \frac{1}{N!}[d^N + N(N-1)(\frac{1}{2} - g)d^{N-1} + ...]. \tag{2.2}$$

In the limit $d \to \infty$, it is easy to see that

$$\frac{1}{2} - g = \lim_{d\to\infty} \frac{d}{N(N-1)} \left[\frac{N!\ D_N(g)}{d^N} - 1\right]. \tag{2.3}$$

While in the above we have obtained a relation that determines g, it is not yet useful since $D_N(g)$ is not a meaningful quantity for systems in continuum. However, recall that a regulated definition of the dimension of the Hilbert space is given by the N-particle partition function,

$$D_N = \lim_{\beta\to 0} Z_N = \lim_{\beta\to 0} Tr(e^{-\beta H_N}), \tag{2.4}$$

where $\beta = 1/kT$ is the inverse temperature and H_N is the N-particle Hamiltonian. Thus for dealing with the infinite dimensional Hilbert space for systems in continuum in arbitrary space dimensions, we propose a definition of g as a generalization of the Haldane's definition-

$$\frac{1}{2} - g = C \lim_{\beta\to 0} \frac{Z_1}{N(N-1)} \left[N!\frac{Z_N}{Z_1^N} - 1\right]. \tag{2.5}$$

Physically it is easy to see that at high temperature we are sampling infinitely many states and is thus equivalent to taking to the continuum limit.

The constant C is an overall constant of proportionality which should be fixed so as to have a consistent definition of g. In general C depends on the system geometry and the dimension of space but should be independent of statistics . For illustration assume the particles to be non-interacting fermions or bosons. The canonical partition function for N particles [36] in arbitrary space dimensions, is given by

$$Z_N^{B,F} = (\pm)^N \sum_{\mathcal{P}} \prod_l \frac{1}{n_l!} [\pm \frac{Z_1(l\beta)}{l}]_l^n, \tag{2.6}$$

where the $\pm$ signs correspond to bosons or fermions and the sum over $\mathcal{P}$ is given as the number of partitions of N such that $\sum_{l=1}^N n_l l = N$, n and l being positive integers. Since we are interested in the high temperature limit, it is sufficient to expand Z_N to the next to leading order in powers of Z_1, that is

$$Z_N^{B,F}(\beta) = \frac{1}{N!} \left[Z_1^N(\beta) \pm Z_1(2\beta) Z_1^{N-2}(\beta) \frac{N(N-1)}{2} + ...\right]. \tag{2.7}$$

A simple derivation of this expansion is given in Appendix B. Notice that the second term in eq.(2.7) involves $Z_1(2\beta)$ since the contribution to this term comes when two particles are in the same energy level. Substituting eq.(2.7) in eq.(2.5) and taking the high temperature limit we obtain

$$\pm\frac{1}{2} = C \lim_{\beta\to 0} \frac{Z_1}{N(N-1)} \left[N!\frac{Z_N^{B,F}}{Z_1^N} - 1\right] \quad \Longrightarrow \quad C = \lim_{\beta\to 0} \frac{Z_1(\beta)}{Z_1(2\beta)}. \tag{2.8}$$

In determining C we made use of the fact $g = \mp 1$ for Boson or Fermion. In either case the constant C is the same and infact it should be the same for all g for a consistent definition of the statistics parameter. For example for a particle confined in a box in η dimensions we have $Z_1(\beta) = (L/\lambda)^\eta$, where $\lambda = \sqrt{2\pi\hbar^2\beta/m}$ is the thermal wave length. Therefore the constant $C = 2^{\eta/2}$.

It is easy to see why the g given by the state counting definition, eq.(2.3), and through a regulated definition involving partition functions, eq.(2.5), should differ by a proportionality factor C. Notice that the expansion of D_N involves the single particle space dimension d. The analogous expansion for $Z_N(\beta)$ should therefore be interms of $Z_1(\beta)$ since $d = \lim_{\beta\to 0} Z_1(\beta)$. In actual fact, however, the expansion for $Z_N(\beta)$ involves, in the next-to-leading-order term which is relevent here, the factor $Z_1(2\beta)$.

2.2 Harmonically confined systems

To test the utility of eq.(2.5), we first consider the case of identical particles in η space dimensions. To be specific consider a system of N particles of mass m confined in an oscillator potential given by

$$V(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) = \frac{1}{2} m\omega^2 \sum_{i=1}^{\infty} r_i^2, \tag{2.9}$$

where ω is the oscillator frequency and $\vec{r}_i$ are single particle coordinates. We emphasise that the choice of the confining potential is not unique but merely acts as a regulator.

Using the spectrum of a particle moving in an oscillator potential in two dimensions,

$$E_{n,l} = \hbar\omega(2n + |l| + 1),$$

where n is the radial quantum number and l is the angular momentum quantum number in two-dimensions. The single particle partition function in η space dimensions is given by

$$Z_1(\beta) = \frac{e^{-\beta\hbar\omega\eta/2}}{(1 - e^{-\beta\hbar\omega})^\eta}. \tag{2.10}$$

The N-particle partition function is then given by eq.(2.6) as before. Substituting eq.(2.7) and eq.(2.10) in eq.(2.5) it is easy to show that for the oscillator potential

$$\frac{1}{2} - g = \pm\frac{C}{2}\frac{1}{2^\eta}. \tag{2.11}$$

Since $g = 0, 1$ for bosons and fermions once again we have a consistent definition of C given by

$$C = \lim_{\beta\to 0} \frac{Z_1(\beta)}{Z_1(2\beta)} = 2^\eta, \tag{2.12}$$

where η is the space dimension as noted before. Note that this value of C is different from the one obtained for particles in a box.

We are now in a position to apply the definition of g to systems of identical interacting particles in the continuum limit.

> **Theorem:** In the high temperature limit the statistical parameter g is entirely determined by the second virial coefficient alone provided all the virial coefficients in the virial expansion of the equation of state are finite.

Proof: For simplicity we consider all particles to be confined in an oscillator potential as mentioned before. A glance at the definition of g through eq.(2.5) for two particles ($N = 2$), immediately suggests that g in this case is determined by the second virial coefficient in the virial expansion of the equation of state of the sytem. We will now elaborate on this connection. To begin with consider a system of interacting particles in two space dimensions so as to have well defined factors throughout. (The proof can be generalized to arbitrary space dimensions.) The grand partition function of the system is given by

$$Z_G(z, V, \beta) = \sum_{N=0}^{\infty} z^N Z_N(V, \beta); \quad z = e^{\beta\mu}, \tag{2.13}$$

where z is the fugacity parameter and μ is the chemical potential of the system. In the dilute limit, the equation of state of the system in fugacity expansion may be written as[72]

$$\begin{aligned} \beta P &= \lim_{V\to\infty}[\frac{1}{V}\ ln(Z_G)] = \sum_{l=1}^{\infty} b_l z^l, \\ \rho &= \lim_{V\to\infty}[\frac{1}{V}\ z\frac{\partial ln(Z_G)}{\partial z}] = \sum_{l=1}^{\infty} l b_l z^l, \end{aligned} \tag{2.14}$$

where b_l's are the expansion coefficients determined completely by $Z_1, Z_2, \cdots, Z_l$ using eq(2.13) in eq(2.14) and P, ρ respectively refer to the pressure and density in the dilute limit. Thus the general expression for b_l can be obtained and is given by

$$b_l = (Z_1)^{l-1} \sum_{\{m_i\}} (-1)^{(\sum_i m_i - 1)} (\sum_i m_i - 1)! \prod_i \frac{(Z_i/Z_1^i)^{m_i}}{m_i!}. \tag{2.15}$$

The summation over m_i is constrained by

$$\sum_{i=1}^{l} i m_i = l.$$

Using eq.(2.15) we have, for example

$$\begin{aligned} b_1 &= 1 \\ b_2 &= \frac{Z_1}{2}(2\frac{Z_2}{Z_1^2} - 1) \\ b_3 &= \frac{Z_1^2}{3}(3\frac{Z_3}{Z_1^3} - 3\frac{Z_2}{Z_1^2} + 1). \end{aligned} \tag{2.16}$$

We may also write the equation of state as an expansion in powers of average density of the system, $\rho = N/V$ by eliminating z using eqs.(2.14), as

$$\beta\frac{P}{\rho} = \frac{\sum_l b_l z^l}{\sum_l l b_l z^l} = \left[1 + \sum_{l=1}^{\infty} a_{l+1}(\rho\lambda^2)^l\right], \tag{2.17}$$

where $\lambda = \sqrt{2\pi\hbar^2\beta/m}$ is the thermal wave length. The virial coefficients a_l are related to the coefficients b_k, some times called cluster expansion coefficients[72], in the fugacity expansion

$$a_1 = b_1 = 1; \quad a_2 = -b_2; \quad a_3 = 4b_2^2 - 2b_3; \quad \text{etc.} \tag{2.18}$$

Thus the high temperature limit of the virial expansion of the equation of state exists if all the virial coefficients are finite in this limit. In order to proceed with the computation let us assume that all the a_l's are finite and hence the virial expansion exists in the high temperature limit. To relate these virial coefficients to the statistics parameter g, first consider an expansion for the factor Z_N/Z_1^N which appears in eq.(2.5). In the high temperature, or $\beta \to 0$, limit we may write

$$\frac{Z_N}{Z_1^N} = \frac{1}{N!} + f_2^{(N)}(\beta\hbar\omega)^2 + f_3^{(N)}(\beta\hbar\omega)^4 + ..., \tag{2.19}$$

where $f_k^{(N)}$ are coefficients in the expansion which are as yet undetermined. The superscript N indicates that these are expansion coefficients corresponding to Z_N. They are in general functions of the interaction strength and may also depend on N. Notice that the high temperature expansion of Z_N/Z_1^N involves expansion in powers of $(\beta\omega)^2$ since we considering N particles confined in an oscillator potential in two space dimensions. In other dimensions this factor is appropriately modified while the proof goes through. The leading term is simply the classical term. Substituting the expression for Z_N/Z_1^N in eq.(2.19) in eq.(2.15), we obtain

$$b_l = \frac{1}{(\beta\hbar\omega)^{2(l-1)}}[\sum_{n=0}^{l-2} \frac{f_2^{(l-n)}}{l!}(-1)^{l-1}(\beta\omega)^2 + ...]. \tag{2.20}$$

If we now demand that b_l 's are finite, hence all virial coefficients a_l are finite, for all l in the high temperature limit, all the coefficients in above expansion up to $(\beta\omega)^{2(l-2)}$ must vanish. Therefore

$$\sum_{n=0}^{l-2} \frac{f_2^{(l-n)}}{l!}(-1)^{l-1} = 0, \tag{2.21}$$

which is the term of relevence to us. Using the above identity it is easy to see that

$$f_2^{(3)} = f_2^{(2)}; \quad f_2^{(4)} = f_2^{(3)} - f_2^{(2)}/2 = f_2^{(2)}/2; \quad \text{etc} \tag{2.22}$$

Therefore the general expression for $f_2^{(N)}$ is

$$f_2^{(N)} = \sum_{n=1}^{N-2} \frac{f_2^{(N-n)}}{n!}(-1)^{(n-1)} = N(N-1)\frac{f_2^{(2)}}{N!}, \tag{2.23}$$

where all $f_2^{(N)}$ are now related to $f_2^{(2)}$ corresponding to the two particle partition function. Substituting eq.(2.23) and eq.(2.19) in eq.(2.5), we find

$$\frac{1}{2} - g = 4f_2^{(2)}, \tag{2.24}$$

where we have used the value of C which is equal to 4 in two space dimensions with harmonic oscillator potential. In order to calculate $f_2^{(2)}$ recall the definition of the second virial coefficient a_2[72, 73, 74]

$$a_2 = \frac{Z_1}{2}\left[1 - 2\frac{Z_2}{Z_1^2}\right]. \tag{2.25}$$

Substituting for Z_2/Z_1^2 we find

$$a_2 = -b_2 = -4f_2^{(2)}. \tag{2.26}$$

We therefore now have a unique relation between the FES parameter g and the second virial coefficient a_2 in the high temperature limit,

$$\frac{1}{2} - g = -4a_2. \tag{2.27}$$

A few remarks are in order here. Notice that in order to determine $f_2^{(N)}$ in terms of $f_2^{(2)}$ it is sufficient to demand that the coefficient of the leading divergence in b_l be zero in the high temperature limit. This will uniquely fix the statistics parameter g interms of a_2 and is a much weaker condition than demanding all b_l's and hence all the virial coefficients a_l's are finite . Nevertheless, there is no way the weaker condition can be imposed in physical systems except to demand the stronger condition that all the virial coefficients be finite in the high temperature limit. The Haldane exclusion statistics parameter g is completely determined by the high temperature limit of the second virial coefficient of the system of interacting particles in the continuum which admits a virial expansion in this limit.

Notice that the final result is independent of the oscillator potential that was used as a regulator since the oscillator parameter ω always occurs in combination with β as $\beta\omega$. Thus the high temperature limit and $\omega \to 0$ limits can not be separated. In other words, the dilute limit corresponding to $\omega \to 0$ is the same as imposing the high temperature limit in the particular case of oscillator regularisation.

2.3 Distribution function for ideal gas of FES

One of the main difficulties with anyons introduced through exchange statistics is that we still do not know the statistical mechanics of an anyon gas. The difficulty mainly stems from the fact that the quantum mechanics of many anyons is still not a completely solved problem. Unlike the case of anyons, Haldane's approach of generalising the Pauli exclusion principle independent of the space dimension, allows us to define the statistical mechanics of an ideal gas of FES particles. The formulation of statistical mechanics of a gas of particles obeying fractional exclusion statistics is due to the important contributions by Ramanathan[42], Wu[55] and Isakov[56]. Indeed the distribution function of particles obeying ideal FES is also the distribution function of a gas of anyons confined to the lowest Landau level as was shown earlier by Dasniéres de Veigy and Ouvry[57]. The interpretation of exclusion statistics for anyons will be discussed later where this connection will be shown explicitly.

In the following we follow closely the derivation of the distribution function as given by[1] Wu[55]. We consider a system of non-interacting identical particles with total energy E confined to a volume V which is very large. Since both the number of particles and the volume is very large, we may divide the energy spectrum in to groups of levels or *cells* each with an arbitrary number of single particle levels $d_i >> 1$ and an average energy ϵ_i in the ith cell. A given distribution of particles may have N_i particles in the ith cell. An ideal gas, by definition, is such that the distribution conforms to

$$E = \sum_i N_i \epsilon_i; \quad \sum_i N_i = N, \tag{2.28}$$

where the energy is simply a sum of the single particle energies.

One may then consider, following the standard procedure[72], the grand canonical ensemble at a temperature T and with chemical potential μ. The grand partition function is given by

$$Z = \sum_{\{N_i\}} D(\{N_i\}) \exp[\sum_i N_i \beta(\mu - \epsilon_i)], \tag{2.29}$$

where $\beta = 1/kT$ and k is the Boltzmann constant,

$$D(\{N_i\}) = \prod_i \frac{(d_i + (1-g)(N_i-1))!}{N_i!(d_i - 1 - g(N_i-1))!} \tag{2.30}$$

[1]The original derivation given by Wu is even more general than the one given here including many species and accounting for possible mutual statistics.

is the number of distinct microstates associated with the distribution set $\{N_i\}$. When d_i and N_i are large, we expect the sum in eq.(2.30) to have a sharp peak around the set of most probable particle numbers $\{N_i\}$. The average *occupation number* in the ith cell is defined $n_i = N_i/d_i$ and using the Stirling formula for evaluating the factorials we have

$$\begin{aligned}\log D(\{N_i\}) &\approx \sum_i d_i[-n_i \log n_i - (1-gn_i)\log(1-gn_i)] \\ &\quad +[1+(1-g)n_i]\log[1+(1-g)n_i]. \end{aligned} \tag{2.31}$$

The most probable distribution is determined by the condition

$$\frac{\partial}{\partial n_i}[\log D(\{N_i\}) + \sum_i d_i n_i \beta(\mu - \epsilon_i)] = 0. \tag{2.32}$$

Thus we have

$$n_i \exp[\beta(\epsilon_i - \mu)] = [1+(1-g)n_i]^{1-g}[1-gn_i]^g. \tag{2.33}$$

Following Wu[55], we define the function

$$w = \frac{1}{n} - g \tag{2.34}$$

and the statistical distribution for a single species is given by the average occupation number

$$n_i = \frac{1}{w(e^{\beta(\epsilon_i - \mu)}) + g}, \tag{2.35}$$

where the function $w(x)$ satisfies the functional equation

$$w(x)^g(1+w(x))^{(1-g)} = x = \exp[\beta(\epsilon - \mu)]. \tag{2.36}$$

Note that the eq.(2.36) obeyed by the Wu function w is highly non-linear for arbitrary g. However, it is easy to see that the distribution function reduces to the Bose and Fermi cases for $g = 0$ and $g = 1$ respectively. Boltzmann distribution is recovered when $x = \exp[\beta(\epsilon - \mu)]$ is very large such that $w(x) = x$ and neglecting g compared to w we have

$$n = \exp[-\beta(\epsilon - \mu)] \tag{2.37}$$

as to be expected. This is the limit at low densities for any statistics. It is also obvious that since x is non-negative, so is w and infact w is a monotonically increasing function of x . At zero temperature, it follows from eq.(2.35) that

$$\begin{aligned} n_i &= 0, \quad \text{if} \quad \epsilon_i > E_F; \\ n_i &= 1/g, \quad \text{if} \quad \epsilon_i < E_F, \end{aligned} \tag{2.38}$$

where E_F is the Fermi energy (chemical at zero temperature). In general at arbitrary temperatures

$$n_i \leq \frac{1}{g} \tag{2.39}$$

which is an expression of the generalised exclusion principle for exclusion statistics. It is also quite striking that at $T = 0$, FES particles exhibit a *Fermi* surface. This fact dictates the low temperature thermodynamics of the system when the number of particles is fixed. In particular it also implies that the system does not condense for any finite value of g however small it is [75].

For $g = 1/2$ it reduces to a simple form

$$n_i = \frac{1}{\sqrt{1/4 + \exp[2\beta(\epsilon_i - \mu)]}}. \tag{2.40}$$

Aoyama[76] has obtained exact solutions for $g = 1/m$ where $m > 1$ is a positive integer in terms of hypergeometric function given by

$$\begin{aligned} w(g,x) &= (1 - \frac{1}{m})[{}_{m-1}F_{m-2}(\frac{1}{m}, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-2}{m}; \frac{1}{m-1}, \frac{2}{m-1}, \ldots, \\ &\quad \frac{m-2}{m-1}; \frac{-mx^m}{(m-1)^{m-1}}) - 1] \end{aligned} \tag{2.41}$$

It is also interesting to note that there is a duality property[64, 77], reminiscent of the one that is present in Chern-Simons models. This property relates the statistics with g and $1/g$. For example using eq.(2.33) for g and $1/g$, it can be shown that

$$1 - g\, n_g(\beta(\epsilon - \mu)) = \frac{1}{g} n_{1/g}(-\frac{\beta}{g}(\epsilon - \mu)) \tag{2.42}$$

2.4 Properties of a homogeneous gas

Now consider an ideal gas of particles obeying exclusion statistics in η space dimensions confined in a volume V which is large. The volume element may be written as

$$d^\eta p = \rho(\epsilon) d\epsilon, \tag{2.43}$$

where $\epsilon = p^2/2m$ and

$$\rho(\epsilon) = \frac{(2m\pi)^{\eta/2}}{\Gamma(\eta/2)} \epsilon^{\eta/2-1} \tag{2.44}$$

denotes the density of states of a free gas of particles with mass m. At zero temperature the number of particles and the energy are given by

$$\begin{aligned} N &= \frac{V}{g} \int_0^{\epsilon_g} \rho(\epsilon)\, d\epsilon = \frac{V}{g} \frac{(2m\pi)^{\eta/2}}{\Gamma(\eta/2+1)} \epsilon_g^{\eta/2}, \\ E &= \frac{V}{g} \int_0^{\epsilon_g} \epsilon\, \rho(\epsilon)\, d\epsilon = \frac{V}{g} \frac{(2m\pi)^{\eta/2}}{\Gamma(\eta/2+2)} \frac{\eta}{2} \epsilon_g^{\eta/2+1}, \end{aligned} \tag{2.45}$$

where ϵ_g is the Fermi energy at zero temperature when the statistics parameter is g. In the above we have used the fact that at zero temperature all levels up the ϵ_g are occupied with $1/g$ as the occupancy.

The energy per particle at zero temperature for an ideal gas is therefore given by

$$\frac{E}{N} = \frac{\eta}{\eta + 2}\, \epsilon_g = g^{2/\eta} \frac{\eta}{\eta + 2}\, \epsilon_F \tag{2.46}$$

which shows the scaling behaviour with g of the energy per particle of fermions at $g = 1$. Infact this equation can be used to determine g for gases which differ from the ideal fermion gas behaviour. We will discuss this situation again in the context of the behaviour of cold fermionic atoms.

The main advantage of Wu's distribution, however, is the calculation of the bulk properties of the gas as a function of the temperature, and this we proceed to do now. For a given density of single-particle states $\rho(\epsilon)$, we have

$$N = \int_0^\infty \frac{\rho(\epsilon)d\epsilon}{(w+g)}, \quad E = \int_0^\infty \frac{\epsilon\rho(\epsilon)d\epsilon}{(w+g)}. \tag{2.47}$$

We may convert the variable of integration from ϵ to w using

$$d\epsilon = \frac{1}{\beta}\frac{w+g}{w(1+w)}dw. \tag{2.48}$$

Thus after some straight forward algebra we have the set of coupled equations:

$$\frac{\eta}{2}\left(\frac{1}{\beta\epsilon_F}\right)^{\eta/2}\int_{w_0}^\infty \frac{dw}{w(1+w)}\left[\ln\left\{\left(\frac{w}{w_0}\right)^g\left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^{\eta/2-1} = 1. \tag{2.49}$$

$$\frac{E}{N\epsilon_F} = \frac{\eta}{2}\left(\frac{1}{\beta\epsilon_F}\right)^{\eta/2+1}\int_{w_0}^\infty \frac{dw}{w(1+w)}\left[\ln\left\{\left(\frac{w}{w_0}\right)^g\left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^{\eta/2}, \tag{2.50}$$

where w_0 is the value of w when $\epsilon = 0$ and ϵ_F is the Fermi energy of the system at $T = 0$. For any choice of g, eq. (2.49) is solved at a given $(\beta\epsilon_F)$ for w_0 numerically, and this w_0 is used in Eq.(2.50) next to obtain $(E/N\epsilon_f)$. It also follows from eq.(2.36) that the chemical potential μ at temperature T obeys the relation

$$-\beta\mu = g\log w_o + (1-g)\log(1+w_0) \tag{2.51}$$

In the foregoing discussion we have taken the case of free gas for which the dispersion relation is of the form $\epsilon \sim p^2$. Extension to more general dispersion of the form $\epsilon \sim p^\sigma$ is achieved by replacing $\eta/2$ with η/σ.

As in the previous section, the grand partition function of an ideal gas with FES, may be written in the standard form as

$$Z_G = \sum_{N=0}^\infty e^{\beta\mu N} Z_N; \quad z = e^{\beta\mu}. \tag{2.52}$$

The canonical partition Z_N is given by

$$Z_N = \sum_{\{n_k\}} g(\{n_k\})e^{-\beta E(\{n_k\})}\ ; \quad \sum_k n_k = N \tag{2.53}$$

where $g(\{n_k\})$ denotes the degeneracy of states at a given energy E. For Fermi or Bose gas $g(\{n_k\}) = 1$ and the summation in eq.(2.52) reduces to a factorised form for these cases.

$$Z_F = \prod_{k=1}^\infty (1 + ze^{-\beta\epsilon_k}); \quad Z_B = \prod_{k=1}^\infty (1 - ze^{-\beta\epsilon_k})^{-1}. \tag{2.54}$$

In the case of FES it is not so obvious that it can be written in a factorised form. We thus make the product ansatz for FES and show that it is indeed consistent

$$Z_G = \prod_{k=1}^\infty (1 + w_k^{-1}); \quad \ln Z_G = \sum_{k=1}^\infty \ln(1 + w_k^{-1}), \tag{2.55}$$

where $w_k = w(\epsilon_k, g)$ and $w_k^g(1+w_k)^{1-g} = e^{\beta(\epsilon_k-\mu)}$. Furthermore it is also consistent with the definition of average occupation number defined as

$$\bar{n}(\epsilon_k, g) = -\frac{1}{\beta}\frac{\partial \ln Z_G}{\partial \epsilon_k} = \frac{1}{w(\epsilon_k, g) + g}. \tag{2.56}$$

The thermodynamic potential $\Omega(\mu, T)$ is obtained by using the expression $(\partial\Omega/\partial\mu) = -N$. Using eq.(2.47) for N and using the fact

$$\frac{\partial w}{\partial \mu} = -\beta \frac{w+g}{w(1+w)}$$

we have

$$\Omega(\mu, T) = -\frac{1}{\beta}\ln Z_G = -\frac{1}{\beta}V\int_o^\infty d\epsilon\ \rho(\epsilon)\ln(1+\frac{1}{w}). \tag{2.57}$$

It is easy to see that the thermodynamic potential may also be obtained by replacing the summation in eq.(2.55) by the corresponding continuum approximation using $\sum_k \Rightarrow V\int d\epsilon\ \rho(\epsilon)$, where $\rho(\epsilon)$ is the density of states. and V denotes the volume. At this stage it is not necessary to specify the dimension of the space which shall remain arbitrary.

We now turn our attention to an important theorem first proved by Robert May [78] way back in 1964:

> **Theorem:** In two dimensions the specific heat $C_V(T, N)$ for an ideal gas of Fermions is identical with that for an ideal Bose gas for all T and N.

That this is true despite the differences in the distribution functions of the two systems at low temperatures is surprising at the outset. The fact that the specific heat is independent of statistics also agrees with the well known fact that bose gas does not condense in two dimensions. We may now ask the question if this is true for a gas of particles obeying FES, that is for arbitrary g. Indeed the theorem of May [78] is true for all g provided the density of states is constant.

We shall now prove this theorem for all g. We assume that the density of states is constant (as, for example, particles confined in an oscillator trap in one dimension or free gas in two dimensions). In the continuum limit we have

$$\ln Z_G = V\int_0^\infty d\epsilon \ln[1 + w(\epsilon, g)^{-1}]\ , \tag{2.58}$$

up to an irrelevant constant factor which is not important for the analysis that follows. Noting that

$$\beta d\epsilon = (\frac{g}{w} + \frac{1-g}{1+w})dw$$

we split the integral in eq.(2.58) into two parts, as

$$\begin{aligned} \ln Z_G &= V\left[g\int_{w(0)}^\infty dw\frac{1}{\beta w}\ln(1+w(\epsilon,g)^{-1}) + (1-g)\int_{w(0)}^\infty dw\frac{1}{\beta(1+w)}\ln(1+w(\epsilon,g)^{-1})\right]\ , \\ &= V\left[gI_1 + (1-g)I_2\right]\ , \end{aligned} \tag{2.59}$$

and the lower limit $w(0)$ is determined by the eq.(2.51).

We now rewrite the integrals I_1, I_2 as follows: Consider I_1 first and rewrite

$$w = w(0)e^{\beta\epsilon} = e^{\beta(\epsilon-\mu_F)}\ \Rightarrow\ \frac{dw}{w} = \beta d\epsilon,$$

where μ_F is a parameter that replaces w(0). Substituting this change of variable in the integral I_1 we have

$$I_1 = \int_0^\infty d\epsilon \ln[1 + e^{-\beta(\epsilon-\mu_F)}] \tag{2.60}$$

Next consider I_2 and rewrite

$$1 + w = w(0)e^{\beta\epsilon} = e^{\beta(\epsilon-\mu_B)} \;\Rightarrow\; \frac{dw}{1+w} = \beta d\epsilon$$

where now μ_B is the parameter that replaces w(0). Substituting this in the integral I_2 we have

$$I_2 = -\int_0^\infty d\epsilon \ln[1 - e^{-\beta(\epsilon-\mu_B)}] \tag{2.61}$$

Substituting eqs.(2.60) and (2.61) back in eq.(2.59) we have

$$\ln Z_G = V\left[g\int_0^\infty d\epsilon\ \ln(1 + e^{\beta(\epsilon-\mu_F)}) \ -\ (1-g)\int_0^\infty d\epsilon\ \ln(1 - e^{\beta(\epsilon-\mu_B)})\right] . \tag{2.62}$$

If we now identify μ_F (μ_B) with the chemical potential of a fermionic system (bosonic system) we have

$$\ln Z_G = g\ \ln Z_F \ +\ (1-g)\ \ln Z_B\ . \tag{2.63}$$

valid for a system with constant density of states. This was first proved in the context of a one dimensional exactly solvable model [61] but has a more general validity. We thus have the important result that, for constant density of states, the grand canonical partition function of a system of FES particles, in any space dimension, is given by

$$Z_G(\beta,\mu) = [(Z_F(\beta,\mu_F)^g(Z_B(\beta,\mu_B))^{1-g}] \tag{2.64}$$

where Z_F and Z_B are the grand canonical partition functions for the fermions and bosons respectively. Since $w(0)^g(1+w(0))^{1-g} = e^{-\beta\mu}$, for the chemical potentials we have

$$\mu = g\mu_F + (1-g)\mu_B, \tag{2.65}$$

where μ is the chemical potential of the exclusion statistics particles. In the case of systems with constant density of states it can be computed from the chemical potential μ_F and μ_B of Fermi and Bose systems. These in turn are determined by the condition,

$$\bar{N} = \frac{1}{\beta}\frac{\partial ln\ Z^F}{\partial\mu_F} = \frac{1}{\beta}\frac{\partial ln\ Z^B}{\partial\mu_B},$$

where $\bar{N}$ is the average number of particles in the system.

Since the thermodynamic potential, for arbitrary g, is given by $\Omega = -kT\ln Z_G$, using eq.(2.63) we have

$$\Omega = g\Omega_F + (1-g)\Omega_B \tag{2.66}$$

for all g. Now we may use the standard form of the thermodynamic potential for the ideal Fermi and Bose gas in two dimensions, namely

$$\Omega_{B,F}(\mu,T) = (\mp)\frac{1}{\beta}\frac{V}{\lambda^2}\int_0^\infty d\epsilon\ \ln[1 \mp e^{-\beta(\epsilon-\mu)}], \tag{2.67}$$

where $\lambda^2 = 2\pi\hbar^2\beta/m$ is the thermal wavelength. Following May [78], we define T_0 as

$$(\frac{T}{T_0})^{1/2} = \frac{\text{mean interparticle separation}}{\text{thermal wavelength}} = \frac{1}{\sqrt{\rho\lambda^2}}, \tag{2.68}$$

where ρ is the density and T_0 is the temperature which divides the classical regime ($T > T_0$) from the regime of quantum statistics ($T < T_0$). The total energy of the system can be shown to be

$$E_F(N,T) = E_B(N,T) + \frac{1}{2}NkT_0, \tag{2.69}$$

where we have made use of the equation for the energy in eq.(2.47). Thus the energy of the Bose and Fermi systems, in two dimensions, differ by a quantity proportional to N and independent of T. This result is exact. From the definition of the specific heat we have

$$C_V(N,T) = (dE/dT)_{N,V} \tag{2.70}$$

and it immediately follows that

$$[C_V(N,T)]_B = [C_V(N,T)]_F = [C_V(N,T)]_g \tag{2.71}$$

thus proving the theorem of May for arbitrary values of g.

We may also realise the result in the dilute, low density and high temperature, limit. The virial expansion in terms of average density $\rho = N/V$ for a gas in two dimensions for Fermi and Bose gas is given by

$$\beta\Omega_{B,F} = -N[1 \mp \frac{1}{4}(\rho\lambda^2) + \frac{1}{36}(\rho\lambda^2)^2 + \cdots], \tag{2.72}$$

where except the second term all other terms are independent of statistics of the particles. Therefore for a gas of particles obeying FES in two dimensions we have

$$\beta\Omega = -N[1 + \frac{1}{4}(2g-1)(\rho\lambda^2) + \frac{1}{36}(\rho\lambda^2)^2 + \cdots]. \tag{2.73}$$

Except the leading correction all other terms are common to all statistics. Therefore the specific heat in the dilute limit is given by

$$\beta[C_V]_g = -Nk[1 - \frac{1}{36}(\rho\lambda^2)^2 + \cdots], \tag{2.74}$$

where the leading term is the classical equipartition value, $Nk/2$ for each degree of freedom. Obviously this is a special case since the first order correction vanishes and other terms are common to all statistics characterised by the statistical parameter g. It is interesting to note that in the extreme relavistic limit the specific heat is independent of statistics in *one dimension* which is the analogue of the non-relativistic particles in two dimensions as the density of states are constant[78].

Before we leave this section, few remarks are in order. In the above analysis we have emphasised the thermodynamics of a gas of FES particles with constant density of states because of its interesting behaviour. More general cases have been dealt in great detail by many authors. The thermodynamics of an FES gas was first discussed by Nayak and Wilczek[64] who also pointed out the problem of the existence of negative probabilities. A detailed analysis of the thermodynamics properties was presented by Isakov etal [79] (emphasising the low temperature and low density expansions). In particular the fact that the specific heat with constant density of states is independent of statistics has been noted in Refs.[76, 79]. A detailed and careful analysis of the quantum statistical mechanics of an ideal FES gas may also be found in Ref.[80] and more recently in Refs.[81, 82].

Chapter 3

Microscopic interpretation of FES

If we attempt to interpret the distribution function in eq.(2.35) as arising from the statistical mechanics of a single mode with Boltzmann statistical weight $f_n \exp(-n\beta\epsilon)$ for the mode occupied by n particles, then it is easy to show that some of the coefficients f_n are invariably negative when g is not zero or unity. The problem with such negative probabilities was first pointed out by Nayak and Wilczek[64] and further elaborated by Polychronokos [65] for all values of g (except at $g = 0, 1$). This raises the problem of interpreting these negative probabilities. It has also been speculated that these negative probabilities are an essential feature of nontrivial exclusion statistics[65]. This also raises the question if indeed exclusion statistics is fundamentally flawed. We will try to address this question from a microscopic point of view. In this chapter we first discuss the origin of these negative probabilities and discuss the resolution which leads to the microscopic interpretation of exclusion statistics. We formulate the rules of occupancy such that the negative probabilities are avoided.

3.1 Origin of negative probabilities

Following Nayak and Wilczek [64], We illustrate the problem in the simple case of $g = 1/2$: Defining $f_n \exp(-n\beta(\epsilon - \mu))$ as the probability for an n-fold occupancy we may write

$$\frac{\sum_n n f_n \exp(-\beta n(\epsilon - \mu))}{\sum_n f_n \exp(-\beta n(\epsilon - \mu))} = \frac{1}{\sqrt{1/4 + \exp[2\beta(\epsilon_i - \mu)]}}. \tag{3.1}$$

Normalising with $f_0 = 1$, we find that $f_1 = 1, f_2 = 1/2, f_3 = 1/8, f_4 = 0, f_5 = -1/128, \ldots$. Except for $g = 0, 1$, invariably the coefficients become negative for some n at a given value of g. Interestingly, in their original paper Nayak and Wilczek correctly point out that the negative probabilities appear in a natural and meaningful way: *For bosons and fermions the fundamnetal assumption of symmetry or antisymmetry of the wave-function holds rigorously and locally in momentum space. This is enough to allow one to derive the appropriate statistical distribution for an ideal gas locally in phase space.* However, in the case of exclusion statistics for non-trivial g, the exclusion principle operates on states of nearby energy. Thus one can not assign a probability for a single state or cell unless one takes the cell size to be very large. If we do so by force, it will result in over counting states and the negative probabilities occur in order to correct for this over counting.

In order to clarify the origin of negative probabilities or weights, we first discuss an equation and its solution due to the famous mathematician Ramanujan[83]. Ramanujan considered the following equation:

$$aqX^p - X^q + 1 = 0, \tag{3.2}$$

where a may be complex and p, q are positive. This equation can be mapped on to Wu's equation (2.36) as a particular case.

The most general solution for X^d is,

$$X^d \;=\; \sum_{N=0}^{\infty} C_N(p,q,d)a^N, \tag{3.3}$$

where $C_0(p,q,d) = 1$ and $C_1(p,q,d) = d$ and

$$C_N(p,q,d) = \frac{d}{N!}\prod_{j=1}^{N-1}(d+Np-jq), \quad N \geq 2. \tag{3.4}$$

Now consider Wu's equation and substitute $X = 1 + \frac{1}{w}$. Then the equation (2.36) reads

$$e^{-\beta(\epsilon-\mu)}X^{1-g} - X + 1 = 0 \tag{3.5}$$

which is the same equation as Ramanujan equation (3.2) with

$$p = 1-g; \quad q = 1; \quad a = e^{-\beta(\epsilon-\mu)}.$$

We can straight away apply Ramanujam's solution and obtain

$$1 + \frac{1}{w} = \sum_{N=0}^{\infty} C_N(1-g,1,1)\, e^{-N\beta(\epsilon-\mu)}, \tag{3.6}$$

where $C_0 = C_1 = 1$ and

$$C_N(1-g,1,1) = \; P_N \;=\; \prod_{m=2}^{N}(1-\frac{gN}{m}) \tag{3.7}$$

Note that when $g = 0$ we have $C_N = 1$ for all N and therefore

$$1 + \frac{1}{w} = \sum_{N=0}^{\infty}\exp(-N\beta(\epsilon-\mu)) = \frac{1}{1-\exp(-\beta(\epsilon-\mu))}. \tag{3.8}$$

Furthermore we have for $g = 1$, $C_N = 0$ for all $N \geq 2$ since $m = N$ at least for the last term of the product in eq.(3.7). We therefore obtain the solution

$$1 + \frac{1}{w} = 1 + \exp(-\beta(\epsilon-\mu)). \tag{3.9}$$

These are indeed the correct solutions for $g = 0$ and $g = 1$. There is no problem with negative probabilities or indeed locality in phase space for these two cases and one can make the cell size as small as possible.

However, for arbitrary values of g, the probabilities P_N become negative when $gN > m$ as can be seen from eq.(3.7). This is indeed the problem of negative weights pointed out by Nayak and Wilczek. This problem was further elaborated by Polychronakos[65]. Suppose we choose $p = (1-g)$ and $q = 1$ in eq.(3.4), we have

$$C_N(1-g,1,d) \;=\; d\frac{(d+(1-g)N-1)!}{N!(d-gN)!} \tag{3.10}$$

which is clearly different from the dimension formula of Haldane. Polychronakos obtained the above multiplicity of states by placing them on a circle or a periodic lattice and implementing the *minimum distance* g rule. This does not alter the statistical behaviour of the system since it leads to the same statistical distribution as in eq.(2.35). One can also derive the Haldane dimension, D_N, formula using the above method with the restriction that any two particles are g sites apart but without the restriction of periodicity. The above formula also correctly reproduces the bosonic and fermionic dimension formula for $g = 0$ and $g = 1$ respectively. In the limit $d >> 1$, however, it is easy to see that

$$C_N(1-g,1,d) = D_N(g,d) + O(\frac{1}{d^2}).$$

Recall that d as usual stands for the number of available single particle states. Therefore in the continuum limit, when $d \to \infty$, the C_N and D_N are approximately the same and lead to the same statistical behaviour although the microscopic interpretations are different. The problem with negative probabilities however persists in both cases.

The grand canonical partition function of the system may be written as,

$$Z_G \;=\; (1+w^{-1})^d \;=\; \sum_{N=0}^{\infty} C_N(1-g,1,d)e^{-\beta N(\epsilon-\mu)}, \tag{3.11}$$

where w satisfies the equation (2.35). We have also assumed that all the energy levels are degenerate with energy given by ϵ. Note that this is an exact expression and no assumption is required on the single particle dimension d. The negative weights therefore arise [64, 65], when one insists on expanding $1 + w^{-1}$ in powers of $e^{-\beta\epsilon}$.

Few remarks are in order here: The negative probabilities arise because of our insistence on the factorization[66] implied in eq.(3.11). Here we assign a statistical weight to a given state (or cell) independent of the weights corresponding to other states. The over counting resulting from this unconstrained sum is compensated by the occurence of negative weights. This is a feature that arises from the fact that, unlike in the case of Fermi or Bose statistics where the exclusion principle is stated with a single level in mind, the Haldane exclusion principle acts across a set of levels. We will show in the subsequent sections that particles obeying fractional exclusion statistics can be characterised by constraints on the sets of occupation numbers. Once these constraints are obeyed there are no negative probabilities. However, if we relax these constraints (as in the case of Wu distribution function where only maximal occupancy of a state is relevant), the negative weights occur to compensate for the resulting overcounting.

Indeed this is they way we encounter negative probabilities in other systems in physics: for example, in gauge theories, they arise in the ghost sectors. Ghosts come from the Jacobian associated with nonlinear gauges which essentially ensure the correct counting of the states. Another example is that of the Wigner distribution funciton in quantum mechanics which is not positive definite precisely because some quantum mechanical constraints are relaxed. We should mention here that Isakov [68] proposed an alternative formulation based on the variable number of single particle states, which in turn depend on the total number of particles, as a way of avoiding the problem of negative probabilities.

3.2 Canonical Partition Function

Chaturvedi and Srinivasan[66] provided an elegent microscopic interpretation of the semion case where they prove that the factorisation implied in eq.(3.11) is not possible in exclusion

statistics if the probabilities are to be positive definite. We reproduce their arguement as originally given in [66] and then consider a realisation of the same in an interacting model.

Consider a system of N identical particles each of which can occupy d states with energies $\epsilon_1, \ldots, \epsilon_d$. The canonical partition function then has the following structure

$$Z_N(x) = \sum_{\{n_i\}} f(n_1, \cdots, n_d)\, x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}; \quad \sum n_i = N, \tag{3.12}$$

where x denotes the set of all $x_i; i = 1, \cdots, d$ and $x_i \equiv \exp(-\beta\epsilon_i)$. Thus the set $\{n_i\}$ defines a partition of N and the sum is over all such allowed partitions of N.

A consistent statistics is defined provided the weight has the following properties:

1. $f(n_1, \cdots, n_d) \geq 0$
2. $f(n_1, \cdots, n_d)$ is a symmetric function of its arguements.

While the first condition is necessary, the second one may be relaxed. The statistics obeyed by identical particles is defined by the function $f(n_1, \cdots, n_d)$. Since $f(n_1, \cdots, n_d)$ is a symmetric function of its arguements, we may rewrite the sum over all possible combinations of n_i contained in eq.(3.12) in terms of a sum over the partitions of N.

$$Z_N(x) = \sum_{\lambda} f(\lambda_1, \cdots, \lambda_d)\; m_\lambda(x) \quad , \tag{3.13}$$

where $\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_d)$, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_d$ is a partition of N and $m_\lambda(x_1, \cdots, x_d)$ denotes the monomial symmetric function[84] corresponding to the partition which we denote by λ

$$m_\lambda(x) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_M^{\lambda_M} \quad . \tag{3.14}$$

The sum on the rhs of eq.(3.14) is over all distinct permutations of $(\lambda_1, \cdots, \lambda_d)$. Since we have written Z_N in the form given by eq.(3.13) using the symmetric condition (2) it is clear that each choice of $f(\lambda_1, \cdots, \lambda_d)$ satisfying the positivity condition in (1) above defines the statistics of identical particles under consideration. For example we have

$$\begin{aligned} f_B(\lambda) &= 1\ ; \\ f_{MB}(\lambda) &= \frac{1}{\lambda_1! \cdots \lambda_d!}\ ; \qquad (3.15) \\ f_F(\lambda) &= 1 \text{ for } \lambda = (1^N) \text{ zero otherwise}\ , \qquad (3.16) \end{aligned}$$

define Bose, Maxwell-Boltzmann and Fermi statistics respectively. In the case of Fermions the notation implies that only distinct partitions without repetitions is allowed in the energy distribution. Thus the N fermions are distributed in d levels such each level contains utmost one fermion. Setting $x_1 = x_2 = \cdots, x_d = 1$ in eq.(3.13) we obtain the formula for the statistical weights

$$D_N(d) \equiv Z_N(1^M) = \sum_{\lambda} f(\lambda)\; m_\lambda(1^d) \quad . \tag{3.17}$$

The number $m_\lambda(1^d)$ is given by

$$m_\lambda(1^d) = \frac{d!}{m_1! m_2! \cdots} \quad , \tag{3.18}$$

where $m_1, m_2 \cdots$ denote the multiplicities of $\lambda_1, \lambda_2, \cdots$ in the partition λ.

The problem of negative probabilities can restated in terms of the $f(\lambda)$: Suppose we demand that

$$f(\lambda_1, \cdots, \lambda_d) = P_{\lambda_1}\ P_{\lambda_2}\ \cdots\ P_{\lambda_d}, \tag{3.19}$$

where the weights P_λ are defined by eq.(3.7). In this case there are no occupancy restrictions though $f(\lambda)$ are not necessarily positive definite since the weights P_n can take both postitive and negative values. This then violates the assumption (1) required in order to define valid statistics. Further, in this case the $f(\lambda)$ are factorisable in terms of the weights. Indeed, it is this last property of the $f's$ that one has to give up in order to have positive weights and consistent statistics. Fortuitously it so happens that this problem does not arise in the case of standard Bose, Fermi and Maxwell-Boltzmann statistics.

The formula expressing $D_N(d)$ in terms of $f(\lambda)$ is used to calculate $D_N(d)$ for a given statistics, that is for given $f(\lambda)$. Here we would do the reverse and use it to fix $f(\lambda)$ for the given $D_N(d)$ and hence obtain a detailed specification of the underlying statistics. For the semion case, this strategy, as we shall see, determines $f(\lambda)$ uniquely.

After these generalities, we now turn our attention to the D_N for exclusion statistics as given in eq.(1.6): we find that for $g = 1/m$, $D_N^{(1/m)}(d)$ is an integer for arbitrary d if and only if m divides $N-1$ that is $N = pm+1, p = 0, 1, 2, \cdots$. Restricting only to these values of N, we may rewrite eq.(1.6) for $g = 1/m$ as

$$D_{mp+1}^{(1/m)}(d) = \begin{pmatrix} d+(m-1)p \\ mp+1 \end{pmatrix} \ . \tag{3.20}$$

We may now ask the question whether $D_{mp+1}^{(1/m)}(d)$ may be written in the following form:

$$D_{mp+1}^{(1/m)}(d) = \sum_{\substack{\lambda \\ \lambda_1 \le m}} f_{1/m}(\lambda)\ m_\lambda(1^d) \ \ , \tag{3.21}$$

to determine $f_{1/m}(\lambda)$. Since λ_1 is greater than all other λ's, the restriction $\lambda_1 \le m$ on the partitions that contribute to the rhs of eq.(3.21) is to incorporate the fact that the maximum permissible occupancy when $g = 1/m$ is m.

First let us consider the semion statistics which corresponds to choosing $m = 2$. Setting $m = 2$ in eq.(3.21) and using the explicit expression for $D_{2p+1}^{(1/2)}(d)$ given in eq.(3.20) we obtain

$$\begin{pmatrix} d+p \\ 2p+1 \end{pmatrix} = \sum_{\substack{\lambda \\ \lambda_1 \le 2}} f_{1/2}(\lambda)\ m_\lambda(1^d) \ \ .) \tag{3.22}$$

As an example consider, for simplicity, the case $p = 1$ or $N = 3$. In this case the only permissible partitions are (1^3) and $(2,1)$ and hence eq.(3.22) becomes

$$\begin{pmatrix} d+1 \\ 3 \end{pmatrix} = f_{1/2}(1^3)\ \ \frac{d!}{3!(d-3)!} + f_{1/2}(2,1)\ \ \frac{d!}{(d-2)!} \quad , \tag{3.23}$$

where we have substituted the values of the m's using eq.(3.18). Comparing the powers of d on both sides one obtains

$$f_{1/2}(1^3) = 1 \ \ ; \ \ f_{1/2}(2,1) = \frac{1}{2} \quad . \tag{3.24}$$

which are both positive. Let us now consider p to be arbitrary. The permissible partitions in this case are

$$(1^{2p+1}), (2, 1^{2p-1}), (2^2, 1^{2p-3}), \cdots, (2^p, 1).$$

By introducing the notation $\lambda(0) \equiv (1^{2p+1})$, $\lambda(q) \equiv (2^q, 1^{2(p-q)+1})$; $q = 1, \cdots, p$ and using eq.(3.18), eq.(eq3.8) may be written as

$$\left(\begin{array}{c} d+p \\ 2p+1 \end{array} \right) = \sum_{q=0}^{p} f_{1/2}(\lambda(q)) \; \frac{d!}{q!(2p-2q+1)!(d-2p+q-1)!} \quad . \tag{3.25}$$

On cancelling the factor $d(d-1)\cdots(d-p)$ on both sides, eq.(3.25) may be written in the following form

$$\left(\begin{array}{c} d+p \\ p \end{array} \right) = \sum_{q=0}^{p} f_{1/2}(\lambda)(q) \left(\begin{array}{c} p \\ q \end{array} \right)^{-1} \left(\begin{array}{c} 2p-q+1 \\ q \end{array} \right) \left(\begin{array}{c} 2p+1 \\ q \end{array} \right) \left(\begin{array}{c} d-p-1 \\ p-q \end{array} \right) \quad . \tag{3.26}$$

Using the identity

$$\left(\begin{array}{c} n+m \\ p \end{array} \right) = \sum_{q=0}^{p} \left(\begin{array}{c} n \\ q \end{array} \right) \left(\begin{array}{c} m \\ p-q \end{array} \right) \quad , \tag{3.27}$$

and taking $n = 2p+1$ and $m = d-p-1$, we find that the f's are uniquely determined:

$$f_{1/2}(\lambda(q)) = \left(\begin{array}{c} p \\ q \end{array} \right) \left(\begin{array}{c} 2p-q+1 \\ q \end{array} \right)^{-1} \quad . \tag{3.28}$$

The canonical partition function for $N = 2p+1$ semions is thus given by

$$Z^{(1/2)}_{2p+1}(x) = \sum_{q=0}^{p} \left(\begin{array}{c} p \\ q \end{array} \right) \left(\begin{array}{c} 2p-q+1 \\ q \end{array} \right)^{-1} m_{\lambda(q)}(x) \quad . \tag{3.29}$$

This expression for the canonical partition function for the semion statistics is central to the microscopic interpretation of the exclusion statistics.

Few comments are in order here:

- It is clear from eq.(3.29) that no negative fs appear and therefore the semion statstics is as legitimate a statistics as Bose, Maxwell-Boltzmann or Fermi. The main difference with these well known cases is that, unlike Bose, Fermi, or Maxwell Boltzmann statistics, $f_{1/2}(\lambda_1, \lambda_2, \cdots, \lambda_d)$ does not factorise as $P_{\lambda_1} P_{\lambda_2} \cdots P_{\lambda_d}$ as given by eq.(3.7). As a consequence the grand-canonical partition function does not have a factorised form. However while the factorised form is useful, it does not form the basis for defining the statistics. When the factorised form is imposed, negative probabilities occur.

- From the structure of the canonical partition function it is clear that the semion statistics is characterized by occupancy restrictions-the maximal occupancy of a state is two particles and if a state is fully occupied the next level can be filled without any restriction. However, a partially filled level puts further restrictions on the subsequent levels being occupied.

Chaturvedi and Srinivasan[66] have also discussed the the possibility of accomodating the semion statistics in a generelised scheme based on the permutation group and conclude that it is *not possible.*

Until now we have assumed the existence of fractional exclusion statistics as determined by the interpolating dimension formula given by Haldane. In the next section we consider an abinitio physical realisation of this statistics in an interacting system where the quasiparticles behave as ideal exclusion statistics particles. We further show that not only the weights are positive definite but may also be determined for all other fractions and not just for semions.

3.3 Realisation in an interacting system

As emphasised in Chapter 1, it is our view that any realization of fractional exclusion statistics must have its origins in systems of interacting particles. The expectation is that under certain conditions systems of interacting particles which obey Fermi or Bose statistics may be described in terms of quasiparticles (or quasiholes) which obey fractional statistics. As emphasised earlier, a crucial property of exclusion statistical interaction is that it should cause shifts in single particle energies at all scales[54]. This property is realised in a large class of one dimensional models of interacting fermions where Fermi liquid theory breaks down[46, 63]. In fact it has been shown exactly that quasi particles with nontrivial exclusion statistics exist in a class of models that are solved by Bethe ansatz[60]. In particular, we concentrate on the properties of the quasiparticles of the Calogero-Sutherland model (CSM)[39, 40, 41]. It has been shown that the quasiparticles of the CSM behave like ideal exclusion statistics particles [56, 60, 61, 62]. The main feature of CSM is that the total energy of the many-body system can be written in terms of single quasi-particle energies which involve shifted momenta and these shifts contain the information about the exclusion statistics of the quasiparticles. In this section we analyse these shifted momenta and make explicit connection with the formula in eq.(1.6). We then use them to obtain constraints on the allowed set of occupation numbers. These are what we refer to as the counting rules that reproduce the formula in eq.(1.6). The statistical mechanics of the system obeying these constraints is then the same as that defined by Wu [55] and all statistical weights are shown to be positive, thus providing a resolution of the problem of negative probabilities[64].

We begin with the trignometric Sutherland model[41] of an N-particle system on a ring of unit radius. The Hamiltonian is given by,

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{\hbar^2}{m}\sum_{j<i}\frac{2g(g-1)}{\sin^2[(x_i-x_j)/2]}\ , \tag{3.30}$$

where g is the interaction parameter. This notatioin is deliberate as we will soon identify this with the statistical parameter of the exclusion statistics. While the model can be applied to both interacting bosons and fermions, we choose to work in the fermionic basis here after. The energy of an N-fermion state may be written in terms of shifted momenta as

$$E = \frac{\hbar^2}{2m}\sum_{i=1}^{\infty} k_i^2 n_i, \tag{3.31}$$

where $n_i = 0, 1$ and the shifted momenta k_i (also called pseudo momenta in Ref.[62]) are given by

$$k_i \ = \ m_i - (1-g)\frac{(N_i^- - N_i^+)}{2}, \tag{3.32}$$

where m_i are distinct integers, $N_i^{-(+)}$ are the number of particles with shifted momenta less (greater) than k_i. Note that we could have also started with the Calogero-Sutherland model with harmonic confinement. We discuss this in the next chapter. The results below follow analogously with the proviso that we have shifted energies instead of shifted momenta. We choose to work with shifted momenta here since generalizations to higher dimensions may be easier if one were to derive the constraints in momentum variables instead of assuming specific form of confinement.

First we establish the relationship between the shifted momenta given above and the Haldane's dimension formula eq.(1.6). Consider the above system with an upper and lower cutoff on the momenta, k_{max} and k_{min} respectively. We divide this range of momenta into

cells of unit length (the first and last cells could be smaller) and define the occupancy of the j^{th} cell, n_j to be the number of particles with momenta k_i, such that $j+1 > k_i \geq j$. We identify single particle space dimension d with the number of cells in the range, i.e $d = k_{max} - k_{min}$, where d may be fractional. If we now denote the range of the m_is by d_F, we have

$$d_F = m_{max} - m_{min} = d + (1-g)(N-1), \tag{3.33}$$

where N is thenumber of particles in the cell. Since there exists an m_i for every k_i, the total number of states in the range $k_{max} - k_{min}$ is the same as that between $m_{max} - m_{min}$. The total number of states is then the number of ways N distinct integers can be picked from d_F distinct integers, i.e ${}^{d_F}C_N$, as in fermionic description. Substituting for d_F from the above expression we immediately reproduce the Haldane dimension formula in Ref.[54].

In order to obtain the counting rules for occupancy we will first derive three properties of the set of momenta $\{k_i\}$. If k_i are ordered such that they increase with increasing i, then we have, $k_{i+1} - k_i = m_{i+1} - m_i - (1-g)$. If $g < 1$, then it follows that $m_{i+1} > m_i$. Further, if $m_{i+1} - m_i = 1$ then $k_{i+1} - k_i = g$ and if $m_{i+1} - m_i > 1$ then $k_{i+1} - k_i > (1+g) > 1$ because m_i's are integers.

We can then draw the following three conclusions from the properties of the shifted momenta k_i:

1. The ordering in k_is is the same as the ordering in m_is.
2. "Close packed" m_is with unit spacing correspond to "close packed" k_is with spacing g.
3. The gaps between any two non-close packed k_is is greater than 1. Therefore all the k_is in any cell are close packed.

We now come to the question as to what are the constraints on the sets of occupation numbers $\{k_i\}$. For example, if $g = 0$, there are no constraints as in the bosonic case. If $g = 1$ the constraints are $n_j \leq 1$ as in the case of fermions. For any other g, one obvious constraint come from the second property derived above, namely the occupancy of the j-th cell $n_j \leq \frac{1}{g}$ which specifies the maximum occupancy of a given cell assumed to be of unit spacing. This is the same constraint one derives from the distribution function of Wu (2.35) at zero temperature. An important departure from the usual bosonic and fermionic case is that the cell size is important and cannot be arbitrarily taken to zero as in the case of bosons and fermions [64].

There are further constraints on the occupancy. To formulate them we use the third property. Let k_L be the lowest momentum in the j^{th} cell. Then from the second and third property, it follows that

$$k_L + g(n_j - 1) < j + 1. \tag{3.34}$$

We can write k_L as $k_L = j + f(k_L)$, where $f(k_L)$ denotes the fractional part of k_L, that is, $0 \leq f(k_L) < 1$. We then have,

$$f(k_L) + g(n_j - 1) \ < \ 1. \tag{3.35}$$

From equation (3.32), we can express $f(k_L)$ as a function of the occupation numbers,

$$f(k_L) \ = \ f\left[-(1-g)\frac{[N_{cj}^- - N_{cj}^+ - (n_j - 1)]}{2}\right], \tag{3.36}$$

where $N_{cj}^- = \sum_{l<j} n_l$ and $N_{cj}^+ = \sum_{l>j} n_l$. Equations (3.35) and (3.36) then constitute a set of constraints on the occupation numbers.

We will now show that these form a complete set of constraints. Namely, given any set of occupation numbers, $\{n_j\}$, that satisfies the constraints, there exists a set of momenta, $\{k_i\}$, that realizes it. To do this, consider a set $\{n_j\}$, where $j_{min} \leq j \leq j_{max}$. The lowest value of the momentum in the j^{th} cell is uniquely determined by the occupation numbers through equation(3.36). Because of the third property, all the other momenta are also uniquely determined. Hence we have shown that there are no more constraints. Equations (3.35) and (3.36) form a complete set of constraints. Note also that the above logic implies that there is a one to one correspondence between the sets of occupation numbers, $\{n_j\}$, that satisfy the constraints (3.35) and (3.36)and the sets of momenta, $\{k_i\}$, that satisfy equation (3.32).

We can now remove the scaffolding of the Sutherland Model that we started with and *define* exclusion statistics system by the above constraints. The connection to the dimension formula in eq.(1.6) established earlier implies that

$$\sum_{\{n_i\}} F(\{n_i\}) \;=\; D_N(g,d), \tag{3.37}$$

where $N = \sum_j n_j$ and $F(\{n_i\}) = 1$ if $\{n_i\}$ satisfy the constraints and zero otherwise. Note that the weights now are positive definite. There are no negative weights once the constraints are imposed.

Next, we construct some simple examples from the above counting rules. For simplicity we look at occupation numbers for special values of $g = 1/m$ where m is an integer. The rules formulated above for the occupation number of exclusion particles may be combined and restated thus:

> *Let $m = 1/g$, and let N_i be the number of particles in the occupied states below some ith level, $N_i = \sum_{j<i} n_j$. Then an occupation $n_i (n_i \leq m)$ is allowed iff $(N_i \; mod \; m) \leq (m - n_i)$.*

This rule now includes all the three constraints stated above and may be regarded as the statement of *Generalised Pauli principle*.

To see how this rule is implemented, consider a system of N-particles spread over d states. In order that D_N is an integer, we choose $N = mp + 1$, where p is an integer. Since $N \leq md$, we have $p < d$. We shall divide these d states into cells. An allowed configuration may be represented as a string of numbers $(n_1, n_2, n_3, ...)$, where each $n_i \leq m$ denotes the occupancy of levels ordered from left to right. Instead of dealing with a configuration where all N particles are spread over d states (some which may be empty), we can simplify the discussion by considering one cell at a time. Each cell may now have a partition of m. This allows us to fill the subsequent cells without reference to the previous cell according to the counting rules since $N_i \; mod \; m = 0$. We now fill each cell with a partition of m which is allowed by the rules given above. This then generates all possible allowed configurations whose sum is given by D_N.

If, in particular, we are interested in expectation values of symmetric functions of n_i, we can work with symmetrised weights. Consider a symmetric operator $O(\{n_i\})$. The expectation value of this operator may be written as,

$$< O(\{n_i\}) >= \frac{\sum_{\{n_i\}} F_s(\{n_i\}) O(\{n_i\})}{\sum_{\{n_i\}} F_s(\{n_i\})}, \tag{3.38}$$

where

$$F_s(\{n_i\}) \;=\; \frac{1}{d!} \sum_p F(p_{n_i}). \tag{3.39}$$

Here p stands for all permutations of the allowed configurations. Every allowed configuration in $\{n_i\}$ may be characterised by the multiplicities q_n, namely a given allowed configuration may be written as a string, $m^{q_m}(m-1)^{q_{m-1}}...1^{q_1}$, where $q_1 + 2q_2 + ... + mq_m = N$. We may now also allow any permutation of these occupancies (with zeros added to make up d-states). The dimension of the N-particle space may then be written as

$$D_N(g,d) = \sum_{\{q_n\}} f_m^N(q_1, q_2, ...) \quad {}^dC_q, \tag{3.40}$$

where $q = \sum_{n=1}^{m} q_n$. The new weights f are defined as,

$$f_m^N(q_1, q_2, ...q_m) = \frac{M_a(q_1, q_2, ...q_m)}{M_t(q_1, q_2, ..., q_m)}, \tag{3.41}$$

where M_a are allowed configurations after symmetrising and M_t is the total number of configurations for a given set of q's which define a configuration. We shall clarify this now with specific examples.

The case of semion: We first consider the case of semion, that is $g = 1/2$ or equivalently $m = 2$. The results obtained here are exactly the same as those obtained from a more general microscopic interpretation of FES in the previous section by Chaturvedi and Srinivasan[66]. The maximal occupancy of a state in this case is 2. Hence allowed occupancy of a state is 2 or 1. Zeros may occur any where without changing the rules. Let us implement this in the specific case of $d = 4, N = 5$, say. In this case the allowed configurations are given by the strings (2210),(2111),(1121). In the first configuration, zero can be anywhere and therefore there are four configurations. Notice that a string of the form (1211) or (1112) violates the counting rules. Therefore counting all the allowed configurations we obtain $D_5(1/2, 4) = 6$. This is exactly what one gets from the Haldane formula.

Further if we symmetrise each of these allowed configurations, then the new weights may be computed using eq.(3.41). In the specific case of $m = 2$, we have

$$M_t(q_1, q_2) = {}^{q_1+q_2}C_{q_2}, \qquad M_a(q_1, q_2) = {}^{p}C_{q_2}, \tag{3.42}$$

where p is defined through the equation $N = 2p + 1$. The corresponding f is therefore given by,

$$f_2^N = \frac{{}^{p}C_{q_2}}{{}^{q_1+q_2}C_{q_2}} \tag{3.43}$$

Note that these weights, whether in the symmetrised form or unsymmetrised form, are positive definite. Further, this is exactly the formula derived by Chaturvedi and Srinivasan[66] in their microscopic analysis of Haldane statistics for semions.

It is important to stress the differences in these two approaches- in their analysis Chaturvedi and Srinivasan start from a formulation of the statistical mechanics of a system by removing factorizability of the weights as a criterion. They derive the expression for the weights in eq.(3.43) by imposing the conditions positivity and the requirement of symmetry (all configuratons which are permutations of each other carry the same weight). Our starting point is the Sutherland model. We derive our rules from the properties of shifted momenta. After removing this scaffolding, we obtain not only positive definite weights for each configuration but when symmetrised they reproduce the results of Chaturvedi and Srinivasan.

The case with $g = 1/3$ or $m = 3$: The maximal occupancy of a state in this case is 3. The allowed configurations for each cell are (3),(21),(12),(111). That is we can form a string of allowed configuration with any of these cells in any order to make up N particles. Any number of zeros may be added in between to make up a total of d-states.

As in the semion case we may consider expectation values of symmetric functions of n_i. Following the same procedure we can derive the symmetrized weights f_3^N defined in eq.(3.41). Since, $m = 3$, we have

$$M_t(q_1, q_2, q_3) = {}^{q_1+q_2+q_3}C_{q_3}\ {}^{q_1+q_2}C_{q_2}, \qquad M_a(q_1, q_2, q_3) = {}^{p}C_{q_3}\ {}^{p-q_3}C_{q_2}\ (2)^{q_2}, \tag{3.44}$$

where p, as before, is defined through the equation $N = 3p + 1$. The corresponding weight f is therefore given by,

$$f_3^N = \frac{{}^{p}C_{q_3}\ {}^{p-q_3}C_{q_2}\ (2)^{q_2}}{{}^{q_1+q_2+q_3}C_{q_3}\ {}^{q_1+q_2}C_{q_2}} \tag{3.45}$$

These weights are again positive definite. Chaturvedi and Srinivasan[66] also suggest how their method may be extended beyond the semion case which they considered in detail. However, this extension requires additional conditions which are not imposed in the semion case. In contrast, the rules as derived from the point of view of an exactly solvable model are completely specified independent of the actual value of g (or m). There is an algorithm to derive f_m^N for arbitrary m though this gets complicated for larger m. A further generalisation of this approach applicable to $g = l/m$ for all number of particles N and arbitrary number of states d has been discussed by Bergere[85]. In here all fractional weights over symmetrised configurations are given in full generality.

To summarise, the origin of the negative probabilities in exclusion statistical system are well understood. In our approach we have chosen an unusual starting point in an equation and its solution given by Ramanujan. This starting point makes precise the statements about the occurrence of negative probabilities. Further, we have formulated a counting principle which reproduces the Haldane dimension formula. It can therefore be used to define exclusion statistics purely in terms of state counting. The negative probabilities discussed in literature[64] can be understood as arising when the system constrained by the counting rules is replaced by an unconstrained one. The negative weights then compensate for the introduction of unphysical configurations. This is therefore exactly analogous to other situations in physics where negative probabilities arise, for example, the ghosts and negative norm states in gauge theories or as in the case of Wigner distribution in quantum mechanics.

Chapter 4

Models of ideal FES in one dimension

In the previous chapter we discussed the microscopic interpretation of exclusion statistics and its realisation in a one-dimensional exactly solvable model. In fact this not an isolated model since it has been recognised in general that models with the inverse square law interaction, like the Calogero-Sutherland Model (CSM)[39, 40, 41], Haldane-Shastry[86, 87] model and other related models behave very much like ideal gases which obey fractional statistics[88, 89]. In fact, all established models of FES, that is the ones that can be exactly mapped to FES, are one dimensional or effectively one dimensional like anyons in the lowest landau level. In this chapter, we concentrate on the CSM, which is a model defined in the continuum and is exactly solvable. Just as the Hamiltonian of the particle-flux composite describes the anyons under exchange in two dimensions, the CSM Hamiltonian provides the framework for discussing fractional statistics particles in one dimension. This is infact a very well studied model. The spectrum and the thermodynamics of particles described the CSM Hamiltonian are well known. However the ideal exclusion statistical interpretation of the quasi-particles of this system was brought out much later[60, 61, 62]. Bernard and Wu[60] showed that the thermodynamic Bethe ansatz equations for one-dimensional integrable many-body systems may be reinterpreted so that they code exclusion statistics. Using a completely different approach, Ha[62] showed the same result by computing the exact dynamical density-density correlation function. In this chapter we follow yet another approach used by us[61] to show that the interacting fermions of CSM can be mapped *exactly* to a system of "non-interacting" quasi-particles obeying FES. Though the model, being one dimensional, may appear "gedanken" but it provides the template for analysing models in higher dimensions as we shall see subsequently.

In chapter 2, we have already discussed the distribution function, given in eq.(2.35) for particles obeying FES as derived by Ramanathan[42], Wu[55], Isakov[56] and Ouvry[57]. Wu derived the result for a system with flat dispersion and assuming that Haldane exclusion principle holds (as in the case of anyons in a magnetic field in two dimensions confined to the lowest Landau level[57]). Furthermore, Wu also analysed the more general case of many species with different energies ϵ_i and had shown that the above distribution holds when the matrix of statistical parameters g_{ij} is of the form $g\delta_{ij}$. If the species index i could be identified with the momentum k, then indeed it could be the general form for the ideal exclusion statistics. We will do precisely this in this chapter and demonstrate [61] by a first principles calculation that this is so for model hamiltonian systems in one space dimension. In particular we analyse the (CSM)[39, 40, 41] and the quasiparticles in the Luttinger model[58, 59]. The inverse square interaction can thus be looked upon as a pure statistical interaction in one dimension where the statistical parameter is related to the interaction strength. In this chapter we discuss the mapping of interacting particles of the CSM to a system of non-interacting quasi-particles obeying FES. We follow the method

outlined by us in Ref.[61]. Alternative approaches may be found in [60, 62].

4.1 The Calogero-Sutherland model

The hamiltonian of the system of interacting particles in this model is given by,

$$H = \sum_{i=1}^{N} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y_i^2} + \frac{1}{2}\omega^2 m^2 y_i^2 \right] + \frac{\hbar^2}{m} \lambda \sum_{i<j=1}^{N} (y_i - y_j)^{-2}, \tag{4.1}$$

where the particles, whose positions are denoted by y_i, are confined in a harmonic well and the thermodynamic limit is obtaining by taking $\omega \to 0$. The particles can also be put on a circle with $|y_i - y_j|$ replaced by the chord length $|sin\frac{\pi}{L}(y_i - y_j)|$ in the interaction term as was done in eq.(3.30). At special values of the coupling the model can be mapped on to particular matrix models[90]. While all these models have fractional exclusion statistics in the sense defined by Haldane, not all of them may be regarded as ideal in the thermodynamic limit unlike the CSM. For convenience, here after, we use the dimensionless variable $x_i = \sqrt{\frac{m\omega}{\hbar}} y_i$.

The two-body problem :

In order to determine the relationship between the interaction parameter λ and the statistical parameter g we first consider the two-body problem, the minimum necessary to bring in statistical nature of the interaction. The hamiltonian can be written as

$$H = \hbar\omega \sum_{i=1}^{2} \left[-\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\omega^2 x_i^2 \right] + \lambda \frac{1}{(x_1 - x_2)^2} \tag{4.2}$$

We make the following ansatz for the solution:

$$\psi = X^{\alpha/2} Y^{\gamma}\, \phi(x_1, x_2)\, e^{-(x_1^2 + x_2^2)/2}, \tag{4.3}$$

where

$$X = (x_1 - x_2)^2, \quad Y = (x_1 - x_2).$$

The Gaussian term is the usual factor one encounters in any harmonic potential problem and arises from the long distance behaviour of the wave function. As we will see below the Jastrow factors X, Y are necessary to have the correct short distance behaviour or a regular solution. The reason for keeping both X and Y is that it allows one to switch the basis from bosonic to fermionic by just setting γ appropriately. Substituting this in the Schroedinger equation

$$H\psi = E\psi$$

we obtain an equation for the function ϕ after taking out the long and short range parts of ψ as

$$H_2 \phi = E\phi$$

where the reduced hamiltonian H_2 is given by

$$\begin{aligned} H_2 &= -\frac{1}{2}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right] + \left[x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right] \\ &+ \frac{1}{X}\left[\lambda - (\alpha+\gamma)(\alpha+\gamma+1) + (\alpha+\gamma) Y \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) \right] + (\alpha+\gamma+1). \end{aligned} \tag{4.4}$$

It is clear that $\psi(x_1, x_2)$ is a solution with well defined eigenvalues provided

$$\lambda = (\alpha + \gamma)(\alpha + \gamma + 1) \tag{4.5}$$

in which case the reduced hamiltonian becomes

$$H_2 = -\frac{1}{2}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] + \left[x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right] + \frac{Y}{X}(\alpha+\gamma)(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}) + (\alpha + \gamma + 1). \tag{4.6}$$

The equation becomes separable if

$$(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2})\phi = 0.$$

This will yield a special class of solutions. However the spectrum obtained therefrom remains exact [91]. The solution for ϕ is given by

$$\phi(x_1, x_2) = \frac{1}{N!}\sum_{m=0}^{N} {}^N C_m \; H_m(x_1) H_{N-m}(x_2) \tag{4.7}$$

which is a symmetric function of x_1, x_2 and H_m is a Hermite polynomial. The full solution is therefore given by eq.(4.3) and the energy eigenvalues are given by

$$E_N = \hbar\omega(N + \alpha + \gamma + 1) \; , \tag{4.8}$$

where N is the principle quantum number.

Let us consider the fermionic basis first $\gamma = 1$. The spectrum of bosons and fermions are then given by

$$E_N = \hbar\omega(N + \alpha + 2), \tag{4.9}$$

where α is a constant shift in energy and can be calculated in terms of the interaction strength

$$2\alpha = -1 \pm \sqrt{1 + 4\lambda} \tag{4.10}$$

For bound states α should be real and we get the condition $\lambda \geq -1/4$.

Since the energy spectrum is known we can calculate the second virial coefficient exactly. We use this approach to relate the interaction parameter λ to the statistics parameter g. The second virial coefficient for a system in one dimension is given by

$$a_2 = \frac{Z_1}{2}\left[1 - 2\frac{Z_2}{Z_1}\right], \tag{4.11}$$

where Z_1, Z_2 are single particle and two particle partition functions respectively. In the high temperature limit we have

$$a_2 = \frac{\alpha}{2} + \frac{1}{4} \tag{4.12}$$

Since it is also well known that the virial coefficients are finite for the CSM system in the high temperature limit, we can directly apply the definition of the FES parameter g through the second virial coefficient to the CSM given in eq.(2.27):

$$\frac{1}{2} - g = -2a_2 \quad \Rightarrow g = \alpha + 1 = (1 \pm \sqrt{1 + 4\lambda})/2. \tag{4.13}$$

Since $\lambda > -1/4$, the lower branch gives $\alpha < -1/2$ while the upper branch gives $\alpha > 1/2$. Consequently we have $g \leq 1/2$ in the lower branch and $g > 1/2$ in the upper branch.

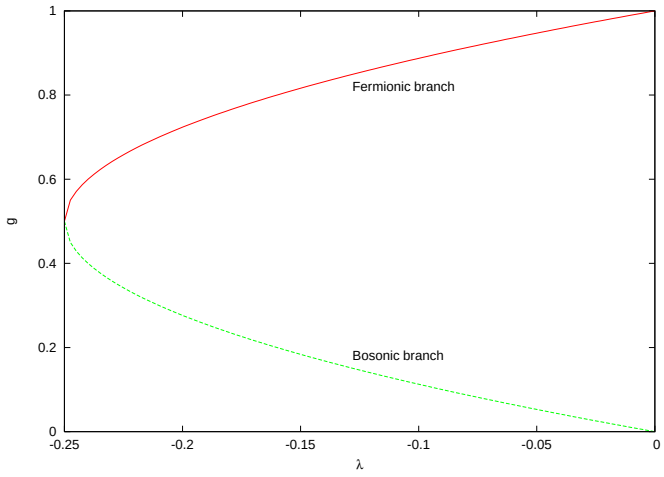

Figure 4.1: Statistics parameter g shown as a function of the interaction parameter λ. The upper and lower branches are identified to include the noninteracting fermion ($g = 1$) and boson ($g = 0$) limits withing this variation. λ has no upper limit for the fermionic branch.

Therefore the $g = 1/2$ semion point is somewhat special. If we now demand that the noninteracting fermionic value of $g = 1$ be contained in the solution, we should choose the upper branch for fermions, see Fig.(4.1). Note that the above equations imply that α may be negative and may even go up to -1 since the wave function is well behaved. However $g \geq 0$ for all allowed values of α and we have the relationship between the interaction parameter λ and the statistical parameter g given by

$$\lambda = g(g-1) \tag{4.14}$$

Similar considerations apply in the bosonic basis with $\gamma = 0$ and $E = N+\alpha+1$. Following the steps outlined for fermionic basis and using the second virial coefficient we find

$$\frac{1}{2} - g = -a_2 \quad \Rightarrow g = \alpha = (1 \pm \sqrt{1+4\lambda})/2 \tag{4.15}$$

and the interaction parameter is related to the statistics parameter g through eq.(4.14). The range of variation of g in the bosonic basis is also shown in Fig.(4.1).

Many-body problem As noted before, the spectrum of CSM hamiltonian in eq.(4.1) is exactly known. The states can be labelled by a set of occupation numbers $\{n_k\},\ k = 1, ..., \infty$. The energy is given by,

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_k n_k + \hbar\omega\alpha \frac{N(N-1)}{2}, \tag{4.16}$$

where $\epsilon_k = k\hbar\omega,\ N = \sum_{k=1}^{\infty} n_k$. We have substracted out the zero point energy in the above expression for the spectrum of the CSM Hamiltonian.

Hereafter we choose to work with the Fermionic basis. Independent of which basis we are working, it is easy to see that the Hamiltonian may always be written as,

$$\frac{H}{\hbar\omega} = \sum_{i=1}^{N} \left[-\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\omega^2 x_i^2 \right] + g(g-1) \sum_{i<j=1}^{N} (x_i - x_j)^{-2}, \tag{4.17}$$

where we have replaced the interaction parameter λ by the statistical parameter g as obtained in the two-body analysis. The spectrum of states may then be written as

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_k n_k - (1-g)\hbar\omega \frac{N(N-1)}{2}, \tag{4.18}$$

where n_k are the fermionic occupancies (0,1). Therefore the spectrum as a function of g remains the same once we correct for the difference in the oscillator energies for bosons and fermions. For all practical purposes involving only the spectra, one may assume that the spectra is valid for all $g \geq 0$. In the following discussions we work in the fermionic basis with this proviso.

The energy functional can also be written as,

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_k n_k - \hbar\omega(1-g) \sum_{k_1<k_2=1}^{\infty} n_{k_1} n_{k_2}. \tag{4.19}$$

The exact spectrum of the model is thus reproduced by an effective hamiltonian of quasiparticle with constant density of states and constant Landau parameters[92]. As we had discussed earlier it is this scale invariant energy shift that is the basic reason for the occurrance of nontrivial exclusion statistics.

Thus the particles in the CSM are FES particles in the sense of Haldane with the exclusion statistics parameter equal to g. We will now show this more directly by deriving the formula for the N-particle Hilbert space dimension. The approach is similar to the one given in Chapter 3, where we argued using the trigonometric Sutherland model using shifted moments due to interaction. To this end, we define $N(\epsilon, 0)$ as the number of particles with energy $\epsilon_k < \epsilon$, ie., $N(\epsilon, 0) = \sum_{k=1}^{\infty} \theta(\epsilon - \epsilon_k) n_k$, where $\theta(x) = 0$ for $x \leq 0$ and 1 for $x > 0$. We may now define the shifted single particle energies as

$$\epsilon_A(k, g) = \epsilon_k - \hbar\omega(1-g)N(\epsilon_k, 0). \tag{4.20}$$

We identify this shifted energy with the single particle energy since the total energy can now be written as

$$E[\{n_k\}] = \sum_{k=1}^{\infty} \epsilon_A(k, g) n_k \tag{4.21}$$

In Fig.(4.1) we display the single particle energies as a function of g starting from a fermionic ground state where all the levels up to a given cutoff are occupied (see the figure on the left).

Notice at $g = 1$, the occupencies are 0 or 1 corresponding to fermionic occupancies and it can be arbitrary at the bosonic end, $g = 0$. While for $g = 0$ or $g = 1$, we can define these occupencies without any reference to the *cell size*, in order to define the possible occupation numbers for arbitrary g, we choose the oscillator frequency ω to provide the size[64]. This does not alter the results at boson and fermion ends. For simplicity we look at occupation numbers for special values of $g = 1/m$ where m is an integer. For the ground state spectrum displayed in Fig.(4.1), the occupation of each cell of size ω is maximal, that is, it is equal to m. However, when one removes particles below the cutoff, as shown in Fig.(4.1) to the right, the occupation can be less than or equal to m. The position of the holes also changes because of the spectral flow in g. The rule that emerges for occupation number of quasiparticles follows the generalised Pauli principle stated in chapter 3.

If we allow arbitrary occupation numbers up to a maximum of m without this restriction, then one is lead to statistics different from that of Haldane, namely the Gentile statistics[93].

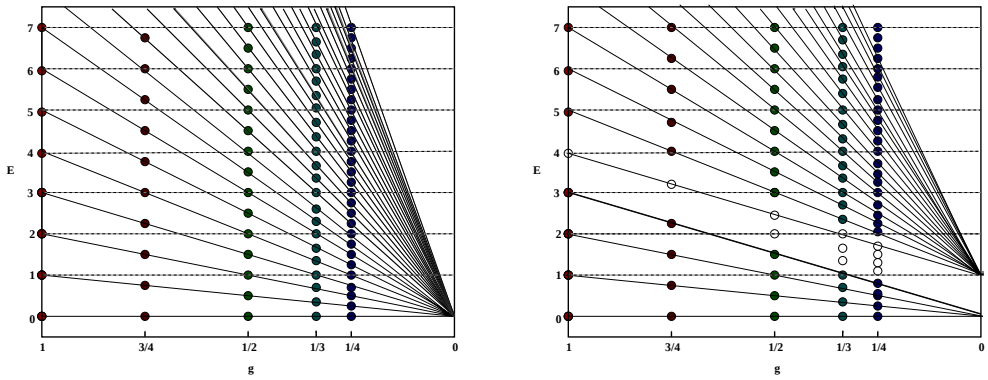

Figure 4.2: The ground state of many fermions in CSM as a function of statistics parameter g is shown on the left. The figure on the right shows an excited state configuration with a hole in the fermionic end. The noninteracting fermion ($g = 1$) and boson ($g = 0$) limits are also shown.

Now assume that there are some d states up to the cutoff and a total of N particles. Using rules of occupancy, one can count for each value of m (or g), the total number of states for the N particle system. Indeed it turns out to be the dimension of the N particle Hilbert space given by Haldane, namely

$$D_N(g) = \frac{(d + (1-g)(N-1))!}{N!(d-1-g(N-1))!}.$$

In the next section, it will be shown that even the distribution function is valid for CSM quasiparticles in the thermodynamic limit.

4.2 The distribution function of CSM particles

The distribution function for exclusion statistics particles is given by eq.(2.35). In deriving the distribution function, the statistics of the particles was assumed as given by Haldane. We now show that the interacting particles in CSM may be thought of fractional exclusion particles with the same distribution function. In this section we first define quasiparticle number densities and show that their thermal average is exactly given by the distribution function in eq.(2.35).

The number of particles with energy less than ϵ, from eq.(4.21), is given by

$$N(\epsilon, g) = \sum_{k=1}^{\infty} \theta(\epsilon - \epsilon_A(k, g)) n_k.$$

The quasiparticle number density in the thermodynamic limit is then given by

$$n_A(\epsilon, g) = \lim_{\Delta\epsilon \to 0} \lim_{\omega \to 0} \frac{N(\epsilon + \Delta\epsilon, g) - N(\epsilon, g)}{\Delta\epsilon}, \tag{4.22}$$

where the limit $\omega \to 0$ is to be taken first.

We will now derive a differential equation for $\bar{N}(\epsilon, g)$, the thermal average of $N(\epsilon, g)$ and solve it to obtain the distribution function of CSM quasiparticles whose dispersion is given by eq.(4.20). Consider a state with N-particles labelled by $\{k_i\}, i = 1, ..., N$ ordered such that $k_{i+1} > k_i$. Thus for this state $N(\epsilon_{k_i}, 0) = i - 1$. The i^{th} shifted energy is

$$\epsilon_A(k_i, g) = \epsilon_{k_i} - \omega(1-g)(i-1).$$

Note that $\epsilon_A(k_{i+1}, g) - \epsilon_A(k_i, g) \geq \omega g > 0$. Thus the quasiparticle energies $\epsilon_A(k_i, g)$ also increase monotonically with i. We therefore have

$$N(\epsilon_{k_i}, g) = N(\epsilon_{k_i}, 0) = i - 1 \tag{4.23}$$

To get some feel for the number density, let us consider the ground state of the N particle system. The set $\{k_i\}$ is (1,2,3,...,N). The corresponding set of shifted energies is $(\omega, (1 + g)\omega, (1 + 2g)\omega, \cdots, (1 + (N - 1)g)\omega)$. The number of particles in the interval ϵ to $\epsilon + \omega$ is then

$$\begin{aligned} n_A(\epsilon, g) &= \frac{1}{g} \ , \quad \epsilon < \epsilon_F = g\bar{\rho} \\ &= 0 \ , \quad \epsilon \geq \epsilon_F \ , \end{aligned} \tag{4.24}$$

where $\bar{\rho} \equiv \omega N$. This is the same as the zero temperature limit of the distribution function in eq.(2.35)(see also Fig.(4.1)).

Next we compute how the quasiparticle density changes with g. From eqs.(4.22) and (4.20) we see that as g increases the particles move to the right in ϵ space with a *velocity* given by $\omega N(\epsilon, g)$. Thus the number of particles crossing a point ϵ when g increases by Δg, is given by the velocity at the point of crossing multiplied by the density at that point. We thus obtain the differential equation for the density flow

$$\frac{\partial \rho(\epsilon, g)}{\partial g} = \rho(\epsilon, g) \frac{\partial \rho(\epsilon, g)}{\partial \epsilon}, \tag{4.25}$$

where $\rho(\epsilon, g) = \omega N(\epsilon, g)$. Denoting the thermal average of $\rho(\epsilon, g)$ as $\bar{\rho}(\epsilon, g)$ and using the fact that in the thermodynamic limit, $\bar{\rho^2}(\epsilon, g) = (\bar{\rho}(\epsilon, g))^2$, neglecting fluctuations, we obtain the differential equation

$$\frac{\partial \bar{\rho}(\epsilon, g)}{\partial g} = \bar{\rho}(\epsilon, g) \frac{\partial \bar{\rho}(\epsilon, g)}{\partial \epsilon}. \tag{4.26}$$

The distribution function $\bar{n}_A(\epsilon, g)$ can be obtained from the solution of eq.(4.26) by using

$$\bar{n}_A(\epsilon, g) = \frac{\partial \bar{\rho}(\epsilon, g)}{\partial \epsilon}. \tag{4.27}$$

Thus we need the solution to eq.(4.26), with the boundary condition

$$\left. \frac{\partial \bar{\rho}(\epsilon, g)}{\partial \epsilon} \right|_{g=1} = \frac{1}{e^{\beta\epsilon} + 1} \tag{4.28}$$

so as to match the result at the fermionic end.

We will now show that eq.(4.26) along with the boundary condition (4.28) is satisfied by

$$\bar{\rho}(\epsilon, g) = \int_0^{\epsilon} d\epsilon' \frac{1}{w(\epsilon', g) + g} \ , \tag{4.29}$$

where $w(\epsilon, g)$ is determined through eq.(2.36) and is consistent with the Wu distribution function through eq.(4.27). By changing variables from ϵ to w in eq.(4.29) using eq.(2.36), the integral can be written as

$$\bar{\rho}(\epsilon, g) = \frac{1}{\beta} \int_{w(0)}^{w(\epsilon)} dw \frac{1}{w(w + 1)} \ . \tag{4.30}$$

The integration can be done easily and we get

$$\bar{\rho}(\epsilon, g) = \frac{1}{\beta}(ln(\frac{w(\epsilon)}{1+w(\epsilon)}) - ln(\frac{w(0)}{1+w(0)})) \tag{4.31}$$

The condition $\bar{\rho}(\infty, g) = \bar{\rho}$ and the fact that $\lim_{\epsilon\to\infty} w(\epsilon, g) \to \infty$ implies that $w(0)$ is independent of g and

$$e^{-\beta\mu} = w(0)^g(1+w(0))^{1-g}. \tag{4.32}$$

Furthermore it is easy to verify that

$$\frac{\partial}{\partial g}w(\epsilon) = \frac{w(w+1)}{(g+w)}\bar{\rho}(\epsilon, g), \tag{4.33}$$

$$\frac{\partial}{\partial \epsilon}w(\epsilon) = \beta\frac{w(w+1)}{(g+w)}. \tag{4.34}$$

Using these results it can be verified that the form of $\bar{\rho}(\epsilon, g)$ in eq.(4.29) does satisfy eq.(4.26). Thus the CSM quasiparticle distribution function is given by

$$\bar{n}_A(\epsilon, g) = \frac{1}{w(\epsilon, g) + g} \tag{4.35}$$

which is exactly the distribution function derived by Wu [55].

It is important to note that the starting point of the above derivation is a system of interacting fermions in CSM. We first obtained the total energy of the interacting system as a sum of single particle, albeit shifted, energies. This allowed us to formulate the basic differential equation for the number density as a function of g and energy whose solution is the distribution function of derived by Wu by assuming exclusion statistics from the beginning. The result indeed proves that the CSM quasi-particles are indeed ideal particles obeying exclusion statistics.

4.3 Partition function and equation of state

Recall that the grand canonical partition function of a system with flat distribution is given by eq.(3.11). We first write this in general for a system with arbitrary dispersion before we apply it to CSM. The grand partition function of an ideal gas with FES, may be written in the standard form as

$$Z_G = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N. \tag{4.36}$$

The canonical partition Z_N is given by

$$Z_N = \sum_{\{n_k\}} g(\{n_k\}) e^{-\beta E(\{n_k\})} \; ; \quad \sum_k n_k = N \tag{4.37}$$

where $g(\{n_k\})$ denotes the degeneracy of states at a given energy E. For Fermi or Bose gas $g(\{n_k\}) = 1$ and the summation in eq.(4.36) reduces to a factorised form for these cases. The additivity property of the quasi particle energies in eq.(7.33) immediately suggests that the grand partition function of CSM should be expressible in a factorized form. The canonical partition function for the spectrum in eq.(7.32) is given by

$$Z_N = e^{\beta\omega(1-g)\frac{N(N-1)}{2}} Z_N^F, \tag{4.38}$$

where Z_N^F is the N particle fermion partition function. Setting $g = 0$, the bosonic partition function is obtained,

$$Z_N^B = e^{\beta\omega\frac{N(N-1)}{2}} Z_N^F, \tag{4.39}$$

Combining eqs.(7.35) and (7.36) we may write the canonical partition function for CSM as,

$$Z_N = (Z_N^F)^g (Z_N^B)^{1-g}. \tag{4.40}$$

The grandpartition function may also be written in the form,

$$Z_G = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N = \sum_{N=0}^{\infty} (e^{\beta\mu_F N}(Z_N^F))^g (e^{\beta\mu_B N}(Z_N^B))^{1-g}, \tag{4.41}$$

where we have set the chemical potential

$$\mu = g\mu_F + (1-g)\mu_B.$$

In the thermodynamic limit, the sum is saturated at the value of $N = \bar{N}$ where

$$\bar{N} = \frac{1}{\beta}\frac{\partial ln Z}{\partial \mu}.$$

Neglecting fluctuations, the grand partition function can then be written as,

$$Z_G(\beta,\mu) = (Z^F(\beta,\mu_F))^g (Z^B(\beta,\mu_B))^{1-g}. \tag{4.42}$$

This is exactly the grand partition function of a system obeying FES at constant density of states as seen from eq.(2.64) where we showed that the grand canonical partition function for a gas in two dimensions, that is for constant density of states, can be written in the form given by eq.(4.42).

We now consider the equation of state of the quasiparticles in CSM. This has already been done by Sutherland[40] who however chose to work with the fugacity expansion. The coefficients of terms in the fugacity expansion in general depend on g and appear quite complicated. However, the coefficients of the virial expansion in average density $\bar{\rho}$, in the dilute limit, are extremely simple. From the product form of the partition function in eq. (4.42), the virial coefficients in CSM can be computed exactly using the well known virial expansion for the Fermi and Bose systems with oscillator regularisation. The pressure is given by

$$P = g(P_F) + (1-g)(P_B), \tag{4.43}$$

where

$$\beta P_F = \bar{\rho}\left[1 + \frac{2\pi\beta\bar{\rho}}{4} + \sum_{n=2}^{\infty}\frac{\mathcal{B}_n}{(n+1)!}(2\pi\beta\bar{\rho})^n\right] \tag{4.44}$$

and

$$\beta P_B = \bar{\rho}\left[1 - \frac{2\pi\beta\bar{\rho}}{4} + \sum_{n=2}^{\infty}\frac{\mathcal{B}_n}{(n+1)!}(2\pi\beta\bar{\rho})^n\right]. \tag{4.45}$$

Here B_ns are the Bernoulli numbers. Notice the difference between P_F and P_B entirely comes from the second term while all the other virial coefficients are identical. When combined in eq.(4.43) we have the virial expansion for the system obeying CSM model. Thus the equation of state in the dilute limit is given by

$$\beta P = \bar{\rho}\left[1 + \frac{2\pi\beta\bar{\rho}}{4}(2g-1) + \sum_{n=2}^{\infty}\frac{\mathcal{B}_n}{(n+1)!}(2\pi\beta\bar{\rho})^n\right]. \tag{4.46}$$

The information about the statistics of the system is entirely given by the second virial coefficient, $a_2 = \frac{1}{2}(g - 1/2)$ while all other virial coefficients are independent of g. It is interesting to note that the interaction affects only the second virial coefficient. As we had shown [54], it is the second virial coefficient that determines the exclusion statistics. Thus the $\frac{1}{r^2}$ interaction in this system modifies the equation of state in a minimal way. It can therefore be thought of as a purely statistical interaction in CSM. It also follows that the theorem of May[78] that the specific heat is independent of statistics is also valid for quasiparticles of CSM. One should however be careful in interpreting these results in the thermodynamic limit due to the presence of the confinement potential [94, 95].

The equation of state for gases obeying FES for arbitrary dispersion and arbitrary dimension is given by Isakov et al [79]. In particular they also find that the equation of state of a free gas of particles obeying FES in two dimensions has similar expansion as in eq.(4.46). The statistical parameter dependence is contained in the second virial coefficient where as all other virial coefficients are independent of g. In fact they are identical to the virial coefficients given by eq.(4.46). This should not be surprising since the density of states is constant in both cases although they refer to two different physical models. As we showed in the beginning of this section, the product form of the partition function (see eq.(4.42) follows from the fact that the density of states is constant and it is also true of quasiparticles of CSM. It is this property that underlies the interesting consequences for the equation of state.

4.4 Quasiparticles in the Luttinger Model

We will now consider quasiparticles in the Luttingermodel. We note that this spectrum (eq.(7.32))is identical to the spectrum of quasi-particle states in a gaussian theory of compact bosons[54, 91] with radius $R = 1/\sqrt{(}g)$ with the following identification- while k in the above equation is a state label in CSM, in the gaussian theory of compact bosons k refers to the box quantized momenta. The discussion in the subsequent sections thus applies to this case also and we will elaborate on these cases in the next section. As can be seen from eq.(7.32), the effect of the interaction is that each particle shifts the energy of every other particle by a constant $\omega(g-1)$. The low energy physics of this model maps on to the massless Thirring model. As is well known[58, 59], the model is exactly solvable. The theory can be written completely in terms of the left(L) and right(R) currents satisfying the algebra

$$[J_n^r, J_m^{s\dagger}] = n\delta_{mn}\delta_{rs},$$

where $n, m = 1, 2, ...$ and $r, s = L, R$. We also have the zero modes

$$[\phi_0^r, J_0^s] = i\delta_{rs}.$$

The hamiltonian is

$$H = \frac{2\pi v_F}{L}\sum_r \left[\frac{1}{2}J_{0r}^2 + \sum_{n=1}^{\infty} J_n^r J_n^{r\dagger}\right]. \tag{4.47}$$

Quasiparticles are created by the action of the vertex operators $V_r^\dagger(x) =: e^{-iq^r\phi^r(x)} :$ on the ground state. The $\phi^r(x)$ are the bosonic phase fields given by,

$$\phi^r(x) = \phi_0^r(x) + \frac{2\pi}{L}\sigma_r\left[xJ_{0r} + \sum_{n=1}^{\infty}\left(\frac{e^{-i\sigma_r\frac{2\pi}{L}nx}}{in}J_n^r + hc\right)\right], \tag{4.48}$$

where $\sigma_{L,R} = \pm 1$. The periodicity properties of the compact zero modes $\phi_0^{L,R}$ constrain the allowed values of q_r to be $q_r = (NR + \sigma_r \frac{M}{2R})$ and R is the radius of the field ϕ. It is a function of the interaction strength. Our convention correspond to $R = 1$ for the case of noninteracting fermions. The constraints on q_r imply that quasiparticles in the left sector must be created along with quasiparticles in the right sector (except for special values of R). However because the hamiltonians of the two sectors are completely decoupled, it is consistent to analyse the spectrum of the two sectors independently. We will therefore focus on the left sector alone.

We first consider the space of one quasiparticle states which we define to be the space of the states $|x>= V_L^\dagger(x)|0>$. These are not a linearly independent set. From the form of $V_L^\dagger(x)$, it follows that $|x + L>= e^{i\alpha\pi}|x>$, where $\alpha = q_L^2$. Therefore, we have the expansion $|x>= \sum_{n=1}^{\infty} e^{ik_n x}|n>$, where $k_n = \frac{2\pi}{L}(n + \frac{\alpha}{2})$. From the form of $V_L^\dagger(x)|0>$ it follows that $n > 0$. It can be easily shown that $|n>$ form an orthogonal set of eigenstates of the Hamiltonian with eigenvalues $E_n^{(1)} = v_F k_n$. The single particle partition function is then given by,

$$Z_1 = \frac{e^{-\tilde{\beta}\alpha/2}}{1 - e^{-\tilde{\beta}}} = e^{-\tilde{\beta}\alpha/2} Z_1^B; \qquad \tilde{\beta} = \frac{2\pi}{L}\beta v_F. \tag{4.49}$$

Next we come to the N quasiparticle states which we define to be the span of $|\{x_n\}>= \prod_{n=1}^N V_L^\dagger(x_n)|0>$. From the fact that $V_L^\dagger(x)V_L^\dagger(y) = e^{i\alpha\pi}V_L^\dagger(y)V_L^\dagger(x)$, it follows that these many quasiparticle states pick up a phase $e^{i\alpha\pi}$ under the exchange of two of the coordinates. We normal order the vertex operators to obtain, $|\{x_n\}>= \prod_{n>m}\left[sin\left(\frac{\pi(x_n - x_m)}{L}\right)\right]^\alpha : \prod_{n=1}^N V_L^\dagger(x_n) : |0>$. Again from the form of the vertex operators it follows that we have the expansion,

$$|\{x_n\}>= \prod_{n>m}^{N}\left[sin\left(\frac{\pi(x_n - x_m)}{L}\right)\right]^\alpha \sum_{\{k_n\}} \phi_N^B(\{k_n\}|\{x_n\})|\{k_n\}>, \tag{4.50}$$

where $\phi_N^B(\{k_n\}|\{x_n\})$, are symmetrized N particle plane waves with momenta $\{k_n\}$, where

$$k_n = \frac{2\pi}{L}(n + \frac{N\alpha}{2}); \quad n > 0.$$

The states $|\{k_n\}>$ can be shown to be eigenstates of the hamiltonian with energy $E^N(\{k_n\}) = v_F \sum_{n=1}^N k_n$. This exactly reporduces the spectrum of states one gets in the CSM(see eq.(4.20)). We can show that $|\{k_n\}>$ are a linearly independent set of states. The states with the same energy are however not orthogonal.

The N-quasiparticle partition function is then

$$Z_N = e^{-\tilde{\beta}N^2\alpha/2} Z_N^B,$$

where Z_N^B is the N-particle bosonic partition function. We can now exactly compute g using the high temperature expansion for Z_N^B and obtain $g = \alpha$. Thus the exchange and exclusion statistics parameters are identical for these models. Note that the exchange statistics in one dimensions is somewhat arbitary. We could have changed it by multiplying the vertex operators with suitable cocycle factors. The exclusion statistics parameter is however unambigous and unique. When $R = \frac{1}{\sqrt{2}}$, the theory is equivalent to the low energy physics of the SU(2) symmetric quantum antiferromagnetic chain. We then have the spinon excitations with $\alpha = 1/2$. Thus we recover Haldane's result that $g = 1/2$ for this case.

The above example gives a clear insight into the mechanism of the phenomenon. What is happening is that the addition of a quasiparticle causes a phase shift of every other

quasiparticle, resulting in an energy shift of $\frac{\alpha\pi v_F}{L}$ per particle . When we count the dimension of the single particle space with a fixed (smooth) cutoff, there are α states missing. The important thing here is that the all single particle levels shift up by the same amount however high the energy. This is why we get g to be well defined and nontrivial in the cutoff going to infinity limit. We will come back to this intuitive but physical arguement again in the last section.

Chapter 5

Exchange and Exclusion Statistics

As noted in Chapter 1, a way of characterising the statistics of identical particles is through their properties under exchange as in the case of ideal fermions and bosons. In general we characterise the particles as bosons or fermions if under the exchange of any two particles in an identical particle system, the wave function is either symmetric or antisymmetric. Pauli exclusion principle naturally follows from the antisymmetry of the fermionic wave function. In this chapter we look at the inter-relation between exclusion and exchange statistics in two-space dimensions.

Topological considerations allow us to generalise the definition of exchange in two space dimensions. In 1977, Leinaas and Myrheim[1] showed that in two space dimensions it is possible to have particles obeying intermediate statistics different from the well known Bose-Einstein and Fermi-Dirac statistics. Later, Wilczek[3] coined the name *anyons* for particles obeying these peculiar statistics under exchange.

It is beyond the scope of this book to discuss the quantum mechanics and statistical mechanics of anyons in detail and can be found elsewhere[6, 7, 9]. Recent review on this topic by Ouvry[10] clarifies a number of issues including a detailed discussion of anyons in the lowest Landau level which is of relevance here. We first develop the concept of anyons in two dimensions through multi-valued wave functions. The exchange statistics of anyons then involves a continuous parameter, α–the so called statistical parameter. It is the peculiarity of two dimensions that the quantisation does not depend of the particular value of α, unlike in higher dimensions where we require α to be an integer.

In the following, by *anyons* we mean a quantum mechanical system of N particles in two dimensions with wave functions which have a stipulated multi-valuedness to be specified below. To make this explicit[9], let us denote a generic multi-valued wave function as $\psi(\vec{r}_1, ..., \vec{r}_N)$, where $\vec{r}_i$ denotes the position vector of a particle. Let $[P_{ij}]_\gamma$ denote the operation of taking the ith particle coordinate around the jth coordinate along a closed path γ. The path γ does not enclose any other particle coordinate and is taken in an anti-clockwise sense, say. Then let us stipulate that under such an operation ψ acquires a phase namely

$$[P_{ij}]_\gamma \psi(\vec{r}_1, ...\vec{r}_N) = exp(i2\pi\alpha)\psi(\vec{r}_1, ...\vec{r}_N). \tag{5.1}$$

If a path γ encloses other particle coordinates as well then such a path can be broken into a set of closed paths each of which encloses exactly one particle. Applying the stipulation above, one can compute the total phase, for such a path. If the sense of the path is reversed then $\alpha \to -\alpha$. Clearly the phase acquired depends only on the homotopy class of the path (i.e., it is the same for two paths γ and γ' if γ and γ' can be continuously deformed into each other)- α is then the statistics parameter. Because of the periodicity of the phase in eq.(5.1), the parameter α may be restricted as $0 \leq \alpha \leq 2$

Let us introduce the complex notation for particle coordinates: $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$. Clearly z_{ij}, where $z_{ij} = z_i - z_j$, has the property that if z_j is taken around z_i, z_{ij}^{α} changes by $\exp(i2\pi\alpha)$. This allows us to write any generic wave function satisfying eq.(5.1) as,

$$\psi(z_i, \bar{z}_i) = \left[\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2}\right] \tilde{\psi}(z_i, \bar{z}_i), \tag{5.2}$$

with the bracketed expression being a phase that encodes the property of the wave function ψ under exchange completely. Thus we have the freedom to choose $\tilde{\psi}(z_i, \bar{z}_i)$ as a single valued function. In particular we may choose $\tilde{\psi}$ to be either bosonic (symmetric) or fermionic (antisymmetric) under exchange.

Clearly,

$$\nabla_k \psi(z_i, \bar{z}_i) = \prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2} \left[\nabla_k \tilde{\psi}(z_i, \bar{z}_i) + \nabla_k ln \left(\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2}\right) \tilde{\psi}(z_i, \bar{z}_i)\right]. \tag{5.3}$$

which can be rewritten as,

$$\nabla_k \psi(z_i, \bar{z}_i) = \prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2} \left[\nabla_k \tilde{\psi}(z_i, \bar{z}_i) + i\alpha \sum_{j\neq k} \frac{\hat{z} \times \vec{r}_{kj}}{\mid \vec{r}_{kj} \mid^2} \tilde{\psi}(z_i, \bar{z}_i)\right]. \tag{5.4}$$

Since $\tilde{\psi}$ is single valued, the right hand side of the above equation has exactly the same multivaluedness as the left hand side. In other words we have,

$$\nabla_k \left[\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\alpha/2} \tilde{\psi}(z_i, \bar{z}_i)\right] = \left[\prod_{i<j} \left(\frac{z_{ij}}{\bar{z}_{ij}}\right)^{\frac{\alpha}{2}}\right] D_k \tilde{\psi}, \tag{5.5}$$

where

$$D_k \tilde{\psi} = [\nabla_k + \vec{A}_k] \tilde{\psi}$$

and

$$A_k(\vec{r}_k) = i\alpha \sum_{j\neq k} \frac{\hat{z} \times \vec{r}_{kj}}{\mid \vec{r}_{kj} \mid^2}.$$

The multi-valued wave function ψ can be interchanged with a single valued bosonic (or fermionic) wave function but at the cost of introducing a singular vector potential.

Thus any higher order differential operators on ψ can be written in terms of corresponding covariant differential operators on the single-valued wave function $\tilde{\psi}$. In particular a Hamiltonian operator, typically $-\sum_i \nabla_i^2 + V$ can be written similarly. An eigenvalue equation written in terms of ψ can then be recast as a corresponding equation in terms of $\tilde{\psi}$ involving the covariant derivative. Note that the field strength therefore is singular involving Dirac $\delta(\vec{r}_{jk})$. As pointed out by Ouvry[10], because of the singular nature of the vector potential, the perturbation theory is not well defined and a renormalisation procedure has to be implemented by adding a counter term. Equivalently we can define the configuration space of N particles with all the coincident points removed, that is the configuration space $Q^N = (R_2)^N - \Delta$.

Although both formulations–single valued or multi-valued– are equivalent, dealing with operators on multi-valued wave functions is much less transparent than dealing with operators on single-valued wave functions. Naive commutation rules, symmetries that one would expect by looking at an operator on single valued functions are not at all true in general for the "same" differential operators acting on multi-valued wave function.

Considering eigenvalue problem in terms of $\tilde{\psi}$ has other advantages too. Since all the subtleties of multivaluedness are equivalently transcribed in terms of additional "interaction" terms (the so called statistical interactions), the eigenvalue problem is amenable to solution in principle. One is also on firmer ground in doing usual algebraic manipulations with operators. With these in mind we will work with single-valued wave functions with "statistical interactions".

As a first step one would like to understand the system of *free anyons*. However, the statistical interaction falls off as $\mid r_{ij} \mid^{-2}$ as $\mid r_{ij} \mid \to \infty$. So one is not sure whether the Hamiltonian with only statistical interactions has only discrete eigenvalues. An oscillator potential ensures discrete spectrum without introducing a finite size. One could take some other confining potential but in the limit $\alpha \to 0$ one should know the spectrum. One then has hope of doing at least the perturbative analysis [26, 28]. Since the statistical interaction depends only on relative separations, the Centre of Mass (CM) dynamics should play a trivial role and oscillator potential also allows a separation of CM and relative coordinate dynamics. Bearing these things in mind, we choose the oscillator potential without further justification as in previous chapters. In order to derive the thermodynamic properties of a system of anyons there exist well defined methods of eliminating the dependence on the oscillator frequency [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

The Hamiltonian we consider is – after carrying out the usual scaling of variables – in terms of dimensionless quantities

$$H = \hbar\omega[\frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i=1}^{N} r_i^2 - \alpha \sum_{j>i=1}^{N} \frac{\ell_{ij}}{r_{ij}^2} + \frac{\alpha^2}{2}\sum_{i\neq j,k}^{N} \frac{\vec{r}_{ij}.\vec{r}_{ik}}{r_{ij}^2 r_{ik}^2}], \tag{5.6}$$

where

$$\ell_{ij} = (\vec{r}_i - \vec{r}_j) \times (\vec{p}_i - \vec{p}_j).$$

and all distances have been expressed in units of $\sqrt{\frac{\hbar}{m\omega}}$. Notice that the statistical interaction is independent of the centre of mass coordinates. In the quantum analysis, the Hamiltonian is considered to act on wave functions which vanish suitably at the coincident points effectively removing the diagonal points from the configuration space.

We first briefly summarise the results in the case of two anyons which is exactly solvable and use the eigenvalue spectrum to derive results that relate exchange statistics of anyons with their exclusion property.

5.1 Spectrum of two-anyons

The quantum Hamiltonian in the case of two anyons may be written as

$$H = H_{cm} + H_{rel}, \tag{5.7}$$

where H_{cm} is the Hamiltonian that describes the dynamics of the centre-of-mass coordinate $\vec{R} = (\vec{r_1} + \vec{r_2})/2$. Since the statistical interaction is translation invariant, the centre-of-mass part is independent of this. As a result the spectrum of this Hamiltonian is the same as the spectrum of a two dimensional oscillator. That is,

$$E_{cm} = \hbar\omega_{cm}[2n_{cm} + |l_{cm}| + 1]; \quad n_{cm} = 0, 1, 2, \cdots, l_{cm} = 0, \pm 1, \pm 2, \cdots \tag{5.8}$$

Here n_{cm} and l_{cm} denote the radial and angular momentum quantum numbers of the centre-of-mass excitations and $\omega_{cm}^2 = 2\omega^2$.

The relative Hamiltonian is given by,

$$H_{rel} = \hbar\omega[p^2 + r^2 - \alpha\frac{l}{r^2} + \frac{\alpha^2}{2}\frac{1}{r^2}], \tag{5.9}$$

where p is the momentum operator and $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative position vector. The eigenvalue equation is easily solved by noting that the additional α dependent terms, may be combined with the centrifugal barrier with l shifted by $l - \alpha$. The spectrum of the Hamiltonian is then given by,

$$E_{rel} = \hbar\omega[2n + |l - \alpha| + 1]; \quad n = 0, 1, 2, 3, \cdots;\ l = 0, \pm1, \pm2, \cdots \tag{5.10}$$

where n and l denote the radial and angular momentum quantum numbers which label the state. Together with the spectrum of H_{cm}, this provides the complete spectrum of two anyons in an oscillator potential. Indeed this is the only case that can be solved exactly. When l is even(odd) the corresponding wave functions are symmetric(antisymmetric) and therefore the wave function corresponds to a system of two bosons (fermions) interacting via the statistical interaction.

5.2 Statistical parameters g and α

We now use the spectrum of two anyons to derive the second virial coefficient of anyons and relate it to the exclusions behaviour of anyons. In order to apply the results, however, we need to assume that the virial expansion exists for anyon gas. This is a nontrivial assumption since it is not conclusively proved that all the virial coefficients are finite for anyon gas. However the second virial coefficient is easily calculated and it has also been proved that the third virial coefficient is finite[33, 34, 35, 32, 37]. If indeed all virial coefficients are finite, we can relate the exclusion statistics parameter g to the exchange statistics parameter α through the second virial coefficient. We do this now.

The spectrum of two anyons confined in an oscillator potential is given by eq.(5.8) and eq.(5.10) broken into centre of mass and relative parts. The centre of mass contribution does not depend on α the statistical parameter α. This information is sufficient to calculate the partition function of two anyons and hence relate the statistical parameters in exchange and exclusion statistics. In chapter 2 we obtained the relationship

$$\frac{1}{2} - g = 4 \lim_{\beta\to 0} \frac{Z_1}{2}\left[2\frac{Z_2}{Z_1^2} - 1\right], \tag{5.11}$$

where $\beta = 1/kT$ as before and the factor 4 is due to the regularisation using oscillator potential. The single particle partition function Z_1 is given by

$$Z_1 = \sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty} e^{-\hbar\omega\beta(2n+|l|+1)} = \frac{e^{-\hbar\omega\beta}}{(1 - e^{-\hbar\omega\beta})^2}, \tag{5.12}$$

which is of course independent of the interaction parameter α.

Using the spectrum of two anyons we have

$$Z_2 = Z_1 \sum_{n=0}^{\infty}\sum_{l'=-\infty}^{\infty} e^{-\hbar\omega\beta(2n+|l-\alpha|+1)}, \tag{5.13}$$

where the prime over l is used to restrict the summation over l even or odd depending on whether the calculation is done in the bosonic or fermionic basis, that is bosons or fermions interacting through the anyonic potential.

We now calculate the two-particle partition function in the bosonic basis by restricting l to be even in eq.(5.13). Carrying out the summations We have

$$Z_2 = Z_1 \frac{e^{-\hbar\omega\beta}}{1-e^{-2\hbar\omega\beta}}\left[e^{-\hbar\omega\beta\alpha} + \frac{e^{-2\hbar\beta\omega}}{1-e^{-2\hbar\beta\omega}}\right]. \tag{5.14}$$

Substituting Z_2 and Z_1 in eq.(5.11) and taking the high temperature limit we have

$$\frac{1}{2} - g = \frac{1-4\alpha+2\alpha^2}{2}, \tag{5.15}$$

where we recognise that the rhs is related to the second virial coefficient of anyons. It should be noted that the high temperature limit is equivalent to letting $\omega \to 0$ (or the area $A \to \infty$) and the result obtained above is independent of the trap frequency which simply acts as a regulator. Thus we have for anyons the relationship between the exchange and exclusion statistical parameters

$$g = \alpha(2-\alpha). \tag{5.16}$$

This relation reproduces the correct values of g for bosons and fermions. It is however interesting that the exchange statistics parameter α is not the same as the exclusion statistics parameter g, for intermediate values of α. In fact the relationship is not even linear, unlike in the one dimensional systems discussed in the last chapter, where it turned out that $g = \alpha$, where α refers to the interactions strength.

It is also interesting to note that the exclusion statistical parameter g is related to the two types of energy shifts produced in the spectrum in the presence of α. For example, for $l \leq 0$, the two anyon energy levels move up by α and for $l > 0$, the levels move down by $-\alpha$. Equivalently if we add the state $l = 0$ for the set of levels with $l > 0$, then the shift $-\alpha$ may be reinterpreted as $2-\alpha$. Note that all the calculations presented here are done in the bosonic basis where l is even including the $l = 0$ state. As we argued in the case of CSM, the parameter g is indeed related to the energy shift when all levels shift up(or down) no matter where the cutoff lies. In one dimensional systems such a shift immediately gives g in terms of the shift. In the case of anyons, however, it is not clear why g is a product of the two types of shifts in the energy spectrum. As has been commented in the literature[64], this is perhaps due to the fact that anyons may be considered as interacting exclusion statistics particles.

Next we consider anyons in the presence of a uniform magnetic field B while keeping the oscillator confinement. We are specifically interested in anyons in the lowest Landau level (LLL). The single particle spectrum in the presence of a uniform magnetic field is given by

$$E^{(1)}_{n,l} = (2n + |l| + 1)\hbar\Omega - l\hbar\omega_c, \tag{5.17}$$

where

$$\Omega = \sqrt{\omega_c^2 + \omega^2}$$

and $\omega_c = e|B|/2m$ denotes the cyclotron frequency of the magnetic field and ω as usual denotes the strength of oscillator confinement.

The single particle partition function for a particle moving in a uniform magnetic field is given by

$$Z_1(\beta) = \frac{e^{-\beta\hbar\Omega}}{(1-e^{-\beta\hbar\Omega_+})(1-e^{-\beta\hbar\Omega_-})}, \tag{5.18}$$

where $\Omega_\pm = \Omega \pm \omega_c$.

In order to compute g we need to know the two particle spectrum. The centre of mass energy is given by eq.(5.17). The energy spectrum solved in terms of the relative coordinate is given by

$$E^{(2)}_{n,l} = (2n + |l - \alpha| + 1)\hbar\Omega - (l - \alpha)\hbar\omega_c, \tag{5.19}$$

In particular when $\alpha = 0$ the above energy corresponds to the spectrum of two bosonic oscillators in a uniform magnetic field in relative coordinates when l is even. In calculating the partition function we use the bosonic basis and hence the sum is confined to even l only. The result can be easily translated to the fermionic basis by replacing α by $1 - \alpha$ which is equivalent to taken l odd. The two particle partition function is then given by

$$Z_2 = Z_1 \frac{e^{-\beta\hbar\Omega}}{1 - e^{-2\beta\hbar\Omega}} \left[e^{-\beta\hbar\Omega_+ \alpha} + \frac{e^{-\beta\hbar\Omega_+(2+\alpha)}}{1 - e^{-2\beta\hbar\Omega_+}} + \frac{e^{-\beta\hbar\Omega_-(2-\alpha)}}{1 - e^{-2\beta\hbar\Omega_-}} \right]. \tag{5.20}$$

and the second virial coefficient[7] is given by,

$$a_2 = -\frac{1}{4} - 2 \lim_{\beta\to 0} \left[\frac{Z_2(\alpha) - Z_2(0)}{Z_1} \right], \tag{5.21}$$

where the first term is the bosonic virial coefficient and the contribution due to anyonic interaction is given by the second term.

Now consider the high temperature limit, ie $\beta \to 0$. The problem now involves two scales, namely the oscillator frequency and the cyclotron frequency of the magnetic field. To begin with we might set both $\beta\Omega_+ << 1$ and $\beta\Omega_- << 1$. Immediately we see, expanding the terms in the square brackets to the order β, a_2 reproduces the second virial coefficient of the anyon gas, namely

$$a_2(\alpha) = -\frac{1 - 4\alpha + 2\alpha^2}{4}, \tag{5.22}$$

and the corresponding value of statistical parameter g given in eq.(5.16). In this case both ω and ω_c act merely as regulators to produce the anyon gas result at high temperatures.

But the case of interest to us now is when the trace over levels is confined to a single (lowest) Landau level. This can be achieved by taking the cyclotron frequency $\omega_c >> \omega$. Thus we demand, that $\beta\Omega_+ >> 1, \beta\Omega_- << 1$. Immediately we find that

$$a_2 = -e^{-\beta\Omega}(1/2 - \alpha), \tag{5.23}$$

where we have kept the overall exponential factor as it is. This factor arises from the contribution of the zero point energy to the partition function. Obviously since $\beta\tilde{\omega} >> 1$, we get an absurd result for a_2. It is easy to see why- the existence of the zero point energy ensures that there is an energy gap and naively taking the limit $\beta\Omega >> 1$ implies that the cutoff in energy stays inside the gap. To recover the trace over lowest Landau level, one should move the cutoff above the gap, or equivalently subtract the zero point energy so the energy cutoff for the purposes of tracing lies above the LLL. This is achieved by simply taking out the exponential factor in a_2 and hence we have the simple result

$$a_2 = -(1/2 - \alpha). \tag{5.24}$$

By using the definition of g in eq.(2.5), we immediately obtain

$$\frac{1}{2} - g = \frac{C}{2}[\frac{1}{2} - \alpha], \tag{5.25}$$

where we have retained the factor C which is given by 2^η, where η is the space dimension. Although we started with anyons in two space dimensions confined in a magnetic field as well

as oscillator potential, restricting the trace to LLL is equivalent to treating this system as an one-dimensional system for the purposes of state counting. Therefore we set $\eta = 1(C = 2)$ and immediately obtain

$$g = \alpha. \tag{5.26}$$

We also wish to point out that unlike the case of anyon gas, where we had to assume the existence of the virial expansion, the above result is exact without requiring any assumption about the finiteness of the virial coefficients in the high temperature limit. This is so since the energy of the N-particle bosonic ground state in the presence statistical interaction is exactly known. Basically in the N-particle sector the angular momentum J in the absence of the statistical interaction gets shifted to $J-\nu\frac{N(N-1)}{2}$ in the presence of statistical interactions which also results in energy shifts. The factor ν depends on how the wave functions approach zero as two particle coordinates coincide and is completely determined by α, the strength of the statistical interaction. The form of the LLL spectrum is then very similar to that of the Calogero Model (for details of this mapping see Ref.[57]).

It may be of interest to point out that the result for g given above also depends on whether the magnetic field is pointing in the positive z-direction or negative z-direction. In the former case the ground state is degenerate with all the positive angular states populating the state. The energy shift is given by $-\alpha$ in the presence of the statistical interaction for all $l > 0$. This is equivalent to a shift $2 - \alpha$ if we count the states in the partition function by including also the $l = 0$ state. In this case therefore $g = 2 - \alpha$. This is exactly the result proved numerically by Johnson and Canright[45] for FQHE system where they find that for quasi-particles of statistics -1/3, $g = 2 - 1/3$. The second case yields $g = \alpha$ as noted above.

It is therefore of interest to study the second virial coefficient and hence g as a function of temperature since the results for anyons in LLL and the anyon gas can both be obtained in different limits. We have numerically evaluated the second virial coefficient exactly as a function of temperature (β) and the results are shown in Fig.(5.2) for fixed values of $\omega << \omega_c$. It is easily seen that there exist two clear flat regions corresponding to $\beta\omega_c \to 0$ (anyon gas) and $\beta\Omega_+ >> 1, \beta\Omega_- << 1$ (anyons in a magnetic field). The value of the second virial coefficient at the plateau regions correspond to the two cases discussed above.

We would like to stress here that this does not mean that g as defined by us is a temperature dependent property. Nor is it a high temperature property of the system. The high temperature limit of the partition function is taken in order to count all the states. All states, at low and high energy, contribute equally to the partition function in this limit. The Hamiltonian serves the purpose of ordering the states according to energy. To define finite difference of two divergent series it is necessary to order them. Energy gives us a physical basis for doing so. Thus g does not depend on the details of the Hamiltonian but only on the ordering. However g can be scale dependent. This can be seen and understood in the $\alpha = 0$ case (free bosons) of the above example (see Fig.(5.2). At high temperatures ($\beta\omega_c << 1$), the magnetic field is irrelevant and the system behaves like a free Bose gas in two dimensions. In the range $\beta\omega << 1, \beta\omega_c >> 1$, the system effectively behaves like a one dimensional free Bose gas. Thus the second virial coefficient and hence g have the two corresponding plateaus. Exactly the same thing happens for the non-zero values of α also. Therefore g can be scale dependent. In fact this is how we expect non-trivial values of g to appear in physical systems of interacting fermions. Though the full Hilbert space may obey the dimension formula corresponding to fermions ($g = 1$), the low energy sector could deviate from it and correspond to some non-trivial value of g.

Before closing, we remark that we have a nontrivial and interesting result for anyon gas, where g is determined by the exchange parameter α as in other one-d examples. The anyon gas result is surprising since for lattice anyon gas, Haldane argued that the particles are

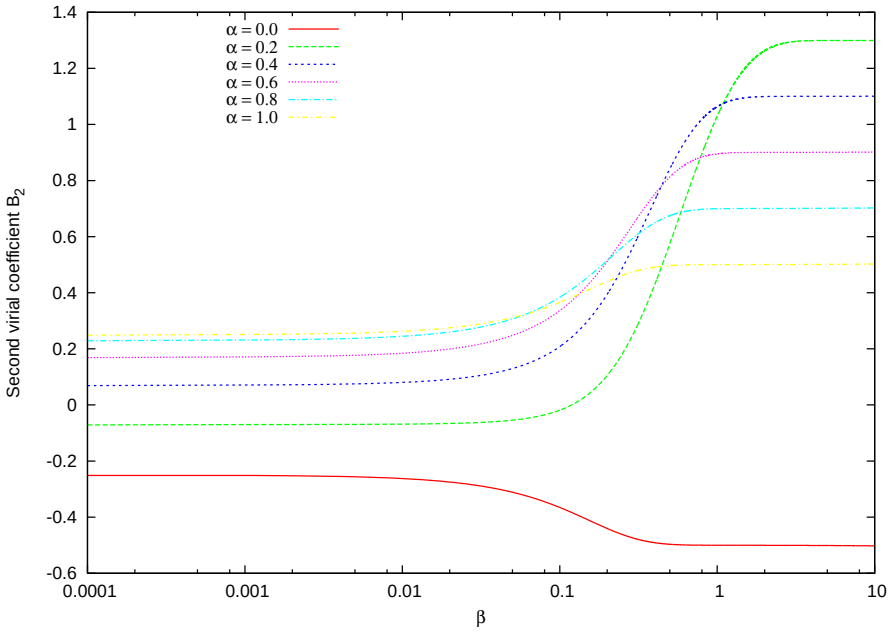

Figure 5.1: The second virial coefficient of anyons as a function of inverse temperature. The flat regions correspond to the anyon gas and the anyons in LLL limits. In the anyon gas case we obtain $g = \alpha(2-\alpha)$ and in the region dominated by anyons in LLL we have $g = 2-\alpha$. We have chosen $\omega = 0.1, \Omega_c = 5.0$ for numerical computation of the second virial coefficient.

classified as hard core bosons (hence fermions) since the coupling of the particles to the Chern-Simons gauge field does not affect the Hilbert space dimension. Obviously the above result shows that this is not so in the continuum.

Chapter 6

Models of FES in two and higher dimensions

In chapter 4 we discussed the microscopic interpretation of exclusion statistics and its realisation in a class of one-dimensional exactly solvable models. While this is so in one dimension, until now there is no exactly solvable model in higher dimensions where such a mapping can be realised. However an approximate description may be obtained in two dimensions though the problem is yet to be solved in three dimensions. Using the Thomas-Fermi approximation, we show [96] that an interacting two dimensional gas of fermions may be described in terms of ideal fractional exclusion statistics at zero and finite temperatures when the interaction has a short-range component.

As pointed out, in principle, the exclusion statistics is applicable to particles in any spatial dimension but the best known examples are the mathematical models in one dimension discussed in chapter 4. The first calculation for a two-dimensional realistic system in this context was done by Johnson and Canright[45], who demonstrated, by exact diagonalisation of a small number of interacting electrons, that the bulk excitations in FQHE liquids exhibit Haldane statistics. In this chapter, we show that under certain conditions, a two-dimensional interacting gas of fermions in its ground state may exhibit this property of exclusion. The conditions are shown to be favourable for electrons in a quantum dot. In this case, we show that the dominant effect of the interaction may be incorporated in the fractional statistics of the gas. If the residual interactions are neglected, then that the system also obeys Haldane statistics at finite temperature. This opens up the exciting possibility that the bulk properties of a mesoscopic two-dimensional system may be understood by regarding it as an almost ideal fractional statistics gas confined in a potential well.

The claims made here are based on the Thomas-Fermi (TF) method[97]. Being a mean-field method, it cannot reproduce two-body correlations, but is successful in giving a good estimate of bulk properties like the ground-state energy and the single-particle spatial density. It has previously been applied with success to atoms[98], nuclei[99], and metal clusters[100]. In two-dimensions, TF yields an accurate approximation to the total energy of a many-anyon system[101]. For an ideal gas obeying the generalised exclusion statistics, TF calculation has been shown to yield the exact answer for the energy in the large-N limit[75]. It is therefore reasonable to expect that the method gives meaningful answers.

6.1 Semiclassical approach in 2-d

We start by constructing the energy density functional for the ground state energy of a system of interacting spin-half fermions. Consider the N-fermion Hamiltonian in two dimensions:

$$H = \frac{1}{2m^*}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} V_1(r_i) + \sum_{j<k} V_2(|\vec{r}_j - \vec{r}_k|), \tag{6.1}$$

where V_1 is a one-body confining potential whose specific form is not crucial at present and V_2 is the two body potential which is repulsive. In a mean-field theory, the expression for the energy at zero-temperature is given by,

$$\begin{aligned} E = & \int d^2r \left[\frac{\hbar^2}{2m^*}\tau(r) + V_1(r)\rho(r)\right. \\ & + \frac{1}{2}\{\rho(r)\int d^2r'\rho(r')V_2(|\vec{r} - \vec{r}\,'|) \\ & \left. - C\int d^2r'|\rho(r,r')|^2 V_2(|\vec{r} - \vec{r}\,'|)\}\right], \end{aligned} \tag{6.2}$$

where $\rho(r)$ is the spatial single-particle density, $\tau(r)$ is the kinetic energy density and $\rho(r,r')$ is the density matrix. In the above we have taken into account the effect of both direct and exchange terms in the interaction energy. For more details see Ref.[102]. The factor $1/2$ is the correction due to over-counting of pairs. The constant C is determined by the spin polarisation of the gas: for unpolarised electrons, it is $1/2$, whereas for a fully polarised system, it is 1. For arbitrary polarization $P = \frac{N_+ - N_-}{N}$, where $N_\pm$ is the number of up or down spins, the factor $C = \frac{1+|P|}{2}$. The spatial density is normalized such that $N = \int d^2r\rho(r)$. In the Thomas-Fermi method, the kinetic energy density $\tau(r)$ is itself expressed in terms of the density $\rho(r)$ and its gradients. The energy and the density are determined self-consistently by a variational principle. In two-dimensions, the TF expression for kinetic energy density is given by

$$\tau(r) = \pi\rho^2(r),$$

taking into account the spin-degeneracy factor of 2. In this case, there is no gradient correction in the bulk up to $O(\hbar^2)$. However, there are edge corrections when the sample is of finite size[103].

Next consider the energy due to the two body interactions. The matrix elements of the direct term is,

$$\sum_{i,j} < ij|V_2|ij > = \int \rho(r_1)\rho(r_2)V(|\vec{r}_1 - \vec{r}_2|)d^2r_1 d^2r_2, \tag{6.3}$$

where the sum (here as well as in what follows) is over the *occupied single-particle states* only. The matrix elements of the exchange term is

$$\sum_{i,j} < ij|V_2|ji > = \int |\rho(r_1,r_2)|^2 V(|\vec{r}_1 - \vec{r}_2|)d^2r_1 d^2r_2, \tag{6.4}$$

where $\rho(r_1,r_2) = \sum_i \psi_i^*(r_1)\psi_i(r_2)$. At this stage, it is useful to perform the density-matrix expansion following Skyrme[104]. Defining $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$ and expanding the density up to this order in $\vec{r}$, we obtain

$$\rho(\vec{r}_1) = \rho(\vec{R} + \vec{r}/2) = \rho(\vec{R}) + (\vec{r}.\nabla)\rho + \frac{1}{2}(\vec{r}.\nabla)^2\rho + \ldots \tag{6.5}$$

The direct matrix element may then be written as,

$$\sum_{i,j} < ij|V_2|ij > \;\; = \;\; \int d^2r V_2(r) \int d^2R \rho^2(R)$$
$$-\frac{1}{4} \int d^2r \; r^2 \; V_2(r) \int d^2R (\nabla \rho(R))^2 + \ldots, \tag{6.6}$$

Similarly the density matrix $\rho(\vec{r}_1, \vec{r}_2)$ may be expanded up to second order in $\vec{r}$ about $\vec{R}$,

$$\rho(\vec{r}_1, \vec{r}_2) = \sum_i \psi_i^*(\vec{R} + \vec{r}/2)\psi_i(\vec{R} - \vec{r}/2)$$
$$= \sum_i \Big[\psi_i^*(\vec{R})\psi_i(\vec{R})$$
$$+ \frac{1}{16} r^2 (\psi_i^* \nabla^2 \psi_i + (\nabla^2 \psi_i^*)\psi_i - 2\nabla\psi_i^*.\nabla\psi_i)\Big], \tag{6.7}$$

and the exchange contribution to second order is given by,

$$\sum_{i,j} < ij|V_2|ji > = \int d^2r V_2(r) \int d^2R \rho^2(R)$$
$$-\frac{1}{2} \int d^2r \; r^2 \; V_2(r) \int d^2R \tau(R)\rho(R) + \ldots. \tag{6.8}$$

Furthermore, the kinetic energy density is defined as

$$\tau = -\frac{1}{4} \sum_i (\psi_i^*(\nabla^2\psi_i) + (\nabla^2\psi_i^*)\psi_i) + \frac{1}{2} \sum_i (\nabla\psi_i^*).(\nabla\psi_i). \tag{6.9}$$

Often the kinetic energy density is defined either by the first term or by the second term in the above equation without the over all 1/2. What we naturally get in the expansion is an average of both these commonly used forms. We have computed each one of these forms exactly using harmonic oscillator wave functions for a few particles. While the the first and second terms show oscillations around the smooth TF density, the definition given above almost precisely coincides with the TF density even with as little as two particles.

We note that the leading terms in both direct and exchange terms are the same (proportional to ρ^2). For spin-half fermions the interaction energy is given by,

$$\sum_{i,j} [< ij|V_2|ij > - \delta_{m_i,m_j}\delta_{m_j,m_i} < ij|V_2|ji >], \tag{6.10}$$

where m_i is the spin projection. Summing over all particle indices immediately gives a factor $(1 + |P|)/2$ for the exchange contribution, where P is the spin polarization of the system. Therefore, if there is no other degree of freedom, or if the spins are all polarized, the contribution from the leading terms to the interaction energy vanishes as it happens in FQHE systems. However for the unpolarized 2-D electron systems there is a factor half for exchange contribution.

Here after we concentrate on the unpolarized case. Combining all the contributions the total energy of the system is given by,

$$E \;\; = \;\; \int d^2r \left[\frac{\hbar^2}{2m^*}\pi\rho^2(r) + V_1(r)\rho(r) + \frac{1}{4}\rho^2(r)M_0\right.$$
$$\left. + \frac{1}{8}[(\pi\rho^3(r) - (\nabla\rho(r))^2]M_2 + \ldots\right], \tag{6.11}$$

where

$$M_n = \int d^2 r V_2(r) r^n$$

are the moments of the two body potential. Note that we obtain an expression similar to the above if we use an expansion of the form[105]

$$V_2(r) = \sum_{j=0} c_j b^{2j} \nabla^{2j} \delta^2(\vec{r}), \tag{6.12}$$

where b is the range of the potential and c_j are related to the j-th moment of the potential V_2 as

$$M_{2j} = 2^{2j} j! c_j b^{2j}.$$

The spatial density is now determined by the variation $\delta(E - \mu N) = 0$, where μ is the chemical potential at zero temperature. The variation immediately gives the equation for the density

$$\begin{aligned} & \frac{\pi\hbar^2}{m^*}\left[1 + \frac{m^* M_0}{2\pi\hbar^2}\right]\rho(r) + \frac{3\pi M_2}{8}\rho^2(r) + \frac{M_2}{4}\nabla^2\rho(r) \\ = \;& \mu - V_1(r). \end{aligned} \tag{6.13}$$

In the large-N limit we expect the density in the bulk to be approximately constant. We can therefore neglect the derivative term in this limit. Further if the potential is extremely short-ranged, the term proportional to the second moment of the potential may also be neglected. (We will elaborate on these approximations shortly.) Then the density is given by

$$\begin{aligned} \rho_0(r) &= \frac{m}{\pi\hbar^2\, g}(\mu - V_1(r)), \quad r \le r_0 \\ &= 0, \qquad\qquad\quad r > r_0, \end{aligned} \tag{6.14}$$

where r_0 is the classical turning point defined by $\mu = V_1(r_0)$ and

$$g = 1 + \frac{m^* M_0}{2\pi\hbar^2} \tag{6.15}$$

is now the exclusion statistics parameter (same as g in Haldane's definition) as we show below. In the effective range expansion (6.12), $c_0 = M_0$. The expression for ρ_0 in Eq.(6.14) may be interpreted as if the fermions in the one-body confining potential V_1 are noninteracting, but that they *obey the generalised exclusion statistics for occupancy* at zero temperature:

$$\begin{aligned} n(\epsilon) &= \frac{1}{g}, \quad \epsilon < \mu \\ &= 0, \quad \epsilon > \mu. \end{aligned} \tag{6.16}$$

This may be easily seen as follows. For noninteracting fermions, the Thomas-Fermi density of states $g(\epsilon)$ in an external potential $V_1(r)$ is

$$g(\epsilon) = 2\int \frac{d^2 r d^2 p}{(2\pi\hbar)^2}\, \delta\left(\epsilon - \frac{p^2}{2m^*} - V_1(r)\right). \tag{6.17}$$

The over-all factor of two on the right-hand side is due to the spin degeneracy. Using the new occupancies given by Eq.(6.16), we get

$$\begin{aligned} N &= \frac{1}{g}\int_0^\mu g(\epsilon) d\epsilon \\ &= \frac{1}{g}\int 2\, \frac{d^2 r d^2 p}{(2\pi\hbar)^2}\; \theta\left(\mu - \frac{p^2}{2m^*} - V_1(r)\right). \end{aligned} \tag{6.18}$$

The function $\theta(y) = 1$ for $y > 0$, and zero otherwise. Now performing the p-integration immediately yields the total number of particles, with density $\rho_0(r)$ given by Eq.(6.14). Indeed we have now the precise condition under which ideal exclusion statistics is realised within the framework of the Thomas-Fermi method.

In the more realistic situation, the higher moments may not be neglected, and the system is a non-ideal fractional statistics gas. In the thermodynamic limit, we may write

$$\rho(r) = \rho_0(r) \left[1 - \frac{3m^* M_2}{8\hbar^2 g} \rho_0(r) + \dots \right], \tag{6.19}$$

where $\rho_0(r)$, given by Eq.(6.14), is the density for the ideal FES case. Note that $M_2 = 4c_1 b^2$ where b is the range of the potential. The typical densities in two dimensional systems of interest is of the order of $10^{-5}/\mathring{A}^2$. Using the values of $m^* = 0.067 m_e$, which is the effective electron mass in GaAs materials, and $g \geq 1$ (but not very large), it is easy to estimate that the second term becomes important only for ranges of the order of $100\mathring{A}$ or above. Another way to view the problem is to regard the short-range part of the two-body interaction, which dominates M_0, to alter the statistics only. The long-range part of V_2, giving the higher moments, modifies the self-consistent mean field. Consider for example the electrons in two dimensional quantum dots. The two body potential is usually taken to be the Coulomb interaction and the confining potential of the device is modelled by the oscillator potential. However, it is expected that the effective two-body interaction after averaging over the probability densities in the direction perpendicular to the plane will be more complicated. Many qualitative features of the system may be explained by several choices of the potential. As in the case of FQHE liquids, we assume that the model interaction has a short range part $V_{2s}(r)$ and a long range part $V_{2l}(r)$. We use the moments expansion for the short-range part and neglect the effect of higher moments. The self-consistent equation for the density is then given by,

$$\begin{aligned} \rho(r) &= \frac{m^*}{\pi\hbar^2 g}(\mu - U(r)), \quad r \leq r_0 \\ &= 0, \qquad\qquad\qquad r > r_0, \end{aligned} \tag{6.20}$$

where the mean TF potential is defined as

$$U(r) = V_1(r) + \int d^2 r \rho(r') V_{2l}(|\vec{r} - \vec{r}\,'|). \tag{6.21}$$

The equation further simplifies for circularly symmetric density. Expanding the potential in partial waves,

$$V_{2l}(|\vec{r} - \vec{r}\,'|) = \frac{1}{\pi} \sum_{m=0}^{\infty} v_m(r, r') \cos m(\theta - \theta'),$$

the TF potential reduces to,

$$U(r) = V_1(r) + \int r' dr' \rho(r') v_0(r, r'). \tag{6.22}$$

In the above equation we have ignored the exchange effects which are not important for the long range potentials. Thus the Eq.(6.22) is the self consistency condition to determine the density $\rho(r)$, and in general is not solvable analytically.

6.2 TF at finite temperature

Finally we consider briefly the finite temperature problem using the Thomas-Fermi method. We restrict our attention to the case where the two body potential is extremely short-ranged and regard the system as ideal. The temperature T is expressed in units of the Boltzmann constant, so that it has the dimensions of energy. The one-body potential is now temperature-dependent, and is given by

$$\begin{aligned} V(r,T) &= V_1(r) + \frac{M_0}{2}\rho(r,T) \\ &= V_1(r) - (1-g)\frac{\pi\hbar^2}{m^*}\,\rho(r,T)\,, \end{aligned} \tag{6.23}$$

where g is the statistics parameter defined by Eq.(6.15). We have assumed that the external potential $V_1(r)$ is temperature independent. In the above equation, the density $\rho(r,T)$ for the fermions is obtained from the relation (including the spin-degeneracy of 2)

$$\rho(r,T) = \frac{2}{(2\pi\hbar)^2}\int \frac{d^2p}{\exp[(p^2/2m^* + V - \mu)/T] + 1}\,, \tag{6.24}$$

and the chemical potential is determined by $N = \int d^2r\rho(r,T)$. The $p-$integration above may be done analytically, giving

$$\rho(r,T) = \frac{m^*T}{\pi\hbar^2}\,\ln\left(1 + \exp[-(V-\mu)/T]\right)\,. \tag{6.25}$$

This is inverted to give

$$\frac{\mu}{T} = [V + \frac{\pi\hbar^2}{m^*}\rho]/T \;+\; \ln\left(1 - \exp(-\pi\hbar^2\rho/m^*T)\right)\,. \tag{6.26}$$

Substituting for V above from Eq.(6.23), we get

$$\begin{aligned} \frac{\mu}{T} &= \left(V_1(r) + g\frac{\pi\hbar^2}{m^*}\rho(r,T)\right)/T \\ &+ \ln\left(1 - \exp(-\pi\hbar^2\rho/m^*T)\right)\,. \end{aligned} \tag{6.27}$$

For a gas in the thermodynamic limit, we set $V_1(r) = 0$ above. Further, the spatial density ρ may be expressed as $2\rho_0$, where ρ_0 is the density for spin-less partcles. Then Eq.(6.27) reduces to the form

$$\frac{\mu}{T} = g\frac{2\pi\hbar^2}{m^*T}\rho_0 + \ln\left(1 - \exp(-2\pi\hbar^2\rho_0/m^*T)\right)\,. \tag{6.28}$$

Note that this is precisely the equation derived by Wu[55] (see his Eq.(23)) for a two-dimensional gas obeying the statistics

$$n(\epsilon) = \frac{1}{w(\exp(\epsilon-\mu)/T) + g}\,, \tag{6.29}$$

with $w(x)$ satisfying the functional equation

$$w^g(1+w)^{1-g} = \exp(\epsilon-\mu)/T\,. \tag{6.30}$$

Here, as in our case, $g = 1$ corresponds to free fermions.

We have thus shown that in the large-N limit, ideal exclusion statistics may be realised in a system of spin-half fermions with very short-range interactions. Note that this situation is peculiar to two dimensions since both the leading term in the moments expansion and the kinetic energy density have the same dependence on the spatial density.

6.3 Bosons in a Trap

The results obtained for a gas of fermions may equally well apply to the interacting bosons in a quasi-two dimensional harmonic trap. Following Bhaduri et al [106] we consider bosons which are trapped in a two dimensional oscillator potential and interacting via a repulsive zero-range pair potential of a fixed strength. This is similar to the case of fermions discussed above, but the interaction is now repulsive instead of being attractive. Obviously there is no phase transition for such a system, no matter how weak the repulsion is.

Though the traps are usually three dimensional, we consider the limit of the oscillator frequencies in which $\omega_x = \omega_y = \omega << \omega_z$. For low enough temperatures, we may assume that the excitations are much higher and restrict the Hilbert space by setting the oscillator quanta in the $z-$ direction to be $n_z = 0$. Then the quantum mechanical behaviour of a system of N bososn is determined effectively by a two dimensional Hamiltonian given by

$$H = \frac{1}{2m}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N}\frac{1}{2}m\omega^2 r_i^2 + \frac{2\pi\hbar^2}{m}g\sum_{j<k}\delta(\vec{r}_j - \vec{r}_k), \tag{6.31}$$

where straightaway we taken the interaction to be a short range repulsive interaction. The dimensionless coupling constant g may be related to $s-$wave scattering length a in three dimensions since the strength of the three dimensional delta function pseeudo-potential is given by $\frac{4\pi\hbar^2 a}{m}$. The effective two dimensional interaction given in eq.(6.31) may be written as

$$g = \sqrt{\frac{2}{\pi}}\frac{a}{b_z}, \tag{6.32}$$

where $b_z = \sqrt{\hbar/m\omega_z}$ is the length scale of confinement in the $z-$ direction. For realistic parameters, generally, $g << 1$. For temperatures well above the temperature T_0[1], the one body potential generated by the zero-range interaction is given by

$$U(\rho(r)) = \frac{2\pi\hbar^2}{m}g\rho(r). \tag{6.33}$$

Using the Thomas-Fermi approximation we have the self-consistent equation for the density at finite temperature given by

$$\rho(r) = \frac{1}{2\pi\hbar^2}\int d^2p\ \frac{1}{\left[\exp\left(\beta(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{2\pi\hbar^2}{m}g\rho(r) - \mu)\right) - 1\right]}, \tag{6.34}$$

where μ is the chemical potential for the bosonic system determined by the condition

$$N = \int d^2r\rho(r,T), \tag{6.35}$$

and β is the inverse temperature. The integration in (6.34) can be done analytically and the number density is given by

$$\rho(r) = -\frac{m}{2\pi\hbar^2\beta}\ln\left(1 - \exp[-(\frac{1}{2}m\omega^2 r^2 + \frac{2\pi\hbar^2}{m}g\rho(r) - \mu)\beta]\right)\ . \tag{6.36}$$

[1]As there is no phase transition, T_0, denotes the temperature below which there is macroscopic occupation of the ground state.

Solving eq.(6.34) and eq.(6.35) self consistently for fixed system size N. For non-zero and positive values of g the equations can be solved self consistently. Taking $T \to 0$ limit of eq.(6.34), we obtain the spatial density within the classical turning point r_0 given by

$$\rho_0(r) = \frac{m}{2\pi\hbar^2 g}\left(\mu - \frac{1}{2}m\omega^2 r^2\right), \tag{6.37}$$

where $r_0 = \sqrt{2\mu/m\omega^2}$ and $N = \mu^2/2g(\hbar\omega)^2$ which is the same as the result one would obtain at zero temperature starting from the Gross-Pitaevskii density functional[107]. We note that this result for the density at zero temperature is the same as that obtained in eq.(6.20) starting from a system of interacting fermions. Thus the density smoothly interpolates from $g = 1$ fermionic limit to $g = 0$ bosonic limit.

We have thus shown, in the TF limit, a two dimensional Fermi or Bose gas with short range repulsive interaction has the same number density of an ideal FES gas. A more direct statistical mechanics derivation based on counting of states has been used by Hansson et al [108] to prove rigorously the same result.

6.4 FES in arbitrary dimensions

Even before the FES was introduced by Haldane, the germ of the concept of exclusion statistics was already evident in systems of particles interacting by a pair potential of the form $g(g-1)/r^2$ in one dimension. Here r is the relative distance between particles. Since the wave function vanishes as $\psi \to r^g$ at short distances, we have $g = 0$ for bosons and $g = 1$ for fermions. Interestingly, the spectrum of the interacting case for $g = 2$ could be obtained by requiring that the particles neither occupy the same orbital nor neighbouring orbital in momentum space. This in a sense is generalised Pauli principle, more exclusive than the exclusion principle for fermions albeit in one dimension. While the proposal of Haldane has the advantage generalisation to higher dimensions, the introduction of the statistical parameter is adhoc.

Recently, Sutherland[109, 110] has proposed a microscopic theory of exclusion statistics based on an analogy with the exactly solvable integrable models in one dimension. A similar proposal was made by Baskaran[111]. Unlike the semiclassical realisation discussed earlier in this chapter, here there is no Hamiltonian. The approach relies on the fact that in one dimension exclusion in momentum space can be achieved by defining momentum shifts on the asymptotic momentum as realised in the case of a spinless fermion on a ring interacting with invere-square interaction. We briefly comment on this approach.

To begin with we consider a system of spinless particles (interacting fermions or bosons) in a η dimensional box of length L in each direction with periodic boundary conditions. The momenta of non-interacting particles in the box is denoted by

$$\vec{p}_i^{\,0}, \quad i = 1, ..., N. \tag{6.38}$$

Let us assume that the effect interaction is to shift these mometa,

$$\vec{p}_i = \vec{p}_i^{\,0} + \frac{\gamma}{2}\left(\frac{2\pi\hbar}{L}\right)\vec{f}(\vec{p}_i), \quad i = 1, ..., N. \tag{6.39}$$

where γ is a dimensionless constant. The shifts are such that the translation invariance is kept intact and are dimension dependent.

In one dimension, we choose the function

$$f(p_i) = \sum_{j\neq i}\frac{p_i - p_j}{|p_i - p_j|}$$

which is merely the sign of the difference in momenta. This corresponds to the solution of the Sutherland model for particles on a circle interacting via inverse square of the distance between them. For arbitrary dimensions η we may surmise the following general form:

$$\vec{f}(\vec{p}_i) = (\frac{2\pi\hbar}{L})^{\eta-1} \sum_{j\neq i} \frac{\vec{p}_i - \vec{p}_j}{|\vec{p}_i - \vec{p}_j|^\eta}, \tag{6.40}$$

where the displacement field $\vec{f}$ is assumed to be given as the sum of displacements due to all other momenta. The ground state and excitations of such a system as also its thermodynamics in two dimensions has been discussed in detail by Sutherland[109]. Our approach here is to demonstrate the connection to exclusion statistics, that is the relation between the interaction parameter γ and the exclusion statistics parameter g directly using the method outlined in chapter 2.

We first consider the case of two particles interacting such that the momentum shifts are given by eq.(6.40). We therefore have

$$\begin{aligned} \vec{p}_1 &= \vec{p}_1^{\,0} + \frac{\gamma}{2}(\frac{2\pi\hbar}{L})\ \vec{f}(\vec{p}_1), \\ \vec{p}_2 &= \vec{p}_2^{\,0} + \frac{\gamma}{2}(\frac{2\pi\hbar}{L})\ \vec{f}(\vec{p}_2), \end{aligned} \tag{6.41}$$

where $\vec{f}(\vec{p}_1) = -\vec{f}(\vec{p}_2) = \vec{f}$ by virtue of eq.(6.40). Define a new set of momenta

$$\begin{aligned} \vec{p} &= \frac{\vec{p}_1 + \vec{p}_2}{\sqrt{2}} = \vec{p}^{\,0} \\ \vec{q} &= \frac{\vec{p}_1 - \vec{p}_2}{\sqrt{2}} = \vec{q}^{\,0} + \frac{\gamma}{\sqrt{2}}(\frac{2\pi\hbar}{L})\ \vec{f}(\vec{q}), \end{aligned} \tag{6.42}$$

where $\vec{p}^{\,0} = (\vec{p}_1^{\,0} + \vec{p}_2^{\,0})/\sqrt{2}$, $\vec{q}^{\,0} = (\vec{p}_1^{\,0} - \vec{p}_2^{\,0})/\sqrt{2}$ and

$$\vec{f}(\vec{q}) = (\frac{\pi\hbar}{\sqrt{2}L})^{\eta-1} \frac{\vec{q}}{|\vec{q}|^\eta}. \tag{6.43}$$

Note that the displacement field is a solution of the differential equation

$$\vec{\nabla}.\vec{f} = (\frac{\pi\hbar}{\sqrt{2}L})^{\eta-1}\ \delta(\vec{q}) \tag{6.44}$$

analogous to the fundamental equation in electro-statics.

The single-particle partition function is given by

$$Z_1(\beta) = (\frac{L}{2\pi\hbar})^\eta \int_{-\infty}^{\infty} d\vec{p}\ \exp(-\beta p^2/2m) = (\frac{L}{\sqrt{2\pi\hbar^2\beta/m}})^\eta = (\frac{L}{\lambda})^\eta, \tag{6.45}$$

where $\lambda = \sqrt{2\pi\hbar^2/mkT}$ is the thermal wavelength.

The two-particle partition function in the bosonic basis is given by

$$Z_2(\beta) = \frac{1}{2}\left[(\frac{L}{2\pi\hbar})^{2\eta} \int d\vec{p}^{\,0} d\vec{q}^{\,0} \exp[-\beta(p^2+q^2)/2m] + Z_1(2\beta)\right], \tag{6.46}$$

where $Z_1(2\beta) = (L/\sqrt{2}\lambda)^\eta$. Switching over to shifted momenta in the integral we have

$$Z_2(\beta) = \frac{1}{2}\left[(\frac{L}{2\pi\hbar})^{2\eta} \int d\vec{p} d\vec{q}\ J\ \exp[-\beta(p^2+q^2)/2m] + Z_1(2\beta)\right], \tag{6.47}$$

where the Jacobian of the transformation is given by

$$J = 1 - \gamma(\frac{\pi\hbar}{\sqrt{2}L})^{\eta}\ \delta(\vec{q}) + O(\gamma^2) \tag{6.48}$$

upto order γ. The integration can be done easily to this order and we obtain

$$Z_2(\beta) = \frac{1}{2}\left[Z_1^2(\beta) - \gamma(\frac{L}{\sqrt{2}\lambda})^{\eta} + Z_1(2\beta) + O(\gamma^2)\right]. \tag{6.49}$$

In order to obtain the relation between the exclusion statistics parameter g and γ to first order in γ we use the fundamental equation eq.(2.5) given in chapter 2 for particles in a box. We have

$$\frac{1}{2} - g = \lim_{\beta\to 0}\ 2^{\eta/2}\ \frac{Z_1}{2}\left[2\frac{Z_2}{Z_1^2} - 1\right]. \tag{6.50}$$

Substituting for Z_2 from eq.(6.49) and simplifying we obtain

$$\frac{1}{2} - g = \frac{1}{2} - \gamma \tag{6.51}$$

and therefore the interaction parameter as defined through momentum shifts is the same as the exclusion statistics parameter

$$g = \gamma \tag{6.52}$$

The interpretation of g as generalised Pauli principle immediately follows from the above result. A simple interpretation interms of the number of excluded states may be shown as below:

Note that because of the shifted momentum

$$\vec{q}^{\,0} = \vec{q} - \gamma(\frac{\pi\hbar}{\sqrt{2}L})^{\eta}\ \frac{\vec{q}}{q^{\eta}}$$

we have for the modulus the relation,

$$(q^0)^2 = \left[q - \gamma(\frac{\pi\hbar}{\sqrt{2}L})^{\eta}\frac{q}{q^{\eta}}\right]^2$$

Thus the value of q never approaches zero unlike q^0 which can be zero. As a result, in the q space a void is created which leads to a certain number of excluded states given by,

$$\text{Number excluded states} = \frac{\gamma(\pi\hbar/\sqrt{2}L)^{\eta}}{(\pi\hbar/\sqrt{2}L)^{\eta}} = \gamma = g,$$

where we have appropriately scaled the box length using the definition of $\vec{q}$. Thus the number of forbidden or excluded states is proportional to the statistics parameter g as it should be. Thus we have the simple interpretation of the statistics parameter g as a measure of the number of excluded states indicative of the generalised Pauli principle.

Chapter 7

Fluctuations in Models of FES

In previous chapters we have discussed the basic idea of exclusion statistics and how it may be realised in models of interacting fermions or bosons. In this chapter we discuss an interesting application of these ideas to the number fluctuation in canonical and grand canonical ensembles when the generalised Pauli principle operates. It is well-known that the number fluctuation in the grand canonical ensemble, which is directly proportional to the compressibility, diverges for an ideal bose gas as $T \to 0$ often referred to as *fluctuation catastrophe*. In the following we show that this divergence is removed when the gas of atoms interact in one dimension through an inverse square two-body interaction. In two dimensions, similar results are obtained using a self-consistent Thomas-Fermi (TF) model for a repulsive zero-range interaction. As seen before, both models may be mapped on to a system of non-interacting quasi-particles obeying the Haldane-Wu exclusion statistics.

We also calculate the number fluctuation from the ground state of the gas in these interacting models, and compare the grand canonical results with those obtained from the canonical ensemble.

To begin with, consider an ideal bose gas at low temperatures. In the grand canonical ensemble (GCE), the compressibility and the number fluctuation of this system diverge at low temperatures. To quote Landau and Lifshitz [112]:

> ...in a bose gas at temperature $T < T_c$, the pressure is independent of the volume, i.e. the compressibility becomes infinite. Accordingly...this would imply that the fluctuations of the number of particles also become infinite. This means that, in calculating fluctuations in a gas obeying bose statistics, *the interactions between the particles cannot be neglected at low temperatures, however weak this interaction may be. When the interactions, which must exist in any actual gas, is taken into account, the resulting fluctuations are finite.*

One purpose of this chapter is to demonstrate the validity of this statement in one and two dimensional traps for a special class of interactions which may be thought of as statistical interactions in the sense of exclusion statistics. As shown in Chapter 3 and 4, in one dimension, the quantum many-body problem of particles in a harmonic oscillator interacting with an inverse square two-body potential is exactly solvable [39, 40, 41]. Moreover, it is known that the global properties of these interacting bosons are the same as those of non-interacting particles obeying the Haldane-Wu generalized exclusion statistics [38, 55, 61, 62]. Using this mapping, we show that the number fluctuation is finite as $T \to 0$, no matter how weak the interaction strength is. In this model, since the exact correlation function is known for some specific strength parameters of the interaction, it is also possible to verify explicitly the well-known relation between its integral and the number fluctuation.

In two dimensions, there is no suitable exactly solvable model for our purpose. We therefore consider the mean-field model of bosons trapped in a harmonic oscillator, and interacting pair-wise with a zero-range repulsive pseudo-potential [106, 113] discussed earlier in Chapter 6. In the absence of this interaction, the number fluctuation starts diverging at $T = T_c$. When the interaction is present, however weak, we show that the compressibility and hence the number fluctuation of the system is finite right down to $T = 0$. Moreover, they are shown to be identical to those of a collection of non-interacting particles obeying exclusion statistics (often called geons or haldons in the literature).

Another quantity of some interest is the number fluctuation of particles from the *ground state* of the system, which is present even when the total number of particles in the trap is fixed. Consider a dilute gas of bosons in a trap at $T = 0$. The system is in its ground state. When a certain amount of excitation energy is given to the system, it may be absorbed in many possible ways, so that the number of bosons remaining in the ground state is not fixed. This number fluctuation for non-interacting bosons in a harmonic trap has been calculated by a number of authors [114, 115, 116, 117, 118] as a function of the excitation energy or temperature. Thermodynamic identities and particle number fluctuations in weakly interacting BEC have also been analysed when the particle number is fixed[119]. For ideal bosons, the number fluctuation from the ground state diverges at low temperatures in GCE, but this can be avoided using more careful canonical, or microcanonical treatment. When the inverse square pair-wise interaction is used in one dimension, this divergence is again removed. Further, in this interacting model, we can also perform the canonical ensemble (CE) calculations, and compare with the GCE results. We find that even though the ground state number fluctuation goes to zero as $T \to 0$ in both GCE and CE, the very low-temperature behaviors are rather different. At higher temperatures the results from CE and GCE tend to coincide. Similar studies can be made for fermionic systems at low temperatures. Since the interactions that we have used are of statistical character [61, 120], our results interpolate from bosonic to fermionic behavior with the variation in the strength of the interaction.

In the next section we discuss the total number fluctuation in GCE for interacting models and how the fluctuation catastrophe may be avoided. We first discuss the fluctuations in one dimension, and later the fluctuation in two dimensions. Subsequently we discuss the ground state number fluctuations for the one-dimensional interacting model in GCE and CE and compare the results. Particular attention is paid to the low temperature behavior (for some low-temperature expansions see Appendix C).

7.1 Fluctuations in GCE

The number fluctuation in a gas in the GCE formalism is defined by $(\delta N)^2 = (\langle N^2\rangle - \langle N\rangle^2)$, where the angular brackets denote ensemble averaging. For an ideal bose gas,

$$(\delta N)^2 = \sum_{k=0}^{\infty} \langle n_k\rangle(\langle n_k\rangle + 1), \tag{7.1}$$

where the single particle occupancy $\langle n_k\rangle$ at a given temperature T and energy ϵ_k are given by the bose distribution function

$$\langle n_k\rangle = \frac{1}{\exp[\beta(\epsilon_k - \mu)] - 1}. \tag{7.2}$$

Here $\beta = 1/k_B T$ and k_B is the Boltzmann constant, and μ denotes the chemical potential. The isothermal compressibility χ_T of a gas of density ρ_0 is related to the number fluctuation

of the system in GCE :

$$\frac{(\delta N)^2}{N} = T\rho_0\chi_T \ . \tag{7.3}$$

The problem with the number fluctuation in a bose gas in GCE is obvious from Eq. (7.1). If there is BEC, then a macroscopic fraction of the particles occupy the ground state for $T < T_c$, so that $(\delta N_0)^2 \simeq N^2$. Even if there is no BEC, the same is the case as $T \to 0$. In the thermodynamic limit, therefore, the fluctuation diverges below the critical temperature, or in any case at $T = 0$. This is manifestly incorrect, since at $T = 0$, all the bosons are in the ground state, and the number fluctuation should vanish. In the presence of interactions, however, this number fluctuation is expected to be finite. We demonstrate this by analyzing two models in one and two dimensions which are also templates for FES.

7.1.1 Fluctuations in a one dimensional model

We first consider the exactly solvable one dimensional Calogero-Sutherland Model (CSM) of a system of interacting particles with the Hamiltonian [39, 40, 41]

$$H = \sum_{i=1}^{N}\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}m\omega^2 x_i^2\right] + \frac{\hbar^2}{m}\sum_{i<j=1}^{N}\frac{g(g-1)}{(x_i - x_j)^2}, \tag{7.4}$$

with the dimensionless coupling parameter $g \geq 0$. The particles are confined in a harmonic well and the thermodynamic limit is obtained by taking $\omega \to 0$ as $N \to \infty$, with $\omega N =$ *constant*. In the thermodynamic limit, the properties of the system are translationally invariant, and would be the same if the particles were on a line, or a circle, instead of a harmonic confinement. To make the problem well-defined quantum mechanically, we have to demand that the wave functions go to zero as $|x_i - x_j|^g$ whenever two particles i and j approach each other. Since the particles cannot cross each other, we may choose the wave function to be either symmetric (bosonic) or antisymmetric (fermionic). For $g = 0$ and 1, the model describes free bosons and free fermions respectively.

Using the relation between the integral of the correlation function and the number fluctuation, we now show that the number fluctuation vanishes at zero temperature in the above interacting model unlike the ideal boson result in GCE. If $\nu(r)$ denotes the two-particle ground state density-density correlation function in the ground state, with $r = |x_1 - x_2|$, then the number fluctuation is related to the correlation function as [112]

$$\frac{(\delta N)^2}{N} - 1 = -\int_{-\infty}^{\infty}\nu(x)dx. \tag{7.5}$$

Note that the ground state correlation function $\nu(r)$ is defined only for $r \geq 0$. However, in computing the above integral it is necessary to assume $\nu(r)$ to be even function, and extend the domain of integration to negative values of r[121]. Unlike the one-particle off-diagonal density matrix, $\nu(r)$, by definition, is related to the diagonal element of the the two-particle density matrix, and is the same in CSM for bosons or fermions. In this section we work in the bosonic basis. The correlation functions are known exactly in the CSM for three values of g independent of whether the particles are bosons are fermions and are given (in the thermodynamic limit) by

$$g = 1: \quad \nu(r) = s(r)^2 = \left[\frac{\sin(\pi r)}{\pi r}\right]^2 \tag{7.6}$$

$$g = 1/2: \nu(r) = s(r)^2 + \frac{ds}{dr}\int_r^{\infty} dt\ [s(t)] \tag{7.7}$$

$$g = 2: \quad \nu(r) = s(2r)^2 - \frac{ds(2r)}{dr}\int_0^{2r} dt\ [s(t)], \tag{7.8}$$

where the Fermi momentum k_F is set equal to π so that the maximum central density is unity. For all three forms of $\nu(r)$ given above, explicit calculations show that

$$\int_{-\infty}^{\infty} \nu(x)dx = \int_{-\infty}^{\infty} [s(r)]^2 dx = 1, \tag{7.9}$$

independent of the value of g. Substituting this result in Eq.(7.5) it follows that for interacting bosons in CSM the fluctuation vanishes identically at $T = 0$.

While we cannot obtain the exact $\nu(r)$ in CSM for all g, the same may be calculated for all values of g in the harmonic lattice approximation. The correlation function so obtained compares very well with the exact correlation functions for $g = 1/2, 1, 2$ and is given by [122]

$$\nu(x) = \rho_0 \sum_{n\neq 0} \Big(\frac{1}{4\pi F(n,0)} \Big)^{1/2} \exp\ \Big[-\frac{(x\rho_0 - n)^2}{4F(n,0)} \Big] - \rho_0, \tag{7.10}$$

where $\rho_0 = N/L$ is the average density and

$$\lambda F(n,0) = \frac{1}{2\pi^2}\int_0^{\pi} dy\ \frac{1-\cos\ (yn)}{y\ -\ y^2/2\pi} = \frac{1}{2\pi^2}\int_0^{2\pi} dy\ \frac{1-\cos\ (yn)}{y}\ . \tag{7.11}$$

The above expression is given for completeness and its exact form is not needed for further calculations. Again integrating over the real line we get a result identical to that obtained using the exact correlation functions in CSM. Thus the fluctuation vanishes identically for all g in this approximation at zero temperature. However, the result does not give any indication of the behavior of the fluctuation at finite temperature. To do this we take recourse to the mapping between the CSM and the exclusion statistics first proposed by Haldane through a generalized Pauli principle[61, 60].

We recall that a crucial property of exclusion statistical interactions is that they should cause shifts in single particle energies at all scales[54]. As noted in the introduction this property is realized by a large class of one dimensional models of interacting fermions where Fermi liquid theory breaks down[46, 63]. In fact it has been shown exactly that quasiparticles with nontrivial exclusion statistics exist in a class of models that are solved by the Bethe ansatz[60]. Thus the result obtained below should be, in principle, valid in a large class of models with interactions. In particular, as we have demonstrated in chapters 3 and 4, the interacting particles of CSM may be regarded as ideal exclusion statistics particles[61]. The thermodynamic properties of an ideal gas of exclusion statistics particles have been investigated widely[64, 79, 123, 76]. The distribution function has is ofcourse given by

$$\langle n(\epsilon)\rangle = \frac{1}{(w(\epsilon)+g)}, \tag{7.12}$$

where $w(\epsilon)$ is the solution of the equation

$$w(\epsilon)^g (1+w(\epsilon))^{(1-g)} = e^{\beta(\epsilon-\mu)} \tag{7.13}$$

At zero temperature we have,

$$\langle n\rangle = \frac{1}{g}, \quad for\ \ \epsilon_k \leq \mu \tag{7.14}$$

and zero otherwise.

Note that the distribution function reduces to the usual Fermi and Bose distribution functions for $g = 1$ and $g = 0$ respectively and in general g is regarded as the exclusion statistics parameter. Indeed, as demonstrated before the statistical parameter g is precisely the interaction strength as given in Eq. (7.4) in one dimension. However, the distribution function as given above is valid in general and not necessarily restricted to one dimensional models. The following discussion is therefore used for illustration in the case of one-dimensional model but not restricted to this case alone.

The number fluctuation at a given energy ϵ_k is given by,

$$(\delta n_k)^2 = T\frac{\partial\langle n_k\rangle}{\partial\mu}. \tag{7.15}$$

Substituting for $\langle n_k\rangle$ from Eq. (7.12), we have for total number fluctuation[64],

$$\begin{aligned}(\delta N)^2 &= \sum_{k=0}^{\infty} w_k(1+w_k)\langle n_k\rangle^3 \\ &= \sum_{k=0}^{\infty}\langle n_k\rangle(1-g\langle n_k\rangle)(1+(1-g)\langle n_k\rangle).\end{aligned} \tag{7.16}$$

The number fluctuation vanishes at $T \to 0$ since $n_k \to 1/g$ below the Fermi energy and zero otherwise. This result holds no matter how weak the interaction strength is. However, at $g = 0$, the bosonic limit, the number fluctuation diverges as noted earlier. In this exactly solvable model, we have thus shown that interactions do remove the fluctuation catastrophe encountered in the ideal Bose gas.

While these results have been derived in one dimension, extension to higher dimension is non-trivial since there is no suitable exactly solvable many body model. However, it has been shown that models with short range interactions in two dimension may be regarded as obeying exclusion statistics in the mean-field picture[96]. We discuss the fluctuation in these models in the next section.

7.1.2 Fluctuations in a two dimensional model

We consider a two-dimensional system of bosons interacting via a zero-range repulsive pseudo-potential as in chapter 6. The quantum dynamics is then approximated by the following Hamiltonian

$$H = \sum_{i=1}^{N}\left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 r_i^2\right) + \frac{2\pi\hbar^2}{m}\tilde{g}\sum_{i<j}^{N}\delta(\mathbf{r}_i - \mathbf{r}_j)\ , \tag{7.17}$$

where the momenta and coordinates are planar vectors. The one-body potential generated by the above zero-range interaction (including exchange) is

$$U(n(\mathbf{r})) = \frac{2\pi\hbar^2}{m}g\ n(\mathbf{r}), \qquad g = 2\tilde{g}\ , \tag{7.18}$$

where $n(\mathbf{r})$ is the local number density of the system. In two-dimensions, $g \geq 0$ plays the role of the statistical parameter, with $g = 0$ for non-interacting bosons. At finite temperature, for $T > T_c$, the Thomas-Fermi approximation yields[106, 113]

$$n(\mathbf{r}) = \int \frac{d^2p/(2\pi\hbar)^2}{\left[\exp[(\frac{p^2}{2m} + V(r) - \mu)\beta] - 1\right]}\ , \tag{7.19}$$

where the Thomas-Fermi mean potential $V(r)$ is given by

$$V(\mathbf{r}) = V_0(r) + \frac{2\pi\hbar^2}{m} g\ n(\mathbf{r})\ . \tag{7.20}$$

Here $V_0(r)$ is the one-body harmonic trap. Note that Eq.(7.19) is valid only in the absence of a condensate. It has been shown [106, 113]. however, that for a nonzero positive g, a self-consistent solution of this equation satisfying $\int n(r)d^2r = N$ may be obtained right down to T=0. This solution has a lower free energy than the one with a condensate [125], so we may take $T_c = 0$ for $g > 0$.

The momentum integration may be done analytically:

$$n(r) = -\frac{m}{2\pi\hbar^2\beta} \ln\left[1 - \exp[-\beta(V(r) - \mu)]\right]\ . \tag{7.21}$$

The local number fluctuation (between r and $r + dr$) in GCE is given by

$$(\delta N)^2 = T\frac{\partial n(r)}{\partial \mu}, \tag{7.22}$$

where the coefficient of the temperature T on the rhs is related to the compressibility. Taking the derivative of the local density with respect to the chemical potential we have

$$(\delta N)^2 = T\frac{m}{2\pi\hbar^2}\ \frac{1}{\exp[(V(r) - \mu)\beta] - 1 + g}\ , \tag{7.23}$$

Note that μ is a function of temperature, and is determined by the condition that $\int n(r)d^2r = N$, and in the thermodynamic limit it approaches the lowest energy eigenstate as the temperature goes to zero. A few remarks on the thermodynamic limit are in order: The thermodynamic limit is reached when $N \to \infty$ and $\omega \to 0$. However, in the limit of no confinement the density of states becomes a constant and there is no critical temperature below which condensation takes place. Preserving the density of states as in a two dimensional oscillator, the condensation temperature of an *ideal* bose gas is given by

$$T_c^{(0)} = (6/\pi^2)^{1/2}\ N^{1/2}\omega.$$

Thus the limit $N \to \infty$ and $\omega \to 0$ is obtained keeping $T_c^{(0)}$ constant. In an ideal bose gas ($g = 0$), no self-consistent solution of Eq. (7.21) can be found for a fixed N below this temperature $T_c^{(0)}$. However, when $g > 0$ no matter how small, the self-consistent solution of Eq. (7.21) may then be found for all $T > 0$.

In this limit the fluctuation is given by,

$$(\delta N)^2 = T\frac{m}{2\pi\hbar^2}\frac{1}{\exp[(\frac{2\pi\hbar^2}{m}gn - \mu)\beta] - 1 + g}\ , \tag{7.24}$$

where n is the constant density given in the thermodynamic limit. (For the one-dimensional case, this was denoted by ρ_0 earlier.)

In the absence of interaction, $g = 0$, the chemical potential μ goes to zero at $T = T_c$ and the above expression diverges as expected. However, as in the one dimensional case, the fluctuation remains finite and approaches zero as $T \to 0$ when g is finite, however weak the interaction may be.

The same result may be obtained in the non-interacting exclusion statistics description in two-dimensions. The local density as a function of the radial coordinate is given by [106],

$$n(\mathbf{r}) = \int \frac{d^2p/(2\pi\hbar)^2}{[w + g]}\ , \tag{7.25}$$

where the local variable $w(p,r)$ is defined through Wu's equation within TF approximation

$$w^g(1+w)^{1-g} = \exp[\beta(\frac{p^2}{2m} + V_0(r) - \mu)], \tag{7.26}$$

and g is the exclusion statistics parameter which we will identify with the interaction strength in the mean-field picture. We have retained the trap potential V_0 as in the interacting picture. Once again the momentum integration may be done easily and we obtain,

$$n(r) = \frac{m}{2\pi\hbar^2\beta} \ln\left[\frac{1+w_0}{w_0}\right] , \tag{7.27}$$

where the local variable $w_0(r)$ is determined through

$$w_0^g(1+w_0)^{1-g} = \exp[\beta(V_0(r) - \mu)]. \tag{7.28}$$

The fluctuation is then found by using Eq. (7.22) and we have

$$\delta N^2 = T\frac{\partial n(r)}{\partial \mu}, \tag{7.29}$$

$$= T\frac{m}{2\pi\hbar^2}\frac{1}{w_0+g} . \tag{7.30}$$

In the bosonic limit taking $g = 0$ in Eq. (7.28), we have $w_0 = \exp[\beta(V_0-\mu)]-1$. Substituting this in the above expression for the fluctuation in the thermodynamic limit it is easy to see that at T_c there is divergence. However, for positive definite g there is no divergence. Furthermore the equivalence between the non-interacting exclusion statistics picture and the mean-field description is established using the following relationship

$$w_0(r) = \exp[\beta(V(r) - \mu)] - 1, \tag{7.31}$$

where $V(r)$ is the self consistent mean-field potential. Substituting this in Eq.(7.30), and taking the thermodynamic limit yields Eq.(7.24). Note that this equivalence allows one to calculate fluctuations in either the mean field picture or in the non-interacting exclusion statistics picture. Indeed this holds for the computation of other global thermodynamic quantities as well.

7.2 Ground state fluctuation in the canonical ensemble

The total number fluctuation defined in the previous section in the GCE is obtained by summing over all the single particle states. In the canonical ensemble however, the number of particles is fixed and therefore the fluctuation in particle number has to be defined with respect to a reference state. One way of defining the fluctuation is to look at the ground state occupancy as a function of temperature, which is present even when the total number of particles in the trap is fixed. At $T = 0$ all the particles are in the ground state. At a nonzero temperature (or excitation energy), there are many ways of exciting the particles from the ground state, leading to a fluctuation in the ground state population. This number fluctuation for non-interacting bosons in a harmonic trap has been calculated by a number of authors [114, 115, 116, 117, 118] as a function of the excitation energy or temperature. This has also been calculated for fermions in the CE by us in a previous publication[124].

Unlike the case of GCE for bosons, the ground state fluctuation in the CE is finite at all temperatures.

We may extend the analysis of fluctuations in CE to particles interacting via the inverse square pair-wise interaction in one dimension. It is more convenient to perform the calculation in the fermionic basis, although the formulae given here are applicable for both interacting fermions, or interacting bosons. In this section, quantities like energy and ground state number fluctuation of the interacting system with interaction strength g will be denoted by a bracketed superscript, e.g. $E^{(g)}$, $((\delta N_0)^2)^{(g)}$. We recall that the spectrum of the CSM Hamiltonian given in Eq. (7.4) is exactly known. The states may be labeled by a set of fermionic occupation numbers $\{n_k\},\ k = 1, ..., \infty,\ n_k = 0, 1$. The energy $E^{(g)}$ of the system in the fermionic basis is then given by,

$$E^{(g)}\{n_k\} = \sum_{k=1}^{\infty} \epsilon_k n_k - \omega(1-g)\frac{N(N-1)}{2}, \tag{7.32}$$

where $\epsilon_k = (k - \frac{1}{2}\hbar\omega)$ denotes the harmonic oscillator energy levels and $N = \sum_{k=1}^{\infty} n_k$. As can be seen from Eq. (7.32), the effect of the interaction is that each particle shifts the energy of every other particle by a constant $\hbar\omega(g-1)$. The energy functional can also be written as

$$E^{(g)}\{n_k\} = \sum_{k=1}^{\infty} \epsilon_k n_k - \omega(1-g) \sum_{k_1<k_2=1}^{\infty} n_{k_1} n_{k_2}. \tag{7.33}$$

The exact spectrum of the model is thus reproduced by an effective Hamiltonian of quasiparticles with constant density of states and constant Landau parameters. As mentioned before, this scale invariant energy shift is the basic reason for the occurrence of nontrivial exclusion statistics where g plays the role of exclusion statistics parameter with $g = 0, 1$ for bosons and fermions.

The general canonical partition function in any basis is written in the occupation number representation as[61]

$$Z_N^{(g)} = \sum_{\{n_k\}} e^{-\beta E^{(g)}\{n_k\}}. \tag{7.34}$$

Using the energy spectrum in CSM given in Eq. (7.32), the N-particle partition function in this one dimensional model is given by

$$Z_N^{(g)} = e^{\tilde{\beta}(1-g)\frac{N(N-1)}{2}} Z_N^F, \tag{7.35}$$

where $\tilde{\beta} = \beta\hbar\omega$ and Z_N^F is the N particle fermion partition function. Setting $g = 0$, the bosonic partition function is obtained,

$$Z_N^B = e^{\tilde{\beta}\frac{N(N-1)}{2}} Z_N^F. \tag{7.36}$$

Combining Eqs. (7.35) and (7.36) we may write the partition function for CSM as,

$$Z_N^{(g)} = (Z_N^F)^g (Z_N^B)^{1-g}. \tag{7.37}$$

The canonical partition function given above is exact in CE and may be used for calculating the thermodynamic properties of the system in CSM within the canonical ensemble formalism. The moments of the occupation number are related to the partition function by

$$\langle n_k \rangle^{(g)} = \frac{1}{Z_N^{(g)}} y_k \frac{\partial Z_N^{(g)}}{\partial y_k}, \tag{7.38}$$

$$\langle n_k^2 \rangle^{(g)} = \frac{1}{Z_N^{(g)}} y_k \frac{\partial}{\partial y_k}\left(y_k \frac{\partial Z_N^{(g)}}{\partial y_k} \right), \tag{7.39}$$

where $y_k = \exp(-\beta\epsilon_k)$. Therefore, it follows that

$$\langle(\delta n_k)^2\rangle^{(g)} = y_k \frac{\partial\langle n_k\rangle^{(g)}}{\partial y_k}. \tag{7.40}$$

Using Eqs. (7.37)-(7.39), $\langle n_k\rangle^(g)$ can be expressed in terms of those of fermions and bosons via:

$$\langle n_k\rangle^{(g)} = g\langle n_k\rangle^F + (1-g)\langle n_k\rangle^B \tag{7.41}$$

Unlike $\langle n_k\rangle^{(g)}$, $\langle n_k^2\rangle^{(g)}$ does not have a simple form as Eq. (7.41). However, the expression for the fluctuation in the occupation number does:

$$((\delta n_k)^2)^(g) = g((\delta n_k)^2)^F + (1-g)((\delta n_k)^2)^B. \tag{7.42}$$

Eq. (7.42) gives only the fluctuation in the occupation of a given level k, while the quantity we are seeking is the *ground state* number fluctuation. The latter is formally defined in any ensemble as:

$$(\delta N_0)^2 = \sum_k (\delta n_k)^2 = \sum_k (\langle n_k^2\rangle - \langle n_k\rangle^2) \tag{7.43}$$

where the sum k runs over only the levels which are completely occupied at zero temperature. Thus, in an ab initio calculation, one would formally sum over the quasiparticle levels which are occupied at $T = 0$ to get $((\delta N_0)^2)^(g)$. Fig.(4.1) shows the level flow in CSM as a function of g obtained from Eq. (7.33) at $T = 0$. It can be seen that as g changes from the fermionic to the bosonic end, the number of levels contributing to the ground state remains constant, while the Fermi energy decreases accordingly. This means that one may obtain $((\delta N_0)^2)^{(g)}$ by simply substituting the ground state fluctuations for fermions and bosons. *ie:*

$$((\delta N_0)^2)^{(g)} = g((\delta N_0)^2)^F + (1-g)((\delta N_0)^2)^B \tag{7.44}$$

Chapter 8

Cold atoms and FES

In previous chapters we have discussed systems, in detail, in one- and two-dimensions where the interactions may be treated as statistical. This allows us to treat these interacting many-body systems as systems of quasi-particles which obey fractional exclusion statistics as proposed by Haldane at least to the leading order. In this chapter we consider a realistic three-dimensional system which at the outset appear to incorporate features of exclusion statistics. Main advantage with this system is that it allows comparison with experiments as well as with realistic many-body calculations.

We first consider a gas of neutral fermionic atoms at ultra-low temperatures, with the attractive interaction tuned to Feshbach resonance. We calculate, the variation of the chemical potential and the energy per particle as a function of temperature by assuming the system to be an ideal gas obeying the Haldane-Wu fractional exclusion statistics. The results for the untrapped gas compare favourably with the recently published Monte Carlo calculations of two groups. For a harmonically trapped gas, the results agree with experiment, and also with other published work.

Next We consider a few-particle system of trapped neutral fermionic atoms at ultra-low temperatures. We calculate the energies and the spatial densities of the few-body systems using a generalisation of the extended Thomas-Fermi (ETF) method, and assuming the particles obey the Haldane-Wu fractional exclusion statistics (FES) at unitarity. The semi-classical FES results are then found to be consistent with the Monte-Carlo calculations of the above authors, but can hardly be distinguished from their over all scaling of the ETF result at unitarity.

8.1 Cold fermionic atoms in a trap

There has been a lot of interest in a dilute gas of neutral fermionic atoms at ultra-cold temperatures both experimentally [126, 127, 128] and theoretically [129, 130, 131, 132]. In general, the low-energy properties of the gas are determined by the scattering length a, the number density n, and the temperature T of the gas (the effective range r_0 is small, so that $r_0/|a| \to 0$ as a becomes large). When the attractive interaction between the atoms is increased continuously by magnetic tuning from weak to strong, the scattering length a goes from a small negative to a small positive value. In between, there is a zero-energy two-body bound state, and $|a|$ is infinite. The gas is said to be at unitarity in this situation, and the only length scale a drops out. The behaviour of the gas is expected to be universal at unitarity [129, 130, 131, 132].

Experimentally, if the temperature is small enough, a BCS superfluid is observed at the weak end, and a BEC condensate of dimers at the strong end [126, 127, 128]. This was

predicted long back by Leggett [133], who extended the BCS formalism in a novel fashion to analyse the physical situation. The BCS to BEC transition is found to be smooth, with no discontinuity in properties across the unitary point. There has been much interest amongst theorists to calculate the properties of the gas in the unitary regime ($k_f|a| >> 1$), where $k_f = (3\pi^2 n)^{1/3}$ is the Fermi wave number of the noninteracting gas. This is a challenging task, since there is no small expansion parameter, and a perturbative calculation cannot be done.

In particular, at $T = 0$, the energy per particle of the gas is calculated to be

$$\frac{E}{N} = \xi \, \frac{3}{5} \frac{\hbar^2 k_f^2}{2M}, \tag{8.1}$$

where $\xi \simeq 0.44$ [134, 135]. The experimental value is about 0.5, but with large error bars [136, 137]. Recently, there have been two Monte Carlo (MC) finite temperature calculations [138, 139] of an untrapped gas at unitarity, where various thermodynamic properties as a function of temperature have been computed. For a harmonically trapped gas, there are experimental results [140], as well as theoretical calculations [141].

In the unitary regime, the thermodynamic properties have both bosonic and fermionic features [138], and it is natural to ask if in this regime the quasi-particles obey a statistics which is possibly intermediate between Fermi and Bose statistics. In a recent paper[142], on very general grounds, it was suggested that at unitarity, so far as average properties of the system are concerned, it should behave like an ideal gas obeying the generalised exclusion statistics of Haldane [38]. The definition of the statistical parameter, denoted by $g(> 0)$ as before, is based on the rate at which the number of available states in a system of fixed size decreases as more and more particles are added to it. As noted earlier, the statistical parameter g assumes values 0 and 1 for bosons and fermions respectively, because the addition of one particle reduces the number of available states by g.

At the outset we remark that there is no quantative calculations to provide a microscopic theory mapping the interacting cold atoms to the quasi-particles obeying excusion statistics. However we can advance several qualitative arguements to provide the rationale for using Haldane-Wu statistics at unitarity as follows:

- As is well known, Haldane-Wu statistics is realised by the Calogero-Sutherland model in one dimension [61]. The potential and kinetic energy both scale the same way in this model, and both the energy densities scale as n^3. Similarly, fermions in two dimensions interacting with a zero-range potential have their kinetic and potential energy densities scale as n^2, obeying Haldane-Wu statistics [96].

  Likewise, a strong hint that Haldane statistics may be realised for the system under consideration comes from the observation that the kinetic and potential energies scale the same way (see Eq.8.1) when there is no length scale left from the interaction at $T = 0$. While this scaling behaviour at $T = 0$ is a necessary condition for systems obeying Haldane statistics, it does not by itself imply that all systems showing scaling behaviour obey Haldane statistics at finite temperatures.

- In the present case, a compelling evidence comes from the fact that the second virial coefficient of the gas at unitarity is temperature independent [143]. In exclusion statistics, the scale-invariant interaction between atoms alters the ideal Fermi (Bose) values of the (exchange) second virial coefficient $+(-)2^{-5/2}$ by adding an interacting part [54].

- The zero temperature calculations with Haldane statistics yields results consistent with density functional approach of Papenbrock[144].

The above arguments are heuristic and indicative. A quantitative understanding can be obtained only when the effective interaction is known fully. In the absence of such a theory, in here we pursue a phenomenological approach where we assume the validity of exclusion statistics on the average for quasi-particles which are otherwise non-interacting. The effect of interaction is entirely subsumed in defining the statistics of the quasi-particles.

The value of the statistical parameter g for the unitary gas is deduced from theory using Eq.(8.1), fitting the parameter $\xi = 0.44$. The value of g thus determined remains the same independent of the nature of confinement as it should since the microscopic origin of the value of g depends only on the interaction between fermions and not on how the system is prepared experimentally. The application of the finite temperature distribution function [55] then enables us to calculate the temperature dependence of the energy per particle, and the chemical potential of the unitary gas. The results for both trapped and untrapped gases are in good agreement with experiment, and MC calculations.

Let us first estimate the value of the statistical parameter g by using the following considerations: For Haldane-Wu statistics, the distribution function (or occupancy factor) in a single particle state with energy ϵ_i is $f_i = (w_i + g)^{-1}$, where w_i obeys the relation

$$w_i^g (1 + w_i)^{1-g} = exp[(\epsilon_i - \mu)\beta] \ , \tag{8.2}$$

where $\beta = 1/T$, T being the temperature in units of the Boltzmann constant. Note from the above that for $g = 0$ and 1, the distribution function f_i reduces to the familiar bosonic and fermionic forms. It is also clear that for $T = 0$, the occupancy factor is

$$\begin{aligned} f_i(T=0) &= \frac{1}{g} \ , \quad \epsilon_i < \mu, \\ f_i(T=0) &= 0 \ , \quad \epsilon_i > \mu \ . \end{aligned} \tag{8.3}$$

Now consider N interacting fermionic atoms mapped to this statistics at $T = 0$ in a large volume V. The new Fermi momentum $\tilde{k_f}$ is determined from the relation

$$N = V \frac{1}{g} \frac{2}{(2\pi)^3} \int_0^{\tilde{k_f}} 4\pi k^2 dk \ , \tag{8.4}$$

where we have included a spin degeneracy factor of 2. The modified Fermi momentum $\tilde{k_f}$, from above. is $\tilde{k_f} = g^{1/3} k_f$, where k_f is the fermi momentum of the noninteracting Fermi gas. It also follows that the energy per particle of the unitary gas is given by

$$\frac{E}{N} = g^{2/3} \ \frac{3}{5} \frac{\hbar^2 k_f^2}{2M} \ . \tag{8.5}$$

Comparing with Eq.(8.1), we see that $\xi = g^{2/3}$, and choosing $g = 0.29$ gives the generally accepted value of $\xi = 0.44$. This therefore fixes the only free parameter in the model, namely, g and it should be valid independent of temperature and the nature of confinement as it is the parameter which determines the statistics of quasi-particles.

8.2 Homogeneous gas

The main advantage of the phenomenological model, however, is the calculation of the bulk properties of the gas as a function of the temperature, and this we proceed to do now. We follow the well known method (see for example the paper by Aoyama [76]) for this purpose. For a given density of single-particle states $D(\epsilon)$, we have

$$N = \int_0^\infty \frac{D(\epsilon) d\epsilon}{(w + g)}, \quad E = \int_0^\infty \frac{\epsilon D(\epsilon) d\epsilon}{(w + g)}. \tag{8.6}$$

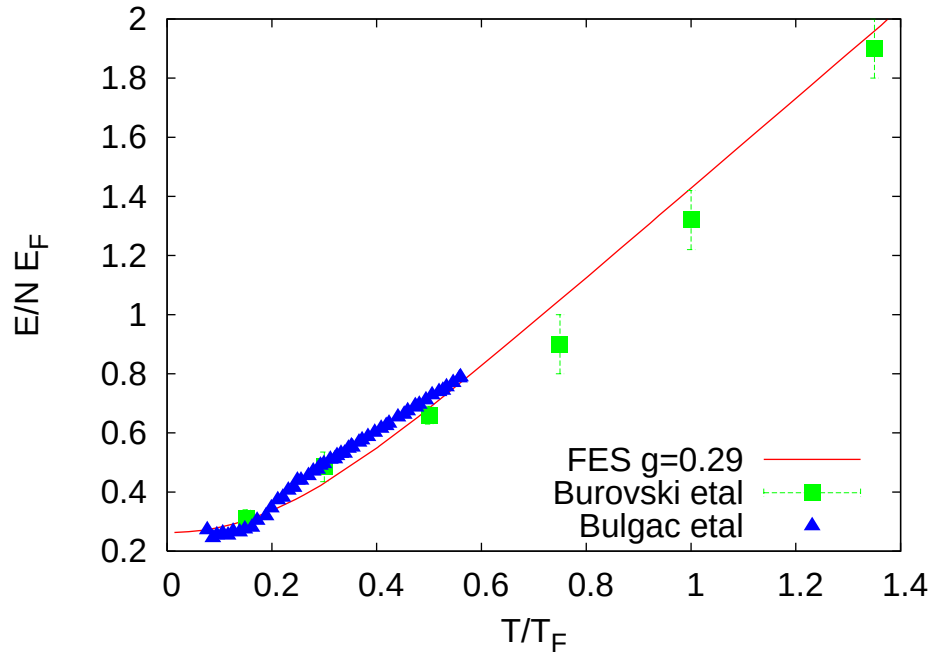

Figure 8.1: Plot of the energy per particle as a function of temperature. Both the abscissa and ordinate are in units of the free Fermi energy. The solid line corresponds to our calculations with $g = 0.29$. The solid squares (green) with error bars are the MC calculations of [139], and the triangles (blue) are the MC calculations of [138].

For the 3−dimensional gas, $D(\epsilon) = C\sqrt{\epsilon}$, where the constant $C = \frac{3}{2}N\epsilon_f^{-3/2}$. Furthermore, $\epsilon_f = \frac{\hbar^2 k_f^2}{2M}$ is the Fermi energy of the noninteracting Fermi gas. Changing the variable from $d\epsilon$ to dw, and using the relation involving w's given above, one gets after some algebra

$$\frac{3}{2}\left(\frac{T}{\epsilon_f}\right)^{3/2}\int_{w_0}^{\infty}\frac{dw}{w(1+w)}\left[\ln\left\{\left(\frac{w}{w_0}\right)^{g}\left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^{1/2} = 1, \tag{8.7}$$

$$\frac{E}{N\epsilon_f} = \frac{3}{2}\left(\frac{T}{\epsilon_f}\right)^{5/2}\int_{w_0}^{\infty}\frac{dw}{w(1+w)}\left[\ln\left\{\left(\frac{w}{w_0}\right)^{g}\left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^{3/2}. \tag{8.8}$$

In the above, w_0 is the value of w at $\epsilon = 0$. For our choice of $g = 0.29$, the Eq. (8.14) is solved at a given (T/ϵ_f) for w_0 numerically, and this w_0 is used in Eq.(8.15) next to obtain $(E/N\epsilon_f)$. From the definition of w_0, it also follows that the chemical potential μ at temperature T obeys the relation

$$\frac{\mu}{\epsilon_f} = -\frac{T}{\epsilon_f}\left[g\ln w_0 + (1-g)\ln(1+w_0)\ \right]. \tag{8.9}$$

The results for the energy per particle and the chemical potential (in units of the noninteracting Fermi energy ϵ_f) are plotted in Fig.8.1 and Fig.8.2 respectively.

The results are not sensitive to the fine-tuning of the statistical parameter g. In Fig. 8.1, we also show, for comparison, the recent MC calculated points of Bulgac *et al.*[138] and Burovski *et al.* [139]. It will be seen that the agreement is very good, especially for energy per particle, although the chemical potential μ as calculated by us starts to differ from Burovski *et al* result for $T/\epsilon_f > 0.8$). The Quantum Monte-Carlo calculations for μ between $T/T_F = 0.8$ and 1, despite the large error bars, appear to have a pronounced break in the slope of the μ vs. T curve. The reason for this is not clear. Our results, on the other hand, smoothly go over to the classical behaviour, as is to be expected.

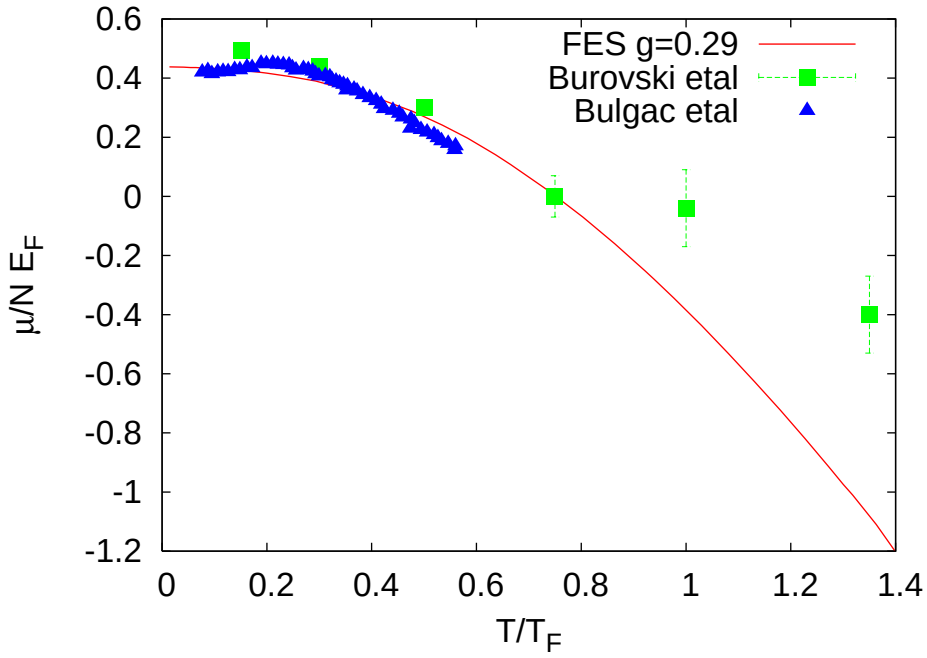

Figure 8.2: Same as in Fig.(8.1) for chemical potential plotted as a function of temperature.

8.3 Trapped gas in harmonic oscillator

The finite temperature results are easily generalised for fermions in harmonic trap. Consider the fermions at $T = 0$. The density of states $D(\epsilon)$, including a spin degeneracy factor of 2, is $\epsilon^2/(\hbar\omega)^3$, where the oscillator parameter is defined as $\omega = (\omega_x\omega_y\omega_z)^{1/3}$. It follows immediately that $\tilde{\epsilon_f} = (3gN)^{1/3}\hbar\omega$, and the energy $E = g^{1/3}(3N)^{4/3}/4\ \hbar\omega$. These results are the same as the Thomas-Fermi density functional approach of Papenbrock [144]. We can easily extend these results to finite temperatures using this density of states in Eq.(8.6).

$$1 = 3\left(\frac{T}{\epsilon_f}\right)^3 \times \int_{w_0}^{\infty} \frac{dw}{w(1+w)} \left[\ln\left\{\left(\frac{w}{w_0}\right)^g \left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^2, \tag{8.10}$$

$$\frac{E}{N\epsilon_f} = 3\left(\frac{T}{\epsilon_f}\right)^4 \times \int_{w_0}^{\infty} \frac{dw}{w(1+w)} \left[\ln\left\{\left(\frac{w}{w_0}\right)^g \left(\frac{1+w}{1+w_0}\right)^{1-g}\right\}\right]^3. \tag{8.11}$$

The expression for μ remains the same as Eq.(8.9), although the numerical values of w_0 as a function of T are quite different from the unconfined gas. We present the results for average energy in the trap in Fig.8.3, using the same value of $g = 0.29$ since the statistical parameter depends only on the mutual interaction and not on the nature of confinement. It will be seen that the agreement with experimental data of Kinast et al [140] as well as the many body calculation of Hu et al [141] is very good.

Other thermodynamic quantities, like the specific heat and the entropy, could be readily calculated. However we note that the model cannot yield the two- or many-particle correlation functions. In this regard, the situation is similar to the one-dimensional Calogero-Sutherland model [39, 40, 41], which can be mapped on to a system of quasi-particles which obey Haldane-Wu statistics [75, 61, 62, 56]. But this does not help in obtaining the correlation functions, for which the full many-body calculation has to be done. Moreover, the ideal Haldane-Wu gas cannotdescribe super-fluidity. Therefore, the main usefulness of the present approach is its ability to calculate the temperature-dependence of various bulk properties of a unitary gas with just one free parameter, namely the statistical parameter g.

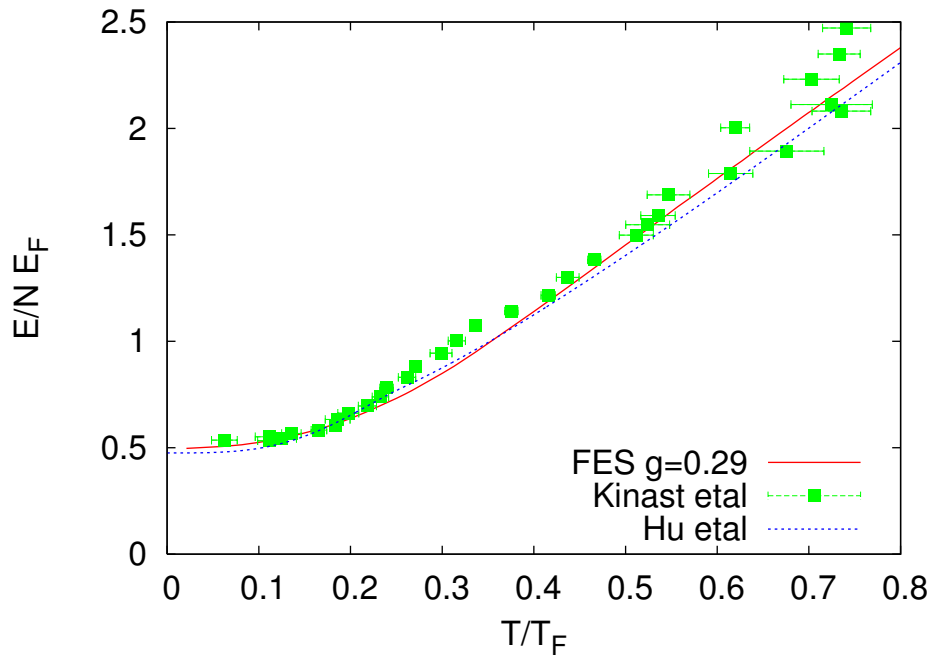

Figure 8.3: Plot of the energy per particle as a function of temperature for the confined gas. Both the abscissa and ordinate are in units of the free Fermi energy. The solid line corresponds to our calculations with $g = 0.29$ and the dashed lines corresponds to the calculations presented in Ref.[141]. The experimental data is taken from Kinast et al [140].

8.4 Many-particle ground state

Chang and Bertsch [145] have recently presented an *ab initio* Green's Function Monte-Carlo (GFMC) calculation of the energy and density of $N = 2 - 22$ trapped fermionic atoms in a harmonic potential. The atoms are interacting via a short range central two-body potential, with its strength adjusted to yield a zero-energy two-body bound state in free space. The many-body properties of this system are expected to be independent of the shape of the two-body interaction in such a set-up.

In the previous sections [142] we obtained the energy per particle and the chemical potential of a noninteracting gas of atoms at finite temperatures obeying fractional exclusion statistics (FES). As is clear we assumed that at unitarity, the effect of the interaction could be simulated by FES for the bulk properties of the system. The results, with the choice of one free parameter in FES, were found to be in good agreement with theoretical MC calculations for a free gas [138, 139] and the experimental results in a trap [140, 141]. Since the number of atoms, N, was taken to be large, no finite-N corrections were needed in our semiclassical calculations. This is not the case for the ground state energies, where N is taken to be small. The purpose of this section is therefore to test whether the FES hypothesis gives improved results when the finite-N corrections are incorporated in our calculations. The account given here is taken from Ref.[149]

In the next section, we first summarise the TF and ETF results. Both these have limitations at the classical turning point, where the spatial density behaves discontinuously. To rectify this, we make use of a modified semiclassical method [146] that gives a continuous variation of the density across the turning point. Our semiclassical results incorporating FES are next compared with the Green Function Monte-Carlo (GFMC) calculations of Chang and Bertsch [145] for fermions trapped in a three dimensional oscillator potential. We find that FES results are consistent with the many-body GFMC results.

8.5 Semiclassical Calculations incorporating FES

It also follows from Eq.(8.1) that the scaling factor ξ in a Fermi gas is related to the statistical parameter g by the relation $\xi = g^{2/3}$. In a three-dimensional isotropic harmonic trap, a similar scaling of the TF expression gives [144], in units of $\hbar\omega$,

$$E_{TF} = \frac{\xi^{1/2}}{4}(3N)^{4/3} \ . \tag{8.12}$$

In FES, an identical relation is obtained, with $\xi^{1/2}$ replaced by $g^{1/3}$. The scaled TF spatial density is also identical to the FES expression when this replacement is made :

$$\rho_{TF}(r) = \frac{1}{3\pi^2 g}(\frac{2}{l^2})^{3/2}\left((3gN)^{1/3} - \frac{1}{2}\frac{r^2}{l^2}\right)^{3/2} \ , \tag{8.13}$$

where $l = \sqrt{(\hbar/m\omega)}$. The above expression is valid for $r \leq r_0$, where $r_0 = \sqrt{2}l(3gN)^{1/6}$ is the classical turning point. For $r > r_0$, the TF density is zero. To implement finite-N corrections, one has to consider ETF [147]. Chang and Bertsch [145] scale the energy expression for ETF by the same over all factor as in TF (denoted by ETF'), where as FES yields a different expression [148] (in units of $\hbar\omega$):

$$E'_{ETF} = \xi^{1/2}\left(\frac{(3N)^{4/3}}{4} + \frac{(3N)^{2/3}}{8} + ...\right); \tag{8.14}$$

$$E_{ETF} = \left(g^{1/3}\frac{(3N)^{4/3}}{4} + g^{-1/3}\frac{(3N)^{2/3}}{8} + ...\right) \ . \tag{8.15}$$

Although ETF gives a reasonable description of the smooth part of the energy, it fails to do so for the spatial density. In fact, the ETF density diverges at the turning point. To give a consistent description of both the energy and the spatial density, we adopt a method where a selective summation of the higher order gradient terms of the Wigner-Kirkwood series is made [146]. For a harmonic trapping potential $V(r)$, retaining terms up to third order in β, the Bloch density $C(r,\beta)$ incorporating FES is given by

$$C(r,\beta) = \frac{1}{4\pi^2 g}(\frac{2m}{\hbar^2\beta})^{3/2} \left(1 - \frac{\hbar^2\beta^2}{12m}\right)\exp\left[-\beta V + \beta^3\frac{\hbar^2}{24m}(\nabla V)^2\right] \ . \tag{8.16}$$

The spatial density is obtained by taking the inverse Laplace transform of $C(r,\beta)/\beta$ with respect to the chemical potential μ, which we denote by $\tilde{\rho} = \mathcal{L}_\mu^{-1}[C(r,\beta)/\beta]$. It is the cubic term in the exponent that makes the density continuous across the classical turning point. Similarly, the energy is given by $\tilde{E} = \mu N - \mathcal{L}_\mu^{-1}C(r,\beta)/\beta^2$. The inverse Laplace transformations are carried out by the saddle-point method. The quality of the approximation is tested by applying the method to N noninteracting spin-1/2 fermions ($g = 1$) in a harmonic potential. The result for the energy is plotted as a function of N is plotted in Fig.8.4. To facilitate the comparison, the TF energy is subtracted out from the quantum as well as the semiclassical results for E_{ETF} and $\tilde{E}$. Note that the shell effects in the energy as well as the density are not reproduced in the semiclassical calculations.

In Fig.8.5, we compare the GFMC results [145] of the energy for $N = 2 - 22$ atoms with the various semiclassical calculations. For the latter, the scaling factor in Eq.(8.14) is taken to be $\xi = 0.48$, that corresponds to $g = 1/3$ for the ETF Eq.(8.15), and also for

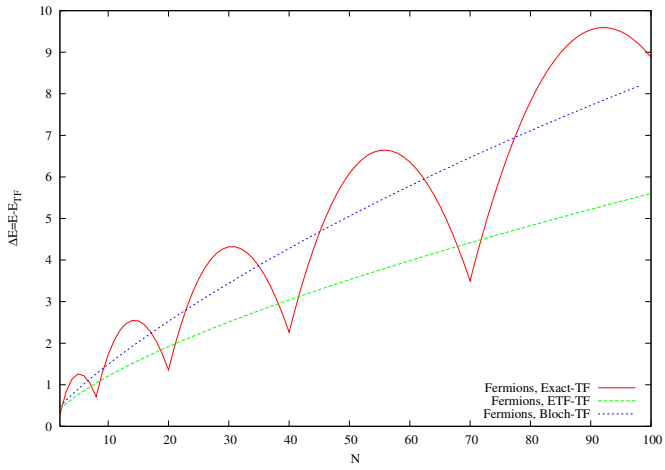

Figure 8.4: Plot of the energy, $E - E_{TF}$ vs particle number for fermions (i.e., $g = 1$). The red (solid) curve corresponds to the exact calculation in a harmonic oscillator while the green (dashed) and blue (dotted) curves correspond the calculations based on ETF and resummation methods.

$\tilde{E}$. Our choice of $\xi = 0.48$ is very close to that of [145], and corresponds to a g not too different from the value of 0.29 chosen earlier [142]. It is seen from Fig.8.5 that all the semiclassical methods fare well, and it is not possible to distinguish the scaled results from the FES ones. A clearer comparison is made in the figure on the right in Fig.8.5, where

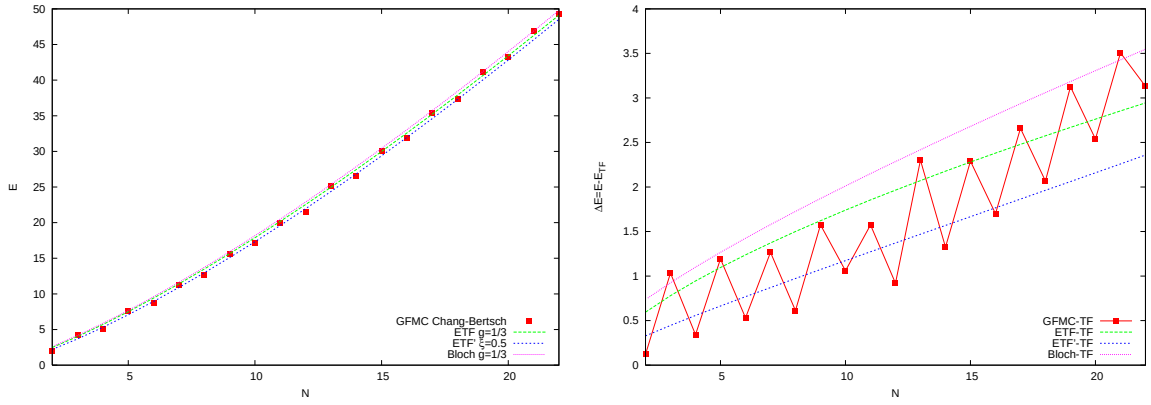

Figure 8.5: Left: Plot of the energy vs particle number with $g = 1/3$. The data points refer to the GFMC calculation of Chang and Bertsch[145]. The pink(dotted) curve corresponds to the energy calculation using the resummation method. The green (dashed) and blue(short-dashed) curves correspond to the standard ETF calculation with FES (see Eq.(8.15)) and the ETF' approximation given by Chang and Bertsch (see eq.(8.14)). Figure on the right shows the energy after subtracting the TF contribution as in Fig.1 but choosing $g = 1/3$.

the large TF term given by Eq.(8.12) is subtracted out from the energies. Even then, it is not possible to assert the relative superiority of over all scaling to FES. We suggest that a distinction may possibly be made if a larger range of N values are spanned by a MC many-body calculation. An interesting aspect of GFMC results (figure on the right in Fig.2) is the odd-even oscillations in energy. In Fig.8.6 the calculated density for $N = 20$ particles is plotted using the resummation method and is compared with the TF density. Although there is not much to choose between the ETF and the resummation results for the energy, the density in the latter case is distinctly superior and appears to agree with the smoothed

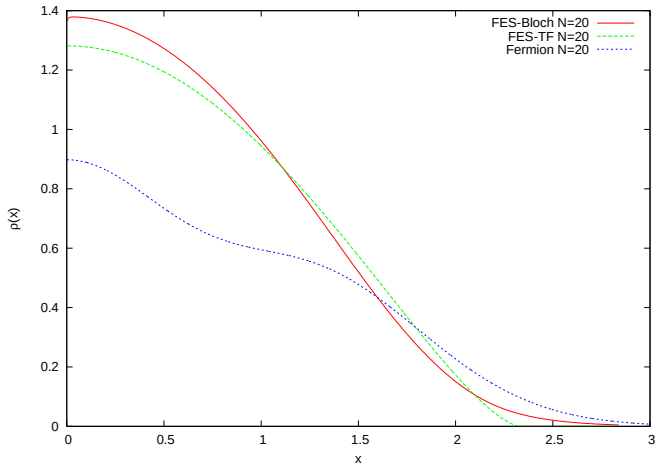

Figure 8.6: Plot of radial density for $N = 20$ as a function of the scaled distance x. Shown are the density calculated using resummation method (red-solid) and the TF density (green-dashed) with $g = 1/3$. The density of fermions in a harmonic oscillator is also shown for comparison (blue-dotted).

part of the density calculated by the the GFMC method. It also reproduces the tail beyond the turning point, which is not possible in TF or ETF approximations.

One may also be tempted to calculate the exact energy of an N particle system using the harmonic oscillator shells with redefined occupancies according to FES. For example the $L = 0$ level then has an occupancy of 6 instead of 2 at $g = 1/3$. Indeed this gives reasonable agreement with the energy of the N particle system, for example for $N = 22$ the GFMC calculation gives an energy of 49.3 where as the naive calculation with FES occupancies gives 49 in units of $\hbar\omega$. However, the exact density calculated using the corresponding wavefunctions has the wrong behaviour for shell effects. Thus FES framework can be used only in an average sense and should not be interpreted literally for all N.

To summarise, We have considered a few particle system of trapped and interacting neutral fermionic atoms at ultra-low temperatures. The energy and spatial density of this system is calculated semiclassically assuming the particles obey the Haldane-Wu fractional exclusion statistics (FES) at unitarity. The semiclassical FES results are consistent with the Monte-Carlo calculations of Chang and Bertsch [145], but can hardly be distinguished from the over all scaling of the noninteracting energy that is commonly used at unitarity. However, it is interesting to note that both at finite temperature [142] and at zero temperature the FES frame work yields reasonably good results for the smooth parts of the energy and the spatial density.

Chapter 9

Discussion

We have thus presented a detailed discussion of the generalized Pauli principle as enunciated by Haldane in his seminal paper[38]. Much of this presentation is based on our own work in this area. We have also tried to include discussions of the relevant details from other important contributions as needed. One of the importants conclusions that emerges is that we may look upon certain types interactions as giving rise to new types of statistics, where the word statistics has been used purely in the sense of restrictions on occupation numbers. Given a cell or unit in the energy space, the number of single particle states in the cell depend on the nature of the interaction between particles and hence the occupencies. This fact alone is not sufficient to generalise the occupation probabilities. If however, the occupancies are independent of where the cell is located in the full energy space, one expects a generalised Pauli principle to hold. Haldane's idea realises such a situation. There have also been other complementary ideas one of which due to Polychronakos is discussed in the appendix A.

We would like to emphasise this crucial property of exclusion statistical interactions it should cause shifts in single particle energies at all scales[54]. This property is realized by a large class of one dimensional models of interacting fermions where Fermi liquid theory breaks down [46, 61]. In fact it has been shown exactly that quasiparticles with nontrival exclusion statistics exist in a class of models that are solved by the Bethe ansatz. In particular, as we have demonstrated, the quasiparticles of the Calogero-Sutherland model (CSM) behave like ideal exclusion statistics system.

One immediate casuality of the exclusion statistics is the fact the now we can not associate an occupation probability for a single state as the exclusion acts across levels. If we did this in an unconstrained manner, then some of these probabilities may turn out to be negative. However, as we have shown the particles obeying fractional exclusion statistics may be characterised by constraints on the sets of occupation numbers. There are no negative probabilities if these constraints are obeyed. If these constraints are relaxed then the negative weights arise in order to compensate for the resulting over counting. The negative probabilities discussed in literature can be understood as arising when the system constrained by the counting rules is replaced by an unconstrained one. The negative weights then compensate for the introduction of unphysical configurations. This is therefore exactly analogous to other situations in physics where negative probablities arise, for example, the ghosts and negative norm states in gauge theories or as in the case of Wigner distribution in quantum mechanics.

Before we end the discussion, we would also like to mention other interesting applications of exclusion statistics, not envisaged in the original suggestion. These refer to the application of exclusion statistics in real three dimensional space without actually invoking the quantum statistics for the particles. The residual entropy of ice is thought of as a minifestation of FES in real three dimensional space [150]. Each hydrogen atom in ice can occupy one of

the two equvalent bonding sites between neighbouring oxygen atoms. However, there is a chemical constraint known as the *ice rule* which reduces the number of allowed configurations. Jisoon Ihm[150] proposed that this reduction may be due to fractional statistics. In yet another interesting application of exclusion statistics in real space, Riccardo et al[151] have proposed a new description of the theory of adsorption. The experimentally measured adsorption isotherms agree very well the proposed new theoretical description based on exclusion statistics.

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Appendix A

Other forms exclusion statistics

Haldane proposed the generalisation of Pauli prinicple. As already demonstrated the particles which obey Fractional Exclusions Statistics may be regarded as the quasi-particles of an interacting system like CSM. Though the interacting particles may be fermions (or bosons), the properties of the system may be realised through ideal quasi particles which obey FES.

Polychronakos [152] proposed another form for the exclusion statistics which is appealing in its simplicity. The starting point of Polychronakos is also the same as Haldane, namely that there is a change in the single particle space when new particles are added

$$g = -\Delta d/\Delta N. \tag{A.1}$$

However, there are many ways such a generalised principle can be realised through counting of many body states.

In this appendix we argue that the classical CSM may provide a realisation of exclusion statistics as proposed by Polychronakos.

A.1 Exclusion statistics of Polychronakos

The combinatorial formula for putting N identical particles in d states as suggested by Polychronakos is given by

$$D_N(\alpha) = \frac{d(d-\alpha)(d-2\alpha)...(d-\alpha N)}{N!}. \tag{A.2}$$

This can be thought of as follows: The first particles has d states to chose from, the next has $d-\alpha$ states to choose from and the last particle has $d-\alpha N$ states to occupy. The factor $N!$ is introduced to avoid overcounting. Thus the counting is essential classical and $\alpha = 1, 0, -1$ correspond respectively to Fermi, Boltzmann and Bose statistics. Unlike in the case of Haldane's proposal, the classical statistics limit is defined by $\alpha = 0$. Interestingly, there are no negative weights as in the case of FES. An important difference with FES ala Haldane is in the counting method- Haldane's starting point is the counting of many body states which is akin to Bose statistics, that is it is essential quantum. However in Polychronakos method of counting one can start from the Boltzman or classical counting, when $\alpha = 0$ and use the blocking to reproduce the quantum statistics.

Most interestingly it can be shown that the distribution function of the system is given by

$$\overline{n} = \frac{1}{\exp(\beta(\epsilon-\mu)) + \alpha} \tag{A.3}$$

which is the simplest possible generalisation of the Fermi and Bose distribution. This distribution function was first proposed by Acharya and Narayanaswami[153] in the context of anyons without any reference to the counting of states or FES. One can also obtain all the virial coefficients for a gas obeying the Exclusion statistics of Polychronakos (PES).

Appendix B

Boson and Fermion partition functions

In this appendix we derive two useful identities for bosonic and fermionic partition functions. Consider first the grand canonical partition function for bosons

$$Z^B(\beta, z) = \sum_{N=0}^{\infty} z^N Z_N^B, \tag{B.1}$$

where $z = e^{\beta\mu}$ is the fugacity and Z_N is the canonical partition function for N bosons. Therefore

$$Z_N^B(\beta) = \frac{1}{N!}\partial_z^N Z^B|_{z=0}, \tag{B.2}$$

where $\partial_z = (\partial/\partial z)$. If n labels the single particle modes then

$$Z^B = \prod_n \sum_{m=0}^{\infty} z^m e^{-m\beta E_n} = \prod_n \frac{1}{1 - z\ e^{-\beta E_n}} = e^{-S}, \tag{B.3}$$

where

$$S = \sum_n \ln(1 - z\ e^{-\beta \mathrm{E}_n}) = -\sum_n \sum_{m=1}^{\infty} \frac{z^m}{m}\ e^{-m\beta E_n} = -\sum_{m=1}^{\infty} \frac{z^m}{m}\ Z_1(m\beta). \tag{B.4}$$

Therefore we have

$$\partial_z^m S = -(m-1)! Z_1(m\beta) \tag{B.5}$$

and

$$S|_{z=0} = 0; \quad Z|_{z=0} = 1.$$

Using eq.(B.3) we have

$$\begin{aligned} \partial_z Z^B &= -Z\partial_z S \\ \partial_z^2 Z^B &= Z((\partial_z S)^2 - \partial_z^2 S) \\ \partial_z^3 Z^B &= -Z((\partial_z S)^3 - 3\partial_z^2 S(\partial_z S) + \partial_z^3 S), \end{aligned} \tag{B.6}$$

and in general

$$\partial_z^m Z^B = (-1)^m Z\left[(\partial_z S)^m - a(m)\partial_z^2 S(\partial_z S)^{m-2} + \cdots\right], \tag{B.7}$$

We can compute the coefficients $a(m)$ recursively noting that

$$\begin{aligned} \partial_z^{m+1} Z^B &= (-1)^{m+1} Z^B\left[(\partial_z S)^{m+1} - a(m+1)\partial_z^2 S(\partial_z S)^{m-1} + \cdots\right], \\ &= (-1)^{m+1} Z^B\left[(\partial_z S)^{m+1} - (a(m)+m)\partial_z^2 S(\partial_z S)^{m-1} + \cdots\right], \end{aligned} \tag{B.8}$$

which gives the recursion relation

$$a(m+1) = a(m) + m = \frac{m(m-1)}{2} + c. \tag{B.9}$$

The constant $c = 0$ since $a(2) = 1$. Thus we have

$$\partial_z^m Z^B = (-1)^m Z^B \left[(\partial_z S)^m - \frac{m(m-1)}{2} \partial_z^2 S (\partial_z S)^{m-2} + \cdots \right]. \tag{B.10}$$

Using eq.(B.2) and eq.(B.10) we obtain the desired result for the canonical partition function of bosons

$$Z_N^B = \frac{1}{N!} \left[(Z_1(\beta))^N + \frac{N(N-1)}{2} \, (Z_1(\beta))^{N-2} \, Z_1(2\beta) + \cdots \right]. \tag{B.11}$$

We follow the same steps in the fermionic case: The grand partition function of a system of fermions is given by

$$Z^F(\beta, z) = \sum_{N=0}^{\infty} z^N Z_N^F. \tag{B.12}$$

Therefore

$$Z_N^F(\beta) = \frac{1}{N!} \partial_z^N Z^F|_{z=0}, \tag{B.13}$$

Furthermore

$$Z^F = \prod_n \sum_{m=0}^{\infty} z^m e^{-m\beta E_n} = \prod_n (1 + z \; e^{-\beta E_n}) = e^{-S}, \tag{B.14}$$

where

$$S = -\sum_n \ln(1 + z \; e^{-\beta \mathrm{E}_n}) = \sum_n \sum_{m=1}^{\infty} (-1)^m \frac{z^m}{m} \; e^{-m\beta E_n} = \sum_{m=1}^{\infty} (-1)^m \frac{z^m}{m} \; Z_1(m\beta). \tag{B.15}$$

Making the above changes in the fermionic case and following the steps as in the case of the bosonic partition function, we obtain the result

$$Z_N^F = \frac{1}{N!} \left[(Z_1(\beta))^N - \frac{N(N-1)}{2} \, (Z_1(\beta))^{N-2} \, Z_1(2\beta) + \cdots \right]. \tag{B.16}$$

Appendix C

Low temperature expansions in GCE and CE

The low temperature behaviour of thermodynamic quantities in GCE are well known for bosonic systems. However, a comparison between GCE and CE calculations at low temperatures is not usually discussed for either Bose or Fermi systems. Further, making use of some asymptotic expansions, the CE fluctuation for bosons was earlier found to be linear right down to $T = 0$. However, we give here the expansion of the fluctuation squared at low temperature in power of x, where $x = e^{\tilde{\beta}}$, and show that the CE fluctuation of bosons is in fact exponential at very low T. In GCE only expansion for fermions is possible, since the fluctuation tends to infinity at low temperature for bosons. Both expansions are possible in CE.

Grand Canonical Ensemble: In GCE the (fermionic) occupation number is:

$$\langle n_k \rangle_{GCE} = \frac{1}{x^{(\mu-\epsilon_k)}+1} = \frac{1}{x^{(\mu-k-1/2)}+1} \tag{C.1}$$

for a one dimensional system. The ground state number fluctuation squared is given by:

$$\langle (\delta N_0)^2 \rangle_{GCE} = \sum_{k=0}^{k_F} \frac{x^{(\mu-k-1/2)}}{\left[x^{(mu-k-1/2)}+1\right]^2} \tag{C.2}$$

where k_F is the Fermi level. At low temperatures, for the one-dimensional harmonic oscillator, $\mu \approx \mu_0 = N$. Therefore,

$$\langle (\delta N_0)^2 \rangle_{GCE} = \sqrt{x} - 2x + 4x^{3/2} - 4x^2 + 6x^{5/2} - 8x^3 + 8x^{7/2} - 8x^4 ... \quad , \quad N \geq 4. \tag{C.3}$$

Note that the first few terms are independent of N.

Canonical Ensemble: In CE the first and second moments of the occupation number are known [154]:

$$\langle n_k \rangle = \frac{1}{Z_N} \sum_{j}^{N} (\pm)^{j+1} x^{j\epsilon_k} Z_{N-j}, \tag{C.4}$$

$$\langle n_k^2 \rangle = \frac{1}{Z_N} \sum_{j}^{N} (\pm)^{j+1} \left[j \pm (j-1)\right] x^{j\epsilon_k} Z_{N-j}, \tag{C.5}$$

where the upper and lower signs refer to bosons and fermions respectively. Summing over the ground states up the Fermi level gives the fermionic ground state number:

$$\langle N_0\rangle_{CE} = \frac{1}{Z_N}\sum_{j}^{N}(-1)^{j+1}x^{j/2}\frac{1-x^{jN}}{1-x^j}\ Z_{N-j}, \tag{C.6}$$

where we let $\epsilon_k = k - 1/2$, with $k = 1, 2, ...$ Therefore,

$$\begin{aligned}\langle(\delta N_0)^2\rangle_{CE} &= \sum_{k=1}^{N}\langle n_k\rangle - \sum_{k=1}^{N}\langle n_k\rangle^2 \\ &= x + 2x^4 + ...\ \ , \quad N \geq 4,\end{aligned} \tag{C.7}$$

where we have used $\langle n_k\rangle = \langle n_k^2\rangle$ and again the first few terms are independent of the system size N.

For bosons the ground state consists of one single lowest level, the low temperature expansion of the number fluctuation is given by

$$\begin{aligned}\langle(\delta N_0)^2\rangle_{CE} &= \langle n_0^2\rangle - \langle n_0\rangle^2 \\ &= x + 3x^2 + 4x^3 + 7x^4 + ...\ \ , \quad N > 4.\end{aligned} \tag{C.8}$$

Again as in the fermionic case the first few terms in the low temperature expansions are independent of the system size. Indeed it is interesting to note that in CE, the fluctuations in both the systems approach zero as $T \to 0$ in exactly identical fashion.

Appendix D

Fermi golden rule

We generalize the Fermi Golden Rule for quasiparticles obeying the new statistics and apply it to the calculation of a relaxation time of a nuclear spin in a metal[155]. Both the applications are qualitative, with a view to explore qualitative changes from the standard results.

We now consider generalization of the well-known Fermi Golden Rule, for particles obeying the fractional exclusion statistics. Consider a two-particle scattering process $k + p \to k' + p'$, in one dimension. Let the two particles be distinct. We have in mind processes like the electron-nuclear interactions in metals. According to the Fermi Golden Rule, the transition probability per unit time for such a process is given by

$$w_{i\to f} = 2\pi| < i|H_{int}|f > |^2\delta(E_i - E_f),$$

where H_{int} is the interaction Hamiltonian and the delta function ensures energy conservation between the initial and the final states. Typically in electron-nuclear interactions (as in magnetic relaxation in solids) one can neglect the nuclear recoil to a very good approximation and therefore one can replace $\delta(E_i - E_f)$ by $\delta(E_k - E_{k'})$, k and k' being the initial and final momenta of the scattered electron. We work in this limit to keep the calculation simple and also because this is the relevant limit for many physical applications. The total transition probability per unit time at nonzero temperature is then given by[(17)]

$$W_{i\to f} = 2\pi \int dkdk'| < i|H_{int}|f > |^2\delta(E_k - E_{k'})\rho(k)\rho(k')f(E_k)F(E_{k'}),$$

where $\rho(k)$ denotes the density of states in the momentum space, $f(E)$ is as in (4) and $F(E)$ is what was denoted by $F(f)$ earlier, see (6). Performing one of the two integrations by using the delta function, we obtain

$$W_{i\to f} = 2\pi \int dE| < i|H_{int}|f > |^2(\rho(E))^2 f(E)F(E). \tag{D.1}$$

In order to evaluate the integral in (13), we note that

$$f(E)F(E) = (f(E))^2\ e^{(E-E_F)/T}, \tag{D.2}$$

where E_F is the Fermi energy. Notice that in the low-temperature limit we have

$$f(E) = 1/\alpha \qquad E \le E_F,$$

and zero otherwise. Thus at low enough temperatures, the system for an arbitrary α (except very close to the Bosonic end) does exhibit a Fermi surface. We will be using this fact later

in the calculation. Now for reasons that will become clear shortly, we wish to obtain an expression for the derivative df/dE. Using (4-5), we get

$$\frac{df}{dE} = -(f(E))^2 \frac{d\omega}{dE} = -(f(E))^2 \frac{\omega(1+\omega)}{T(\alpha+\omega)}. \tag{D.3}$$

Substituting $(f(E))^2$ from (15) into (14), we get

$$f(E)F(E) = -\frac{df}{dE}\frac{T(\alpha+\omega)}{\omega(1+\omega)} e^{(E-E_F)/T}. \tag{D.4}$$

We now consider the low-temperature ($T \ll E_F$) limit of the process under consideration. In this limit, $f(E) \approx \theta(E_F - E)/\alpha$, and hence, $df/dE \approx -\delta(E_F - E)/\alpha$. Substituting this in (16), we find

$$f(E)F(E) \approx \frac{T(\alpha+\omega(E))}{\alpha\omega(E)(1+\omega(E))}\delta(E_F - E). \tag{D.5}$$

Substituting (17) in (13) and performing the energy integration we get the Generalized Fermi Golden Rule:

$$W_{i\to f} = 2\pi| < i|H_{int}|f > |^2(\rho(E_F))^2 \frac{T(\alpha+\omega(E_F))}{\alpha\omega(E_F)(1+\omega(E_F))}.$$

In the special case when $\alpha = 1$ (Fermions), we have, from (5), $\omega(E_F) = 1$, and we get

$$W^F_{i\to f} = 2\pi| < i|H_{int}|f > |^2(\rho(E_F))^2 T.$$

We can therefore, in general, write

$$W^\alpha_{i\to f} = W^F_{i\to f} \frac{\alpha+\omega(E_F)}{\alpha\omega(E_F)(1+\omega(E_F))}.$$

We may now apply this result to specific cases. A straightforward application is to the calculation of a relaxation time of a nuclear spin in a metal[156]. The relaxation time for arbitrary α is then given by[17]

$$\frac{1}{\tau}(\alpha) = \frac{1}{\tau}(Fermions)\frac{\alpha+\omega(E_F)}{\alpha\omega(E_F)(1+\omega(E_F))}.$$

Thus the change due to fractional statistics is simply given by a multiplicative factor which depends on α. In particular, at $\alpha = 1/2$, this multiplicative factor can be explicitly calculated, and we get

$$\frac{1}{\tau}(\alpha) = \sqrt{5}\frac{1}{\tau}(Fermions).$$

Notice that in deriving the Generalized Fermi Golden Rule we have made a number of simplifying assumptions. Real systems are likely to be more complicated. Nevertheless, the above derivation probably indicates the correct direction in which the transition rates move when fractional exclusion statistics particles are involved.

Appendix E

Thomas-Fermi for contact interaction: Phase space approach

We give below a phase space formalism of TF with delta function interaction in 2-dimensions. The results are similar to the real space approach given in chapter 6, but is done in such a way that it can be adopted in higher dimensions also.

Consider a model of two species of fermions (may be spin-half system) in two dimensions interacting via a delta function interaction. The Hamiltonian is given by,

$$H = \sum_{i,\sigma}[\frac{1}{2m}p_{i\sigma}^2 + V(x_{i\sigma})] + u\sum_{i,j}\delta(x_{i\frac{1}{2}} - x_{j,-\frac{1}{2}}), \tag{E.1}$$

where $i, j = 1, 2;\ \sigma = 1/2, -1/2$. V denotes the one body potential.

Now introduce the phase space density by the following definition:

$$\rho_\sigma(p, x) = (2\pi\hbar)^2 \sum_i \delta^2(p - p_{i\sigma})\delta^2(x - x_{i\sigma}) \tag{E.2}$$

The Hamiltonian can then be written as,

$$H = H_0 + H_I \tag{E.3}$$

where

$$H_0 = \frac{1}{(2\pi\hbar)^2}\int d^2p\ d^2x \sum_\sigma [\frac{1}{2m}p^2 + V(x)]\rho_\sigma(p, x) \tag{E.4}$$

and

$$H_1 = \frac{1}{(2\pi\hbar)^4}\int d^2p\ d^2p'\ d^2x\ u\rho_\sigma(p, x)\rho_{-\sigma}(p', x) \tag{E.5}$$

The interaction Hamiltonian can be rewritten as

$$H_1 = \frac{1}{(2\pi\hbar)^2}\int d^2p\ d^2x\ \sum_\sigma U_\sigma(p, x)\rho_\sigma(p, x) \tag{E.6}$$

where

$$U_\sigma(p, x) = \frac{1}{(2\pi\hbar)^2}\int^p d^2p'\ u\rho_{-\sigma}(p', x) \tag{E.7}$$

We thus have

$$H = \frac{1}{(2\pi\hbar)^2}\int d^2p\ d^2x \sum_\sigma [\frac{1}{2m}p^2 + U_\sigma(p, x) + V(x)]\rho_\sigma(p, x) \tag{E.8}$$

We can now define new coordinates on the phase space with shifted momenta as follows:

$$p_{A\sigma}^2 = p^2 + 2mU_\sigma(p,x); \quad x_{A\sigma} = x \tag{E.9}$$

Note that this is exactly analogous to the shifted energies that we had defined in the CSM in one dimension. The above change of variables can now be made in the equation (8) above. The Jacobian of transformation is easily computed-

$$p_A dp_A = pdp + m\frac{dU_\sigma(p,x)}{dp}dp$$

and we have

$$d^2p_{A\sigma} = \frac{d^2p}{1+\frac{2mu}{(2\pi\hbar)^2}\rho_{-\sigma}(p,x)} \tag{E.10}$$

We can redefine the density such that,

$$\rho_\sigma^A(p,x) = \frac{\rho_\sigma(p,x)}{1+\frac{2mu}{(2\pi\hbar)^2}\rho_{-\sigma}(p,x)} \tag{E.11}$$

Making these change of variables, the original Hamiltonian can be written as

$$H = \frac{1}{(2\pi\hbar)^2}\int d^2p\ d^2x \sum_\sigma [\frac{1}{2m}p^2 + V(x)]\rho_\sigma^A(p,x) \tag{E.12}$$

Thus we have mapped the interacting system to a non-interacting Hamiltonian with the density redefined as above. The integration over p should be done carefully since there is a hole at the centre defined by the interaction. To find the ground state density, we have to minimise the energy as given above subject to the constraint

$$\rho_\sigma(p,x) \leq 1$$

since the original particles are fermions. Assuming the densities for both spins to be equal, the solution is

$$\rho_\sigma^A(p,x) = \frac{1}{\alpha}\theta(\mu - (\frac{p^2}{2m} + V(x))) \tag{E.13}$$

where

$$\alpha = (1 + \frac{2mu}{(2\pi\hbar)^2}) \tag{E.14}$$

This is exactly like the solution we had for the two-dimensional delta function obtained in chapter 6. However, the method given above may be carried out for three-dimension also with some changes.