Chapter 2 Hausdorff measure and dimension

The notion of dimension is central to fractal geometry. Roughly, dimension indicates how much space a set occupies near to each of its points. Of the wide variety of ‘fractal dimensions’ in use, the definition of Hausdorff, based on a construction of Carathéodory, is the oldest and probably the most important. Hausdorff dimension has the advantage of being defined for any set, and is mathematically convenient, as it is based on measures, which are relatively easy to manipulate. A major disadvantage is that in many cases it is hard to calculate or to estimate by computational methods. However, for an understanding of the mathematics of fractals, familiarity with Hausdorff measure and dimension is essential.

2.1 Hausdorff measure

Recall that if U is any non-empty subset of n-dimensional Euclidean space, $\mathbb{R}^n$, the *diameter* of U is defined as $|U| = \sup\{|x - y| : x, y \in U\}$, i.e. the greatest distance apart of any pair of points in U. If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F, i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \leqslant |U_i| \leqslant \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of F.

Suppose that F is a subset of $\mathbb{R}^n$ and s is a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\right\}. \tag{2.1}$$

Thus we look at all covers of F by sets of diameter at most δ and seek to minimize the sum of the sth powers of the diameters (figure 2.1). As δ decreases, the class of permissible covers of F in (2.1) is reduced. Therefore, the infimum $\mathcal{H}^s_\delta(F)$ increases, and so approaches a limit as $\delta \to 0$. We write

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F). \tag{2.2}$$

Fractal Geometry: Mathematical Foundations and Application. Second Edition Kenneth Falconer
 ISBNs: 0-470-84861-8 (HB); 0-470-84862-6 (PB)

Figure 2.1 A set F and two possible δ-covers for F. The infimum of $\Sigma|U_i|^s$ over all such δ-covers $\{U_i\}$ gives $\mathcal{H}^s_\delta(F)$

This limit exists for any subset F of $\mathbb{R}^n$, though the limiting value can be (and usually is) 0 or ∞. We call $\mathcal{H}^s(F)$ the *s-dimensional Hausdorff measure* of F.

With a certain amount of effort, $\mathcal{H}^s$ may be shown to be a measure; see section 1.3. It is straightforward to show that $\mathcal{H}^s(\emptyset) = 0$, that if E is contained in F then $\mathcal{H}^s(E) \leqslant \mathcal{H}^s(F)$, and that if $\{F_i\}$ is any countable collection of sets, then

$$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} F_i\right) \leqslant \sum_{i=1}^{\infty} \mathcal{H}^s(F_i). \tag{2.3}$$

It is rather harder to show that there is equality in (2.3) if the $\{F_i\}$ are disjoint Borel sets.

Hausdorff measures generalize the familiar ideas of length, area, volume, etc. It may be shown that, for subsets of $\mathbb{R}^n$, n-dimensional Hausdorff measure is, to within a constant multiple, just n-dimensional Lebesgue measure, i.e. the usual n-dimensional volume. More precisely, if F is a Borel subset of $\mathbb{R}^n$, then

$$\mathcal{H}^n(F) = c_n^{-1}\mathrm{vol}^n(F) \tag{2.4}$$

where c_n is the volume of an n-dimensional ball of diameter 1, so that $c_n = \pi^{n/2}/2^n(n/2)!$ if n is even and $c_n = \pi^{(n-1)/2}((n-1)/2)!/n!$ if n is odd. Similarly,

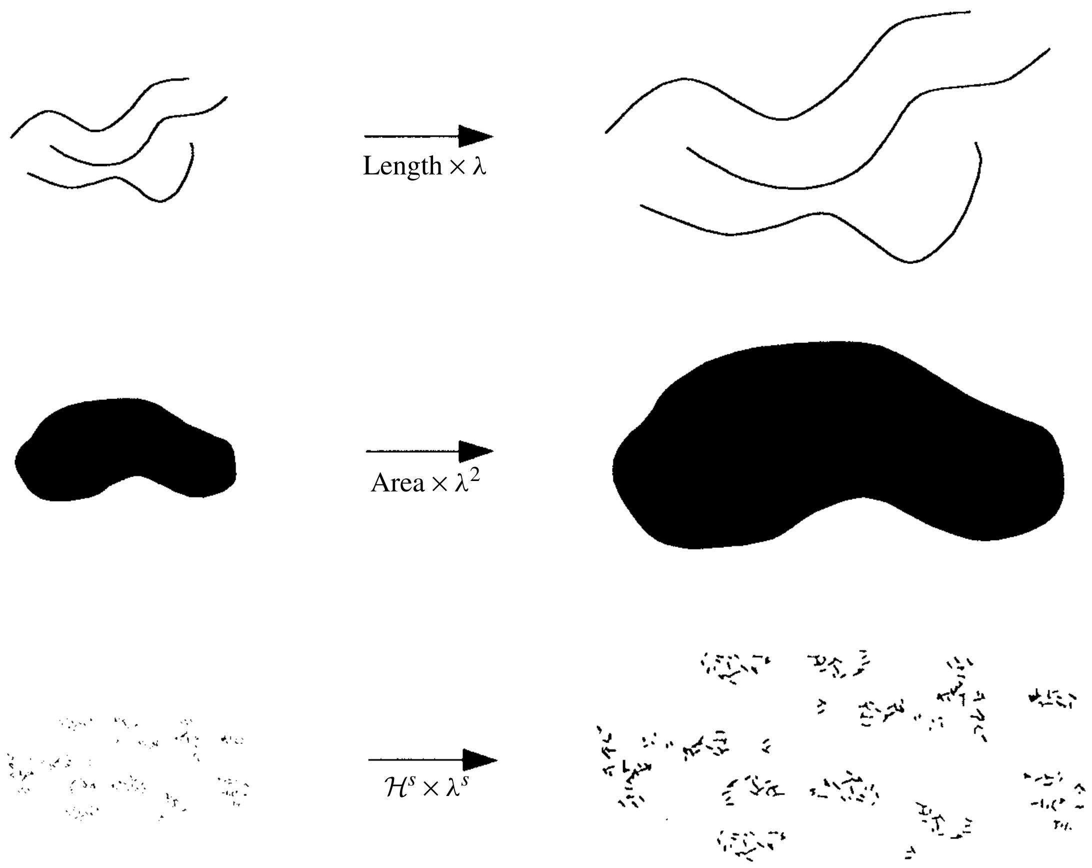

Figure 2.2 Scaling sets by a factor λ increases length by a factor λ, area by a factor λ^2, and s-dimensional Hausdorff measure by a factor λ^s

for 'nice' lower-dimensional subsets of $\mathbb{R}^n$, we have that $\mathcal{H}^0(F)$ is the number of points in F; $\mathcal{H}^1(F)$ gives the length of a smooth curve F; $\mathcal{H}^2(F) = (4/\pi) \times \text{area}(F)$ if F is a smooth surface; $\mathcal{H}^3(F) = (6/\pi) \times \text{vol}(F)$; and $\mathcal{H}^m(F) = c_m^{-1} \times \text{vol}^m(F)$ if F is a smooth m-dimensional submanifold of $\mathbb{R}^n$ (i.e. an m-dimensional surface in the classical sense).

The scaling properties of length, area and volume are well known. On magnification by a factor λ, the length of a curve is multiplied by λ, the area of a plane region is multiplied by λ^2 and the volume of a 3-dimensional object is multiplied by λ^3. As might be anticipated, s-dimensional Hausdorff measure scales with a factor λ^s (figure 2.2). Such scaling properties are fundamental to the theory of fractals.

Scaling property 2.1

Let S be a similarity transformation of scale factor $\lambda > 0$. If $F \subset \mathbb{R}^n$, then

$$\mathcal{H}^s(S(F)) = \lambda^s \mathcal{H}^s(F). \tag{2.5}$$

Proof. If $\{U_i\}$ is a δ-cover of F then $\{S(U_i)\}$ is a $\lambda\delta$-cover of $S(F)$, so

$$\Sigma|S(U_i)|^s = \lambda^s \Sigma|U_i|^s$$

so

$$\mathcal{H}^s_{\lambda\delta}(S(F)) \leqslant \lambda^s \mathcal{H}^s_\delta(F)$$

on taking the infimum. Letting $\delta \to 0$ gives that $\mathcal{H}^s(S(F)) \leqslant \lambda^s \mathcal{H}^s(F)$. Replacing S by S^{-1}, and so λ by $1/\lambda$, and F by $S(F)$ gives the opposite inequality required. □

A similar argument gives the following basic estimate of the effect of more general transformations on the Hausdorff measures of sets.

Proposition 2.2

Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}^m$ be a mapping such that

$$|f(x) - f(y)| \leqslant c|x - y|^\alpha \quad (x, y \in F) \tag{2.6}$$

for constants $c > 0$ and $\alpha > 0$. Then for each s

$$\mathcal{H}^{s/\alpha}(f(F)) \leqslant c^{s/\alpha}\mathcal{H}^s(F). \tag{2.7}$$

Proof. If $\{U_i\}$ is a δ-cover of F, then, since $|f(F \cap U_i)| \leqslant c|F \cap U_i|^\alpha \leqslant c|U_i|^\alpha$, it follows that $\{f(F \cap U_i)\}$ is an ε-cover of $f(F)$, where $\varepsilon = c\delta^\alpha$. Thus $\sum_i |f(F \cap U_i)|^{s/\alpha} \leqslant c^{s/\alpha} \sum_i |U_i|^s$, so that $\mathcal{H}^{s/\alpha}_\varepsilon(f(F)) \leqslant c^{s/\alpha}\mathcal{H}^s_\delta(F)$. As $\delta \to 0$, so $\varepsilon \to 0$, giving (2.7). □

Condition (2.6) is known as a *Hölder condition of exponent α*; such a condition implies that f is continuous. Particularly important is the case $\alpha = 1$, i.e.

$$|f(x) - f(y)| \leqslant c|x - y| \quad (x, y \in F) \tag{2.8}$$

when f is called a *Lipschitz mapping*, and

$$\mathcal{H}^s(f(F)) \leqslant c^s \mathcal{H}^s(F). \tag{2.9}$$

In particular (2.9) holds for any differentiable function with bounded derivative; such a function is necessarily Lipschitz as a consequence of the mean value theorem. If f is an isometry, i.e. $|f(x) - f(y)| = |x - y|$, then $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$. Thus, Hausdorff measures are translation invariant (i.e. $\mathcal{H}^s(F + z) = \mathcal{H}^s(F)$, where $F + z = \{x + z : x \in F\}$), and rotation invariant, as would certainly be expected.

2.2 Hausdorff dimension

Returning to equation (2.1) it is clear that for any given set $F \subset \mathbb{R}^n$ and $\delta < 1$, $\mathcal{H}^s_\delta(F)$ is non-increasing with s, so by (2.2) $\mathcal{H}^s(F)$ is also non-increasing. In fact, rather more is true: if $t > s$ and $\{U_i\}$ is a δ-cover of F we have

$$\sum_i |U_i|^t \leqslant \sum_i |U_i|^{t-s}|U_i|^s \leqslant \delta^{t-s}\sum_i |U_i|^s \tag{2.10}$$

so, taking infima, $\mathcal{H}^t_\delta(F) \leqslant \delta^{t-s}\mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$ we see that if $\mathcal{H}^s(F) < \infty$ then $\mathcal{H}^t(F) = 0$ for $t > s$. Thus a graph of $\mathcal{H}^s(F)$ against s (figure 2.3) shows that there is a critical value of s at which $\mathcal{H}^s(F)$ 'jumps' from ∞ to 0. This critical value is called the *Hausdorff dimension* of F, and written $\dim_H F$; it is defined for *any* set $F \subset \mathbb{R}^n$. (Note that some authors refer to Hausdorff dimension as *Hausdorff–Besicovitch dimension*.) Formally

$$\dim_H F = \inf\{s \geqslant 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\} \tag{2.11}$$

(taking the supremum of the empty set to be 0), so that

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leqslant s < \dim_H F \\ 0 & \text{if } s > \dim_H F. \end{cases} \tag{2.12}$$

If $s = \dim_H F$, then $\mathcal{H}^s(F)$ may be zero or infinite, or may satisfy

$$0 < \mathcal{H}^s(F) < \infty.$$

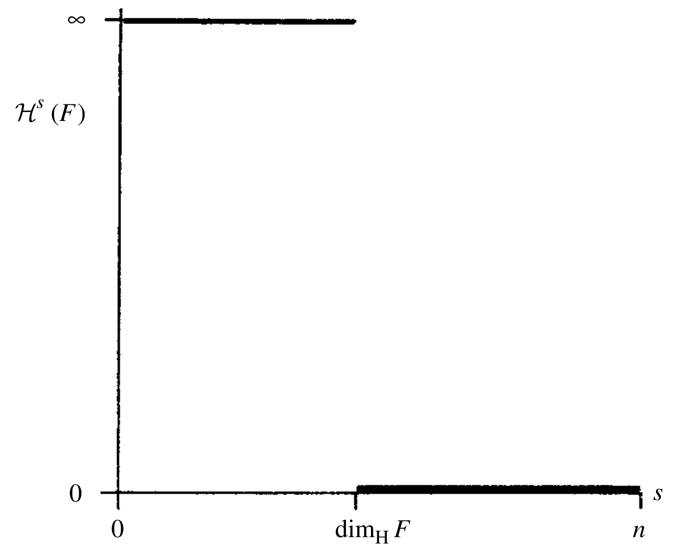

Figure 2.3 Graph of $\mathcal{H}^s(F)$ against s for a set F. The Hausdorff dimension is the value of s at which the 'jump' from ∞ to 0 occurs

A Borel set satisfying this last condition is called an *s-set*. Mathematically, s-sets are by far the most convenient sets to study, and fortunately they occur surprisingly often.

For a very simple example, let F be a flat disc of unit radius in $\mathbb{R}^3$. From familiar properties of length, area and volume, $\mathcal{H}^1(F) = \text{length}\,(F) = \infty$, $0 < \mathcal{H}^2(F) = (4/\pi) \times \text{area}\,(F) = 4 < \infty$ and $\mathcal{H}^3(F) = (6/\pi) \times \text{vol}(F) = 0$. Thus $\dim_{\text{H}} F = 2$, with $\mathcal{H}^s(F) = \infty$ if $s < 2$ and $\mathcal{H}^s(F) = 0$ if $s > 2$.

Hausdorff dimension satisfies the following properties (which might well be expected to hold for any reasonable definition of dimension).

Monotonicity. If $E \subset F$ then $\dim_{\text{H}} E \leqslant \dim_{\text{H}} F$. This is immediate from the measure property that $\mathcal{H}^s(E) \leqslant \mathcal{H}^s(F)$ for each s.

Countable stability. If $F_1, F_2, \ldots$ is a (countable) sequence of sets then $\dim_{\text{H}} \bigcup_{i=1}^{\infty} F_i = \sup_{1 \leqslant i < \infty}\{\dim_{\text{H}} F_i\}$. Certainly, $\dim_{\text{H}} \bigcup_{i=1}^{\infty} F_i \geqslant \dim_{\text{H}} F_j$ for each j from the monotonicity property. On the other hand, if $s > \dim_{\text{H}} F_i$ for all i, then $\mathcal{H}^s(F_i) = 0$, so that $\mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) = 0$, giving the opposite inequality.

Countable sets. If F is countable then $\dim_{\text{H}} F = 0$. For if F_i is a single point, $\mathcal{H}^0(F_i) = 1$ and $\dim_{\text{H}} F_i = 0$, so by countable stability $\dim_{\text{H}} \bigcup_{i=1}^{\infty} F_i = 0$.

Open sets. If $F \subset \mathbb{R}^n$ is open, then $\dim_{\text{H}} F = n$. For since F contains a ball of positive n-dimensional volume, $\dim_{\text{H}} F \geqslant n$, but since F is contained in countably many balls, $\dim_{\text{H}} F \leqslant n$ using countable stability and monotonicity.

Smooth sets. If F is a smooth (i.e. continuously differentiable) m-dimensional submanifold (i.e. m-dimensional surface) of $\mathbb{R}^n$ then $\dim_{\text{H}} F = m$. In particular smooth curves have dimension 1 and smooth surfaces have dimension 2. Essentially, this may be deduced from the relationship between Hausdorff and Lebesgue measures, see also Exercise 2.7.

The transformation properties of Hausdorff dimension follow immediately from the corresponding ones for Hausdorff measures given in Proposition 2.2.

Proposition 2.3

Let $F \subset \mathbb{R}^n$ and suppose that $f : F \to \mathbb{R}^m$ satisfies a Hölder condition

$$|f(x) - f(y)| \leqslant c|x - y|^{\alpha} \qquad (x, y \in F).$$

Then $\dim_{\text{H}} f(F) \leqslant (1/\alpha)\dim_{\text{H}} F$.

Proof. If $s > \dim_{\text{H}} F$ then by Proposition 2.2 $\mathcal{H}^{s/\alpha}(f(F)) \leqslant c^{s/\alpha}\mathcal{H}^s(F) = 0$, implying that $\dim_{\text{H}} f(F) \leqslant s/\alpha$ for all $s > \dim_{\text{H}} F$. □

Corollary 2.4

(*a*) *If $f : F \to \mathbb{R}^m$ is a Lipschitz transformation* (*see* (2.8)) *then* $\dim_{\text{H}} f(F) \leqslant \dim_{\text{H}} F$.

(*b*) *If $f : F \to \mathbb{R}^m$ is a bi-Lipschitz transformation, i.e.*

$$c_1|x-y| \leqslant |f(x)-f(y)| \leqslant c_2|x-y| \qquad (x, y \in F) \tag{2.13}$$

where $0 < c_1 \leqslant c_2 < \infty$, *then* $\dim_H f(F) = \dim_H F$.

Proof. Part (a) follows from Proposition 2.3 taking $\alpha = 1$. Applying this to f^{-1} : $f(F) \to F$ gives the other inequality required for (b). □

This corollary reveals a fundamental property of Hausdorff dimension: *Hausdorff dimension is invariant under bi-Lipschitz transformations*. Thus if two sets have different dimensions there cannot be a bi-Lipschitz mapping from one onto the other. This is reminiscent of the situation in topology where various 'invariants' (such as homotopy or homology groups) are set up to distinguish between sets that are not homeomorphic: if the topological invariants of two sets differ then there cannot be a homeomorphism (continuous one-to-one mapping with continuous inverse) between the two sets.

In topology two sets are regarded as 'the same' if there is a homeomorphism between them. One approach to fractal geometry is to regard two sets as 'the same' if there is a bi-Lipschitz mapping between them. Just as topological invariants are used to distinguish between non-homeomorphic sets, we may seek parameters, including dimension, to distinguish between sets that are not bi-Lipschitz equivalent. Since bi-Lipschitz transformations (2.13) are necessarily homeomorphisms, topological parameters provide a start in this direction, and Hausdorff dimension (and other definitions of dimension) provide further distinguishing characteristics between fractals.

In general, the dimension of a set alone tells us little about its topological properties. However, any set of dimension less than 1 is necessarily so sparse as to be totally disconnected; that is, no two of its points lie in the same connected component.

Proposition 2.5

A set $F \subset \mathbb{R}^n$ *with* $\dim_H F < 1$ *is totally disconnected.*

Proof. Let x and y be distinct points of F. Define a mapping $f : \mathbb{R}^n \to [0, \infty)$ by $f(z) = |z - x|$. Since f does not increase distances, as $|f(z) - f(w)| = \big||z-x| - |w-x|\big| \leqslant |(z-x) - (w-x)| = |z-w|$, we have from Corollary 2.4(a) that $\dim_H f(F) \leqslant \dim_H F < 1$. Thus $f(F)$ is a subset of $\mathbb{R}$ of $\mathcal{H}^1$-measure or length zero, and so has a dense complement. Choosing r with $r \notin f(F)$ and $0 < r < f(y)$ it follows that

$$F = \{z \in F : |z-x| < r\} \cup \{z \in F : |z-x| > r\}.$$

Thus F is contained in two disjoint open sets with x in one set and y in the other, so that x and y lie in different connected components of F. □

2.3 Calculation of Hausdorff dimension—simple examples

This section indicates how to calculate the Hausdorff dimension of some simple fractals such as some of those mentioned in the Introduction. Other methods will be encountered throughout the book. It is important to note that most dimension calculations involve an upper estimate and a lower estimate, which are hopefully equal. Each of these estimates usually involves a geometric observation followed by a calculation.

Example 2.6

Let F be the Cantor dust constructed from the unit square as in figure 0.4. (*At each stage of the construction the squares are divided into* 16 *squares with a quarter of the side length, of which the same pattern of four squares is retained.*) *Then* $1 \leqslant \mathcal{H}^1(F) \leqslant \sqrt{2}$, *so* $\dim_{\mathrm{H}} F = 1$.

Calculation. Observe that E_k, the kth stage of the construction, consists of 4^k squares of side 4^{-k} and thus of diameter $4^{-k}\sqrt{2}$. Taking the squares of E_k as a δ-cover of F where $\delta = 4^{-k}\sqrt{2}$, we get an estimate $\mathcal{H}^1_\delta(F) \leqslant 4^k 4^{-k}\sqrt{2}$ for the infimum in (2.1). As $k \to \infty$ so $\delta \to 0$ giving $\mathcal{H}^1(F) \leqslant \sqrt{2}$.

For the lower estimate, let proj denote orthogonal projection onto the x-axis. Orthogonal projection does not increase distances, i.e. $|\mathrm{proj}\, x - \mathrm{proj}\, y| \leqslant |x - y|$ if $x, y \in \mathbb{R}^2$, so proj is a Lipschitz mapping. By virtue of the construction of F, the projection or 'shadow' of F on the x-axis, proj F, is the unit interval [0, 1]. Using (2.9)

$$1 = \text{length}\,[0,1] = \mathcal{H}^1([0,1]) = \mathcal{H}^1(\mathrm{proj}\, F) \leqslant \mathcal{H}^1(F). \qquad \square$$

Note that the same argument and result hold for a set obtained by repeated division of squares into m^2 squares of side length $1/m$ of which one square in each column is retained.

This trick of using orthogonal projection to get a lower estimate of Hausdorff measure only works in special circumstances and is not the basis of a more general method. Usually we need to work rather harder!

Example 2.7

Let F be the middle third Cantor set (*see figure* 0.1). *If* $s = \log 2/\log 3 = 0.6309\ldots$ *then* $\dim_{\mathrm{H}} F = s$ and $\frac{1}{2} \leqslant \mathcal{H}^s(F) \leqslant 1$.

Heuristic calculation. The Cantor set F splits into a left part $F_{\mathrm{L}} = F \cap [0, \frac{1}{3}]$ and a right part $F_{\mathrm{R}} = F \cap [\frac{2}{3}, 1]$. Clearly both parts are geometrically similar to F but scaled by a ratio $\frac{1}{3}$, and $F = F_{\mathrm{L}} \cup F_{\mathrm{R}}$ with this union disjoint. Thus for any s

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_{\mathrm{L}}) + \mathcal{H}^s(F_{\mathrm{R}}) = (\tfrac{1}{3})^s\mathcal{H}^s(F) + (\tfrac{1}{3})^s\mathcal{H}^s(F)$$

by the scaling property 2.1 of Hausdorff measures. *Assuming that at the critical value* $s = \dim_H F$ *we have* $0 < \mathcal{H}^s(F) < \infty$ (a big assumption, but one that can be justified) we may divide by $\mathcal{H}^s(F)$ to get $1 = 2(\frac{1}{3})^s$ or $s = \log 2/\log 3$.

Rigorous calculation. We call the intervals that make up the sets E_k in the construction of F *level-k intervals*. Thus E_k consists of 2^k level-k intervals each of length 3^{-k}.

Taking the intervals of E_k as a 3^{-k}-cover of F gives that $\mathcal{H}^s_{3^{-k}}(F) \leqslant 2^k 3^{-ks} = 1$ if $s = \log 2/\log 3$. Letting $k \to \infty$ gives $\mathcal{H}^s(F) \leqslant 1$.

To prove that $\mathcal{H}^s(F) \geqslant \frac{1}{2}$ we show that

$$\sum |U_i|^s \geqslant \tfrac{1}{2} = 3^{-s} \tag{2.14}$$

for any cover $\{U_i\}$ of F. Clearly, it is enough to assume that the $\{U_i\}$ are intervals, and by expanding them slightly and using the compactness of F, we need only verify (2.14) if $\{U_i\}$ is a finite collection of closed subintervals of [0, 1]. For each U_i, let k be the integer such that

$$3^{-(k+1)} \leqslant |U_i| < 3^{-k}. \tag{2.15}$$

Then U_i can intersect at most one level-k interval since the separation of these level-k intervals is at least 3^{-k}. If $j \geqslant k$ then, by construction, U_i intersects at most $2^{j-k} = 2^j 3^{-sk} \leqslant 2^j 3^s |U_i|^s$ level-j intervals of E_j, using (2.15). If we choose j large enough so that $3^{-(j+1)} \leqslant |U_i|$ for all U_i, then, since the $\{U_i\}$ intersect all 2^j basic intervals of length 3^{-j}, counting intervals gives $2^j \leqslant \sum_i 2^j 3^s |U_i|^s$, which reduces to (2.14). □

With extra effort, the calculation can be adapted to show that $\mathcal{H}^s(F) = 1$.

It is already becoming apparent that calculation of Hausdorff measures and dimensions can be a little involved, even for simple sets. Usually it is the lower estimate that is awkward to obtain.

The 'heuristic' method of calculation used in Example 2.7 gives the right answer for the dimension of many self-similar sets. For example, the von Koch curve is made up of four copies of itself scaled by a factor $\frac{1}{3}$, and hence has dimension $\log 4/\log 3$. More generally, if $F = \bigcup_{i=1}^m F_i$, where each F_i is geometrically similar to F but scaled by a factor c_i then, provided that the F_i do not overlap 'too much', the heuristic argument gives $\dim_H F$ as the number s satisfying $\sum_{i=1}^m c_i^s = 1$. The validity of this formula is discussed fully in Chapter 9.

*2.4 Equivalent definitions of Hausdorff dimension

It is worth pointing out that there are other classes of covering set that define measures leading to Hausdorff dimension. For example, we could use coverings by spherical balls: letting

$$\mathcal{B}^s_\delta(F) = \inf\{\Sigma|B_i|^s : \{B_i\} \text{ is a } \delta\text{-cover of } F \text{ by balls}\} \tag{2.16}$$

we obtain a measure $\mathcal{B}^s(F) = \lim_{\delta\to 0}\mathcal{B}^s_\delta(F)$ and a 'dimension' at which $\mathcal{B}^s(F)$ jumps from ∞ to 0. Clearly $\mathcal{H}^s_\delta(F) \leqslant \mathcal{B}^s_\delta(F)$ since any δ-cover of F by balls is a permissible covering in the definition of $\mathcal{H}^s_\delta$. Also, if $\{U_i\}$ is a δ-cover of F, then $\{B_i\}$ is a 2δ-cover, where, for each i, B_i is chosen to be some ball containing U_i and of radius $|U_i| \leqslant \delta$. Thus $\Sigma|B_i|^s \leqslant \Sigma(2|U_i|)^s = 2^s\Sigma|U_i|^s$, and taking infima gives $\mathcal{B}^s_{2\delta}(F) \leqslant 2^s\mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$ it follows that $\mathcal{H}^s(F) \leqslant \mathcal{B}^s(F) \leqslant 2^s\mathcal{H}^s(F)$. In particular, this implies that the values of s at which $\mathcal{H}^s$ and $\mathcal{B}^s$ jump from ∞ to 0 are the same, so that the dimensions defined by the two measures are equal.

It is easy to check that we get the same values for Hausdorff measure and dimension if in (2.1) we use δ-covers of just open sets or just closed sets. Moreover, if F is compact, then, by expanding the covering sets slightly to open sets, and taking a finite subcover, we get the same value of $\mathcal{H}^s(F)$ if we merely consider δ-covers by finite collections of sets.

Net measures are another useful variant. For the sake of simplicity let F be a subset of the interval $[0, 1)$. A *binary interval* is an interval of the form $[r2^{-k}, (r+1)2^{-k})$ where $k = 0, 1, 2, \ldots$ and $r = 0, 1, \ldots, 2^k - 1$. We define

$$\mathcal{M}^s_\delta(F) = \inf\{\Sigma|U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \text{ by binary intervals}\} \tag{2.17}$$

leading to the *net measures*

$$\mathcal{M}^s(F) = \lim_{\delta\to 0}\mathcal{M}^s_\delta(F). \tag{2.18}$$

Since any interval $U \subset [0, 1)$ is contained in two consecutive binary intervals each of length at most $2|U|$ we see, in just the same way as for the measure $\mathcal{B}^s$, that

$$\mathcal{H}^s(F) \leqslant \mathcal{M}^s(F) \leqslant 2^{s+1}\mathcal{H}^s(F). \tag{2.19}$$

It follows that the value of s at which $\mathcal{M}^s(F)$ jumps from ∞ to 0 equals the Hausdorff dimension of F, i.e. both definitions of measure give the same dimension.

For certain purposes net measures are much more convenient than Hausdorff measures. This is because two binary intervals are either disjoint or one of them is contained in the other, allowing any cover of binary intervals to be reduced to a cover of *disjoint* binary intervals.

*2.5 Finer definitions of dimension

It is sometimes desirable to have a sharper indication of dimension than just a number. To achieve this let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a function that is increasing and continuous, which we call a *dimension function* or *gauge function*. Analogously to (2.1) we define

$$\mathcal{H}^h_\delta(F) = \inf\{\Sigma h(|U_i|) : \{U_i\} \text{ is a } \delta\text{-cover of } F\} \tag{2.20}$$

for F a subset of $\mathbb{R}^n$. This leads to a measure, taking $\mathcal{H}^h(F) = \lim_{\delta\to 0}\mathcal{H}^h_\delta(F)$. (If $h(t) = t^s$ this is the usual definition of s-dimensional Hausdorff measure.) If h and g are dimension functions such that $h(t)/g(t) \to 0$ as $t \to 0$ then, by an argument similar to (2.10), we get that $\mathcal{H}^h(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$. Thus partitioning the dimension functions into those for which $\mathcal{H}^h$ is finite and those for which it is infinite gives a more precise indication of the 'dimension' of F than just the number $\dim_H F$.

An important example of this is Brownian motion in $\mathbb{R}^3$ (see Chapter 16 for further details). It may be shown that (with probability 1) a Brownian path has Hausdorff dimension 2 but with $\mathcal{H}^2$-measure equal to 0. More refined calculations show that such a path has positive and finite $\mathcal{H}^h$-measure, where $h(t) = t^2 \log\log(1/t)$. Although Brownian paths have dimension 2, the dimension is, in a sense, logarithmically smaller than 2.

2.6 Notes and references

The idea of defining measures using covers of sets was introduced by Carathéodory (1914). Hausdorff (1919) used this method to define the measures that now bear his name, and showed that the middle third Cantor set has positive and finite measure of dimension $\log 2/\log 3$. Properties of Hausdorff measures have been developed ever since, not least by Besicovitch and his students.

Technical aspects of Hausdorff measures and dimensions are discussed in rather more detail in Falconer (1985a), and in greater generality in the books of Rogers (1998), Federer (1996) and Mattila (1995). Merzenich and Staiger (1994) relate Hausdorff dimension to formal languages and automata theory.

Exercises

2.1 Verify that the value of $\mathcal{H}^s(F)$ is unaltered if, in (2.1), we only consider δ-covers by sets $\{U_i\}$ that are all closed.

2.2 Show that $\mathcal{H}^0(F)$ equals the number of points in the set F.

2.3 Verify from the definition that $\mathcal{H}^s(\emptyset) = 0$, that $\mathcal{H}^s(E) \subset \mathcal{H}^s(F)$ if $E \subset F$, and that $\mathcal{H}^s(\bigcup_{i=1}^\infty F_i) \leqslant \sum_{i=1}^\infty \mathcal{H}^s(F_i)$.

2.4 Let F be the closed interval $[0, 1]$. Show that $\mathcal{H}^s(F) = \infty$ if $0 \leqslant s < 1$, that $\mathcal{H}^s(F) = 0$ if $s > 1$, and that $0 < \mathcal{H}^1(F) < \infty$.

2.5 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with continuous derivative. Show that $\dim_H f(F) \leqslant \dim_H F$ for any set F. (Consider the case of F bounded first and show that f is Lipschitz on F.)

2.6 Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$, and let F be any subset of $\mathbb{R}$. Show that $\dim_H f(F) = \dim_H F$.

2.7 Let $f : [0, 1] \to \mathbb{R}$ be a Lipschitz function. Writing graph $f = \{(x, f(x)) : 0 \leqslant x \leqslant 1\}$, show that $\dim_H$graph $f = 1$. Note, in particular, that this is true if f is continuously differentiable, see Exercise 1.13.

2.8 What is the Hausdorff dimension of the sets $\{0, 1, 2, 3, \ldots\}$ and $\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ in $\mathbb{R}$?

2.9 Let F be the set consisting of the numbers between 0 and 1 whose decimal expansions do not contain the digit 5. Use a 'heuristic' argument to show that $\dim_H F = \log 9/\log 10$. Can you prove this by a rigorous argument? Generalize this result.

2.10 Let F consist of the points $(x, y) \in \mathbb{R}^2$ such that the decimal expansions of neither x or y contain the digit 5. Use a 'heuristic' argument to show that $\dim_H F = 2\log 9/\log 10$.

2.11 Use a 'heuristic' argument to show that the Hausdorff dimension of the set depicted in figure 0.5 is given by the solution of the equation $4(\frac{1}{4})^s + (\frac{1}{2})^s = 1$. By solving a quadratic equation in $(\frac{1}{2})^s$, find an explicit expression for s.

2.12 Let F be the set of real numbers with base-3 expansion $b_m b_{m-1} \cdots b_1 \cdot a_1 a_2 \cdots$ with none of the digits b_i or a_i equal to 1. (Thus F is constructed by a Cantor-like process extending outwards as well as inwards.) What is the Hausdorff dimension of F?

2.13 What is the Hausdorff dimension of the set of numbers x with base-3 expansion $0 \cdot a_1 a_2 \cdots$ for which there is a positive integer k (which may depend on x) such that $a_i \neq 1$ for all $i \geqslant k$?

2.14 Let F be the middle-λ Cantor set (obtained by removing a proportion $0 < \lambda < 1$ from the middle of intervals). Use a 'heuristic argument' to show that $\dim_H F = \log 2/\log(2/(1-\lambda))$. Now let $E = F \times F \subset \mathbb{R}^2$. Show in the same way that $\dim_H E = 2\log 2/\log(2/(1-\lambda))$.

2.15 Show that there is a totally disconnected subset of the plane of Hausdorff dimension s for every $0 \leqslant s \leqslant 2$. (Modify the construction of the Cantor dust in figure 0.4.)

2.16 Let S be the unit circle in the plane, with points on S parameterized by the angle θ subtended at the centre with a fixed axis, so that θ_1 and θ_2 represent the same point if and only if θ_1 and θ_2 differ by a multiple of 2π, in the usual way. Let $F = \{\theta \in S : 0 \leqslant 3^k\theta \leqslant \pi (\bmod 2\pi)$ for all $k = 1, 2, \ldots\}$. Show that $\dim_H F = \log 2/\log 3$.

2.17 Show that if h and g are dimension functions such that $h(t)/g(t) \to 0$ as $t \to 0$ then $\mathcal{H}^h(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$.