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On Revolutionizing of Quantum Field Theory with Tomitas Modular Theory

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Abstract: In the book of Haag [Ha92] about local quantum field theory the main results are obtained by the older methods of C^*– and W^*–algebra theory. A great advance, especially in the theory of W^*–algebras, is due to Tomita's discovery of the theory of modular Hilbert algebras [To67]. Because of the abstract nature of the underlying concepts, this theory became (except for some sporadic results) a technique for quantum field theory only in the beginning of the nineties. In this review the results obtained up to this point will be collected and some problems for the future will be discussed at the end.

In the first section the technical tools will be presented. Then in the second section the two concepts, the half–sided translations and the half–sided modular inclusions, will be explained. These concepts have revolutionized the handling of quantum field theory. Examples for which the modular groups are explicitly known are presented in the third section. One of the important results of the new theory is the proof of the PCT–theorem in the theory of local observables. Questions connected with the proof are discussed in section four. Section five deals with the structure of local algebras and with questions connected with symmetry groups. In section six a theory of tensor product decompositions will be presented. In the last section problems which are closely connected with the modular theory and should be treated in the future will be discussed.

Contents:

1. Introduction

In this section we start with some statements of general interest, and add the main concepts and notations to be used in this note.

1.1) *Some general remarks*

Shortly after the invention of quantum mechanics, several scientists tried to generalize this theory to systems of infinite many degrees of freedom. (See e.g., P.A.M. Dirac [Dir27], [Dir28], Jordan and Wigner [JW28], Heisenberg and Pauli [HP29], [HP30].) In many of these attempts the authors wanted to incorporate the principle of special relativity at the same time. The combination of these two aspects is called relativistic quantum field theory, for which the term QFT will be used as short form in this note.

Non–relativistic quantum field theory and QFT are usually used in different branches of physics. The area of application for the first is quantum statistical mechanics, solid state physics, and liquids. The latter theory is mainly used for elementary particle physics. Quantum electrodynamics and the standard model are two theories where the concepts of QFT are used. These examples do not imply that the concepts of one form of the field theory can not be useful for the other. The investigation of Bros and Buchholz [BB94] on the relativistic KMS–condition is such a case.

QFT has several different facets:

1. Lagrangean quantum field theory together with perturbation theory.
2. L.S.Z.–theory, which is useful for scattering problems [LSZ55].
3. Wightman's quantum field theory [Wi56] and its derivative, the Euclidean field theory.
4. The theory of local observables in the sense of Araki, Haag and Kastler [Ha92].

The Lagrangean QFT is closest to the physical intuition. But it has the disadvantage that the expressions which appear in this theory have only a formal meaning. Up to now there is no convincing scheme which puts the formal expressions onto a solid and consistent mathematical basis. The existing perturbation and renormalization theory does not, in most cases, indicate anything about the quality of the approximation. Therefore, only comparison with the experiment can indicate the quality of the Lagrange function and the approximation. Not in all cases is one as lucky as in quantum electrodynamics, where the agreement between calculations and experiments is excellent. If, as it is the case in the standard model, the Lagrange function depends on too many parameters, then some sceptics are not satisfied, since some experimentalists say: "With three parameters one can fit an elephant and with a fourth parameter one can make him wiggle with his tail." Probably the right mathematics has still to be invented in order to make Lagrangean QFT acceptable for everyone.

Before and during World War II the perturbation and renormalization theory consisted largely of formal manipulations. This led R. Jost to the sarcastic remark: "In the thirties, under the demoralizing influence of quantum theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets". In the fifties there have been several attempts to put QFT on an axiomatic basis. This was possible since new mathematics had been developed, for instance the theory of distributions (see e.g. L. Schwartz [Schw57], [Schw59]) and the theory of C^*–algebras (see e.g. Naimark [Nai59]). The theory of distributions is needed for the LSZ [LSZ55] and the Wightman [Wi56] approach, and the theory of C*–algebras for the concept of local observables. While the LSZ– and the Wightman formalisms are still close to the ideas of Lagrangean QFT, a new road was taken in the theory of local observables.

Since von Neumann [Neu27], [Neu32] it is known that in quantum mechanics one can replace the unbounded physical observables by bounded functions of them. This has the advantage that, for many problems of general nature, the annoying domain questions disappear. In 1947 Segal [Seg47] proposed to use this method also for QFT. This idea has been taken up by R. Haag, and it developed between 1959 and 1964 [Ha59], [HS62], [HK64] into the theory of local observables.

The increase of knowledge in functional analysis led also to a partial progress in Lagrangean QFT. With the new technique those theories which are superrenormalizable could be rigorously handled. Glimm and Jaffe (see e.g. [GJ85]) have been the main promotors of this subject. The number of scientists who have contributed to this field is enormous, and it is impossible to mention them all.

Reviewing the past, the situation is as follows: The analyticity properties of the Wightman functions allow one to choose the time coordinates to be purely imaginary. The functions obtained in this way are called Schwinger functions. These are (real) analytic for non–coinciding points and, in the case of Bose fields, symmetric in all variables. With help of the Hahn–Banach theorem one can extend these functions to the coincidence points as symmetric distributions. It was the idea of Symanzik [Sym69] to identify these symmetric functions with the vacuum expectation value of a commutative and hence classical field. He also assumed that the representation of this field is on a Hilbert space with positive metric. In so doing the Schwinger functions can be considered as the moments of a positive measure on the space of tempered distributions $\mathcal{S}$. Sinc many approximation theorems exist for positive measures, one can, in favorable situations, first approximate the dynamics on a lattice in a box and take the continuum limit and the limit for the box tending to the whole space.

Unfortunately, the positivity of the Hilbert space for the Wightman theory does not imply that the Schwinger functions define a positive linear functional (on the symmetrized test function algebra). The positivity of the Wightman functional implies only the restricted Osterwalder–Schrader positivity [OS73], [OS75] (see also V. Glaser [Gl74]). This is the positivity condition for non–overlapping functions. If one uses the Osterwalder–Schrader condition also for overlapping functions, then one calls it extended positivity. If a theory fulfils extended Osterwalder–Schrader positivity and Euclidean covariance at the same time, then, by a result of Yngvason [Yng78], the Schwinger functions define a positive functional.

It is well known that broken time reversal (which is the case in nature) is not compatible with a positive measure for describing the Schwinger functions. A generalization would be to work with a signed (complex) measure. Borchers and Yngvason [BY76] have derived necessary and sufficient conditions implying that the Schwinger functions are moments of a complex measure. These conditions are closely related to the existence of the Wilson–Zimmermann [Zi67], [Wil69] decomposition of products of field operators. The restricted Osterwalder–Schrader positivity still has to hold. In my opinion one has to learn to draw conclusions from this condition before one can handle convergence problems for signed measures. It is not known whether or not the Wilson–Zimmermann product expansion holds for every Lagrangean QFT. If this is not the case, one has to generalize the measure theory on Montel spaces (the test function space) as one has generalized the measure theory on $\mathbb{R}^n$ to distributions, except, one must find a completely different method to handle Lagrangean QFT.

In the theory of local observables the theories of von Neumann– and C*–algebras are the main tools for the investigation. In 1967 the theory of von Neumann algebras made a big step forward in Tomita's discovery of the theory of modular von Neumann algebras. In this paper I will focus my attention on results obtained by this new theory. In the theory of local observables, abbreviated QFTLO, many results have been obtained with the standard theory of von Neumann algebras. Most of them are described in the book of R. Haag [Ha92].

This article is structured into several sections. Each of them is centered around one concept or idea. The order of these sections does not follow some logical concept, but is done in such a manner that the number of references to succeeding sections is minimized. Each section is split into subsections. This is done in order to facilitate the search for special topics. The last section is reserved to open problems.

1.2) *Assumptions of the theory of local observables*

The investigations of this paper are based on the following assumptions:
In the theory of local observables one associates to every bounded open region O in Minkowski space $\mathbb{R}^d$ a C^*-algebra $\mathcal{A}(O)$. For any unbounded open set G the C^*-algebra $\mathcal{A}(G)$ is defined as the C^* inductive limit of the $\mathcal{A}(O)$ with $O \subset G$. These algebras are subject to the following conditions:
(1) They fulfil isotony i.e., if $O_1 \subset O_2$ then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
(2) They fulfil locality, i.e. if O_1 and O_2 are spacelike separated regions then the corresponding algebras commute, i.e.

$$A \in \mathcal{A}(0_1),\ B \in \mathcal{A}(O_2) \quad \text{implies} \quad [A, B] = 0.$$

(3) They fulfil translational covariance, i.e. the translation group of $\mathbb{R}^d$ acts as automorphisms on $\mathcal{A}(\mathbb{R}^d)$. For every $a \in \mathbb{R}^d$ there exists an automorphism $\alpha_a \in \operatorname{Aut}\mathcal{A}(\mathbb{R}^d)$ with

$$\alpha_a \mathcal{A}(O) = \mathcal{A}(O + a).$$

A representation π of $\mathcal{A}(\mathbb{R}^d)$ is called a particle representation if:
(i) π is a non–degenerate representation on a Hilbert space $\mathcal{H}$.
(ii) There exists a strongly continuous unitary representation of the translation group

$$a \longmapsto U(a),$$

such that:

(α) The spectrum of $U(a)$ is contained in the forward light–cone.
(β) The representation $U(a)$ implements the automorphism α_a, which means that for every $A \in \mathcal{A}(\mathbb{R}^d)$ one has

$$\mathrm{Ad}\, U(a)\pi(A) := U(a)\pi(A)U^*(a) = \pi(\alpha_a A).$$

(iii) A representation π is called a vacuum representation if:
(α) π is a particle representation.
(β) In $\mathcal{H}$ exists a vector Ω with

$$U(a)\Omega = \Omega \qquad \forall a \in \mathbb{R}^d.$$

In the following we will always deal with vacuum representations and we set

$$\mathcal{M}(O) = \pi(\mathcal{A}(O))''.$$

(γ) We require weak additivity, i.e for every O there holds

$$\{\bigcup_{a\in\mathbb{R}^d} \mathcal{M}(O + a)\}" = \mathcal{M}(\mathbb{R}^d).$$

(4) Very often also the covariance under the whole Poincaré group will be assumed. This means there shall exist a continuous unitary representation $U(\Lambda)$ of the Lorentz group obeying the correct relations with the translations and
(α) $U(\Lambda)\Omega = \Omega$
(β) $U(\Lambda)\mathcal{M}(O)U(\Lambda)^* = \mathcal{M}(\Lambda O)$.

For the physical interpretation of these assumptions see the book of Haag [Ha92] or the lecture notes of Borchers [Bch96].

1.3) *Tomita-Takesaki theory*

As already mentiond this representation is mainly based on the Tomita–Takesaki theory. At the Baton Rouge conference 1967 Tomita [To67] distributed a preprint containing his theory on the standard form of von Neumann algebras. At the same time Haag, Hugenholtz und Winnink [HHW67] published their paper on the description of thermodynamic equilibrium states using the KMS-condition. Probably N. Hugenholtz and M. Winnink have been the first realizing the similarity between certain aspects of their approach and Tomita's theory and hence the importance of this new mathematical theory for theoretical physics. (See e.g. the thesis of M. Winnink [Win68].) But general knowledge became Tomita's theory only by Takesaki's [Tak70] treatment, published in the Lecture Notes in Mathematics. Since then this theory is usually called the Tomita-Takesaki theory.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra acting on this space with commutant $\mathcal{M}'$. A vector Ω is cyclic and separating for $\mathcal{M}$ if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled then a modular operator Δ and a modular conjugation

J is associated to the pair $(\mathcal{M}, \Omega)$ such that:
(i) Δ is self-adjoint, positive and invertible

$$\Delta\Omega = \Omega, \qquad J\Omega = \Omega.$$

(ii) The unitary group Δ^{it} defines a group of automorphisms of $\mathcal{M}$

$$\mathrm{Ad}\,\Delta^{it}\mathcal{M} = \mathcal{M} \qquad \forall t \in \mathbb{R}.$$

This automorphism group will often be denoted as

$$\mathrm{Ad}\,\Delta^{it}A =: \sigma^t(A). \tag{1.3.1}$$

(iii) For every $A \in \mathcal{M}$ the vector $A\Omega$ belongs to the domain of $\Delta^{\frac{1}{2}}$.
(iv) The operator J is a conjugation, i.e. J is antilinear and $J = J^* = J^{-1}$, where J commutes with Δ^{it}. This implies the relation

$$\mathrm{Ad}\,J\Delta = \Delta^{-1}. \tag{1.3.2}$$

(v) J maps $\mathcal{M}$ onto its commutant

$$\mathrm{Ad}\,J\mathcal{M} = \mathcal{M}'.$$

(vi) The operators $S := J\Delta^{\frac{1}{2}}$ and $S^* = J\Delta^{-\frac{1}{2}}$ have the property

$$\begin{aligned} SA\Omega &= A^*\Omega \qquad \forall A \in \mathcal{M}, \\ S^*A'\Omega &= A^{'*}\Omega \qquad \forall A' \in \mathcal{M}'. \end{aligned}$$

This implies that $A\Omega,\ A \in \mathcal{M}$ is in the domain of $\Delta^{1/2}$ and $B\Omega,\ B \in \mathcal{M}'$ is in the domain of $\Delta^{-1/2}$.
(vii) From (iii) one concludes that for $A \in \mathcal{M}$ the vector valued function

$$t \mapsto \Delta^{\mathrm{i}t}A\Omega$$

has an analytic continuation into the strip $S(-\frac{1}{2}, 0) := \{z \in \mathbb{C}; -\frac{1}{2} < \Im m\, z < 0\}$.
Property (vi) implies

$$\Delta^{\mathrm{i}(t-\frac{\mathrm{i}}{2})}A\Omega = \Delta^{\mathrm{i}t}JA^*\Omega, \qquad A \in \mathcal{M}. \tag{1.3.3}$$

For elements $B \in \mathcal{M}'$ Eq. (1.3.1) implies that $\Delta^{\mathrm{i}t}B\Omega$ has an analytic continuation into the strip $S(0, \frac{1}{2})$ and one gets by (vi)

$$\Delta^{\mathrm{i}(t+\frac{\mathrm{i}}{2})}B\Omega = \Delta^{\mathrm{i}t}JB^*\Omega, \qquad B \in \mathcal{M}'. \tag{1.3.3$'$}$$

(viii) Using Eq. (1.3.3) and the fact that J is a conjugation one obtains that for $A, B \in \mathcal{M}$ the function $(\Omega, B\sigma^t(A)\Omega)$ can be analytically continued into the strip $S(-1, 0)$. One finds at the lower boundary the relation

$$(\Omega, B\sigma^{(t-\mathrm{i})}(A)\Omega) = (\Omega, \sigma^t(A)B\Omega) \quad A, B \in \mathcal{M} \tag{1.3.4a}$$

or equivalently

$$(\Omega, B\Delta^{\mathrm{i}(t-\mathrm{i})}A\Omega) = (\Omega, A\Delta^{-\mathrm{i}t}B\Omega) \quad A, B \in \mathcal{M}. \tag{1.3.4b}$$

The last two relations are called the KMS–condition. They characterize the modular group uniquely. If a unitary group fulfils the KMS–condition for $\mathcal{M}$ then it is the modular group of $\mathcal{M}$. (See [KR86] Thm. 9.2.16.)

For the proofs see Takesaki [Tak70] or textbooks as Bratteli and Robinson [BR79] or Kadison and Ringrose [KR86] or S. Stratila [Stl81].

A central role in this theory is played by faithful normal states of von Neumann algebras. As a consequence of the Reeh-Schlieder theorem [RS61] we know that the vacuum-state has this property for every local algebra in quantum field theory.

Not for every von Neumann algebra exist faithful normal states. The generalization of this concept are the weights. With so called normal, faithful, semi–finite weights the Tomita–Takesaki theory can be developed also (see e.g. Haagerup [Hgr75]). The concept of weights will not be explained in the moment, but only when it has to be used. Also the mathematical results obtained by the Tomita–Takesaki theory will be mentioned when needed.

1.4) *Remarks on the edge of the wedge problem*

In this section we want to collect some results from the theory of analytic functions of several complex variables. All the results are given without proofs.

The theory of several complex variables is an important tool in quantum field theory and we assume familiarity with these methods. The situation appearing here (and often in other physical cases) is the edge of the wedge problem. One deals with two analytic functions $f^+(z)$ and $f^-(z)$, $z \in \mathbb{C}^n$ defined in a tube T^+ and $T^- = -T^+$ respectively. The tube T^+ is based on a convex cone $C \subset \mathbb{R}^n$ with apex at the origin and defined by:

$$T(C) = T^+ = \{z \in \mathbb{C}^n; z = x + iy, y \in C, x \in \mathbb{R}^n\}.$$

One assumes that $f^+(z)$ and $f^-(z)$ both have boundary values $f^+(x), f^-(x)$ respectively (in the sense of distributions) and that these boundary values coincide on some open set $G \subset \mathbb{R}^n$. In this situation one knows from the edge of the wedge theorem [BOT58] that both functions are analytic continuations of each other and are analytic also in a complex neighbourhood of G.

1.4.1 Theorem: (Edge of the Wedge)
Denote by B the ball

$$B = \{z; \|z\| := (\sum |z_i|^2)^{1/2} < 1\}$$

and define $B_C^+ = B \cap T(C)$ and $B_C^- = B \cap T(-C)$. Assume $f^+(z)$ and $f^-(z)$ are functions holomorphic in B_C^+ and B_C^- respectively with f^+ and f^- having continuous boundary values at real points $\|x\| < 1$ and assume that these boundary values coincide. Then there exists a complex neighbourhood $\mathcal{N}$ of $\mathbb{R}^n \cap B$ and a function f holomorphic in $B_C^+ \cup B_C^- \cup \mathcal{N}$ such that

$$f = f^+ \text{ on } B_C^+ \quad \text{and} \quad f = f^- \text{ on } B_C^-.$$

In several applications one has functions depending on several real variables. One knows that one can analytically continue in one variable if the others are fixed. One would like to know conditions which imply that one can analytically continue in all variables simultaniously. An important result on this question is the Malgrange–Zerner theorem. (For details see H. Epstein [Ep66].) Since we need the result only for two variables, we will formulate it only for this situation. The generalization to more than two variables is straight forward.

1.4.2 Theorem: (Malgrange–Zerner)
Let $f(x_1, x_2)$ be a continuous function of two variables defined on $(-1,1)\times(-1,1)$. Assume for fixed x_2 the function $f(x_1, x_2)$ has an analytic continuation $f(z_1, x_2)$ holomorphic in $z_1 \in D^+ = \{z; |z| < 1, \Im m\, z > 0\}$, and for fixed x_1 an analytic continuation $f(x_1, z_2)$ holomorphic in $z_2 \in D^+$. Assume $f(z_1, x_2)$ and $f(x_1, z_2)$ are bounded and continuous, i.e. $f(z_1, x_2)$ is a continuous function in x_2 with values in the bounded analytic functions on D^+, and the same for $f(x_1, z_2)$. Then exists a function $f(z_1, z_2)$ holomorphic in some neighbourhood $\mathcal{N} \cap D^+ \times D^+$, where $\mathcal{N}$ is some neighbourhood of $D^+\times(-1,1)\cup(-1,1)\times D^+$. This function has boundary values on $(-1,1)\times(-1,1)$ which coincide with $f(x_1, x_2)$.

The importance of holomorphic functions of several complex variables is the following fact: Not every domain G is a natural domain in $\mathbb{C}^n$. In such a situation every function holomorphic in G can be analytically continued into a larger domain. The domain into which every function, holomorphic in G, can be analytically continued is called the envelope of holomorphy $H(G)$ of G. We will need the tube theorem, the double cone theorem and the Jost–Lehmann–Dyson theorem. The tube theorem can be found in every text book on several complex variables.

1.4.3 Theorem: (Tube Theorem)
Let G be a connected domain $G \subset \mathbb{R}^n$ and let $T(G) = \{z \in \mathbb{C}^n; \Im m\, z \in G\}$. Then

$$H(T(G)) = T(\mathrm{Co}\, G),$$

where $\mathrm{Co}\, G$ denotes the convex hull of G.

Another result of importance in QFT is the double cone theorem discovered independently by Vladimirov [Vl60] and Borchers [Bch61].

1.4.4 Theorem: (Double Cone Theorem)
Let G be a subdomain of $\mathbb{R}^d$, and let $\mathcal{N}(G)$ be some complex neighbourhood of G. Let $\Gamma = T(C) \cup T(-C) \cup \mathcal{N}(G)$ and $H(\Gamma)$ be its envelope of holomorphy. Assume $c, d \in G$ such that $d - c \in C$ and $c + \lambda(d - c) \in G$ for $0 \leq \lambda \leq 1$. Then

$$D_{c,d} \subset H(\Gamma) \cap \mathbb{R}^n,$$

where $D_{c,d}$ denotes the double cone $(c + C) \cap (d - C)$.

We also need a result of Bros, Epstein, Glaser, and Stora [BEGS75], which deals with the edge of the wedge theorem in two variables.

1.4.5 Theorem (Bros, Epstein, Glaser, Stora)
Let T^+ and T^- be tubes based on the first and third quadrant respectively. Assume the coincidence domain is the first quadrant. If a real line $ax_1 + bx_2 = c$, $a, b, c \in \mathbb{R}$ intersects interior of the first quadrant, then all complex, non real points

$$az_1 + bz_2 = c, \quad z_1, z_2 \qquad \text{not both in } \mathbb{R}$$

belong to the envelope of holomorphy of the edge of the wedge problem.

Many results in QFT are based on the Jost–Lehmann–Dyson representation. This characterizes the envelope of holomorphy in case the cone C is the forward light cone and the coincidence domain has some special properties. Jost and Lehmann have solved a special case [JL57]. The general solution is due to Dyson [Dy58]. In this proof one uses tempered distributions. But that the answer is general has first been shown by Bros, Messiah and Stora [BMS61]. For more details on the Jost–Lehmann–Dyson representation see [Bch96] Sect. III.4.

1.4.6 Theorem: (Jost. Lehmann, Dyson))
Define $h(u, m)$ to be the hyperboloid

$$h(u, m) = \{z \in \mathbb{C}^d;\ (z - u)^2 = m^2,\ u \in \mathbb{R}^d, m \in \mathbb{R}\}.$$

Let $G \subset \mathbb{R}^d$ be a domain bounded by two spacelike hypersurfaces. The complement of the envelope of holomorphy of the edge of the wedge problem for

$$G \cup T(V^+) \cup T(-V^+)$$

consists of the closure of the union of all real and complex points of the hyperboloids $h(u, m)$ which do not intersect G.

1.5) *Some notations*

(i) If O is some open domain in the Minkowski space then O' denotes the interior of the spacelike complement of O.

(ii) A domain of special importance is the *wedge*. Such a domain can be characterized in two ways:
(α) First characterization: Let t, s be two perpendicular vectors in $\mathbb{R}^d$. i.e. $(t, s) = 0$, such that $t^2 = 1$ and t belongs to the forward light–cone and $s^2 = -1$ is spacelike. In this situation one defines

$$W(t, s) := \{a \in \mathbb{R}^d; |(a, t)| < -(a, s)\}. \tag{1.5.1}$$

If, for instance, t is the time direction and s is the 1-direction then this becomes $W_R = \{a; |a_0| < a_1\}$.
(β) Second characterization: Every two-plane containing a timelike direction must cut the boundary of the forward light cone in two light rays. Let these light rays be described

by the two lightlike vectors ℓ_1, ℓ_2 belonging to the forwad light–cone. These vectors are different. Now define:

$$W(\ell_1, \ell_2) := \{\lambda_1 \ell_1 - \lambda_2 \ell_2 + \tilde{a}; \lambda_1, \lambda_i > 0, \, (\tilde{a}, \ell_i) = 0, i = 1, 2\}. \tag{1.5.2}$$

It is easy to see that the two definitions result in the same set of wedges. The two definitions coincide if $\{t, s\}$ and $\{\ell_1, \ell_2\}$ span the same two-plane and if $s = \lambda_1 \ell_1 - \lambda_2 \ell_2$ with positive coefficients.

The opposite wedge of a wedge W is the negative of W and it is usually denoted by W'. It is obtained by replacing s by $-s$ in the first description and by interchanging the two lightlike vectors in the second description.

(iii) Given a wedge W there is exactly a one-parametric subgroup of the Lorentz boosts which maps this wedge onto itself. In the above example of the zero– and one–direction the Lorentz transformations are the boosts in the (0,1)-plane. We will write these transformations (in case the wedge is the right wedge W_R in the (0,1)-plane) as

$$\Lambda(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t & 0 & 0 \\ -\sinh 2\pi t & \cosh 2\pi t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.5.3}$$

(iv) Let $\mathcal{A}$ be a C*–algebra and π_1, π_2 be two equally faithful representations. These representations are called *quasi–equivalent* if the isomorphism between $\pi_1(\mathcal{A})$ and $\pi_2(A)$ extends to an isomorphism of the associated von Neumann algebras

$$\pi_1(\mathcal{A})\text{"} \cong \pi_2(\mathcal{A})\text{"}.$$

Two representations π_1 and π_2 of a theory of local observables are called locally normal if $\pi_1(\mathcal{A}(O))$ and $\pi_2(\mathcal{A}(O))$ are quasi–equivalent for every bounded open region O.

(v) Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. The operator

$$S = J\Delta^{1/2}$$

is anti–linear with square $\mathbb{1}$ (on the domain of definition). Since this operator plays an important role in the Tomita–Takesaki theory it will be called the Tomita conjugation of $(\mathcal{M}, \Omega)$.

1.6) *Things not treated*

It is clear that I am not able to handle all subjects of QFTLO which are not in the book of Haag. There is the reason of space, and more important, there are others who are more expert on that particular field than myself.

(i) Low dimensional QFT's:

If the dimension of the Minkowski space is two, then the set of points spacelike to the origin is no longer connected. This has for the definition of statistics the consequence that not only the permutation– but also the braid–group is of importance.

It is well known that in the classical theory the solution of the free wave equation is the sum of two functions depending only on one light–cone coordinate. A similar phenomenon appears in two–dimensional conformal QFT's. This means there exist quantum fields depending only on one of the light–cone coordinates. These are often called right–or left–movers. One can map the real line onto the circle and often one finds that such theory has an additional symmetry, namely the rigid rotation of the circle. Such theories are usually called chiral field theories.

The braid–group and the additional symmetry of chiral field theories opens a "wonderland" of new possibilities. Whether or not it is possible to get some important inspiration for the four–dimensional QFT from these theories will only be answered in the future.

(ii) General relativistic quantum fields:

It is a dream that one day it will be possible to combine quantum field theory with general relativity. As a first step it is probably reasonable to treat the QFT of test particles. These are theories where the quantum fields are influenced by the gravitational field (which is treated classically), but where the energy of the quantum field does not appear as a source of the gravitational field.

The main problem of this theory is the replacement of the spectrum condition. At the moment it is not clear whether or not there exist states describing a finite number of particles. At least in theories with a horizon the Hawking–Unruh effect [Haw75], [Unr76] seems to indicate that no such states exist in this situation. Therefore, the main stream of investigations focus on the aspect that the speed of particles should not be higher than that of light (defined by the gravitational field). These investigations use extensively the theory of wavefront sets.

(iii) Renormalization group:

For a long time the renormalization group method has been used mainly in connection with perturbation theory. This theory is designed in order to understand the physics at very low or very high energies. Not long ago D. Buchholz and R. Verch [BV95] were able to transcribe the renormalization group technique to QFTLO. In this scheme there are no serious obstructions, that means their method uses a sound mathematical basis. In examples they could show that the limiting theories can be different from the original theory. In some cases there is even more than one limiting theory. In my opinion this is an important new aspect of QFT which deserves one's attention. D. Buchholz will give a representation of this theory in the same volume.

An appendix to the references will be added containing a list of papers on the subjects not treated. This incomplete list shall be a help for a start for those interested in some more details on one or more of these fields. I am obliged to K.H. Rehren and R. Verch for preparing these lists.

2. On von Neumann subalgebras

From the axioms of QFTLO there has been extracted a large number of beautiful results. All of them are in accordance with our physical intuition. Examples are the collision theory and the theory of superselection sectors described in the book of Haag [Ha92], or the properties of the spectrum of the translations presented in the lecture notes by Borchers [Bch96].

However, up to now it is not clear how to distinguish the theories with different dynamics from eachother. Since for two different theories the local nets as a whole are not isomorphic to eachother, one should look (as a start) at the embedding of the algebra of one region O_1 into the algebra of a bigger region O_2. What is known about this question will be collected in this section.

2.1) *Order by inclusion and order of modular operators*

Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$ acting on the Hilbert space $\mathcal{H}$. Assume that both algebras have a common cyclic and separating vector Ω. Then one has $\mathcal{N}\Omega \subset \mathcal{M}\Omega$ and hence the Tomita conjugation $S_{\mathcal{M}}$ of $\mathcal{M}$ is an extension of the Tomita conjugation $S_{\mathcal{N}}$ of $\mathcal{N}$.

Dropping the index $\mathcal{M}$ of the Tomita conjugation, the operator S has the following properties (see 1.3):

(i) S is a densely defined closed anti–linear operator with domain of definition $\mathcal{D}(S)$ and $\mathcal{M}\Omega$ is a core for S.
(ii) $S^2 = \mathbb{1}$ on $\mathcal{D}(S)$.
(iii) $\Omega \in \mathcal{D}(S)$ and $S\Omega = \Omega$.

Since S is closed it has a polar decomposition $S = J\Delta^{1/2}$. The modular operator Δ is invertible and J is a conjugation. Eq. (1.3.2) reads:

$$J\Delta J = \Delta^{-1}, \qquad J = J^* = J^{-1}.$$

These properties follow from the condition $S^2 = \mathbb{1}$. (See e.g, Bratteli and Robinson [BR79] Prop.2.5.11.)

Usually a Tomita conjugation will be a densely defined unbounded operator. The best way of describing an unbounded operator X is by its graph. This is the set $\{[\psi, X\psi] \in \mathcal{H}\oplus\mathcal{H}; \psi \in \mathcal{D}(X)\}$. If the operator is closed then the graph of X is a closed linear manifold of $\mathcal{H}\oplus\mathcal{H}$. Therefore, it can be characterized by the projection $P(X)$ onto the graph. The projection $P(X)$ can be written as a two by two matrix $p_{i,k}, i, k = 1, 2$ of operators on $\mathcal{H}$ fulfilling

$$p^*_{i,k} = p_{k,i}, \qquad \sum_j p_{i,j}p_{j,k} = p_{i,k}. \tag{2.1.1}$$

If the operator X is anti–linear then $p_{1,2}$ and $p_{2,1}$ are anti–linear also. The domain of X is given by $\mathcal{D}(X) = p_{1,1}\psi + p_{1,2}\varphi, \psi, \varphi \in \mathcal{H}$ and its range $p_{2,1}\psi + p_{2,2}\varphi$. Therefore, one gets

$p_{2,1} = Xp_{1,1}$ and $p_{2,2} = Xp_{1,2}$. From these relations and from Eq. (2.1.1) one can easily express $p_{i,k}$ in terms of X. Of interest is $p_{1,1}$ which has the form

$$p_{1,1} = (1 + X^*X)^{-1}. \tag{2.1.2}$$

If X_1 is an extension of X then the graph of X is a subset of the graph of X_1. This implies in particular $P(X_1) \geq P(X)$. If E_1 is the projection onto the first Hilbert space then we get $E_1P(X_1)E_1 \geq E_1P(X)E_1$, and with Eq. (2.1.2)

$$(1 + X_1^*X_1)^{-1} \geq (1 + X^*X)^{-1}.$$

The matrix representing the projection onto the graph has been introduced by M.H. Stone [St51]. It is often called the Stone– or characteristic matrix of the operator. More details can be found in A.E. Nussbaum [Nu64].

If the operator X is anti–linear, then one has to replace the second Hilbert space by the conjugate complex Hilbert space. In this case the operators $p_{1,2}$ and $p_{2,1}$ are anti–linear. With this change one can deal with the graph in the same manner as if the operator would be linear. If one feels uneasy with this procedure one can fix a conjugation K on $\mathcal{H}$ and multiply the anti–linear operator X by K. Since KX is a linear operator the usual arguments can be applied. In the case $\mathcal{N} \subset \mathcal{M}$ one obtains

$$(\mathbb{1} + \Delta_{\mathcal{N}})^{-1} \leq (\mathbb{1} + \Delta_{\mathcal{M}})^{-1},$$

or

$$\Delta_{\mathcal{N}} \geq \Delta_{\mathcal{M}}. \tag{2.1.3}$$

This implies in particular that the domain of $\Delta_{\mathcal{N}}^{1/2}$ is contained in the domain of $\Delta_{\mathcal{M}}^{1/2}$. Since the domain of $\Delta_{\mathcal{N}}^{1/2}$ is the range of $\Delta_{\mathcal{N}}^{-1/2}$, the expression

$$\Delta_{\mathcal{N}}^{-1/2}\Delta_{\mathcal{M}}\Delta_{\mathcal{N}}^{-1/2}$$

is a densely defined bounded and hence a closable operator, and one gets

$$\text{closure}\, \Delta_{\mathcal{N}}^{-1/2}\Delta_{\mathcal{M}}\Delta_{\mathcal{N}}^{-1/2} \leq \mathbb{1}. \tag{2.1.4}$$

As an application of this discussion we obtain:

2.1.1 Theorem:

Let $\mathcal{M}_i$ be an increasing family of von Neumann algebras, i.e. $\mathcal{M}_i \subset \mathcal{M}_{i+1}$. Let

$$\mathcal{M} = \{\bigcup_i \mathcal{M}_i\}''.$$

Assume Ω is cyclic and separating for $\mathcal{M}_i$ and for $\mathcal{M}$. Denote by (Δ_i, J_i) and (Δ, J) the modular operators and modular conjugations of $\mathcal{M}_i$ and $\mathcal{M}$ respectively. Then Δ_i converges to Δ in the resolvent sense and J_i converges strongly to J.

A similar result holds for decreasing sequences. This result has first been obtained by D'Antoni, Doplicher, Fredenhagen and Longo [DDFL87].

Proof: $\mathcal{M}_i \subset \mathcal{M}_{i+1}$ implies that S_{i+1} is an extension of S_i. Hence the projections onto the graphs are a monotonic family which converges strongly. Hence by Eq. (2.1.3) also $(1+\Delta_i)^{-1}$ is increasing and converges strongly. Therefore, by a slight variation of Thm. VIII.19 in [RSi72] the sequence Δ_i converges in the strong resolvent sense. Let us denote the limit by Δ. By repeating the argument with Δ^{-1}, we see that Δ_i^{-1} converges to Δ^{-1} also in the strong resolvent sense. Hence Δ_i^{it} converges strongly to Δ^{it} for every t. (See [RSi72] Thm VIII.20.) In order to demonstrate, that Δ is the modular operator of $\mathcal{M}$, we have to show that it fulfils the KMS–condition. Let $A, B \in \mathcal{M}_i \subset \mathcal{M}_{i+1}$, then one knows for $j \geq i$ that $(\Omega, B\Delta_j^{it}A\omega)$ has an analytic continuation into the strip $S(-1,0)$. These functions have continuous boundary values at $\Im m\, t = -1$ with values $(\Omega, A\Delta_j^{-it}B\Omega)$. Since all these functions are bounded by the value at the boundary and since these converge, we obtain by Vitali's theorem that also the analytic functions converge. Hence Δ fulfils the KMS–condition for the dense subset $\cup\mathcal{M}_i$. Since this subset is $*$–strong dense in $\mathcal{M}$ we obtain by Kaplansky's density theorem that Δ fulfils the KMS–condition for all of $\mathcal{M}$. Hence Δ is the modular operator of $\mathcal{M}$. (See e.g. [KR86] Lemma 9.1.17.) Since the projection onto the graphs converges we know that S_i converges. Because also $\Delta_i^{1/2}$ converges in the strong resolvent sense it follows that J_i converges strongly to J. $\square$

2.2) *The first fundamental relation*

There are other aspects of the relation (2.1.4) which give some more informations. Since the result is needed several times, I quote it as in the report at the IAMP conference in Paris [Bch95]

Theorem A:

Let $\mathcal{M}, \mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω. Denote the modular operators and conjugations by $\Delta_\mathcal{M}, J_\mathcal{M}$ and $\Delta_\mathcal{N}, J_\mathcal{N}$, respectively. Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator with
(i) $V\Omega = \Omega$, and
(ii) $\mathrm{Ad}\, V\mathcal{N} \subset \mathcal{M}$,
then the function $V(t) := \Delta_\mathcal{M}^{-it} V \Delta_\mathcal{N}^{it}$ has the properties:
(a) $V(t)$ is $*$–strong continuous in $t \in \mathbb{R}$.
(b) $V(t)$ possesses an analytic extension into the strip $S(0,\frac{1}{2}) = \{t \in \mathbb{C}; 0 < \Im m\, t < \frac{1}{2}\}$ as holomorphic function with values in the normed space $\mathcal{B}(\mathcal{H})$.
(c) In this strip we have the estimate

$$\|V(\tau)\| \leq 1 \tag{2.2.1}$$

(d) $V(\tau)$ has boundary values at $\Im m\, \tau = 0$ and at $\Im m\, \tau = \frac{1}{2}$ in the $*$–strong topology.
(e) On the upper boundary the value is given by

$$V(t + i\frac{1}{2}) = J_\mathcal{M} V(t) J_\mathcal{N}, \tag{2.2.2}$$

hence by (a) also this function is $*$–strong continuous in t.

2.2.1 Remarks:

(i) With $\mathcal{N}' \supset V^*\mathcal{M}'V$ one obtains

$$V^*(t) = V(-t)^*.$$

Notice that the function $V(\bar{z})^*$ is again an analytic function holomorphic in $S(-\frac{1}{2}, 0) = \{t \in \mathbb{C}; -\frac{1}{2} < \Im m\, t < 0\}$. Therefore, the last relation reads in the complex

$$V^*(z) = V(-\bar{z})^*. \tag{2.2.3}$$

(ii) Inside the strip $S(0, \frac{1}{2})$ the operator function $V(t)$ is an analytic function with values in the normed space $\mathcal{B}(\mathcal{H})$:

Proof: The continuity properties are shown by standard methods. The interesting parts are the analyticity properties. Let us identify for a moment $V\mathcal{N}V^*$ wit $\mathcal{P} \subset \mathcal{M}$. Since $A \to A^\alpha$, $0 \le \alpha \le 1$ is an operator monotone function on positive operators (see e.g. G.K. Pedersen [Ped79] Prop. 1.5.8.) we obtain from Eq. (2.1.3)

$$\Delta_{\mathcal{P}}^\alpha \ge \Delta_{\mathcal{M}}^\alpha, \quad 0 \le \alpha \le 1$$

and hence

$$\text{closure}\,\{\Delta_{\mathcal{P}}^{-\alpha} \Delta_{\mathcal{M}}^{2\alpha} \Delta_{\mathcal{P}}^{-\alpha}\} \le \mathbb{1}, \quad 0 \le \alpha \le \frac{1}{2}.$$

This implies

$$\|\text{closure}\,\Delta_{\mathcal{M}}^{\alpha} \Delta_{\mathcal{P}}^{-\alpha}\| \le 1, \quad 0 \le \alpha \le \frac{1}{2}$$

or

$$\|\text{closure}\,\Delta_{\mathcal{M}}^{-\mathrm{i}(t+\mathrm{i}\tau)} V \Delta_{\mathcal{N}}^{\mathrm{i}(t+\mathrm{i}\tau)} V^*\| \le 1, \quad 0 \le \tau \le \frac{1}{2}.$$

Since V is unitary we obtain (2.2.1).

Next choose $A' \in \mathcal{N}'$ then we get by Eq. (1.3.3')

$$\begin{aligned}\Delta_{\mathcal{M}}^{-\mathrm{i}(t+\frac{\mathrm{i}}{2})} V \Delta_{\mathcal{N}}^{\mathrm{i}(t+\frac{\mathrm{i}}{2})} A'\Omega &= \Delta_{\mathcal{M}}^{-\mathrm{i}(t+\frac{\mathrm{i}}{2})} V \Delta_{\mathcal{N}}^{\mathrm{i}t} J_{\mathcal{N}} A'^*\Omega \\ &= \Delta_{\mathcal{M}}^{-\mathrm{i}(t+\frac{\mathrm{i}}{2})} V \sigma_{\mathcal{N}}^t(j_{\mathcal{N}}(A'^*)) V^*\Omega.\end{aligned}$$

These equations make sense, since $V\sigma_{\mathcal{N}}^t(j_{\mathcal{N}}(A'^*))V^*$ belongs to $\mathcal{P}$ and hence to $\mathcal{M}$, and therefore we get

$$= \Delta_{\mathcal{M}}^{-\mathrm{i}t} J_{\mathcal{M}} V \sigma_{\mathcal{N}}^t(j_{\mathcal{N}}(A')) V^*\Omega = J_{\mathcal{M}} \Delta_{\mathcal{M}}^{-\mathrm{i}t} V \Delta_{\mathcal{N}}^{\mathrm{i}t} J_{\mathcal{N}} A'\Omega.$$

Since $\mathcal{N}'\Omega$ is dense in $\mathcal{H}$ we obtain (2.2.2).

It remains to show the analyticity. Because of Eq. (2.2.1) it is sufficient to show the analyticity for a dense set of matrix elements. If $A \in \mathcal{M}$ and $f(t) \in \mathcal{L}^1$ then one can define

$$\sigma^f(A) := \int f(t)\sigma^t(A)dt.$$

If $f(t)$ has an entire analytic extension the $\sigma^t(\sigma^f(A))$ is an entire analytic operator valued function. It is easy to show that the set of entire analytic $\sigma^f(A), A \in \mathcal{M}$ is a strongly dense subset of $\mathcal{M}$. Choose entire analytic $\sigma^f_{\mathcal{M}}(A) \in \mathcal{M}$ and $\sigma^g_{\mathcal{N}}(B) \in \mathcal{N}$ then $(\Omega, \sigma^f_{\mathcal{M}}(A)V(z)\sigma^g_{\mathcal{N}}(B)\Omega)$ is entire analytic. Therefore, by the density of analytic elements $V(z)$ is analytic in the strip where it is bounded. □

This proof has used ideas of M. Florig [Flo98]. There is a different proof which starts directly from Eqs. (1.3.3) and (1.3.3'). It can be found in [Bch95].

2.3) *Characteristic functions and von Neumann subalgebras*

In the special case $V = \mathbb{1}$ one uses the following notations:

2.3.1 Definition:

Assume $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and Ω is cyclic and separating for both algebras. We set

$$D_{\mathcal{M},\mathcal{N}}(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}. \tag{2.3.1}$$

The function $D_{\mathcal{M},\mathcal{N}}(t)$ satisfies the following relations:

2.3.2 Lemma:

For the function $D(t) := D_{\mathcal{M},\mathcal{N}}(t)$ defined in Eq. (2.3.1) the following holds:

(1) *$D(t)$ is unitary and strongly continuous in t. Moreover $D(0) = \mathbb{1}$.*
(2) *$D(t)\Omega = \Omega$, for all $t \in \mathbb{R}$.*
(3) *$D(t)$ has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ and has strongly continuous boundary values at $\Im m\, t = 0$ and $\Im m\, t = \frac{1}{2}$.*
(4) *$D(t + \frac{\mathrm{i}}{2})$ is unitary and strongly continuous in t.*
(5) *$D(t)$ fulfils the following cocycle relation:*

$$D(s + t) = \sigma_{\mathcal{M}}^{-t}(D(s))D(t). \tag{2.3.2}$$

(6) *For complex values of the arguments one finds*

$$D(t + \frac{\mathrm{i}}{2})^* J_{\mathcal{M}} D(t) = D(t)^* J_{\mathcal{M}} D(t + \frac{\mathrm{i}}{2})$$

is independent of t.
(7) $\mathrm{Ad}\,\{D(t)D(\frac{\mathrm{i}}{2})^*\}\mathcal{M} \subset \mathcal{M}$ *holds for all $t \in \mathbb{R}$.*

Proof: (1) and (2) follow immediately from the definition of $D(t)$. The statements (3) and (4) are nothing else than Thm. A. (5) From the definition of $D(t)$ we obtain $\sigma_{\mathcal{M}}^{-t}(D(s))D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{-\mathrm{i}s}\Delta_{\mathcal{N}}^{\mathrm{i}s}\Delta_{\mathcal{M}}^{\mathrm{i}t}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t} = \Delta_{\mathcal{M}}^{-\mathrm{i}(s+t)}\Delta_{\mathcal{N}}^{\mathrm{i}(s+t)} = D(s + t)$.
(6) From Thm. A we know

$$D(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) J_{\mathcal{N}}. \tag{2.3.3}$$

This implies

$$D(t+\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} D(t) = J_{\mathcal{N}} D(t)^* J_{\mathcal{M}} J_{\mathcal{M}} D(t) = J_{\mathcal{N}},$$

and

$$D(t)^* J_{\mathcal{M}} D(t+\frac{\mathrm{i}}{2}) = D(t)^* J_{\mathcal{M}} J_{\mathcal{M}} D(t) J_{\mathcal{N}} = J_{\mathcal{N}}.$$

This shows (6). To prove (7) we use Eqs. (2.3.1) and (2.3.3) and get

$$\mathrm{Ad}\,\{D(t)D(\frac{\mathrm{i}}{2})^*\}\mathcal{M} = \mathrm{Ad}\,\{\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t} J_{\mathcal{N}} J_{\mathcal{M}}\}\mathcal{M}.$$

Because of $\mathcal{N}\subset\mathcal{M}$ we know $\mathrm{Ad}\,J_{\mathcal{M}}\mathcal{M} = \mathcal{M}' \subset \mathcal{N}'$. Hence $\mathrm{Ad}\,\{J_{\mathcal{N}}J_{\mathcal{M}}\}\mathcal{M}\subset\mathcal{N}$ which implies $\mathrm{Ad}\,\{\Delta_{\mathcal{N}}^{\mathrm{i}t}J_{\mathcal{N}}J_{\mathcal{M}}\}\mathcal{M}\subset\mathcal{N}$. Since $\mathcal{N}\subset\mathcal{M}$ statement (7) is proved. □

Notice that the properties of $D(t)$ described in Lemma 2.3.2 do not contain any reference to the algebra $\mathcal{N}$. Therefore, we introduce the following notation:

2.3.3 Definition:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with a cyclic and separating vector Ω.
1. By $\mathcal{S}ub(\mathcal{M})$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ which have Ω as cyclic vector.
2. An operator–valued function $D(t)$ which fulfils the properties (1)–(7) of Lemma 2.3.2 will be called a characteristic function of $\mathcal{M}$.
3. The set of characteristic functions belonging to $\mathcal{M}$ will be denoted by $\mathcal{C}har(\mathcal{M})$.

2.3.4 Theorem:

Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. Then to every characteristic function $D(t)$ of $\mathcal{M}$ exists a von Neumann subalgebra $\mathcal{N}\in\mathcal{S}ub(\mathcal{M})$ such that $D(t)=\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}$. The correspondence

$$\mathcal{S}ub(\mathcal{M}) \Longleftrightarrow \mathcal{C}har(\mathcal{M})$$

is one to one.

The proof of this theorem will be split into several steps. We start with

2.3.5 Lemma:

Define

$$U(t) = \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) \quad \text{and} \quad K = J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}), \tag{2.3.4}$$

then there holds:

(1) *$U(t)$ is a strongly continuous unitary group.*
(2) *K is a conjugation i.e. $K=K^*=K^{-1}$.*
(3) *K commutes with $U(t)$, which implies that one can write $U(t)=\Delta^{\mathrm{i}t}$ with an invertible operator Δ.*

(4) *The function $D(t)$ can be reconstructed if we know $U(t)$ and K.*

$$D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t} U(t), \quad D(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) K. \tag{2.3.5}$$

Proof: Since $D(t)$ and $\Delta_{\mathcal{M}}^{\mathrm{i}t}$ are both unitary and strongly continuous it follows, that $U(t)$ is unitary and weakly continuous. The unitarity implies that $U(t)$ is strongly continuous. From the cocycle relation (2.3.2) it follows that $U(t)$ is a unitary group. The relation $K = K^*$ is a consequence of property Lemma 2.3.2 (6). Using this again we find $KK = D(\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) = \mathbb{1}$. For proving (3) we reformulate the cocycle relation (2.3.2). It reads $\Delta_{\mathcal{M}}^{-\mathrm{i}t} D(s) \Delta_{\mathcal{M}}^{\mathrm{i}t} = D(t+s) D(t)^*$. If we replace t by $-t$ and s by t we get

$$\Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) \Delta_{\mathcal{M}}^{-\mathrm{i}t} = D(-t)^*. \tag{2.3.6}$$

By analytic continuation of the last but one equation in s we find $\Delta_{\mathcal{M}}^{-\mathrm{i}t} D(\frac{\mathrm{i}}{2}) \Delta_{\mathcal{M}}^{\mathrm{i}t} = D(t + \frac{\mathrm{i}}{2}) D(t)^*$. Using this equation and Lemma 2.3.2 (6) we obtain:

$$\begin{aligned} KU(t) &= J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) = J_{\mathcal{M}} \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t + \frac{\mathrm{i}}{2}) D(t)^* D(t) \\ &= \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) D(t)^* J_{\mathcal{M}} D(t + \frac{\mathrm{i}}{2}) = \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) = U(t) K. \end{aligned}$$

Finally the first relation of Eq. (2.3.5) follows from the definition of $U(t)$. The second relation will be derived by using the independence property of condition (6) of Lemma 2.3.2

$$D(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) D(t)^* J_{\mathcal{M}} D(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) = J_{\mathcal{M}} D(t) K.$$

This shows the lemma. □

Next we want to construct the von Neumann algebra $\mathcal{N}$ or better the algebra $\mathcal{N}'$ which we define

$$\mathcal{N}' = \bigvee_{t \in \mathbb{R}} \operatorname{Ad} U(t) \mathcal{M}'. \tag{2.3.7}$$

This algebra is invariant under $\operatorname{Ad} U(t)$. Now we show that Ω is separating for $\mathcal{N}'$. For this and the following calculation we set $\operatorname{Ad} U(t) = \sigma^t$.

2.3.6 Lemma:

The algebra $K J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}} K$ commutes with $\sigma^t(\mathcal{M}')$ and hence with $\mathcal{N}'$. Since $K J_{\mathcal{M}}$ is unitary and maps Ω onto itself it follows that Ω is cyclic for $\mathcal{N}$.

Proof: Let $A \in \mathcal{M}$ and $B \in \mathcal{M}'$. By using Eqs. (2.3.4) and (2.3.6) we obtain:

$$\begin{aligned} U(t) B U(t)^* K J_{\mathcal{M}} A J_{\mathcal{M}} K &= \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) B D(t)^* \Delta_{\mathcal{M}}^{-\mathrm{i}t} D(\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} J_{\mathcal{M}} A J_{\mathcal{M}} J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \\ &= D(-t)^* \Delta_{\mathcal{M}}^{\mathrm{i}t} B \Delta_{\mathcal{M}}^{-\mathrm{i}t} D(-t) D(\frac{\mathrm{i}}{2})^* A D(\frac{\mathrm{i}}{2}) D(-t)^* D(-t). \end{aligned}$$

Property (7) of Lemma 2.3.2 and Eq. (2.3.6) leads to

$$\begin{aligned}
&= D(-t)^* D(-t) D(\frac{\mathrm{i}}{2})^* A D(\frac{\mathrm{i}}{2}) D(-t)^* \Delta_{\mathcal{M}}^{\mathrm{i}t} B \Delta_{\mathcal{M}}^{-\mathrm{i}t} D(-t) \\
&= D(\frac{\mathrm{i}}{2})^* J_{\mathcal{M}} J_{\mathcal{M}} A J_{\mathcal{M}} J_{\mathcal{M}} D(\frac{\mathrm{i}}{2}) \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) B D(t)^* \Delta_{\mathcal{M}}^{-\mathrm{i}t} \\
&= K J_{\mathcal{M}} A J_{\mathcal{M}} K U(t) B U(-t).
\end{aligned}$$

This shows the lemma. □

From the invariance of $\mathcal{N}'$ and the last lemma we notice for later use

$$[\sigma^{t_1}(A_1'), K\sigma^{t_2}(A_2')K] = 0, \quad A_1', A_2' \in \mathcal{M}';\ t_1, t_2 \in \mathbb{R}. \tag{2.3.8}$$

This follows from $[K, \Delta^{\mathrm{i}t}] = 0$ and $\mathcal{M}' = J_{\mathcal{M}} \mathcal{M} J_{\mathcal{M}}$. Next we want to show that $U(t)$ is the modular group of $\mathcal{N}$. We start with the observation

2.3.7 Lemma:

With $U(t) = \Delta^{\mathrm{i}t}$ we obtain for $A' \in \mathcal{M}'$

$$\Delta^{-\frac{1}{2}} \sigma^t(A')\Omega = K\sigma^t(A'^*)\Omega.$$

Proof: Using Eq. (2.3.4) we get $\Delta^{\mathrm{i}t} A'\Omega = \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) A'\Omega =$ $D(-t)^* \Delta_{\mathcal{M}}^{\mathrm{i}t} A'\Omega$. This expression has an analytic continuation into the strip $S(0, \frac{1}{2})$ and we obtain with the adjoints of Eqs. (2.3.4) and (2.3.5)

$$\begin{aligned}
\Delta^{\mathrm{i}t-1/2} A'\Omega &= D(-t + \frac{\mathrm{i}}{2})^* \Delta_{\mathcal{M}}^{\mathrm{i}t-1/2} A'\Omega = K D(-t)^* J_{\mathcal{M}} \Delta_{\mathcal{M}}^{\mathrm{i}t} J_{\mathcal{M}} A'^*\Omega \\
&= K D(-t)^* \Delta_{\mathcal{M}}^{\mathrm{i}t} A'^*\Omega = K \Delta_{\mathcal{M}}^{\mathrm{i}t} D(t) A'^*\Omega = K U(t) A'^*\Omega,
\end{aligned}$$

and the lemma is proved. □

Next we want to extend Lemma 2.3.7 to all of $\mathcal{N}'$. To this end we recall that $\sigma^s(\sigma^f(A'))$ is entire analytic in s provided $f(t) \in \mathcal{L}^1(\mathbb{R})$ is entire analytic in t.

2.3.8 Lemma:

Let $C' \in \mathcal{N}'$ then we get

$$\Delta^{\mathrm{i}t-\frac{1}{2}} C'\Omega = K\Delta^{\mathrm{i}t} C'^*\Omega.$$

Proof: Choose $A_i \in \mathcal{M}'$ and $f_i \in \mathcal{L}^1(\mathbb{R})$ entire analytic, $i = 1, ..., n$. Then

$$\Delta^{\mathrm{i}t} \sigma^{f_1}(A_1) ... \sigma^{f_n}(A_n)\Omega = \sigma^t\big(\sigma^{f_1}(A_1)\big) ... \sigma^t\big(\sigma^{f_n}(A_n)\big)\Omega$$

can be analytically continued and we obtain with Lemma 2.3.7 and Eq. (2.3.8):

$$\begin{aligned}\Delta^{\mathrm{i}t-\frac{1}{2}}\sigma^{f_1}(A_1)\ldots\sigma^{f_n}(A_n)\Omega &= \sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\ldots\sigma^{f_n}(A_n)\big)\Omega \\ &= \sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\big)\ldots\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_{n-1}}(A_{n-1})\big)\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_n}(A_n)\big)\Omega \\ &= \sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\big)\ldots\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_{n-1}}(A_{n-1})\big)K\sigma^{t}\big(\sigma^{\overline{f}_n}(A_n^*)\big)K\Omega \\ &= K\sigma^{t}\big(\sigma^{\overline{f}_n}(A_n^*)\big)K\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_1}(A_1)\big)\ldots\sigma^{t+\frac{\mathrm{i}}{2}}\big(\sigma^{f_{n-1}}(A_{n-1})\big)\Omega.\end{aligned}$$

Repeating this manipulation we find

$$= K\sigma^{t}\big(\sigma^{\overline{f}_n}(A_n^*)\ldots\sigma^{\overline{f}_1}(A_1^*)\big)\Omega.$$

Since the set $\{\sigma^{f_1}(A_1)\ldots\sigma^{f_n}(A_n),\ n\in\mathbb{N},\ f\in\mathcal{L}^1(\mathbb{R})\text{ entire analytic}\}$ is weakly dense in $\mathcal{N}'$ and the $*$–operation is weakly continuous the lemma is proved. □

Proof of the theorem: In order that $U(t)$ is the modular group of $\mathcal{N}$ we have to show that $U(-t)$ fulfils the KMS–condition for $\mathcal{N}'$. Let $C_1', C_2' \in \mathcal{N}'$ then by Lemma 2.3.8 $(\Omega, C_1'U(t)C_2'\Omega)$ has an analytic continuation into the strip $S(0,\frac{1}{2})$ and we obtain

$$\begin{aligned}&(\Omega, C_1'\Delta^{\mathrm{i}(t+\frac{\mathrm{i}}{2})}C_2'\Omega) = ({C'}_1^*\Omega, K\sigma^t({C'}_2^*)\Omega) = (\sigma^t({C'}_2^*)\Omega, K{C'}_1^*\Omega) = \\ &(\Omega, C_2'\Delta^{-\mathrm{i}t}K{C'}_1^*\Omega) = (\Omega, C_2'\Delta^{-\mathrm{i}t-\frac{1}{2}}C_1'\Omega).\end{aligned}$$

The last expression can again be analytically continued into $S(0,\frac{1}{2})$ and we obtain at the upper boundary $(\Omega, C_2'\Delta^{-\mathrm{i}t}C_1'\Omega)$. This shows the KMS–condition. It remains to show the uniqueness of the mapping. If $D_1(t)$ and $D_2(t)$ are different then follows from the construction used above that the algebras are different. Conversely assume $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{S}ub(\mathcal{M})$ and $D_1(t)$ and $D_2(t)$ coincide. Then $\Delta_1^{\mathrm{i}t}$ and $\Delta_2^{\mathrm{i}t}$ coincide and also J_1 and J_2 coincide by Eq. (2.3.4). This implies that $\mathcal{N}_1 \cap \mathcal{N}_2$ is invariant under $\Delta_1^{\mathrm{i}t} = \Delta_2^{\mathrm{i}t}$. Since $J_1\mathcal{M}'J_1$ is contained in the intersection it follows that Ω is cyclic for $\mathcal{N}_1 \cap \mathcal{N}_2$. Hence $\mathcal{N}_1$ and also $\mathcal{N}_2$ coincide with $\mathcal{N}_1 \cap \mathcal{N}_2$. (See [KR86] Thm. 9.2.36.) Hence the map $\mathcal{S}ub(\mathcal{M}) \Leftrightarrow \mathcal{C}har(\mathcal{M})$ is one to one. □

The content of this subsection is taken from [Bch98c].

2.4) *The second fundamental relation*

There is a second fundamental relation wich has to be used several times also. A special case appeared first in [Bch92]. The present formulation is taken from [Bch95] and this proof is due to M. Florig [Flo98]. It uses only functions of one variable and not of two variables as in the original demonstration.

Theorem B:

Let $\mathcal{M}, \mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω. Let $W(s) \in \mathcal{B}(\mathcal{H})$ be an operator family fulfilling the following requirements with respect to

the triple $(\mathcal{M},\mathcal{N},\Omega)$.
(i) For $s\in\mathbb{R}$ *the operators* $W(s)$ *are unitary and strongly continuous and fulfil the equation* $W(s)\Omega=\Omega$.
(ii) The function $W(s)$ *possesses an analytic continuation into the strip* $S(0,\frac{1}{2})$ *with strongly continuous boundary values.*
(iii) The operators $W(\frac{i}{2}+t)$ *are again unitary.*
(iv) The function $W(\sigma)$ *is bounded, hence* $\|W(\sigma)\|\leq 1$.
(v) For $t\in\mathbb{R}$ *one has* $W(t)\mathcal{N}W(t)^*\subset\mathcal{M}$ *and* $W(\frac{i}{2}+t)\mathcal{N}'W(\frac{i}{2}+t)^*\subset\mathcal{M}'$.
In this situation the modular operator and the transformations $W(s)$ *fulfil the following transformation rules:*

$$\Delta_{\mathcal{M}}^{it}W(s)\Delta_{\mathcal{N}}^{-it}=W(s-t),$$
$$J_{\mathcal{M}}W(s)J_{\mathcal{N}}=W(\frac{i}{2}+s).$$

2.4.1 Remark:

In some applications one has to face the situation that $W(t+\frac{\mathrm{i}}{2})$ has eventually a discontinuity at one point, but all other properties remain valid. Such singularity is harmless. The reason is as follows: The proof of Theorem B is based on the continuation across a line, applied to matrix elements of the operator valued function

$$(t,s)\mapsto\Delta_{\mathcal{M}}^{\mathrm{i}t}W(s+t)\Delta_{\mathcal{N}}^{-\mathrm{i}t}. \tag{2.4.1}$$

These matrix elements have bounded analytic continuations, which are continuous at the boundary of their domain with the possible exception of one point with $\Im m\, t=\mathrm{i}/2$. By the dominated convergence theorem and the boundedness of (2.4.1), this piece-wise continuity is sufficient to ensure coincidence of boundary values in the sense of distributions. The edge-of-the-wedge theorem, Thm.1.4.1, then implies analyticity in the coincidence region, so continuity in the exceptional point holds a fortiori.

Proof: Choose $A\in\mathcal{N}$ and $B\in\mathcal{M}'$ and define for fixed s the two functions of the variable t:

$$F^+(t)=(\Omega,B\Delta_{\mathcal{M}}^{\mathrm{i}t}W(s+t)\Delta_{\mathcal{N}}^{-\mathrm{i}t}A\Omega),$$
$$F^-(t)=(\Omega,A\Delta_{\mathcal{N}}^{\mathrm{i}t}W^*(s+t)\Delta_{\mathcal{M}}^{-\mathrm{i}t}B\Omega).$$

Since $B\in\mathcal{M}'$ and $A\in\mathcal{N}$ and since $W(t)$ has a bounded analytic extension into the strip $S(0,\frac{1}{2})$, also the two functions have bounded extensions, $F^+(t)$ into the strip $S(0,\frac{1}{2})$ and $F^-(t)$ into the strip $S(-\frac{1}{2},0)$. Next we compute the values of $F^\pm$ at the other boundary:

$$\begin{aligned}F^+(t+\frac{\mathrm{i}}{2})&=(\Delta_{\mathcal{M}}^{-\frac{1}{2}}B^*\Omega,\Delta_{\mathcal{M}}^{\mathrm{i}t}W(s+t+\frac{\mathrm{i}}{2})\Delta_{\mathcal{N}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\frac{1}{2}}A\Omega)\\&=(\Omega,\sigma_{\mathcal{M}}^{-t}(j_{\mathcal{M}}(B^*))W(s+t+\frac{\mathrm{i}}{2})\sigma_{\mathcal{N}}^{-t}(j_{\mathcal{N}}(A^*))W(s+t+\frac{\mathrm{i}}{2})^*\Omega),\end{aligned}$$

and

$$\begin{aligned}F^-(t-\frac{\mathrm{i}}{2})&=(\Delta_{\mathcal{N}}^{\frac{1}{2}}A^*\Omega,\Delta_{\mathcal{N}}^{\mathrm{i}t}W(s+t+\frac{\mathrm{i}}{2})^*\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{-\frac{1}{2}}B\Omega)\\&=(\Omega,W(s+t+\frac{\mathrm{i}}{2})\sigma_{\mathcal{N}}^{-t}(j_{\mathcal{N}}(A^*))W(s+t+\frac{\mathrm{i}}{2})^*\sigma_{\mathcal{M}}^{-t}(j_{\mathcal{M}}(B^*))\Omega).\end{aligned}$$

By the assumption about the mapping property of $W(s+t)$ and of $W(s+t+\frac{\mathrm{i}}{2})$ we obtain:

$$F^+(t) = F^-(t), \qquad \text{and} \qquad F^+(t+\frac{\mathrm{i}}{2}) = F^-(t-\frac{\mathrm{i}}{2}).$$

By these coincidences we obtain a periodic entire analytic function. Since this function is bounded by $\max\{\|B^*\Omega\|\|A\Omega\|, \|A^*\Omega\|\|B\Omega\|\}$ it is constant. This implies

$$(\Omega, B\Delta_{\mathcal{M}}^{\mathrm{i}t} W(s+t)\Delta_{\mathcal{N}}^{-\mathrm{i}t} A\Omega) = (\Omega, BW(s)A\Omega).$$

Since Ω is cyclic for $\mathcal{N}$ and for $\mathcal{M}'$ follows the first statement of the theorem. The second statement is the same as Eq. (2.2.2). $\square$

2.5) *Half–sided translations*

From the general theory of von Neumann subalgebras described in subsection 2.3 we turn to special cases. We start with half–sided translations.

2.5.1 Definition:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector $\Omega \in \mathcal{H}$.

1. $\mathcal{H}str(\mathcal{M})^+$ denotes the set of one–parametric continuous unitary groups $U(t)$, $t \in \mathbb{R}$ with the properties:

α. $U(t)$ has a positive generator, i.e. we can write

$$U(t) = \exp\{\mathrm{i}Ht\}, \quad \text{with} \quad H \geq 0.$$

β. $U(t)\Omega = \Omega \quad \forall \quad t \in \mathbb{R}$.

γ. $\mathrm{Ad}\, U(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$.

We call the groups belonging to $\mathcal{H}str(\mathcal{M})^+$ +half–sided translations associated with $\mathcal{M}$.

2. $\mathcal{H}str(\mathcal{M})^-$ denotes the set of one–parametric continuous unitary groups $U(t)$, $t \in \mathbb{R}$ with γ replaced by

γ'. $\mathrm{Ad}\, U(t)\mathcal{M} \subset \mathcal{M}$ for all $t \leq 0$.

We call the groups belonging to $\mathcal{H}str(\mathcal{M})^-$ −half–sided translations associated with $\mathcal{M}$.

In the definition of the +half–sided translations it is not possible to replace $\mathbb{R}^+$ by $\mathbb{R}$ because

$$\mathrm{Ad}\, U(t)\mathcal{M} \subset \mathcal{M} \qquad \forall\, t$$

implies together with the positivity of the spectrum and the invariance of the vacuum $U(t) = \mathbb{1}$ for all $t \in \mathbb{R}$.

An example where half–sided translations appear, is the algebra of the wedge $\mathcal{M}(W)$. If $W = W(\ell_1, \ell_2)$, then the translations along the direction ℓ_1 fulfil the assumptions of +half–sided translations and those along the ℓ_2 direction the assumptions of −half–sided translations.

2.5.2 Theorem:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω and let $U(t) \in \mathcal{H}str(\mathcal{M})^+$. Then holds:

$$\Delta^{\mathrm{i}t} U(s) \Delta^{-\mathrm{i}t} = U(\mathrm{e}^{-\pi t} s),$$
$$J U(s) J = U(-s).$$

This theorem appeared first in [Bch92]. The following proof is based on Thm.B.

Proof: If $U(a)$ fulfils the assumptions of the theorem then it has an analytic continuation into the upper half plane. By assumption $U(a)$ maps $\mathcal{M}$ into itself for positive arguments and hence $U(a)$ maps $\mathcal{M}'$ into itself for negative arguments. Therefore, we can apply Thm.B to the family $W(s) = U(e^{2\pi s})$ and obtain together with the analyticity of $U(a)$

$$\begin{aligned} \operatorname{Ad} \Delta^{it} U(e^{2\pi s}) &= U(e^{2\pi(s-t)}), \\ \operatorname{Ad} \Delta^{it} U(a) &= U(e^{-2\pi t} a), \\ \operatorname{Ad} J U(a) &= U(-a). \end{aligned}$$

This shows the theorem. □

2.5.3 Remarks:

(i) If If $U(t) \in \mathcal{H}str(\mathcal{M})^-$ then one obtains the relations

$$\Delta^{\mathrm{i}t} U(s) \Delta^{-\mathrm{i}t} = U(\mathrm{e}^{2\pi t} s),$$
$$J U(s) J = U(-s).$$

(ii) For a wedge $W(\ell_1, \ell_2)$ the two lightlike directions span the characteristic two–plane of the wedge. If $\mathbf{x}$ is in this plane then one finds the transformation formula

$$\Delta^{\mathrm{i}t} U(\mathbf{x}) \Delta^{-\mathrm{i}t} = U(\Lambda(t)\mathbf{x})$$

where $\Lambda(t)$ are the Lorentz boosts of the wedge described in Eq. (1.5.3).
(iii) Let $U(t) \in \mathcal{H}str(\mathcal{M})^+$ and define $\mathcal{N} = \Delta^{\mathrm{i}1} \mathcal{M} \Delta^{-\mathrm{i}1}$ then one finds by the last theorem

$$\Delta^{\mathrm{i}t} \mathcal{N} \Delta^{-\mathrm{i}t} \subset \mathcal{N} \quad \text{for} \quad t \leq 0.$$

2.6) *Half–sided modular inclusions*

The last point of Remark 2.5.3 led H.W. Wiesbrock [Wie93], [Wie97] to introduce the concept of half–sided modular inclusions.

2.6.1 Definition:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector $\Omega \in \mathcal{H}$. The modular operator and conjugation of this pair will be denoted by Δ and J.

1. By $\mathcal{H}smi(\mathcal{M})^-$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ with the properties:

α. Ω is cyclic for $\mathcal{N}$. It is also separating for $\mathcal{N}$ since $\mathcal{N} \subset \mathcal{M}$.

β. $\Delta^{\mathrm{i}t}\mathcal{N}\Delta^{-\mathrm{i}t} =: \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{N} \subset \mathcal{N} \quad \text{for} \quad t \leq 0.$

The elements of $\mathcal{H}smi(\mathcal{M})^-$ will be called the von Neumann algebras fulfilling the condition of −half–sided modular inclusion.

2. By $\mathcal{H}smi(\mathcal{M})^+$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ with the properties:

α. Ω is cyclic for $\mathcal{N}$. It is also separating for $\mathcal{N}$ since $\mathcal{N} \subset \mathcal{M}$.

β. $\Delta^{\mathrm{i}t}\mathcal{N}\Delta^{-\mathrm{i}t} =: \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{N} \subset \mathcal{N} \quad \text{for} \quad t \geq 0.$

The elements of $\mathcal{H}smi(\mathcal{M})^+$ will be called the von Neumann algebras fulfilling the condition of +half–sided modular inclusion.

It should be remarked, that one cannot replace $\mathbb{R}^-$ by $\mathbb{R}$ because

$$\mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{N} \subset \mathcal{N} \qquad \forall\, t$$

implies $\mathcal{N} = \mathcal{M}$. The principle of half–sided modular inclusion is closely related to the half–sided translations by the following result:

2.6.2 Theorem:

Let $\mathcal{N} \in \mathcal{H}smi(\mathcal{M})^-$. Then there exists a group $U(t) \in \mathcal{H}str(\mathcal{M})^+$ such that the equation

$$\mathcal{N} = \mathrm{Ad}\,U(1)\mathcal{M}$$

holds.

Thm. 2.6.2 is in some sense the converse of Thm. 2.5.2. In some cases where one can compute the modular group one can find subalgebras fulfilling the conditions of half–sided modular inclusion. In these cases the corresponding half–sided translations are known only if they are geometric groups. But this is not always the case.

Proof: Assume the theorem to be true and assume $\mathcal{N} = U(1)\mathcal{M}U(-1)$ then one has $\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it} = \Delta_{\mathcal{M}}^{-it}U(1)\Delta_{\mathcal{M}}^{it}U(-1) = U(e^{2\pi t}-1)$. Therefore, one has to show that the product $\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it} =: D(t)$ commutes for different values of the arguments. For this one uses Thm.B again. In the situation $\mathcal{N} \subset \mathcal{M}$ one can apply Thm.A with $V = \mathbb{1}$ and will find that $D(t)$ has an analytic continuation into the strip $S(0,\frac{1}{2})$. On both boundaries the expression is unitary. By assumption of the modular inclusion one obtains:

$$\begin{aligned} D(t)\mathcal{N}D(t)^* &\subset \mathcal{N}, \quad \text{for} \quad t \geq 0, \\ D(t)\mathcal{N}'D(t)^* &\subset \mathcal{N}', \quad \text{for} \quad t \leq 0, \\ D(\frac{i}{2}+t)\mathcal{N}'D(\frac{i}{2}+t)^* &\subset \mathcal{N}', \quad \text{for} \quad t \in \mathbb{R}. \end{aligned}$$

The last statements follow from $D(\frac{i}{2}+t) = J_{\mathcal{M}}D(t)J_{\mathcal{N}}$. $J_{\mathcal{N}}$ maps $\mathcal{N}'$ onto $\mathcal{N}$, $D(t)$ maps this into $\mathcal{M}$ and finally $J_{\mathcal{M}}$ maps this into $\mathcal{M}' \subset \mathcal{N}'$. Consequently one can apply Thm.B to the expression

$$W(s) = D(\frac{1}{2\pi}\log(e^{2\pi s}+1)),$$

which leads to the relation

$$\Delta_{\mathcal{N}}^{it} D(\frac{1}{2\pi}\log(e^{2\pi s}+1))\Delta_{\mathcal{N}}^{-it} = D(\frac{1}{2\pi}\log(e^{2\pi(s-t)}+1)).$$

Multiplying this equation from the left with $\Delta_{\mathcal{M}}^{-it}$ and from the right with $\Delta_{\mathcal{N}}^{it}$ then we get with $e^{2\pi x} = e^{2\pi s}+1$

$$\begin{aligned}\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-ix}\Delta_{\mathcal{N}}^{ix} &= D(\frac{1}{2\pi}\log(e^{2\pi(s-t)}+1)+t)) = D(\frac{1}{2\pi}\log(e^{2\pi s}+e^{2\pi t}))\\ &= D(\frac{1}{2\pi}\log(e^{2\pi x}+e^{2\pi t}-1)).\end{aligned}$$

Since this expression is symmetric in x and t we obtain the commutativity of the operator family $D(t)$. If we set $U(e^{2\pi t}-1) = D(t)$ then the above equation reads

$$U(e^{2\pi t}-1)U(e^{2\pi x}-1) = U(e^{2\pi x}+e^{2\pi t}-2).$$

This shows that $U(a)$ is additive for positive arguments and by analytic continuation it follows that it is an additive unitary group with positive generator. It remains to show that $\mathcal{N}$ is of the form $U(1)\mathcal{M}U(-1)$. To this end we introduce:

2.6.3 Definition:

Let $\mathcal{N}$ be a $-$modular inclusion then we set

$$\begin{aligned}\mathcal{N}(e^{-2\pi t}) &= \Delta_{\mathcal{M}}^{it}\mathcal{N}\Delta_{\mathcal{M}}^{-it},\\ \mathcal{N}(-e^{-2\pi t}) &= \{\Delta_{\mathcal{M}}^{it}J_{\mathcal{M}}\mathcal{N}J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}\}'.\\ \mathcal{N}(0) &= \{\bigcup_t \mathcal{N}(e^{-2\pi t})\}''.\end{aligned}$$

Next we will show that this is a good definition.

2.6.4 Lemma:

The von Neumann algebras $\mathcal{N}(t)$, defined above, fulfil the following relations:

$$\begin{aligned} t_1 < t_2 \quad \text{implies} \quad & \mathcal{N}(t_1) \supset \mathcal{N}(t_2),\\ & \mathcal{N}(0) = \mathcal{M},\\ t<0 \quad \text{implies} \quad & \mathcal{N}(t) \supset \mathcal{M},\\ t>0 \quad \text{implies} \quad & \mathcal{N}(t) \subset \mathcal{M}.\end{aligned}$$

Proof: Because of modular inclusion we have $\mathcal{N}(t) \subset \mathcal{N}(1)$ for $t > 1$. Since unitary transformations preserve order we obtain the first statement for positive arguments. Moreover, $\mathcal{N} \subset \mathcal{M}$ implies $\mathcal{N}(t) \subset \mathcal{M}$ for positive t. For negative t we obtain the corresponding statements by the properties of $J_{\mathcal{M}}$. Finally, the algebra $\mathcal{N}(0)$ is a subalgebra of $\mathcal{M}$ which is invariant under the modular group of $\mathcal{M}$, has Ω as cyclic vector and hence coincides with $\mathcal{M}$. (See [KR86] Thm. 9.2.36.) □

Proof of the theorem, continuation: From the observation that $U(a)$ is a continuous group it follows that the family $\mathcal{N}(t)$ is also continuous at zero. Hence we obtain

$$\mathcal{M} = U(-1)\mathcal{N}U(1).$$

This shows the theorem. □

We end this subsection with some uniqueness result which is taken from [Bch93].

2.6.5 Theorem:

Let $\mathcal{M}_a$ and $\mathcal{N}_a$, $a \in \mathbb{R}$ be two families of von Neumann algebras on the Hilbert spaces $\mathcal{H}_m, \mathcal{H}_n$ with the cyclic and separating vector Ω_m, Ω_n, respectively. Assume there are continuous unitary one-parametric groups $U_{\mathcal{M}}(a), U_{\mathcal{N}}(a)$ both fulfil spectrum condition and leave Ω_m, Ω_n unchanged and assume

$$\mathcal{M}_a = U_{\mathcal{M}}(a)\mathcal{M}_0 U_{\mathcal{M}}(-a), \qquad \mathcal{N}_a = U_{\mathcal{N}}(a)\mathcal{N}_0 U_{\mathcal{N}}(-a).$$

Let moreover

$$\mathcal{M}_a \subset \mathcal{M}_b, \qquad \mathcal{N}_a \subset \mathcal{N}_b \quad \text{for} \quad a > b.$$

If there exists a unitary map W with $W\mathcal{H}_n = \mathcal{H}_m$ and $W\Omega_n = \Omega_m$ and in addition

$$\mathcal{M}_0 = W\mathcal{N}_0 W^*, \quad \text{and} \quad \mathcal{M}_1 = W\mathcal{N}_1 W^*,$$

then follows

$$\begin{aligned} \mathcal{M}_a &= W\mathcal{N}_a W^* \qquad \forall \quad a \in \mathbb{R}, \\ U_{\mathcal{M}}(a) &= WU_{\mathcal{N}}(a)W^*. \end{aligned}$$

The same is true if we require that $\mathcal{M}_0$ and $\mathcal{M}_1$ as well as $\mathcal{N}_0$ and $\mathcal{N}_1$ both fulfil modular inclusion for negative arguments of the modular groups.

Proof: The relation $\mathcal{M}_1 = \operatorname{Ad} U_{\mathcal{M}}(1)\mathcal{M}_0$ implies $\Delta^{\mathrm{i}t}_{\mathcal{M}_1} = \operatorname{Ad} U_{\mathcal{M}}(1)\Delta^{\mathrm{i}t}_{\mathcal{M}_0}$. From this one finds with help of Thm. 2.5.2 the relation

$$\Delta^{-\mathrm{i}t}_{\mathcal{M}_0}\Delta^{\mathrm{i}t}_{\mathcal{M}_1} = U_{\mathcal{M}}(\mathrm{e}^{2\pi t} - 1), \quad t \in \mathbb{R}.$$

By similar arguments one gets

$$\Delta^{-\mathrm{i}t}_{\mathcal{N}_0}\Delta^{\mathrm{i}t}_{\mathcal{N}_1} = U_{\mathcal{N}}(\mathrm{e}^{2\pi t} - 1), \quad t \in \mathbb{R}.$$

The assumption $\operatorname{Ad} W\mathcal{N}_i = \mathcal{M}_i$, $i = 0, 1$ implies $\operatorname{Ad} W\Delta^{\mathrm{i}t}_{\mathcal{N}_i} = \Delta^{\mathrm{i}t}_{\mathcal{M}_i}$, $i = 0, 1$ and hence we find $\operatorname{Ad} WU_{\mathcal{N}}(\mathrm{e}^{2\pi t} - 1) = U_{\mathcal{M}}(\mathrm{e}^{2\pi t} - 1)$, $t \in \mathbb{R}$. Since both groups fulfil the spectrum condition we obtain by analytic continuation $\operatorname{Ad} WU_{\mathcal{N}}(a) = U_{\mathcal{M}}(a)$, $a \in \mathbb{R}$. This implies the statement of the theorem. □

2.7) *Remarks, additions and problems*

(I) For the definition of half–sided translations one has used that the group $U(s)$ maps the cyclic and separating vector onto itself and that $U(s)$ has a positive generator. From this one concluded Thm. 2.5.2. The arguments can be reversed and one finds

2.7.1 Theorem:

Let $U(s)$ be a continuous unitary group fulfilling $U(s)\mathcal{M}U(-s) \subset \mathcal{M}$ for $s \geq 0$. Then any two of the three conditions imply the third

a. $U(s) = \mathrm{e}^{\mathrm{i}Hs}$ *with* $H \geq 0$.
b. $U(s)\Omega = \Omega$ *for all* $s \in \mathbb{R}$.
c. $\mathrm{Ad}\,\Delta^{\mathrm{i}t}(U(s)) = U(\mathrm{e}^{-2\pi t}s)$,
$JU(s)J = U(-s)$.

The implication $b + c \longrightarrow a$ has been shown by H.W. Wiesbrock [Wie92] and $a + c \longrightarrow b$ can be found in [Bch98a].

2.7.2 Remark: The conditions a., b., and c. of Thm. 2.7.1 do not imply the relation $\mathrm{Ad}\,U(s)\mathcal{M} \subset \mathcal{M}$ for $s \geq 0$. This is due to the fact that the modular group $\Delta^{\mathrm{i}t}$ does not determine the algebra $\mathcal{M}$. But if we know $\mathrm{Ad}\,U(s_0)\mathcal{M} \subset \mathcal{M}$ for one $s_0 \neq 0$ then one finds $s_0 > 0$ and $\mathrm{Ad}\,U(s)\mathcal{M} \subset \mathcal{M}$ for all $s > 0$. The first line of c. implies the inclusion for a half–line, and the conditions a. and b. imply, together with the proof of Thm. 2.5.2, that this is the positive half–line.

(II) Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Assume there exists a unitary group $U^+(x^+) \in \mathcal{H}str(\mathcal{M})^+$ and a unitary group $U^-(x^-) \in \mathcal{H}str(\mathcal{M})^-$. If these groups commute, then one can construct a two–dimensional theory, which eventually does not fulfil the weak additivity property.

We set:
(α) $U(\mathbf{x}) = U^+(x^+)U^-(x^-)$ where $\mathbf{x} \in \mathbb{R}^2$ and x^+, x^- are the light–cone coordinates. This $U(\mathbf{x})$ fulfils the spectrum condition since U^+ and U^- are half–sided translations.
(β) $\mathcal{M}(W_R) = \mathcal{M}$ and $\mathcal{M}(W_L) = \mathcal{M}'$. The algebras of the shifted wedges are defined by the translations $\mathcal{M}(W_R + \mathbf{x}) = \mathrm{Ad}\,U(\mathbf{x})\mathcal{M}(W_R)$ and $\mathcal{M}(W_L + \mathbf{x}) = \mathrm{Ad}\,U(\mathbf{x})\mathcal{M}(W_L)$.
(γ) Notice that in the two–dimensional Minkowski space a double cone is the intersection of a shifted right–wedge with a shifted left–wedge. For $\mathbf{a} - \mathbf{b} \in W_R$ we put $D_{\mathbf{b},\mathbf{a}} = (W_R + \mathbf{b}) \cap (W_L + \mathbf{a})$ and

$$\mathcal{M}(D_{\mathbf{b},\mathbf{a}}) = \mathcal{M}(W_R + \mathbf{b}) \cap \mathcal{M}(W_L + \mathbf{a}).$$

It is easy to check that this defines a Poincaré covariant net on the two–dimensional Minkowski space. We only do not know whether or not Ω is cyclic for $\mathcal{M}(D_{\mathbf{b},\mathbf{a}})$.

2.7.3 Problem: Can one characterize those algebras $\mathcal{M}$ which fulfil the assumption described under (II) and for which Ω is also cyclic for $\mathcal{M}(D_{\mathbf{b},\mathbf{a}})\Gamma$

(III) The space $\mathcal{C}har(\mathcal{M})$ can easily be furnished with a topology.

2.7.4 Definition:

Let $\mathcal{M}$ be a von Neumann algebra with a cyclic and separating vector Ω. We introduce on $\mathcal{C}har(\mathcal{M})$ the topology τ of simultanious $*$–strong convergence of $D_\alpha(t)$ and $D_\alpha(t+\frac{\mathrm{i}}{2})$, and this uniformly on every compact K of the real line. The neighbourhoods of an element $D(t)$ are given by

$$\begin{aligned}U(\psi_1,...,\psi_n,K,D(t)) = \{D'(t)\in \mathcal{C}har(\mathcal{M});\ &\|(D(t)-D'(t))\psi_i\|+\\ \|(D(t)^*-D'(t)^*)\psi\| + \|(D(t+\frac{\mathrm{i}}{2})&-D'(t+\frac{\mathrm{i}}{2}))\psi\|+\\ \|(D(t+\frac{\mathrm{i}}{2})^*-D'(t+\frac{\mathrm{i}}{2})^*)\psi\|\leq 1,\quad &i=1,...,n;\ t\in K\}.\end{aligned}$$

With this topology one obtains:

2.7.5 Theorem:

The space $\mathcal{C}har(\mathcal{M})$ is τ complete.

For details see [Bch98c].

(IV) Using the modular automorphisms of $\mathcal{M}$ one sees that $\mathcal{S}ub(\mathcal{M})$ contains a continuous family of different elements if it contains a non-trivial element. With help of the Longo endomorphism one can construct a decreasing family (by inclusion) of elements. (For $\mathcal{N}\in\mathcal{S}ub(\mathcal{M})$ the Longo endomorphism applied to $\mathcal{N}$ is $\mathrm{Ad}\,(J_\mathcal{N}J_\mathcal{M})\mathcal{N}$.)

If $\mathcal{N}\in\mathcal{S}ub(\mathcal{M})$, then there is a natural injection of $\mathcal{S}ub(\mathcal{N})$ into $\mathcal{S}ub(\mathcal{M})$. Hence if $\mathcal{S}ub(\mathcal{M})$ is non–trivial it must have a rich structure.

2.7.6 Problems: (α) Since finite algebras have a trace it follows that the set $\mathcal{S}ub(\mathcal{M})$ consists of only one point, namely $\mathcal{M}$ itself. That this is not the case for local algebras first has been shown by Kadison [Ka63] and by Guenin and Misra [GM63]. If the von Neumann algebra is infinite, does then $\mathcal{S}ub(\mathcal{M})$ contain non–trivial pointsΓ
(β) The definition of $\mathcal{S}ub(\mathcal{M})$ (Def. 2.3.3) depends on the cyclic and separating vector Ω. If Ω and Ψ are two different cyclic and separating vectors of $\mathcal{M}$, does this imply that $\mathcal{C}har_\Omega(\mathcal{M})$ and $\mathcal{C}har_\Psi(\mathcal{M})$ are homeomorphicΓ

(V) **2.7.7 Problem:** If the algebra $\mathcal{N}\in\mathcal{S}ub(\mathcal{M})$ is connected with a half–sided translation (or a half–sided inclusion) then the characteristic function $D(t)$ is abelian. Assume $D(t)$ is abelian, do then exist two commuting half–sided translations $U_1\in\mathcal{H}str(\mathcal{M})^+, U_1\in\mathcal{H}str(\mathcal{M})^-$, such that $\mathcal{N}=\mathrm{Ad}\,(U_1(1)U_2(-1))\mathcal{M}$ holdsΓ (One of the factors could be trivial.)

3. On local modular action, examples

Since the modular group of the pair $(\mathcal{M}(O),\Omega)$ is defined but not very concrete, one would like to have examples where this group can be computed explicitely. These are those where the modular group of the algebra, associated with some domain in the Minkowski space, defines a geometric transformation. We start with the result of Bisognano and Wichmann [BW75], [BW76] at which we look in some detail. Afterwards the other examples known up to now will be discussed. Since it promotes the feeling for the modular groups, if they act local, it is interesting to look for other possibilities. As the result of Trebels [Tre97] shows there are no other cases in the vacuum sector.

3.1) *The result of Bisognano and Wichmann for the wedge domain*

The first explicite determination of a modular group is due to Bisognano and Wichmann. They assumed that the local algebras are generated by Wightman fields, and that the Lorentz transformations act on the indices of the fields by finite dimensional representations of the Lorentz group, i.e.

$$U(\Lambda)A_i(x)U^*(\Lambda) = \sum_j D_i^j(\Lambda)A_j(\Lambda x),$$

where $D_i^j(\Lambda)$ is the direct sum of finite dimensional representations. In this situation the theory is also PCT invariant (Jost [Jo57]). Here the case of only one scalar field will be treated. For the general case see [BW76]. All our calculations use the $\mathbb{R}^4$.

3.1.1 Remark:

(1) Let $\Lambda(t)$ as in Eq. (1.5.3) and let the forward tube T^+ be defined by

$$T^+ = \{z; \Im m\, z \in V^+\}$$

Then we have:
For $x \in W_R$ one has $\Lambda(t)x \in T^+$ in the range $-\frac{1}{2} < \Im m\, t < 0$, and if $x \in W_L$, one has $\Lambda(t)x \in T^-$ for $0 < \Im m\, t < \frac{1}{2}$.

For $\Im m\, t = 0$, or $\pm\frac{1}{2}$, the vector $\Lambda(t)x$ belongs again to $\mathbb{R}^4$.

(2) Let $A(x)$ be the field operator, then

$$U(\mathrm{i}y)A(x)\Omega = A(x + \mathrm{i}y)\Omega$$

is defined for $y \in V^+$.

(3) Let $x = (x_0, x_1, x_2, x_3) \in W_R$ then

$$\Lambda(-\frac{\mathrm{i}}{2})(x_0, x_1, x_2, x_3) = (-x_0, -x_1, x_2, x_3),$$

and hence

$$U(\Lambda(-\frac{\mathrm{i}}{2}))A(x)\Omega = A(-x_0, -x_1, x_2, x_3)\Omega.$$

(4) On the other hand the PCT–operator Θ gives

$$\Theta A(x)\Omega = A(-x).$$

This suggests for the modular conjugation the representation

$$J = \Theta U(\pi, e_1) = U(\pi, e_1)\Theta,$$

where $U(\pi, e_1)$ represents the rotation around the x_1–axis and π is the angle of rotation.
(5) If $U(\Lambda(-\frac{\mathrm{i}}{2}))$ is the square root of the modular operator of the wedge–algebra then this leads for any testfunction to the relation

$$JU(\Lambda(+\mathrm{i}\pi))A(f)\Omega = A(\overline{f})\Omega.$$

To show that all the remarks are true we need some notations from the theory of the tensor–algebra.

3.1.2 Notations:

1) $\underline{\mathcal{S}}$ denotes the tensor–algebra generated by $\mathcal{S}(\mathrm{I\!R}^4)$.
a) $\underline{f} \in \underline{\mathcal{S}}$ is a terminating sequence

$$\underline{f} = \{f_0, f_1(x), f_2(x_1, x_2), ..., f_n(x_1, ..., x_n)...\},$$

where $f_0 \in \mathbb{C}, \quad f_i \in \mathcal{S}(\mathrm{I\!R}^{4i})$.
b) Addition is defined component–wise.
c) The product is as usual in tensor products, i.e.

$$(\underline{f} \cdot \underline{g})_j = \sum_{i+h=j} f_i g_h,$$

where $(f_i \cdot g_h)(x_1, ..., x_{i+h}) = f_i(x_1, ..., x_i) g_h(x_{i+1}, ..., x_{i+h})$.
d) The conjugation is defined by

$$f_i^*(x_1, ..., x_i) = \overline{f_i(x_i, ..., x_1)}.$$

2) For $\underline{f} \in \underline{\mathcal{S}}$ we set

$$A(\underline{f}) = \sum_i \int A(x_1)...A(x_i) f(x_1, ...x_i) d^4x_1...d^4x_i,$$

and

$$A(\underline{f})^* = A(\underline{f}^*).$$

As domain of definition for the field operators we choose

$$\mathcal{D} = \{A(\underline{f})\Omega; \underline{f} \in \underline{\mathcal{S}}\}.$$

3) If G is a domain, then we denote by $\mathcal{P}(G)$ the algebra generated by elements $A(f)$, where f has its support in G.

4) We call a point y right of x, if $y \in x + W_R$. If G_1, G_2 are two domains, then we say G_1 is right of G_2 if this is true for all pairs of points in G_1 and G_2.

3.1.3 Lemma:

Assume $G_i \subset W_R$, $i = 1...n$ are open sets such that G_{i+1} is right of G_i, then the vectorvalued distribution, defined on $G_1 \times G_2 \times ... \times G_n$

$$A(\Lambda(t)x_1)A(\Lambda(t)x_2)...A(\Lambda(t)x_n)\Omega,$$

has an analytic continuation in t into the strip $S(-\frac{1}{2}, 0)$. Moreover, the boundary values exist for $\Im m\, t \to 0$ and $\Im m\, t \to -\frac{1}{2}$ and it holds

$$\lim_{\tau \to -\frac{1}{2}} A(\Lambda(t + \mathrm{i}\tau)x_1)...A(\Lambda(t + \mathrm{i}\tau)x_n)\Omega = A(\Lambda(t)x_1^j)...A(\Lambda(t)x_n^j)\Omega$$

with $x^j = (-x_0, -x_1, x_2, x_3)$.

Proof: The spectrum condition implies that

$$A(x_1 + \mathrm{i}y_1)A(x_2 + \mathrm{i}y_2)...A(x_n + \mathrm{i}y_n)\Omega \tag{3.1.1}$$

has an analytic continuation into the domain

$$y_1 \in V^+,\ y_2 - y_1 \in V^+, ..., y_i - y_{i-1} \in V^+.$$

This implies the first statement by the choice of the G_i and Remark 3.1.1.(2). Since the vectorvalued function Eq. (3.1.1) converges if the imaginary parts converge to zero we obtain the second statement by Remark 3.1.1.(1) and (3). □

From this we obtain:

3.1.4 Corollary:

Let $G_1, ..., G_n$ as in Lemma 3.1.3 and

$$\text{support} \quad \underline{f} \subset G_1 \times G_2 \times ... \times G_n,$$

then one has
1) $A(\underline{f})\Omega \in \mathcal{D}(U(\Lambda(t - \mathrm{i}/2))$ *and it holds*
2) $U(\overline{\Lambda}(-\mathrm{i}/2))A(\underline{f})\Omega = JA(\underline{f})^*\Omega$.
Here J denotes the operator introduced in Remark 3.1.1.(4).

Proof: Since the product–functions generate $\mathcal{D}(G_1 \times ... \times G_n)$, it is sufficient to show the corollary for such functions. With $f^j = f(x^j)$ we obtain from the lemma and the definition of J

$$U(\Lambda(\mathrm{i}/2))A(f_1)...A(f_n)\Omega = A(f_1^j)...A(f_n^j)\Omega$$
$$= JA(\overline{f_1})...A(\overline{f_n})\Omega = JA(\overline{f_n})...A(\overline{f_1})\Omega = J\{A(f_1)...A(f_n)\}^*\Omega.$$

The first statement has been schown in the last lemma. □

One remark: Since $U(\Lambda(t))$ is a one–parametric group we obtain by Stone's representation $U(\Lambda(t)) = e^{\mathrm{i}Kt}$, and hence $U(\Lambda(-\mathrm{i}\frac{1}{2})) = e^{K\frac{1}{2}}$. K and $e^{K\frac{1}{2}}$ are selfadjoint operators.

Next we formulate the main result of Bisognano und Wichmann.

3.1.5 Theorem:

Let $A(x)$ be a scalar quantum field. Set $\Delta = U(\Lambda(-\mathrm{i}/2))$ and $J = \Theta U(\pi, e_1)$, as introduced in 3.1.1.(4). Then holds:
(a) $J\mathcal{P}(W_R)J = \mathcal{P}(W_L)$,
(b) $\Delta^{\mathrm{i}t}\mathcal{P}(W_{R,L})\Delta^{-\mathrm{i}t} = \mathcal{P}(W_{R,L}), \quad t \in \mathbb{R}$,
(c) $J\Delta^{\frac{1}{2}}X\Omega = X^*\Omega \quad \forall \quad X \in \mathcal{P}(W_R)$
$J\Delta^{-\frac{1}{2}}Y\Omega = Y^*\Omega \quad \forall \quad Y \in \mathcal{P}(W_L)$
(d) $\mathcal{P}(W_R)\Omega$ *is a core for* $\Delta^{\frac{1}{2}}$.

Statement (a) is Jost's PCT-theorem. Statement (b) is nothing else but the Lorentz covariance of the theory. We have added (d) because this is an important aspect of the Tomita-Takesaki theory. For proving (c) we need some preparation.

3.1.6 Lemma:

Let us denote by $\mathcal{Q}$ the set of operators $A(\underline{f})$ where the f's have the following properties:
(a) To $\underline{f}$ exists a sequence of domains $G_i, i = 1, ..., n$ such that $G_i \in W_R$ and G_{i+1} is right of G_i.
(b) f is a product–function with support of $\underline{f} \subset G_1 \times ... \times G_n$.

Suppose, $\mathcal{Q}\Omega$ is a core for $U(\Lambda(-\mathrm{i}\frac{1}{2}))$, then for every $X \in \mathcal{P}(W_R)$ there holds

$$JU(\Lambda(-\mathrm{i}\frac{1}{2}))X\Omega = X^*\Omega.$$

Proof: Assume $Q \in \mathcal{Q}$ and $X \in \mathcal{P}(W_R)$, then by Corollary 3.1.4 and by part (a) of Thm. 3.1.5 we obtain with $(U(-\mathrm{i}\frac{1}{2}) = U(\Lambda(-\mathrm{i}\frac{1}{2}))$:

$$(X\Omega, U(-\mathrm{i}\frac{1}{2})Q\Omega) = (X\Omega JQ^*\Omega) = (X\Omega JQ^*J\Omega) =$$
$$(JQ\Omega, X^*\Omega) = (JX^*\Omega, Q\Omega).$$

Since by assumption $\mathcal{Q}\Omega$ is a core for $U(-\mathrm{i}\frac{1}{2})$ and since this is a selfadjoint operator it follows for $X\Omega \in D(U(-\mathrm{i}\frac{1}{2}))$ and

$$U(-\mathrm{i}\frac{1}{2})X\Omega = JX^*\Omega.$$

This is equivalent to statement (c) of the theorem. □

Since $\mathcal{Q}$ is a subset of $\mathcal{P}$ it remains to show that $\mathcal{Q}\Omega$ is a core for $U(-\mathrm{i}\frac{1}{2})$.

3.1.7 Lemma:

$\mathcal{Q}\Omega$ is a core for $U(-\mathrm{i}\frac{1}{2})$.

Proof: First we show that $\mathcal{Q}\Omega$ is dense in $\mathcal{H}$. Suppose this is not true then exists a vector $\psi \in \mathcal{H}$ such that $\psi - \mathcal{Q}\Omega$. Let Q_n be the set of polynomials of degree n, then $(\psi, \mathcal{Q}\Omega) = 0$ implies that the distribution $(\psi, A(x_1)...A(x_n)\Omega)$ vanishes. For $x_1, ..., x_n \in G_1 \times ... \times G_n$ the above expression is zero and hence by analytic continuation this holds for all $x_1, ..., x_n$. Since this holds for all n we get $\mathcal{Q}\Omega$ is dense in $\mathcal{H}$.

We know in addition that $\mathcal{Q}\Omega$ is invariant under the Lorentz boosts $U(\Lambda(t))$. Therefore, by Nelsons theorem $\mathcal{Q}\Omega$ is a core for the generator of the Lorentz boosts K. From this one concludes that it is also a core for $e^{\frac{1}{2}K}$. This completes the proof of the theorem. □

3.1.8 Definition:

A representation of a QFTLO fulfils the Bisognano–Wichmann property if the modular group of every wedge acts local, like the associated group of Lorentz boosts, on the underlying space.

3.2) *Other examples*

(i) In a field theory of massless, non–interacting particles every influence travels along the boundary of the light–cone. Therefore, there holds not only spacelike, but also timelike commutativity. This implies that the vector Ω is cyclic and separating also for the algebra of the forward light–cone V^+. 1978 D. Buchholz [Bu78] has determined the modular group for this situation. It coincides with the dilatations.

3.2.1 Theorem:

In a field theory of non–interacting massless particles the modular group of the algebra of the forward light-cone V^+ acts as follows:

$$\Delta^{\mathrm{i}t}_{V^+} = V(\mathrm{e}^{-2\pi t}), \quad \text{where}$$
$$V(\lambda)A(x)V^+(\lambda) = A(\lambda x), \quad \lambda > 0$$

holds. This means $V(\lambda)$ implements the dilatations.

Since the calculation is similar to that of the Bisognano–Wichmann case, it will not be repeated here.

(ii) If the theory is conformally covariant then the algebra of the double cone can be transformed onto the algebra of the wedge or the forward light–cone. Since the modular groups are known for the last two algebras, the modular group for the algebra of the double cone can be obtained by transformation. The result is:

3.2.2 Theorem:

Assume we are dealing with a conformal covariant theory. Let D be the double cone

$$D = \{x : |x_0| + \|\vec{x}\| < 1\}$$

and denote by

$$x^{\pm} = x_0 \pm \|\vec{x}\|.$$

Then the modular group of the pair $(\mathcal{M}(D), \Omega)$ induces on D a geometric transformation given by the formula:

$$x^{\pm}(\lambda) = \frac{-(1 - x^{\pm}) + \mathrm{e}^{-2\pi\lambda}(1 + x^{\pm})}{(1 - x^{\pm}) + \mathrm{e}^{-2\pi\lambda}(1 + x^{\pm})}.$$

The modular group of the double cone has first been computed by Hislop and Longo [HL82].

(iii) The examples treated before and those of the next subsection are based on the vacuum representation. There are also situations where one can compute the modular groups for thermal representations. These investigations are due to Borchers and Yngvason [BY98]. In these cases the domain is the forward light–cone or the wedge in two–dimensional models that factorize in light–cone coordinates. In order that one obtains local action for the modular groups one has to deal with Wightman fields of scale–dimension 1. The results are as follows:

3.2.3 Theorem:

Assume we are dealing with a Weyl system over the two–dimensional Minkowski space that factorize in light–cone coordinates. Let ω be the quasi free KMS state and π the corresponding representation of the Weyl algebra for a field of scale–dimension 1. Then the modular groups of the forward light–cone and the wedge act local on the corresponding algebras. The transformations are:

For the forward light–cone:

$$\mathbf{x} \mapsto \varphi_{V^+}(t, \mathbf{x}), \quad \mathbf{x} \in V^+.$$

For the wedge:

$$\mathbf{x} \mapsto \varphi_W(t, \mathbf{x}), \quad \mathbf{x} \in W.$$

Here t is the element of the modular group and the functions φ are given by:

$$\varphi_{\mathrm{V}^+}(u, \mathbf{x}) = (\varphi_+(u, x^{\mathrm{L}}), \varphi_+(u, x^{\mathrm{R}})),$$
$$\varphi_W(u, \mathbf{x}) = (\varphi_-(u, x^{\mathrm{L}}), \varphi_+(u, x^{\mathrm{R}})),$$

with

$$\varphi_-(u, x) = -\varphi_+(-u, -x),$$
$$\varphi_+(u, x) = \frac{\beta}{2\pi} \log \left\{ 1 + \mathrm{e}^{-2\pi u}(\mathrm{e}^{2\pi x/\beta} - 1) \right\}.$$

3.3) *The counterexamples of Yngvason*

The examples of Yngvason [Yng94] are treated separately, because they show examples of theories with special properties. From the result on half–sided translations (Section 2.5.) we know, that the modular group of the wedge acts on the translations as the Lorentz boosts of the wedge. This might give the impression that the modular group of this algebra acts local. That this is not true in general is shown by the first example. If one defines the algebras of the double cones by intersection then the modular group acts local in the characteristic two–plane of the wedge, but not necessarily local in the perpendicular direction, as shown by the last example.

Suppose Φ is a hermitean Wightman field which transforms covariantly under space-time translations, but not necessarily under Lorentz transformations, and depends only on one light cone coordinate, say x^+. Locality implies that the commutator $[\Phi(x^+), \Phi(y^+)]$ has support only for $x^+ = y^+$. Moreover, from the spectrum condition it follows that the generator for translations in the x^+-direction is positive semi–definite. This implies that the Fourier transform of the two–point function, $\mathcal{W}_2$, defined by $(\Omega, \Phi(x^+)\Phi(y^+)\Omega) = (1/2\pi)\int \exp[ip(x^+ - y^+)]\tilde{\mathcal{W}}_2(p)dp$ has the form

$$\tilde{\mathcal{W}}_2(p) = \theta(p)pQ(p^2) + c\delta(p).$$

In this formula Ω is the vacuum vector, $Q(p^2)$ is a positive, even polynomial in $p \in \mathbb{R}$ and $\theta(s) = 1$ for $s \geq 0$ and zero else, and $c = (\Omega, \Phi(x^+)\Omega)^2 \geq 0$ is a constant. Subtracting $c^{1/2}$ from Φ if necessary, we may drop the $\delta(p)$-term. For simplicity of notation from now on we write x, y instead of x^+, y^+.

The models we consider are generalized free fields with the two point function given above (without the δ-term). They are characterized by the commutation relations

$$[\Phi(x), \Phi(y)] = D\,Q(D^2)\delta(x-y)\mathbb{1},$$

where for convenience we have denoted id/dx by D. Let $\mathcal{H}_{Q,1}$ be the Hilbert space of functions $f(p)$ such that $\int_0^\infty |f(p)|pQ(p)dp < \infty$. Define for $f \in \mathcal{H}_{Q,1}$ the unitary Weyl operators as usual by

$$W(f) = \mathrm{e}^{\mathrm{i}\Phi(f)}.$$

The Weyl relations are

$$W(f)W(g) = \mathrm{e}^{-K(f,g)/2}W(f+g)$$

with

$$K(f,g) = (\Omega, [\Phi(f), \Phi(g)]\Omega) = \int_{-\infty}^{\infty} p\,Q(p^2)\tilde{f}(-p)\tilde{g}(p)dp.$$

It follows that $W(f)$ commutes with $W(g)$ if and only if $K(f,g) = 0$, in particular if f and g have disjoint supports. The Weyl operators are defined on the Fock space $\mathcal{H}_Q$. For our future investigations we can restrict our attention to the one–particle Hilbert space $\mathcal{H}_{Q,1}$.

We know that the modular group of the half line acts as a delatation by the factor $\mathrm{e}^{-2\pi t}$. This amounts in momentum space to a dilatation by the factor $\lambda = \mathrm{e}^{2\pi t}$. If we denote the restriction of the modular group of the positive half line Δ_+^{it} to the one–particle Hilbert space $\mathcal{H}_{Q,1}$ by $V_+(\lambda)$ we must get

$$(V_+(\lambda)\psi)(p) = \lambda\sqrt{\frac{Q(\lambda p)}{Q(p)}}e^{i\chi(p)}\psi(\lambda p),$$

where the phasefactor $\mathrm{e}^{\mathrm{i}\chi(p)}$ has to be determined. If $\psi(p)$ is analytic in the upper half plane then the same must be true for $(V_+(\lambda)\psi)(p)$. This condition can be solved by remembering the structure of $Q(p)$ which permits us to write

$$Q(p) = L(p)L(-p), \qquad \text{with} \qquad L(-p) = L(p)^*.$$

The polynomial $L(p)$ is fixed up to a sign by the requirement that its zeros lie in the closed upper half plane. Hence we find:

$$(V_+(\lambda)\psi)(p) = \lambda\frac{L(-\lambda p)}{L(-p)}\psi(\lambda p).$$

That this is the correct expression for the modular group can be checked by showing that the KMS–condition is fulfilled. For this one uses the analyticity property as well in p as in λ.

In the same manner we obtain for the left half–line

$$(V_-(\lambda)\psi)(p) = \lambda\frac{L(\lambda p)}{L(p)}\psi(\lambda p).$$

Since the algebra and its commutant have the same modular group we see that wedge duality is fulfilled iff $L(p)$ has only real zeros.

The duality condition for bounded intervals is a little more difficult. Yngvason has shown:

The duality condition is violated if $L(p)$ and hence $Q(p)$ is not a constant.

Finally we consider fields in n–dimensional Minkowski space. Guided by the low-dimensional examples considered above we shall compute the modular groups of the wedge algebras for generalized free fields on $\mathbb{R}^n$. We treat the special case where the two–point function has in Fourier space the form

$$\mathcal{W}_2(p) = M(p)d\mu(p),$$

where $d\mu$ is a positive Lorentz invariant measure with support in the forward light cone and M is a polynomial that is positive on the support of $d\mu$. The polynomial M allows a factorization,

$$M(p) = F(p)F(-p),$$

where $F(p)$ is a function (in general not a polynomial) with certain analyticity properties to be specified below.

To describe the properties of F we use the light cone coordinates $x^{\pm} = x^0 \pm x^1$ for $x = (x^0, \ldots, x^n) \in \mathbb{R}^n$ and denote $(x^2, \ldots, x^n)$ by $\hat{x}$. The Minkowski scalar product is

$$\langle x, y \rangle = \frac{1}{2}(x^+ y^- + x^- y^+) - \hat{x} \cdot \hat{y}.$$

The right wedge, W_R, is characterized by $x^+ > 0$, $x^- < 0$; hence the Fourier transform, $\tilde{f}(p) = \int \exp(-i\langle p, x \rangle) f(x) d^n x$ of a test function f with support in W_R has for fixed $\hat{p} \in \mathbb{R}^{n-2}$ an analytic continuation in p^+ and p^- into the half planes $\operatorname{Im} p^+ > 0$, $\operatorname{Im} p^- < 0$. We require for F that $F(\pm p)$ is analytic and that $F(-p)$ is *without zeros* in this domain, with $F(-p) = F(p)^*$ for $p \in \mathbb{R}^n$. There is no lack of polynomials M allowing such a factorization; one example is

$$M(p) = (p^1)^2 + \cdots + (p^n)^2 + m^2$$

with

$$F(p) = \sqrt{\hat{p} \cdot \hat{p} + m^2} + \mathrm{i} p^1 = \sqrt{\hat{p} \cdot \hat{p} + m^2} + \frac{\mathrm{i}}{2}(p^+ - p^-).$$

If $d\mu(p) = \theta(p^0)\delta(\langle p, p \rangle - m^2)$ we can replace the polynomial by $(p^0)^2$. Hence the corresponding generalized free field is nothing but the time derivative $(d/dx^0)\Phi_m(x)$, where Φ_m is the free field of mass m.

By analogy with the first example we define for $\lambda > 0$ the unitary operators $V_R(\lambda)$ on the Fock space $\mathcal{H}$ over the one-particle space $\mathcal{H}_1 = L^2(\mathbb{R}^n, M(p)d\mu(p))$ by

$$V_R(\lambda)\varphi(p) = \frac{F(-\lambda p^+, -\lambda^{-1} p^-, -\hat{p})}{F(-p^+, -p^-, -\hat{p})} \varphi(\lambda p^+, \lambda^{-1} p^-, \hat{p}) \tag{3.3.1}$$

for $\varphi \in \mathcal{H}_1$ and canonical extension to $\mathcal{H}$. Then we define by means of $V_R(\lambda)$ a one parameter group of automorphisms of the von Neumann algebra $\mathcal{M}(W_R)$ on $\mathcal{H}$ generated by the Weyl operators $W(f)$ with $\operatorname{supp} f \subset W_R$. By essentially the same computation that verified the example of the half line one shows that (3.3.1) satisfies the KMS condition and that it is therefore, the modular group defined by the vacuum state on $\mathcal{M}(W_R)$.

For the left wedge $W_L = \{x \mid x^+ < 0, x^- > 0\}$ the corresponding operators are

$$V_L(\lambda)\varphi(p) = \frac{F(\lambda p^+, \lambda^{-1} p^-, \hat{p})}{F(p^+, p^-, \hat{p})} \varphi(\lambda p^+, \lambda^{-1} p^-, \hat{p}).$$

By comparing the two modular groups we see that the field satisfies the wedge duality condition $\mathcal{M}(W_R)' = \mathcal{M}(W_L)$ if and only if $F(p) = F(-p)$ on the support of $d\mu$. This condition is, e.g., violated in the above mentioned example.

This example demonstrates also that the modular group of $\mathcal{M}(W_R)$ may act non–local in the $\hat{x}$-directions. In fact, let f be a test function with compact support in W_R. Under the transformation (7.7) the Fourier transform $\tilde{f}$ is mapped into

$$\tilde{f}_\lambda(p) = \frac{\sqrt{\hat{p} \cdot \hat{p} + m^2} - \frac{\mathrm{i}}{2}(\lambda p^+ - \lambda^{-1} p^-)}{\sqrt{\hat{p} \cdot \hat{p} + m^2} - \frac{\mathrm{i}}{2}(p^+ - p^-)} \tilde{f}(\lambda p^+, \lambda^{-1} p^-, \hat{p}).$$

This is no longer the Fourier transform of a function of compact support in the $\hat{x}$-directions, because it is not analytic in $\hat{p}$. From this lack of analyticity it is not difficult to deduce that $W(f_\lambda)$ does not belong to any wedge algebra generated by the field unless the wedge is a translate of W_R or W_L, but we refrain from presenting a formal proof. The operator $W(f_\lambda)$ is still localized in the x^0, x^1-directions in the sense that it is contained in $\mathcal{M}(W_R + a) \cap \mathcal{M}(W_R + b)'$ for some $a, b \in W_R$.

3.4) *The result of Trebels on local modular action*

In the last subsections we saw that under special assumptions the modular groups of algebras, belonging to definite domains, can be computed. In many of these examples the modular transformations led to geometric transformations of the underlying sets. Therefore, it is natural to ask whether or not there might exist other cases where the modular group of the algebra of a set acts as geometric transformations on the underlying set. It is impossible to answer this question for arbitrary sets. Therefore we restrict the sets to the family of double cones and their limits, i.e., to wedges, forward and backward lightcones. The following results are taken from the thesis of S. Trebels [Tre97].

3.4.1 Definition:

A unitary transformation V which maps $\mathcal{M}(G)$ (G open) onto itself and which maps Ω onto itself is called geometric, causal and order preserving if there exists a one to one map $g : G \longrightarrow G$ with the properties:
(i) $x \in G$ implies $x_g \in G,\ x_{g^{-1}} \in G$.
(ii) $x, y \in G$ and $x - y$ are spacelike, then $x_g - y_g$ and $x_{g^{-1}} - y_{g^{-1}}$ are spacelike.
(iii) $x - y \in V^+$ implies $x_g - y_g$ and $x_{g^{-1}} - y_{g^{-1}}$ belong to V^+.
(iv) For every $G^1 \subset G$ one has

$$\mathrm{Ad}\, V \mathcal{M}(G^1) = \mathcal{M}(G^1_g), \quad \text{with} \quad G^1_g = \{x_g; x \in G\}.$$

Notice that $g \longrightarrow x_g$ maps double cones onto double cones. Since double cones form a base of the topology of $\mathbb{R}^d$ we see that $x \longrightarrow x_g$ is continuous. Our first observation is the following

3.4.2 Lemma:

Let g be a geometric causal and order preserving map of the domain G. If $x, y \in G$ and $x - y$ are lightlike then $x_g - y_g$ are lightlike. (The same holds for g^{-1}.)

Proof: Without loss of generality we might assume $y \in x + \overline{V^+}$.Hence we get by continuity $y_g \in x_g + \overline{V^+}$. From $x \in y - \overline{V^+}$ we find $x_g \in y_g - \overline{V^+}$. Both inclusions can only be true if $x_g - y_g$ are lightlike. □

It is our intention to look at the possible geometric, causal and order preserving maps of the double cone. But, by an order preserving conformal transformation γ we can send the double cone onto the forward light cone. Then $\gamma g \gamma^{-1}$ is a geometric, causal and order preserving map of V^+. These are much easier to handle. If we denote by ℓ

a lightray belonging to the boundary of V^+ then a general lightray in V^+ has the form $a\ell_1 + \rho\ell, \quad a > 0,\ 0 < \rho < \infty$. We show next:

3.4.3 Proposition:

Let g be a geometric, causal and order preserving map of V^+. Then g maps parallel lightrays onto parallel lightrays.

Proof: Let $a\ell_1 + \rho\ell$ be a lightray then we associate to it a half–space

$$H(a\ell_1 + \rho\ell) = \text{closure}\{\bigcup_{\rho>0} a\ell_1 + \rho\ell + V^-\}.$$

It is easy to check that an element $x \in V^+$ belongs to $H(a\ell_1 + \rho\ell)$ iff

$$(\ell, x - a\ell_1) \leq 0. \tag{3.4.1}$$

Next we claim that two lightrays $a\ell_1 + \rho\ell$ and $b\ell_2 + \mu\hat{\ell}$ are parallel, i.e., $\ell = \lambda\hat{\ell}$ iff either $(H(a\ell_1 + \rho\ell) \cap V^+) \subset (H(b\ell_2 + \mu\hat{\ell})) \cap V^+$ or vice versa. The first case happens if $a\ell_1 \in H(b\ell_2 + \mu\hat{\ell})$ and the second if $b\ell_2 \in H(a\ell_1 + \rho\ell)$. Let us look at the first case. By the characterization (3.4.1) we obtain for all $\rho > 0$ the relation $(\hat{\ell}, \rho\ell + a\ell_1 - b\ell_2) \leq 0$. This implies $(\hat{\ell}, \ell) \leq \frac{1}{\rho}(\hat{\ell}, b\ell_2 - a\ell_1)$. Taking the limit $\rho \to \infty$ we obtain $(\hat{\ell}, \ell) \leq 0$. But since ℓ and $\hat{\ell}$ both belong to the boundary of V^+ we have $(\ell, \hat{\ell}) \geq 0$. Both inequalities together imply $(\ell, \hat{\ell}) = 0$ or $\hat{\ell} = \lambda\ell$. Let g be a geometric, causal and order preserving map then g maps a lightray $a\ell_1 + \rho\ell$ onto a lightray $a_g\ell_{1,g} + \rho\ell_g$. Since g maps subsets onto subsets it follows that the image of $b\ell_2 + \rho\ell$ is parallel to $a_g\ell_{1,g} + \rho\ell_g$. □

As a consequence of the last construction we obtain the following result, which requires that the dimension of the Minkowski space is larger than two.

3.4.4 Theorem:

Assume $d > 2$ and let g be a geometric, causal and order preserving map of V^+. Then g maps every straight line of V^+ onto a straight line in V^+.

Proof: In the proof of Prop. 3.4.3 we have introduced the closed half–spaces $H(a\ell_1 + \rho\ell)$ associated with the lightray $a\ell_1 + \rho\ell$. The boundary of this half–space is an affine linear manifold of codimension one. By continuity of g this boundary is mapped onto the boundary of the image. The intersection of such affine manifolds is mapped by g onto the intersection of the images, and hence onto the intersection of affine manifolds. Since every spacelike straight line in V^+ is the intersection of $d - 1$ affine surfaces, we obtain that g maps spacelike straight lines onto spacelike straight lines. Since g^{-1} has the same property we conclude that every spacelike straight line is also the image of such line. Next we want to show that every two–plane containing a timelike direction is mapped onto a two–plane of the same kind. In order to construct such two–plane we take two different lightlike vectors ℓ and $\hat{\ell}$ and define the two–plane by:

$$\{a\ell_1 + \mu\ell + \rho\hat{\ell}; a \text{ fixed},\ \mu > 0, \rho > \rho_0(\mu)\}$$

where $\rho_0(\mu)$ is defined by the condition $a\ell_1 + \mu\ell + \rho_0(\mu)\hat{\ell} \in \partial V^+$. This family of points defines a two–plane intersected with V^+. Since by Prop. 3.4.3 g maps parallel lightrays onto parallel lightrays it follows that g maps the two–plane into a two–plane of the same kind. Since g^{-1} has the same properties it follows that the map surjective. Since every timelike line is the intersection of two such two–planes we see that g maps also timelike straight lines onto timelike straight lines. Since we know by 3.4.2 that g maps also lightlike straight lines onto lightlike straight lines the theorem is proved. □

A straight line $L = \{\lambda e, \lambda > 0, e \in V^+\}$ through the origin of V^+ is characterized by the fact, that for every $x \in V^+$ the set $L \cap (V^+ \cap x - V^+)$ is not empty. Hence in $d > 2$ every geometric, causal and order preserving map sends straight lines through the origin onto straight lines through the origin. For simplifying the further calculation we introduce a fixed coordinate system and assume that a lightlike vector ℓ has the form $(1, \vec{\ell})$ with $\|\vec{\ell}\| = 1$. In the following we will denote the set of geometric, causal and order preserving maps of V^+ by $\mathcal{T}$ and the elements of $\mathcal{T}$ by T. The vector $(1, \vec{0})$ will be denoted by t. If $T \in \mathcal{T}$ then it maps the line $\{\lambda t\}$ onto another line through the origin. By a suitable Lorentz transformation $\Lambda(T)$ we can send this line back to the multiples of t, i.e. $\Lambda(T)T\lambda t = f(\lambda)t$ where $f(\lambda)$ is a monotone increasing function with $f(0) = 0$. By $\mathcal{T}_0$ we will denote the set of elements in $\mathcal{T}$ which maps the straight line characterized by t onto itself. We show next:

3.4.5 Lemma:

Let $T \in \mathcal{T}_0$ then T maps every straight line perpendicular to t onto a straight line perpendicular to t.

Proof: Let e be perpendicular to t. Then a straight line perpendicular to t and in direction of e is of the form $L = x + \lambda e$ where $x \in V^+$ and λ belongs to an appropriate interval. Let $\tau = (x, t)$ and $\mu = \sqrt{\tau^2 - x^2}$ then L lies in the hyperplane through τt which is perpendicular to t. This hyperplane is also characterized by the sphere $S = (\frac{(\tau-\mu)}{2} + \partial V^+) \cap (\frac{3\tau+\mu}{2} - \partial V^+)$. Every straight line passing through two points of this sphere lies in the hyperplane in question. Now $T \in \mathcal{T}_0$ maps the points $\frac{\tau-\mu}{2}t$ and $\frac{3\tau+\mu}{2}t$ onto two points on the t–axis. Therefore the sphere S is mapped onto a sphere characterized by the two points. Since a straight line through two of the points is mapped onto the straight line through the corresponding points it follows that T maps straight lines perpendicular to t onto straight lines perpendicular to t. □

Notice that the $(0,1)$–plane is spanned by the t–axis and the lightrays in the $(0,1)$–plane passing through the t–axis. Therefore, this plane is mapped by elements in $\mathcal{T}_0$ onto a two–plane containing the t–axis. Choosing a suitabte rotation $R(T)$ we can transform this plane into the $(0,1)$–plane. Hence $R(T)T$ maps the t–axis and the $(0,1)$–plane onto itself. The set of elements with this property will be denoted by $\mathcal{T}_1$. Next we show:

3.4.6 Lemma:

The restriction of elements in $\mathcal{T}_1$ to the $(0,1)$–plane define dilatations.

Proof: Define the vector $e := (0, 1, 0, .., 0)$. Choose $\beta < \alpha$ and look at the triangle with the corners

$$\beta t, \frac{\alpha+\beta}{2}t + \frac{\alpha-\beta}{2}e, \alpha t.$$

These points are mapped by $T \in \mathcal{T}_1$ onto

$$f(\beta)t, f(\frac{\alpha+\beta}{2})t + g(\alpha,\beta)e, f(\alpha)t.$$

Since the lines between βt and $\frac{\alpha+\beta}{2}t + \frac{\alpha-\beta}{2}e$ and between αt and $\frac{\alpha+\beta}{2}t + \frac{\alpha-\beta}{2}e$ are lightlike and the same is true after translation we obtain $f(\alpha) - f(\frac{\alpha+\beta}{2}) = f(\frac{\alpha+\beta}{2}) - f(\beta) = g(\alpha,\beta)$. From this we find $f(\frac{\alpha+\beta}{2}) = \frac{1}{2}(f(\alpha) + f(\beta))$. Since f is continuous and monotone this equation has only one solution namely

$$f(\alpha) = \alpha f(1).$$

Hence we get $g(\alpha,\beta) = \frac{\alpha-\beta}{2}f(1)$. Therefore, T restricted to the $(0,1)$–plane is a dilatation. □

Let $T \in \mathcal{T}_1$ then we can find a dilatation $D(T)$ such that $D(T)T$ is the identity on the $(0,1)$–plane. The elements with this property will be denoted by $\mathcal{S}_1$. They have the following property:

3.4.7 Lemma:

The elements of $\mathcal{S}_1$ map the subspace perpendicular to the $(0,1)$–plane onto itself.

Proof: Since every sphere with center on the t–axis is mapped onto the sphere with the same center and the same radius it follows that T does not change the distance from the t–axis. Since T is the identity on the $(0,1)$–plane we can make a Lorentz transformation on this two–plane without changing the arguments. Hence the distance from any axis through zero of the $(0,1)$–plane remains unchanged. But this can only hold if T maps the subspace perpendicular to the $(0,1)$–plane onto itself. □

Combining all the arguments we find:

3.4.8 Theorem:

Every geometric, causal and order preserving map of the forward lightcone is an element of the Lorentz group extended by the dilatation.

Proof: If $d = 3$ then $T \in \mathcal{S}_1$ is in the 2–direction either the identity or the reflection at the (0.1)–plane. If $d > 3$ then there exists a rotation $R(T)$ such that $R(T)T$ does not change the 2–axis. This implies also that $R(T)T$ is the idendity on the $(0,1,2)$–space. Repeating this argument we end up with either the identity or a reflection. This means if T is the orginal transformation then

$$R_n ... D R_1 \Lambda_1 T = \mathbb{1} \quad \text{or} \quad P$$

where P is a reflection. This shows the theorem. □

The modular group is a one–parametric group. This implies that every element is the square of another element. Hence if the group acts geometric and causal on the underlying domain, then it acts automatically order preserving. If the modular group induces a geometric and causal action on the underlying domain then we know from the last theorem that it is a one–parametric subgroup of the $(\frac{d(d-1)}{2}+1)$–dimensional Lie group generated by the Lorentz group and the dilatations. In order to restrict the possibilities we have to use the following properties:
1. The group $g(t)$ is induced by the modular group of $\mathcal{M}(D)$, where D is a double cone. This implies that for $A \in \mathcal{M}(D)$ the expression

$$\Delta^{\mathrm{i}t} A\Omega$$

has an analytic continuation into the strip $S(-\frac{1}{2}, 0)$.
2. We are dealing with a quantum field theory in the vacuum sector. This implies in particular that the translations fulfil spectrum condition.

We want to compare the geometric modular action with the action of the translations. As technical tool we need the following result which can easily be proved with help of the double cone theorem, Thm. 1.4.4. Here we will not present the proof.

3.4.9 Theorem:

Assume we are dealing with a quantum field theory in the vacuum sector, and that the dimension of the Minkowski space is larger than two. Let D_1, D_2 be two double cones with center x_1, x_2 respectively. If $x_1 - x_2$ is lightlike and if $\mathcal{M}(D_1)$ and $\mathcal{M}(D_2)$ commute then the whole quantum field theory is abelian.

We want to look at the modular group of the double cone D. Let $x \in D$ and if $\Delta^{\mathrm{i}t}$ acts geometric and causal on D then $g(t)x$ can be differentiated with respect to t since $g(t)$ is a subgroup of a Lie group. We want to investigate the direction of $g'(t)x$.

3.4.10 Theorem

Assume we are dealing with a quantum field theory in the vacuum sector, and that the dimension of the Minkowski space is larger than two. Let D be a double cone and let $\Delta^{\mathrm{i}t}$ be the modular group of $\mathcal{M}(D)$. Assume this group acts geometric and causal on D. Then for $x \in D$ one has

$$g'(0)x \in \overline{V^-}.$$

Proof: If $g'(0)x = 0$ then the statement holds. If $g'(0)x \neq 0$ let t be a fixed timelike vector in V^+ and choose a lightlike vector $\ell \in \partial V^+$ such that $(\ell, g'(0)x) \neq 0$ where $x \in D$. Let E be the two–plane spanned by t and ℓ. Let s be a spacelike vector in E such that $(t, s) = 0$ and $(\ell, s) < 0$. Denote the second vector in $E \cap \partial V^+$ by ℓ'. Let y be such that $x - y$ is a positive multiple of ℓ and such that $D \cap y + W(\ell', \ell) = \emptyset$. Choose a small double cone D_x such that x is the upper tip of D_x and such that $D_x + a \subset D$ for a in some neighbourhood of zero.

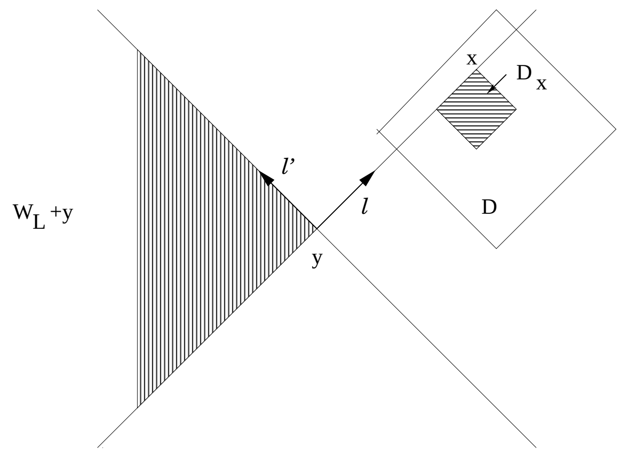

Fig.1: Position of the double cones D, D_x and the wedge $W(\ell', \ell) + y$

Choose $B \in \mathcal{M}(W(\ell', \ell) + y)$ and $A \in \mathcal{M}(D_x)$. Define the two functions

$$f^+(\lambda, \tau) = (\Omega, BU(\lambda t)\Delta^{\mathrm{i}\tau} A\Omega),$$
$$f^-(\lambda, \tau) = (\Omega, A\Delta^{-\mathrm{i}\tau} U(-\lambda t)B\Omega),$$

where $U(x)$ is the representation of the translations. These two functions have the following properties:

(i) By the modular theory $f^+(\lambda, \tau)$ has in τ an analytic continuation into the strip $S(-\frac{1}{2}, 0)$.
(ii) The spectrum condition implies that $f^+(\lambda, \tau)$ has in λ an analytic continuation into the upper complex half–plane $\mathbb{C}^+$.
(iii) $f^-(\lambda, \tau)$ can in τ be analytically continued into $S(0, \frac{1}{2})$
(iv) and in λ into $\mathbb{C}^-$.

Using the Malgrange-Zerner theorem, Thm. 1.4.2, and the tube theorem, Thm. 1.4.3, we see that $f^+(\lambda, \tau)$ has an analytic continuation in both variables into the tube domain $\mathbb{C}^+ \times S(-\frac{1}{2}, 0)$. Correspondingly $f^-(\lambda, \tau)$ has a simultanious continuation into $\mathbb{C}^- \times S(0, \frac{1}{2})$. Now we want to look at the coincidence domain of $f^+(\lambda, \tau)$ with $f^-(\lambda, \tau)$. With help of the edge of the wedge theorem, Thm. 1.4.1, we can analytically continue through the coincidence domain. For its definition we set:

(a) $R = \{\tau;\ g(\tau)D_x \quad \text{is spacelike to} \quad W(\ell', \ell) + y\}$.
(b) For $\tau \in R$ define $\lambda_0(\tau) = \sup\{\lambda;\ g(\tau)D_x + \lambda t \quad \text{is spacelike to} \quad W(\ell', \ell) + y\}$.
(c) For $\tau \in R$ define $\lambda_1(\tau) = \inf\{\lambda;\ g(\tau)D_x + \lambda t \quad \text{is spacelike to} \quad W(\ell', \ell) + y\}$.

Because of locality we get $f^+(\lambda, \tau) = f^-(\lambda, \tau)$ for $\tau \in R$ and $\lambda_1(\tau) < \lambda < \lambda_0(\tau)$. In order to simplify the calculation assume $t^2 = -s^2 = 1$. The vector $g'(0)x$ has the form

$$g'(0)x = \rho t + \mu s + v$$

where v is perpendicular to t and s. The assumption $(g'(0)x, \ell) \neq 0$ implies $\rho - \mu \neq 0$. For small τ we obtain:

$$g(\tau)x = x + \tau(\rho t + \mu s + v) + o(\tau).$$

This implies for small τ:

$$\begin{aligned} \tau \in R \quad &\text{for} \quad \tau > 0, \quad \text{if} \quad \rho - \mu < 0; \\ \tau \in R \quad &\text{for} \quad \tau < 0, \quad \text{if} \quad \rho - \mu > 0. \end{aligned}$$

From this we obtain for $\lambda_0(\tau)$ the following estimate:

$$\begin{aligned} \lambda_0(\tau) &= \tau(\mu - \rho) + o(\tau) \quad \text{for} \quad (\rho - \mu) < 0, \\ \lambda_0(\tau) &= |\tau|(\rho - \mu) + o(\tau) \quad \text{for} \quad (\rho - \mu) > 0. \end{aligned}$$

From the coincidence of the functions $f^+(\lambda, \tau)$ and $f^-(\lambda, \tau)$ we get an opposite edge of the wedge problem, where the local cones are the second and fourth quadrant. Therefore, by the double cone theorem 1.4.4 the tangents at the boundary of the coincidence domain must lie in the first or third quadrant. Now we can compute the tangent at $\tau = 0$ because we have an estimate for $\lambda_0(\tau)$.

For $(\rho - \mu) < 0$ we see that the tangent vector at $\tau = 0$ lies in the first quadrant. But, for $(\rho - \mu) > 0$ this tangent lies in the fourth quadrant.

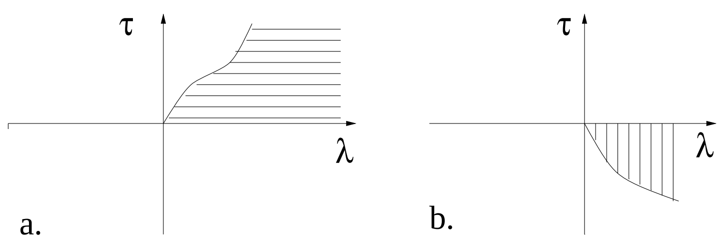

Fig.2: The coincidence domain in the (λ, τ)–plane.
a. $(\rho - \mu) < 0$, b. $(\rho - \mu) > 0$.

Since the case $(\rho - \mu) > 0$ leads to a non–stable situation we conclude that the condition $(\rho - \mu) < 0$ must be fulfilled. This implies that $g'(0)x$ belongs to the half–space $(\rho - \mu) < 0$. Changing now the vector ℓ (There are only a few vectors ℓ for which $(\ell, g'(0)x) = 0$ can hold.) we see that $g'(0)x$ must lie in the intersection of these half–spaces, i.e. in $\overline{V^-}$. □

As a consequence of this result we find:

3.4.11 Theorem:

With the same assumptions as in the last theorem the group $\{g(t)\}$ coincides with the group of Hislop–Longo transformations (up to a positive scale transformation of the group parameter).

Proof: Since the properties of the last theorem are stable under conformal transformations, we will transform the double cone onto the forward lightcone. In this setting we have to show that the group $g(t)$ coincides with the dilatations. If we write $g(t) = \exp\{Mt\}$ then $g'(t)x \in \overline{V^-}$ implies (y, Mx) is smaller zero for all $x, y \in V^+$. By means of the structure of the Lorentz group we find that M is diagonal and hence $M = -m\mathbb{1}$, $m \in \mathbb{R}^+$.

Therefore, the transformed $g(t)$ coincides with the scaled dilatations and consequently the original group coincides with the scaled Hislop–Longo transformations. □

If G is the generator of the Hislop–Longo transformation then we have shown that $g(t)$ is of the form $g(t) = \exp\{mGt\}$ where m is a positive constant. One would like to prove that $m = 1$. To this end one has to use the KMS–condition. (See Sect.1.3.) With the methods available up to now we are not able to give a general proof for the statement $m = 1$. However, if we would deal with a finite number of Wightman fields then the modular transformation would be $\Delta^{\mathrm{i}t}\Phi_k(x)\Delta^{-\mathrm{i}t} = D^j_k(t)\Phi_j(g(t)x)$. Here $D^j_k(t)$ is a finite dimensional representation of the dilatations. In this situation one can at least show that m is bounded by one. We do not want to give the calculations.

3.4.12 Remark: The case $m = 0$ can be excluded. This case would mean that the algebra of every subdomain $D_1 \subset D$ is invariant under the modular group of D. But this implies by the cyclicity of Ω that $\mathcal{M}(D_1)$ and $\mathcal{M}(D)$ coincide. (See [KR86] Thm. 9.2.36.) Such situation is only possible if the theory is abelian.

3.5) *Remarks, additions and problems*

(I) The result of Trebels deals only with double cones. Therefore, it is not possible to argue that the factor m has to be 1. This is due to the fact that the Hislop–Longo transformation $g(t - \mathrm{i}\frac{1}{2})$ maps D to real points but they are not all spacelike with respect to D. If, however, we replace the double cone by the wedge then one can argue that m must be 1.

3.5.1 Problem: Does there exist a convincing argument showing, that m must be 1Γ

(II) In the Trebels situation, the algebra of a sub–double cone with either the same upper or lower tip fulfils the condition of −half–sided +half–sided modular inclusion respectively. If one is dealing with a conformally covariant theory, then the corresponding half–sided translations map, for a proper choosen (finite) group element, the algebra of the double cone onto the algebra of the backward respectively forward light cone.

(III) If the Bisognano–Wichmann property (Def. 3.1.8) is fulfilled only for the subsets of the wedge, then the modular group of the wedge define geometric transformations only for this wedge. This can be extended to geometric transformations of the whole $\mathbb{R}^d$. (See D. Guido [Gui95].)

(IV) As shown by Kuckert [Ku98] the assumptions can be changed. If one replaces the Bisognano–Wichmann property for the wedge by other symmetry conditions, with some locality property, but for the whole space, then one finds that J_W and $\Delta^{\mathrm{i}t}_W$ act local as in the Bisognano–Wichmann situation. A similar result holds for the forward light–cone, provided Ω is cyclic and separating for $\mathcal{M}(V^+)$. In these cases the assumptions are: The symmetry shall map the local net *into* the local net. The associated modular groups shall transform the local algebras in the corresponding manner.

One can replace the transformation property of the local net by transformation properties of localized operators. In this case one has to make more restrictive assumptions on the transformations and the net. For details see [Ku98].

4. The PCT–theorem and connected questions

The PCT–theorem tells us that the product of time reversal, space reflection, and charge conjugation is always a symmetry. Reading the paper of Pauli [Pau55] on this subject one gets the impression that a precurser of the PCT–theorem has been discovered by Schwinger [Schg51]. But it was a mysterious transformation containing the interchange of operators. The first development of the PCT–theorem in the frame of Lagrangean field theory is due to Lüders [Lü54]. This result has triggered the clarification of the connection between spin and statistics and the role of the positive energy. (See W. Pauli [Pau55] and also G. Lüders and B. Zumino [LZ58].)

1957 R. Jost [Jo57] gave a proof of the PCT–theorem in the frame of Wightman's field theory. The beauty of this proof is the clarification of the role of the different conditions one has to impose. These are

1. Covariance of the theory under the (connected part of the) Poincaré group.
2. Positivity of the energy.
3. There are only fields, which transform with respect to finite dimensional representations of the Lorentz group. (Transformation of the index space.)
4. Locality, which means that for spacelike distances the Bose fields commute with all other fields and the Fermi fields anti–commute with eachother.
5. The Minkowski space has even dimensions.
6. To every field in the theory appears its conjugate complex partner.

From the spectrum condition it follows that the Wightman functions have an analytic continuation into the forward tube T_n^+

$$T_n^+ = \{z_1, ..., z_n \in \mathbb{C}^4; \Im m\,(z_i - z_{i+1}) \in V^+\}.$$

Using locality, Poincaré covariance of the theory, and the appearence of only finite dimensional representation of the Lorentz group in the index space, Hall and Wightman [HW57] could show that the analytically continued Wightman functions can be considered as functions on the complex Lorentz group. If the index space transforms under infinite dimensional representation of the Lorentz group then the Hall–Wightman theorem fails because of lack of analyticity. Examples are given by Streater [Str67] and by Oksak and Todorov [OT68]. The Hall–Wightman theorem was the starting point of Jost's investigation. If the Minkowski space has even dimensions then the complex Lorentz group contains the element $-\mathbb{1}$. This transformation is the product of time reversal and space reflection. But there is the time translation e^{iEt} with the positive energy operator. In order to keep the energy positive one has to change i into $-$i. Therefore, the time reversal has to be an antiunitary operator. If Θ is an antiunitary total reflection one obtains for a scalar field

$$\Theta\Phi(x)\Theta = \Phi^*(-x).$$

The passage to the conjugate complex is closely related to the charge conjugation. Therefore, one has to look at the product of C and PT. One remark more to the role of locality: The transition to the conjugate complex interchanges the order of an operator product. At totally spacelike points the original order can be restored. Putting things together one gets the PCT–theorem for scalar fields. The general case needs in addition the handling of finite dimensional matrices which appear with fields of higher spin.

For a long time it was impossible to show the PCT–theorem in the theory of local observables because one did not know the meaning of condition 3 and 6 in the setting of local observables.

A good candidate for the CPT–operator is

$$\Theta = J_W U(R_W(\pi)) \tag{4.0.1}$$

provided the origin is contained in the edge of the wedge. $R_W(\alpha)$ denotes the rotation in the two–plane perpendicular to the characteristic two–plane of the wedge, and J_W the modular conjugation of the algebra of the wedge.

If the Ansatz (4.0.1) is correct, then the representation of the Lorentz group and the modular groups of the wedges have to fit together. Since on the vacuum sector Θ is a geometric transformation, also J_W has to act local. Moreover the transformation $\Theta U(R_W(\pi))$ maps the algebra of the wedge onto the algebra of the opposite wedge. Therefore, the theory has to fulfil wedge duality. First we treat the question of wedge duality and afterwards that of the locality of J_W.

4.1) *The wedge duality*

The problem of this subsection is: When does a Lorentz covariant theory fulfil wedge dualityΓ

The result we present here is essentially a two-dimensional statement. In the proof we can think of sets which are cylindrical in all directions perpendicular to the characteristic two–plane of the wedge. Hence all the expressions depend only on two variables. In this situation we only have two wedges which we call the right wedge W^r and the left wedge W^l. The wedges obtained by applying a shift by a will be denoted by W^r_a and W^l_a respectively. If we denote the double–cones by K then this can be characterized by the intersection of two wedges.

$$K_{a,b} = W^r_a \cap W^l_b, \qquad b - a \in W^r.$$

Let $\mathcal{M}^0_{a,b}$ be the given von Neumann algebra associated with $K_{a,b}$ fulfilling the mentioned assumption. Starting from this we obtain for the wedges the algebras:

$$\begin{aligned} \mathcal{M}^r_a &= \{ \bigcup_{K \subset W^r_a} \mathcal{M}^0_K \}'', \\ \mathcal{M}^l_a &= \{ \bigcup_{K \subset W^l_a} \mathcal{M}^0_K \}''. \end{aligned} \tag{4.1.1}$$

Moreover, we set

$$\begin{aligned} \mathcal{M}'^r_a &= \{ \mathcal{M}^r_a \}' \\ \mathcal{M}'^l_a &= \{ \mathcal{M}^l_a \}'. \end{aligned} \tag{4.1.1$'$}$$

Without loss of generality we can construct a net which might be slightly larger:

$$\mathcal{M}_{a,b} := \mathcal{M}(K_{a,b}) = \mathcal{M}^r_a \cap \mathcal{M}^l_b. \tag{4.1.2}$$

This net fulfils again all requirements listed in the beginning. Moreover, the wedge–algebra constructed with $\mathcal{M}(K)$ coincides with the wedge–algebra constructed with the $\mathcal{M}^0(K)$. In what follows we only will work with the algebras $\mathcal{M}(K)$. Besides $\mathcal{M}_{a,b}$ one can define

$$\mathcal{M}^r_{a,b} := \mathcal{M}^r_a \cap \mathcal{M}'^r_b, \qquad \mathcal{M}^l_{a,b} := \mathcal{M}^l_a \cap \mathcal{M}^l_b. \tag{4.1.2'}$$

In Wightman field theory one is dealing with quantities $\Phi_n(x)$ localized at a point. If x belongs to the right wedge one can analytically continue the expression $U(\Lambda(t))\Phi_n(x)\Omega$ into the strip $S(-\frac{1}{2}, 0)$. This is due to the fact that the representation of the Lorentz group in the index–space is defined for complex Lorentz transformations. The result which one obtains is an element belonging to the left wedge namely $U(\Lambda(t))\Phi_n(-x)\Omega$ (for entire spin). There are two problems if one wants to generalize this:

First our objects are not localized at a point but in bounded domains. Here we will find a natural generalization of the description.

The second problem consists of understanding the exchange of the left and the right wedge by the complex Lorentz transformations because of the following

4.1.1 Remark:

If we are dealing with a von Neumann algebra $\mathcal{M}$ and a one–parametric, strongly continuous group of automorphisms α_t then one can define the analytic elements $\mathcal{M}^{anal}$ for which $\alpha_t A$ has an entire analytic extension. The set $\mathcal{M}^{anal}$ is a $*$–strong dense subalgebra of $\mathcal{M}$ and the elements $\alpha_z A,\ A \in \mathcal{M}^{anal}$ also belong to $\mathcal{M}$.

Therefore, it is not easy to understand why for an element A, localized in the right wedge, the expression $U(\Lambda(-\frac{i}{2}))A\Omega$ can be written as $\hat{A}\Omega$ with an element $\hat{A}$ localized in the left wedge.

First we look at the localization problem. Let $A \in \mathcal{M}$ be a local operator then we denote by K_0 the smallest double–cone such that $A \in \mathcal{M}(K_0)$. By K we denote the translate of K_0 such that the center of K coincides with the origin. Let $K_0 = K + x$ then we can write every localized operator in the form

$$A = A(K, x). \tag{4.1.3}$$

The second problem is much harder and a large part of this subsection is needed to cope with it. The main part of the difficulty is due to the fact that we must start from the assumption that wedge duality is not present. Therefore, to every wedge there are associated two algebras namely the algebra defined in equation (4.1.1) and the commutant of the algebra belonging to the opposite wedge.

In order to get to the opposite wedge one has to look at the analytic extension in the Lorentz transformations. Here we have to cope with a new problem namely we cannot conclude that for sufficiently many elements the analytic extension of the expressions $\mathrm{Ad}\,U(\Lambda(t))A(K,x)$ are bounded. This difficulty is again a consequence of the fact that we do not know the wedge–duality. In order to overcome this problem we have to introduce unbounded operators $X(K,x)$ which are affiliated with the algebra $\mathcal{M}(K+x)$. But with this generalization it will be possible to show that for suitable elements $X(K+x)$, satisfying

$K + x \subset W^r$, the following holds: To $X(K, x)$ there exists an element $\hat{X}(K, -x)$ located in $K - x \subset W^l$ such that the relation

$$U(\Lambda(-\frac{\mathrm{i}}{2}))X(K, x)\Omega = \hat{X}(K, -x)\Omega$$

holds.

From Remark 2.5.3(ii) we know that $\Delta_r^{\mathrm{i}t}$ and $\Delta_l^{\mathrm{i}t}$ act on the translations as the Lorentz transformations. Moreover, the construction of the algebra $\mathcal{M}_a^{r,l}$ (Eq. (4.1.2')) imply the following transformation rules.

$$\begin{aligned}
\Delta_r^{\mathrm{i}t}\mathcal{M}_{a,b}^r\Delta_r^{-\mathrm{i}t} &= \mathcal{M}_{\Lambda(t)a,\Lambda(t)b}^r, \\
\Delta_l^{\mathrm{i}t}\mathcal{M}_{a,b}^l\Delta_l^{-\mathrm{i}t} &= \mathcal{M}_{\Lambda(t)a,\Lambda(t)b}^l, \\
J_r\mathcal{M}_{a,b}^r J_r &= \mathcal{M}_{-b,-a}^r, \\
J_l\mathcal{M}_{a,b}^l J_l &= \mathcal{M}_{-b,-a}^l.
\end{aligned} \tag{4.1.4}$$

These equations permit to compare the Lorentz transformations with the two modular groups. First notice that $U(\Lambda)$ maps the four algebras of the two wedges into themselves and hence $U(\Lambda)$ commutes with the modular groups and the modular conjugations (see e.g. [BR79]). Therefore, we obtain the following representations of the Lorentz group:

$$\begin{aligned}
R(t)\Delta_r^{\mathrm{i}t} &= U(\Lambda(t)) \\
L(t)\Delta_l'^{\mathrm{i}t} &= U(\Lambda(t)).
\end{aligned} \tag{4.1.5}$$

Here Δ' denotes the modular operator of $(\mathcal{M}_0^l)'$. Since $U(\Lambda)$ commutes with the modular groups and acts on the translations in the same manner as the modular goups we obtain the following commutations:

$$\begin{aligned}
{}[R(s), \Delta_r^{\mathrm{i}t}] = [R(s), U(\Lambda)] = [R(s), T(a)] = [R(s), J_r] = 0, \\
{}[L(s), \Delta_l'^{\mathrm{i}t}] = [L(s), U(\Lambda)] = [L(s), T(a)] = [L(s), J_l] = 0.
\end{aligned} \tag{4.1.6}$$

Using the inclusion $\mathcal{M}'^l \supset \mathcal{M}^r$ and $\mathcal{M}'^r \supset \mathcal{M}^l$ we obtain with Thm. A:

4.1.2 Lemma:

As a consequence of Eq. (4.1.5) *we obtain:*

(a) If $A \in \mathcal{B}(\mathcal{H})$ (the set of bounded linear operators on $\mathcal{H}$) and if $L(t)A\Omega$ has a bounded analytic continuation into the strip $S(-\frac{1}{2}, 0)$ then the same is true for $R(t)A\Omega$. If $A \in \mathcal{B}(\mathcal{H})$ is such that $R(t)A\Omega$ has a bounded analytic extension into the strip $S(0, \frac{1}{2})$ then the same holds for $L(t)A\Omega$.

(b) Moreover, we obtain the following identities:

$$\begin{aligned}
J_l L(-\frac{\mathrm{i}}{2}) &= J_r R(-\frac{\mathrm{i}}{2}) \quad \text{on} \quad \mathcal{D}(L(-\frac{\mathrm{i}}{2})), \\
J_l L(\frac{\mathrm{i}}{2}) &= J_r R(\frac{\mathrm{i}}{2}) \quad \text{on} \quad \mathcal{D}(R(\frac{\mathrm{i}}{2})),
\end{aligned}$$

where $\mathcal{D}(X)$ denotes the domain of definition of the operator X.

Proof: Using Eq. (4.1.5) and Thm. A we obtain that the operator valued function

$$D(t) := {\Delta'}_l^{-\mathrm{i}t} \Delta_r^{\mathrm{i}t} = L(t)R(-t)$$

has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$. At the upper boundary we have $D(t + \frac{\mathrm{i}}{2}) = J_l D(t) J_r$. So we obtain $R(t) = D^*(t)L(t)$ and $L(t) = D(t)R(t)$. Since $D(t)$ can be continued into $S(0, \frac{1}{2})$, its adjoint can be continued into $S(-\frac{1}{2}, 0)$. This implies that

$$\mathcal{D}(L(-\frac{\mathrm{i}}{2})) \subset \mathcal{D}(R(-\frac{\mathrm{i}}{2})), \qquad \mathcal{D}(R(\frac{\mathrm{i}}{2})) \subset \mathcal{D}(L(-\frac{\mathrm{i}}{2})). \tag{4.1.7}$$

Therefore,

$$R(t)A\Omega = D^*(t)L(t)A\Omega$$

has an analytic extension into the strip $S(-\frac{1}{2}, 0)$ whenever this holds for $L(t)A\Omega$. By the corresponding arguments we obtain the second relation of (a).

We know the relation $D(\frac{\mathrm{i}}{2}) = J_l J_r$ and obtain

$$J_l J_r = L(\frac{\mathrm{i}}{2})R(-\frac{\mathrm{i}}{2})$$

and thus with the statements about the domains of definition (4.1.7)

$$J_l L(\frac{\mathrm{i}}{2}) = J_r R(\frac{\mathrm{i}}{2}).$$

For the left wedge we have $\mathcal{M}_0^l \subset {\mathcal{M}'}_0^r$ and since the modular operator of the commutant is the inverse of the modular operator of the algebra we obtain the other statements. For later use we retain the relation between the domains which follows from the interchange of right with left. This shows the lemma. □

If $A(K, x)$ is a localized operator such that $K + x \subset W^r$ and such that $U(\Lambda(t))A(K, x)\Omega$ can in t be analytically continued into the strip $S(-\frac{1}{2}, 0)$ then we expect that we can write $U(\Lambda(-\frac{\mathrm{i}}{2}))A(K, x)\Omega$ in the form $\hat{A}(K, -x)\Omega$. This operator should be localized in $K - x$. Since K was a symmetric domain we see that $K - x$ belongs to the left wedge. This shall be shown next. There is however, one problem: At the beginning we do not know wedge–duality. Hence we cannot conclude that there exist elements $A(K, x)$ such that the corresponding operator $\hat{A}(K, -x)$ is bounded. Therefore, we will include unbounded operators in our investigation.

We write $X(K, x)$ for unbounded operators which shall imply that this operator is closable and affiliated with the algebra $\mathcal{M}(K + x)$. Without further mentioning, the domain of definition of $X(K, x)$ and of its adjoint shall contain $\mathcal{M}'(K + x)\Omega$. We always identify $X(K, x)$ with the restriction of X to the domain $\mathcal{M}'(K + x)\Omega$. This has the advantage that we have the transformation

$$T(y)U(\Lambda)X(K, x)U(\Lambda^{-1})T(-y) = X(K_\Lambda, \Lambda x + y).$$

The restriction of the adjoint of $X(K,x)$ to this domain will be denoted by $X^{\dagger}(K,x)$. This definition implies that $X(K,x)\Omega$ belongs for $K+x \subset W^r$ to the domain of the Tomita conjugation S_{K+x} of the algebra $\mathcal{M}(K+x)$, which leads to the relation

$$S_{K+x}X(K,x)\Omega = X^{\dagger}(K,x)\Omega.$$

Next we look at analytic extensions in the Lorentz group. To this end notice that every operator $X(K,x)$ is defined on the vector Ω and we can look at possible analytic extensions of the vector function $U(\Lambda(t))X(K,x)\Omega$. As main tool for the investigation of these expressions we use the groups $R(t)$ and $L(t)$ introduced in Eq.(4.1.5)

4.1.3 Proposition:

(1) *Let $X(K,0)$ be such that for one x with $K+x \subset W^r$ the vector function*

$$U(\Lambda(t))X(K,x)\Omega$$

has a bounded analytic continuation into the strip $S(-\frac{1}{2},0)$ and continuous boundary values at $\Im m\, t = -\frac{1}{2}$. Then

$$L(t)J_l X^{\dagger}(K,0)\Omega \qquad \text{and}$$
$$R(t)J_r X^{\dagger}(K,0)\Omega$$

have bounded analytic continuations into the strip $S(-\frac{1}{2},0)$ and continuous boundary values at $\Im m\, t = -\frac{1}{2}$.

(2) *If for one y with $K+y \subset W^l$ the vector function*

$$U(\Lambda(t))X(K,x)\Omega$$

has a bounded analytic continuation into the strip $S(0,\frac{1}{2})$ and continuous boundary values at $\Im m\, t = \frac{1}{2}$ then

$$L(t)J_l X^{\dagger}(K,0)\Omega \qquad \text{and}$$
$$R(t)J_r X^{\dagger}(K,0)\Omega$$

have bounded analytic continuations into the strip $S(0,\frac{1}{2})$ and continuous boundary values at $\Im m\, t = \frac{1}{2}$.

Proof: It is sufficient to show the first statement. The second follows by symmetry. Let us look at the vector functions

$$F^r(s,t) = R(s)\Delta_r^{\mathrm{i}t}X(K,x)\Omega,$$
$$F^l(s,t) = L(s)\Delta_l'^{\mathrm{i}t}X(K,x)\Omega.$$

For real s these functions can be analytically continued in t into the strip $S(-\frac{1}{2},0)$. For $t = s+t$ we obtain by Eq. (4.1.5) $F^{\#}(s,s+t) = \Delta_{\#}^{\mathrm{i}t}U(\Lambda(s))X(K,x)\Omega$ and since the modular groups commute with the Lorentz transformations we obtain analyticity along the diagonal. Using the Malgrange Zerner theorem, Thm. 1.4.2, we obtain an analytic continuation into the tube with triangular base

$$\Im m\,(s,t) = \{(0,0),\ (-\frac{1}{2},-\frac{1}{2}),\ (0,-\frac{1}{2})\}.$$

This shows by the relation $U(\Lambda(t)) = R(t)\Delta_r^{\mathrm{i}t}$ and the transformation

$$\Delta_r^{\frac{1}{2}} X(K,x)\Omega = J_r X^\dagger(K,x)\Omega$$

that $F^{\#}(s,t-\frac{\mathrm{i}}{2})$ has in s an extension into the strip $S(-\frac{1}{2},0)$. This extension is bounded, since analytic completion does not change the norm of the vector–function. This shows that the expression

$$R(\tau)J_r X^\dagger(K,x)\Omega$$

is defined for $\tau \in S(-\frac{1}{2},0)$ and has continuous boundary values coinciding with those of $R(t)U(\Lambda(-\frac{\mathrm{i}}{2}))X(K,x)\Omega$.

The arguments for $L(t)$ are the same and dont need to be repeated. □

4.1.4 Lemma:

Assume that for one x with $K+x \subset W^r$ the vector function $U(\Lambda(t))X(K,x)\Omega$ has a bounded analytic extension into the strip $S(-\frac{1}{2},0)$ and this function has continuous boundary values at $\Im m\, t = -\frac{1}{2}$. Then the same is true for all x satisfying $K+x \subset W^r$.

The same result is obtained if $K+x$ belongs to the left wedge W^l and if we have an analytic extension into the strip $S(0,\frac{1}{2})$ with continuous boundary values.

Proof: It is sufficient to show the first statement. The other follows by symmetry. By Prop. 4.1.3 we know that $R(t)J_r X^\dagger(K,0)\Omega$ has an analytic extension into the strip $S(-\frac{1}{2},0)$. Since $R(t)$ commutes with the translations we get the analytic extension also for the expression $R(t)J_r X^\dagger(K,y)\Omega$. Choosing now y such that $K+y \subset W^r$ we can use the modular conjugation of the right wedge and obtain

$$R(\tau)J_r X^\dagger(K,y)\Omega = R(\tau)\Delta_r^{\frac{1}{2}} X(K,y)\Omega.$$

Notice that $R(t)$ and Δ_r commute so that by Prop. 4.1.3 $R(t)\Delta_r^{\mathrm{i}t}X(K,y)\Omega$ has an analytic extension into $S(-\frac{1}{2},0)$ with continuous boundary values. □

After this preparation we introduce the following sets:

4.1.5 Definition:

(a) By $\mathcal{A}_r^u$ we denote the set of all bounded or unbounded operators $X(K,0)$ with the properties:

(i) For every x with $K+x \subset W^r$ the vector function

$$U(\Lambda(t))X(K,x)\Omega$$

has a bounded analytic extension into the strip $S(-\frac{1}{2},0)$ and continuous boundary values at $\Im m\, t = -\frac{1}{2}$.
(ii) For every x with $K+x \subset W^l$ the vector function

$$U(\Lambda(t))X^{\dagger}(K,x)\Omega$$

has a bounded analytic extension into the strip $S(0,\frac{1}{2})$ and continuous boundary values at $\Im m\, t = \frac{1}{2}$.
(b) By $\mathcal{A}_l^u$ we denote the set of all bounded or unbounded operators $X(K,0)$ such that $X^{\dagger}(K,0) \in \mathcal{A}_r^u$.

A consequence of the definition is the result

4.1.6 Lemma:

(i) If $X(K,0) \in \mathcal{A}_r^u$ then the vector functions

$$R(t)X(K,0)\Omega \quad \text{and} \quad L(t)X(K,0)\Omega$$

can both be analytically continued into the strip $S(-\frac{1}{2},0)$ and they have continuous boundary values at $\Im m\, t = -\frac{1}{2}$.
(ii) If $X(K,0) \in \mathcal{A}_l^u$ then the vector functions

$$R(t)X(K,0)\Omega \quad \text{and} \quad L(t)X(K,0)\Omega$$

can both be analytically continued into the strip $S(0,\frac{1}{2})$ and they have continuous boundary values at $\Im m\, t = \frac{1}{2}$.

Proof: Using the fact that $X^{\dagger}(K,0)$ belongs to $\mathcal{A}_l^u$ we obtain by Prop. 4.1.3 that $R(t)J_rX(K,0)\Omega$ can analytically be extended into $S(0,\frac{1}{2})$ and has continuous boundary values. Since $R(t)$ commutes with J_r for real t it follows that $X(K,0)\Omega$ belongs to the domain of $R(-\frac{\mathrm{i}}{2})$. This is equivalent with the statement. The other three cases are shown in the same manner. □

Now we are prepared for the main result of this section.

4.1.7 Theorem:

(i) For every $X(K,0) \in \mathcal{A}_r^u$ and every x with $K+x \subset W^r$ there exists an element $\hat{X}(K,0) \in \mathcal{A}_l^u$ such that the following relation holds

$$U(\Lambda(-\frac{\mathrm{i}}{2}))X(K,x)\Omega = \hat{X}(K,-x)\Omega.$$

(ii) For every y with $K+y \in W^l$ and $X(K,0) \in \mathcal{A}_l^u$ there exists an element $\tilde{X}(K,0) \in \mathcal{A}_r^u$ fulfilling the relation

$$U(\Lambda(\frac{\mathrm{i}}{2}))X(K,y)\Omega = \tilde{X}(K,-y)\Omega.$$

Proof: It is sufficient to show (i). The second statement follows by symmetry. Let K be the double–cone $K_{-a,a}$ with $a \in W^r$. Choose an element $B'_r \in \mathcal{M}'^r_{-a-x}$ such that $\mathrm{Ad}\, R(t) B'_r$ is analytic and an element $B'_l \in \mathcal{M}'^l_{a-x}$ such that $\mathrm{Ad}\, L(t) B'_l$ is analytic. These elements are $*$–strongly dense in the respective algebras. For $X(K,0) \in \mathcal{A}^u_r$ we look at the expression

$$(B'_r B'_l \Omega, U(\Lambda(-\frac{\mathrm{i}}{2}))X(K,x)\Omega) = (B'_r B'_l \Omega, R(-\frac{\mathrm{i}}{2}) J_r X^\dagger(K,x)\Omega).$$

By Lemma 4.1.2 $R(t)B'_l\Omega$ is analytic in $S(-\frac{1}{2},0)$. Hence we obtain

$$= (\{\mathrm{Ad}\, R(-\frac{\mathrm{i}}{2}) B'_r\} R(-\frac{\mathrm{i}}{2}) B'_l \Omega, J_r X^\dagger(K,x) J_r \Omega).$$

The operator $J_r X^\dagger(K,x) J_r$ is affiliated with $\mathcal{M}^r(K-x)$. Together with the Remark 4.1.1 this implies

$$= (R(-\frac{\mathrm{i}}{2}) B'_l \Omega, J_r X^\dagger(K,x) J_r R(\frac{\mathrm{i}}{2}) B'^*_r \Omega) = (X^\dagger(K,x) J_r R(\frac{\mathrm{i}}{2}) B'^*_r \Omega, J_r R(-\frac{\mathrm{i}}{2}) B'_l \Omega).$$

The vector $B'^*_r\Omega$ belongs to the domain of $R(\frac{\mathrm{i}}{2})$ by choice of B'_r. Hence by Eq. (4.1.7') this vector belongs also to the domain of $L(\frac{\mathrm{i}}{2})$. The other vector belongs by choice of B'_l to the domain of $L(-\frac{\mathrm{i}}{2})$. Hence Lemma 4.1.2 applies and we obtain

$$= (X^\dagger(K,x) J_l L(\frac{\mathrm{i}}{2}) B'^*_r \Omega, J_l L(-\frac{\mathrm{i}}{2}) B'_l \Omega) = (\mathrm{Ad}\, L(-\frac{\mathrm{i}}{2}) B'_l \Omega, J_l X^\dagger(K,x) J_l L(\frac{\mathrm{i}}{2}) B'^*_r \Omega).$$

By the Remark 4.1.1 we find

$$= (\Omega, J_l X^\dagger(K,x) J_l L(\frac{\mathrm{i}}{2}) B'^*_l B'^*_r \Omega) = (J_l X(K,x)\Omega, L(\frac{\mathrm{i}}{2}) B'^*_l B'^*_r \Omega).$$

Since the translations commute with $L(t)$ it follows by Prop. 4.1.3 and by the definition of $\mathcal{A}^u_r$ that the vector $J_l X(K,x)\Omega$ belongs to the domain of definition of $L(\frac{\mathrm{i}}{2})$ and hence

$$= (L(\frac{\mathrm{i}}{2}) J_l X(K,x)\Omega, B'^*_l B'^*_r \Omega) = (J_l L(-\frac{\mathrm{i}}{2}) X(K,x)\Omega, B'^*_l B'^*_r \Omega).$$

From this transformation we obtain

$$(\sum_i B'^i_r B'^i_l \Omega, U(\Lambda(-\frac{\mathrm{i}}{2}))X(K,x)\Omega) = (J_l L(-\frac{\mathrm{i}}{2}) X(K,x)\Omega, \{\sum_i B'^i_r B'^i_l\}^* \Omega).$$

Since the two vectors $U(\Lambda(-\frac{\mathrm{i}}{2}))X(K,x)\Omega$ and $J_l L(-\frac{\mathrm{i}}{2})X(K,x)\Omega$ are well defined we can pass to the $*$–strong closure of the sums and obtain

$$(A\Omega, U(\Lambda(-\frac{\mathrm{i}}{2}))X(K,x)\Omega) = (J_l L(-\frac{\mathrm{i}}{2}) X(K,x)\Omega, A^*\Omega), \ \forall A \in \mathcal{M}'(K-x).$$

From this we conclude that the two vectors $U(\Lambda(-\frac{i}{2}))X(K,x)\Omega$ and $J_l L(-\frac{i}{2})(K,x)\Omega$ belong to the domain of definition of the Tomita conjugation S_{K-x} and satisfy

$$S_{K-x}U(\Lambda(-\frac{i}{2}))X(K,x)\Omega = J_l L(-\frac{i}{2})X(K,x)\Omega.$$

Hence there exists an operator $\hat{X}(K,-x)$ affiliated with $\mathcal{M}(K-x)$ such that (see e.g. [BR79] Prop. 2.5.9)

$$U(\Lambda(-\frac{i}{2}))X(K,x)\Omega = \hat{X}(K,-x)\Omega \tag{4.1.8}$$

holds. It only remains to show that this operator belongs to $\mathcal{A}_l^u$. From equation (4.1.8) we see that $U(\Lambda(t))\hat{X}(K,-x)\Omega$ has a bounded analytic extension into $S(0,\frac{1}{2})$. From the relation

$$\hat{X}^\dagger(K,-x)\Omega = L(\frac{i}{2})J_l X(K,x)\Omega \tag{4.1.9}$$

we see that the vector $\hat{X}^\dagger(K,-x)\Omega$ belongs to the domain of definition of $L(-\frac{i}{2})$ and hence, as in the proof of Lemma 4.1.4, we conclude that $\hat{X}(K,x)\Omega$ belongs to the domain of $U(\Lambda(-\frac{i}{2}))$. This shows that $\hat{X}(K,-x)$ belongs to $\mathcal{A}_l^u$. □

Before showing the wedge–duality we need an analysis of the map established in Thm. 4.1.7. We start with some notations.

4.1.8 Definition:

For $K+x \in W^r$ and $X \in \mathcal{A}_r^u$ define $\rho^r(X)(K,-x)$ by the formula

$$\rho^r(X)(K,-x)\Omega = U(\Lambda(-\frac{i}{2}))X(K,x)\Omega$$

where the left side is defined by the operator introduced in Thm. 4.1.7. This implies that ρ^r maps $\mathcal{A}_r^u$ into $\mathcal{A}_l^u$.

The map ρ^l is defined correspondingly on the set $\mathcal{A}_l^u$ and it maps it into $\mathcal{A}_r^u$.

In the definition of the operation $X(K,x) \to \hat{X}(K,-x)$ appears the vector x such that $K+x \subset W^r$. First we have to show that the definition of ρ^r is independent of the choice of x.

4.1.9 Lemma:

The map

$$\rho^r(X)(K,0) := \mathrm{Ad}\,T(x)\{\rho^r(X)(K,-x)\}, \quad K+x \subset W^r$$

is independent of the choice of x (provided $K+x \subset W^r$).

The corresponding result holds for ρ^l.

Proof: Assume x_1, x_2 are such that $x_i + K \subset W^r$. Then we can find x_3 such that $x_3 + K \subset W^r$ and $x_i \in x_3 + W^r$ $i = 1,2$. In this situation we obtain $X(K,x_i) =$

$\mathrm{Ad}\, T(x_i - x_3)X(K, x_3)$. Writing ρ_i^r, if we apply ρ^r to an element $X(K, x_i)$, we obtain by using the commutation relations between the translations and the Lorentz transformations

$$\begin{aligned} T(x_i)\rho_i^r(X)(K, -x_i)\Omega &= T(x_i)U(\Lambda(-\frac{\mathrm{i}}{2}))X(K, x_i)\Omega \\ &= T(x_i)U(\Lambda(-\frac{\mathrm{i}}{2}))T(x_i - x_3)X(K, x_3)\Omega \\ &= T(x_3)U(\Lambda(-\frac{\mathrm{i}}{2}))X(K, x_3) = T(x_3)\rho_3^r(X)(K, -x_3). \end{aligned}$$

This calculation shows that ρ_1^r and ρ_2^r coincide. The same transformation holds if we interchange right and left. □

4.1.10 Remark:

The map ρ^r is linear and interchanges the order of factors provided the products appearing in the formulas are defined. The same result is true for ρ^l.

Since this result will not be used later we will not give the proof. It has only been stated in order to show that we do not conflict with the algebraic structure. The operation ρ is closely related with the CPT–operation but we will not introduce such operation. Next we want to look at the adjoint and the inverse.

4.1.11 Lemma:

1) *ρ^l is the inverse of ρ^r i.e.:*

$$\rho^r(\rho^l(X))(K, 0) = X(K, 0).$$

2) *ρ^l is the adjoint of ρ^r i.e.:*

$$\{\rho^r(X)(K, 0)\}^\dagger = \rho^l(X^\dagger)(K, 0).$$

Proof: Choose x, y in such a way that $x + K \subset W^r$ and $y + K \subset W^l$ and $x + y \in W^r$. Take $X(K, 0) \in \mathcal{A}_l^u$ and using the fact that $\rho^l X(K, 0)$ belongs to $\mathcal{A}_r^u$ we obtain

$$\begin{aligned} \rho^r\Big(\rho^l(X)\Big)(K, 0)\Omega &= T(x)U(\Lambda(-\frac{\mathrm{i}}{2}))T(x)\rho^l(X)(K, 0)\Omega \\ &= T(x)U(\Lambda(-\frac{\mathrm{i}}{2}))T(x)T(y)U(\Lambda(\frac{\mathrm{i}}{2}))T(y)X(K, 0)\Omega \\ &= T(-y)U(\Lambda(-\frac{\mathrm{i}}{2}))U(\Lambda(\frac{\mathrm{i}}{2}))T(y)X(k, 0)\Omega = X(K, 0)\Omega. \end{aligned}$$

Since the first and the last expression is affiliated with $\mathcal{M}(K)$ we obtain for $B_1, B_2 \in \mathcal{M}'(K)$ the relation

$$(B_1\Omega, \rho^r\Big(\rho^l(X)\Big)(K, 0)B_2\Omega) = (B_1\Omega, X(K, 0)B_2\Omega).$$

For the left wedge the result is obtained in the same manner. This shows the first statement. For the second statement let us choose x such that $x + K \subset W^r$. Then we obtain

$$\{\rho^r(X)(K,0)\}^\dagger\Omega = \mathrm{Ad}\, T(x)\{\rho^r(X)(K,-x)\}^\dagger\Omega = T(x)\{\rho^r(X)(K,-x)\}^\dagger\Omega.$$

Using Eq. (4.1.9) this can be written as

$$= T(x)L(\frac{\mathrm{i}}{2})J_l X(K,x)\Omega = T(x)L(\frac{\mathrm{i}}{2})J_l T(2x)X(K,-x)\Omega = T(-x)L(\frac{\mathrm{i}}{2})J_l X(K,-x)\Omega.$$

From Def. 4.1.5 we know that $X^\dagger(K,-x)$ belongs to $\mathcal{A}^u_l$. Hence we obtain

$$= T(-x)L(\frac{\mathrm{i}}{2}){\Delta'}_l^{-\frac{1}{2}}X^\dagger(K,-x)\Omega = T(-x)U(\Lambda(\frac{\mathrm{i}}{2}))X^\dagger(K,-x)\Omega$$
$$= T(-x)\rho^l(X^\dagger)(K,x)\Omega = \rho^l(X^\dagger)(K,0)\Omega.$$

Since the first and the last expression are affiliated with $\mathcal{M}(K)$ we obtain for $B_1, B_2 \in \mathcal{M}'(K)$ the relation

$$(B_1\Omega, \{\rho^r(X)(K,0)\}^\dagger B_2\Omega) = (B_1\Omega, \rho^l(X^\dagger)(K,0)B_2\Omega).$$

This shows the lemma. □

We have established a map from a family of operators affiliated with $\mathcal{M}^r_0$ to a family of operators affiliated with $\mathcal{M}^l_0$ and also the inverse of this map. From this result one could derive the wedge–duality if one would know the invariance of these families under the modular automorphism groups σ^t_r and σ^t_l repsectively. We do not know this because σ^t_r maps elements affiliated with the algebra $\mathcal{M}(K), K \subset W^r$ to an element affiliated with $\mathcal{M}^r(K_{\Lambda(t)})$. But for elements belonging to the latter algebra Thm. 4.1.7 has not been proved.

For this reason we will try to "dualize" Thm. 4.1.7 and establish a map from a dense set of ${\mathcal{M}'}^l_0$ to ${\mathcal{M}'}^r_0$. If these sets are invariant under the modular action of the corresponding algebras it will be possible to show that the modular groups are the same. This program can only work with some density requirements. We start with some notation.

4.1.12 Definition:

We introduce the following sets:

$$\begin{aligned}
\mathcal{D}_r :&= \{X(K,x)\Omega; X(K,0) \in \mathcal{A}^u_r,\ K + x \subset W_r\} \\
\mathcal{D}^\dagger_r :&= \{X^\dagger(K,x)\Omega; X(K,0) \in \mathcal{A}^u_r,\ K + x \subset W_r\} \\
\mathcal{D}_l :&= \{X(K,x)\Omega; X(K,0) \in \mathcal{A}^u_l,\ K + x \subset W_l\} \\
\mathcal{D}^\dagger_l :&= \{X^\dagger(K,x)\Omega; X(K,0) \in \mathcal{A}^u_l,\ K + x \subset W_l\}
\end{aligned}$$

In the sequel we will need the sets $\mathcal{D}_r$ and $\mathcal{D}_l$ to define the modular groups uniquely. We put this in the form of a

4.1.13 Requirement:

We introduce the following conditions:

(a) The sets $\mathcal{D}_r$ and $\mathcal{D}_r^\dagger$ are both a core for Δ_r.

(b) The sets $\mathcal{D}_l$ and $\mathcal{D}_l^\dagger$ are both a core for Δ_r^{-1}.

Before coming to the duality result we need to introduce two more sets. They will be the objects of the investigations.

4.1.14 Definition:

(a) By $\mathcal{B}_r^u$ we denote the set of linear operators X affiliated with ${\mathcal{M}'}_0^l$ fulfilling the properties

(i) $L(t)X\Omega$ has in t an analytic continuation into the strip $S(-\frac{1}{2},0)$ with continuous boundary–values at $\Im m\, t = -\frac{1}{2}$ and

(ii) $L(t)X^\dagger\Omega$ has in t an analytic continuation into the strip $S(0,\frac{1}{2})$ with continuous boundary–values at $\Im m\, t = \frac{1}{2}$.

(b) By $\mathcal{B}_l^u$ we denote the set of linear operators X affiliated with ${\mathcal{M}'}_0^r$ fulfilling the properties

(i) $R(t)X\Omega$ has in t an analytic continuation into the strip $S(0,\frac{1}{2})$ with continuous boundary–values at $\Im m\, t = -\frac{1}{2}$ and

(ii) $R(t)X^\dagger\Omega$ has in t an analytic continuation into the strip $S(-\frac{1}{2},0)$ with continuous boundary–values at $\Im m\, t = \frac{1}{2}$.

Note that ${\mathcal{M}'}_0^r$ is invariant under the action of $R(t)$ and hence there are many elements in $\mathcal{B}_l^u$. First we look at the action of the Lorentz group.

4.1.15 Lemma:

Suppose $X \in \mathcal{B}_r^u$ then the vector–function

$$U(\Lambda(t))X\Omega$$

can be analytically continued into $S(-\frac{1}{2},0)$ and has continuous boundary values at $\Im m\, t = -\frac{1}{2}$.

If $X \in \mathcal{B}_l^u$ then we obtain an analytic continuation into $S(0,\frac{1}{2})$ again with continuous boundary values.

Proof: We look at the function

$$F(t,s) := {\Delta'}_l^{it} L(s)X\Omega.$$

By definition of $\mathcal{B}_r^u$ this function has in s an analytic continuation into the strip $S(-\frac{1}{2},0)$ and we obtain

$$F(t, s-\frac{\mathrm{i}}{2}) = L(s-\frac{\mathrm{i}}{2}){\Delta'}_l^{it} X\Omega.$$

Since $X^\dagger\Omega$ is in the domain of $L(\frac{\mathrm{i}}{2})$ i.e., $J_l X^\dagger\Omega$ in the domain of $L(-\frac{\mathrm{i}}{2})$ this function has in t an analytic continuation into $S(-\frac{1}{2},0)$ and we find

$$F(t-\frac{\mathrm{i}}{2}, s-\frac{\mathrm{i}}{2}) = L(s-\frac{\mathrm{i}}{2})\Delta_l^{it} J_l X^\dagger\Omega.$$

Because of Eq. (4.1.5) we obtain by Lemma 4.1.2 the stated result. If X belongs to $\mathcal{B}_l^u$ then the result follows by symmetry. □

Before coming to the duality result we need one more preparation.

4.1.16 Lemma:

Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two commuting algebras with common cyclic and separating vector Ω. Assume X_1 is affiliated with $\mathcal{M}_1$ and X_2 with $\mathcal{M}_2$ then one finds

$$(X_1^\dagger\Omega, X_2\Omega) = (X_2^\dagger\Omega, X_1\Omega).$$

Proof: Let $\overline{X}_1$ be the closure of X_1 it can be written as

$$\overline{X}_1 = V|\overline{X}_1| = V\int_0^\infty \lambda dE_\lambda$$

with V and E_λ belonging to $\mathcal{M}_1$. The requirement that Ω is in the domain of X and of $X^\dagger$ implies that both Ω and $V^*\Omega$ belong to the domain of $|\overline{X}_1|$. Hence to a given $\epsilon > 0$ exists a λ_0 such that

$$\|(X_1 - V|\overline{X}_1|E_{\lambda_0})\Omega\| \le \epsilon, \qquad \|(X_1^\dagger - E_{\lambda_0}|\overline{X}_1|V^*)\Omega\| \le \epsilon.$$

Since $|\overline{X}_1|E_{\lambda_0}$ belongs to $\mathcal{M}_1$ we obtain the estimate

$$|(X_2^\dagger\Omega, X_1\Omega) - (X_1^\dagger\Omega, X_2\Omega)| \le \epsilon(\|X_2^\dagger\Omega\| + \|X_2\Omega\|).$$

Since $X_2\Omega$ and $X_2^\dagger\Omega$ are fixed vectors we obtain the lemma. □

A similar result is the following

4.1.17 Proposition:

Assume X is affiliated with ${\mathcal{M}'}_0^l$ such that $L(t)X\Omega$ has an analytic continuation into $S(-\frac{1}{2}, 0)$ with continuous boundary values and $L(t)X^\dagger\Omega$ can be continued into $S(0, \frac{1}{2})$ again with continuous boundary values then for $B \in \mathcal{M}_0^l$ one finds:

$$(L(-\frac{\mathrm{i}}{2})X\Omega, B^*\Omega) = (B\Omega, L(\frac{\mathrm{i}}{2})X^\dagger\Omega).$$

A corresponding relation holds if we interchange left with right.

Proof: Since $\mathcal{M}_0^l$ is invariant under the action of $L(t)$ there exists a $*$–strong subalgebra of $\mathcal{M}_0^l$ of analytic elements. Let B belong to this set then we find

$$(L(-\frac{\mathrm{i}}{2})X\Omega, B^*\Omega) = (X\Omega, L(-\frac{\mathrm{i}}{2})B^*L(\frac{\mathrm{i}}{2})\Omega) =$$
$$(L(\frac{\mathrm{i}}{2})B\Omega, X^\dagger\Omega) = (B\Omega, L(\frac{\mathrm{i}}{2})X^\dagger\Omega).$$

Since the two vectors $L(-\frac{i}{2})X\Omega$ and $L(\frac{i}{2})X^\dagger\Omega$ are fixed we can go to the $*$–strong closure of the B's and obtain the stated result. The second statement follows by symmetry. $\square$

Using these preparations we obtain

4.1.18 Proposition:

Assume that the Requirement 4.1.13 holds then
(a) to every $X \in \mathcal{B}_r^u$ exists an element $\mu_r(X)$ affiliated with ${\mathcal{M}'}_0^r$ such that the following relation holds:

$$U(\Lambda(-\frac{i}{2}))X\Omega = \mu_r(X)\Omega.$$

(b) To every $Y \in \mathcal{B}_l^u$ exists an element $\mu_l(Y)$ affiliated with ${\mathcal{M}'}_0^l$, so that

$$U(\Lambda(\frac{i}{2}))Y\Omega = \mu_l(Y)\Omega$$

holds.

Proof: By symmetry it is sufficient to show (a). Let $X(K,0) \in \mathcal{A}_r^u$, $x + K \subset W^r$ and $Y \in \mathcal{B}_r^u$ and look at the expression

$$(U(\Lambda(-\frac{i}{2}))Y\Omega, X(K,x)\Omega) = (L(-\frac{i}{2}){\Delta'}_l^{\frac{1}{2}}Y\Omega, X(K,x)\Omega)$$
$$(L(-\frac{i}{2})J_lY^\dagger\Omega, X(K,x)\Omega) = (J_lX(K,x)\Omega, L(\frac{i}{2})Y^\dagger\Omega).$$

Combining Lemma 4.1.16 with Prop 4.1.17 we can commute the operators and we get

$$(L(-\frac{i}{2})Y\Omega, J_lX^\dagger(K,x)\Omega) = (X^\dagger(K,x)\Omega, J_lL(\frac{i}{2})Y\Omega).$$

Since by Requirement 4.1.13 both $\mathcal{D}_r$ and $\mathcal{D}_r^\dagger$ are a core for $\Delta_r^{\frac{1}{2}}$ we see that the two vectors $U(\Lambda(-\frac{i}{2}))Y\Omega$ and $J_lL(\frac{i}{2})Y\Omega$ are in the domain of the Tomita conjugation of ${\mathcal{M}'}_0^r$ and fulfil $SU(\Lambda(-\frac{i}{2}))Y\Omega = J_lL(-\frac{i}{2})X\Omega$. This implies (see [BR79] Prop. 2.5.9) that there exists an operator $\mu_r(Y)$ affiliated with ${\mathcal{M}'}_0^r$ fulfilling the stated relation. $\square$

Next we look at the properties of the maps μ_r and μ_l.

4.1.19 Lemma:

(a) For $X \in \mathcal{B}_r^u$ we obtain $\mu_r(X) \in \mathcal{B}_l^u$.
(b) For $Y \in \mathcal{B}_l^u$ we have $\mu_l(Y) \in \mathcal{B}_r^u$.
(c) The maps μ_r and μ_l are the inverse of each other and hence μ_r maps $\mathcal{B}_r^u$ onto $\mathcal{B}_l^u$ and vice versa.

Proof: (a) The relations

$$\mu_r(X)\Omega = U(\Lambda(-\frac{i}{2}))X\Omega = L(-\frac{i}{2})J_lX^\dagger\Omega$$

show that $\mu(X)\Omega$ is in the domain of $L(\frac{\mathrm{i}}{2})$. Next we get that

$$\{\mu_r(X)\}^\dagger\Omega = J_l L(-\frac{\mathrm{i}}{2})X\Omega = L(\frac{\mathrm{i}}{2})J_l X\Omega$$

belongs to the domain of $L(-\frac{\mathrm{i}}{2})$ and hence $\mu_r(X)$ belongs to $\mathcal{B}_l^u$. Statement (b) is shown in the same way. From $U(\Lambda(\frac{\mathrm{i}}{2}))\mu_r(X)\Omega = U(\Lambda(\frac{\mathrm{i}}{2}))U(\Lambda(-\frac{\mathrm{i}}{2}))X\Omega$ we see that $\mu_l(\mu_r(X)) = X$ holds, which shows (c). □

As further preparation we need

4.1.20 Lemma:

Let $\mathcal{N} \subset \mathcal{M}$ be two von Neumann algebras with common cyclic and separating vector Ω. If $\Delta_{\mathcal{N}}$ and $\Delta_{\mathcal{M}}$ commute then the two von Neumann algebras coincide.

Proof: We look at the expression $D(t) = \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}$. By Lemma 2.3.2 this can be analytically continued into $S(0, \frac{1}{2})$ as a bounded operator–valued function. At the upper boundary one gets

$$D(\frac{\mathrm{i}}{2}) = \Delta_{\mathcal{M}}^{\frac{1}{2}}\Delta_{\mathcal{N}}^{-\frac{1}{2}} = J_{\mathcal{M}}J_{\mathcal{N}}.$$

Since, by assumption, the two modular operators commute it follows that also the two modular conjugations commute. Next we look at the functions

$$\begin{aligned} F^+(t) &= (\Omega, A\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}B\Omega) \\ F^-(t) &= (\Omega, B\Delta_{\mathcal{N}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{\mathrm{i}t}A\Omega) \end{aligned}$$

with $A \in \mathcal{M}'$ and $B \in \mathcal{N}$. By choice of A and B the two functions coincide for real t. Next we look at the boundaries and obtain by the commutativity of the J's

$$\begin{aligned} F^+(t + \frac{\mathrm{i}}{2}) &= (\Omega, A\Delta_{\mathcal{M}}^{\frac{1}{2}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta_{\mathcal{N}}^{\mathrm{i}t}\Delta_{\mathcal{N}}^{-\frac{1}{2}}B\Omega) \\ &= (J_{\mathcal{M}}\Delta_{\mathcal{M}}^{\mathrm{i}t}B\Omega, J_{\mathcal{N}}\Delta_{\mathcal{N}}^{\mathrm{i}t}B^*\Omega) \\ &= (J_{\mathcal{M}}\Delta_{\mathcal{M}}^{\mathrm{i}t}A\Omega, J_{\mathcal{M}}\Delta_{\mathcal{N}}^{\mathrm{i}t}B^*\Omega). \end{aligned}$$

Notice that $\mathrm{Ad}\, J_{\mathcal{N}}(\sigma_{\mathcal{M}}^t(A))$ belongs to $\mathcal{N}$ and $\mathrm{Ad}\, J_{\mathcal{M}}(\sigma_{\mathcal{N}}^t(B^*))$ belongs to $\mathcal{M}'$ so that these operators commute. On the other hand we get

$$\begin{aligned} F^-(t - \frac{\mathrm{i}}{2}) &= (\Omega, B\Delta_{\mathcal{N}}^{-\frac{1}{2}}\Delta_{\mathcal{N}}^{-\mathrm{i}t}\Delta_{\mathcal{M}}^{\frac{1}{2}}\Delta_{\mathcal{M}}^{\mathrm{i}t}B\Omega) \\ &= (J_{\mathcal{N}}\Delta_{\mathcal{N}}^{\mathrm{i}t}B\Omega, J_{\mathcal{M}}\Delta_{\mathcal{M}}^{\mathrm{i}t}B^*\Omega) \\ &= (J_{\mathcal{M}}\Delta_{\mathcal{N}}^{\mathrm{i}t}A^*\Omega, J_{\mathcal{N}}\Delta_{\mathcal{M}}^{\mathrm{i}t}B\Omega). \end{aligned}$$

From this computation we obtain that $F^+(t + \frac{\mathrm{i}}{2})$ and $F^-(t - \frac{\mathrm{i}}{2})$ coincide. Hence we obtain a bounded periodic function which, therefore, is constant. This implies by the cyclicity of Ω that $\Delta_{\mathcal{M}}^{\mathrm{i}t}$ and $\Delta_{\mathcal{N}}^{\mathrm{i}t}$ coincide. But this implies $\mathcal{N} = \mathcal{M}$ (see e.g. [KR86] Theorem 9.2.36) □

After all these preparations we are ready for the main result of this subsection.

4.1.21 Theorem:

Assume that the Requirement 4.1.13 *is satisfied. Then the theory fulfils wedge–duality.*

Proof: Since the modular group $\Delta_l^{\mathrm{i}t}$ commutes with the Lorentz transformations and with $L(t)$ it follows that $\mathcal{B}_r^u$ is invariant under the modular group σ_l^t. Therefore, we look at the expressions

$$\mu_r(\sigma_l^t(X))\Omega = U(\Lambda(-\frac{\mathrm{i}}{2}))\Delta_l^{\mathrm{i}t}X\Omega = \Delta_l^{\mathrm{i}t}U(\Lambda(-\frac{\mathrm{i}}{2}))X\Omega = \Delta_l^{\mathrm{i}t}\mu_r(X)\Omega$$

and

$$\{\mu_r(\sigma_l^t(X)\}^\dagger\Omega = J_lL(-\frac{\mathrm{i}}{2})\Delta'^{\mathrm{i}t}_l X\Omega = \Delta'^{\mathrm{i}t}_l J_lL(-\frac{\mathrm{i}}{2})X\Omega = \Delta'^{\mathrm{i}t}_l\{\mu_r(X)\}^\dagger\Omega.$$

The operator μ_r maps $\mathcal{B}_r^u$ onto $\mathcal{B}_l^u$, so that we obtain

$$\Delta_r^{-\frac{1}{2}}\Delta_l^{\mathrm{i}t}\mu_r(X)\Omega = \Delta_l^{\mathrm{i}t}\Delta_r^{-\frac{1}{2}}\mu_r(X)\Omega.$$

Since the algebra $\mathcal{M}'^r_0$ is invariant under $L(t)$ we see that the elements, which are analytic with respect to the action of $L(t)$, are $*$–strongly dense in $\mathcal{M}'^r_0$. But this implies that $\mathcal{B}_l^u\Omega$ is dense in $\mathcal{H}$ and is a core for the generator of $\Delta_r^{\mathrm{i}t}$. The same holds if we interchange right and left. Since it is also invariant under the action of $\Delta'^{-\frac{1}{2}}_l$ we conclude that the operators $\Delta_r^{-\frac{1}{2}}$ and $\Delta'^{-\frac{1}{2}}_l$ commute and the theorem is a consequence of Lemma 4.1.20. □

Now we are prepared to show the main result.

4.1.22 Theorem:

Given a Lorentz covariant QFTLO in the vacuum sector. Then the following conditions are equivalent:

(1) *The theory fulfils wedge–duality.*
(2) *The set* $\{A(K,x)\}$, *such that*
 (α) $K + x \subset W^r$,
 (β) $U(\Lambda(t))A(K,x)\Omega$ *has a bounded analytic continuation into the strip* $S(-\frac{1}{2},0)$ *with continuous boundary values,*
 (γ) $U(\Lambda(t))A^*(K,-x)\Omega$ *has a bounded analytic continuation into the strip* $S(0,\frac{1}{2})$ *with continuous boundary value,*
 is $*$–strong dense in $\mathcal{M}_0^r$.

Proof: Assume first that we have wedge–duality. Then we have only one modular group and we can write $U(\Lambda(t)) = F(t)\Delta^{\mathrm{i}t}$ where $F(t)$ is a continuous representation of the one–parametric group mapping every $\mathcal{M}(K)$ into itself. Hence there exists a $*$–strong dense subalgebra $\mathcal{M}^{anal}(K) \subset \mathcal{M}(K)$ of elements entire analytic in the action of $F(t)$. Let now $K + x \subset W^r$ and $A(K,x)$ be such an analytic element. Then $U(\Lambda(t))A(K,x)\Omega =$

$F(t)\Delta^{\mathrm{i}t}A(K,x)\Omega$ can be analytically continued into $S(-\frac{\mathrm{i}}{2},0)$. If we look at the operator $A^*(K,-x)$ we obtain the corresponding result for the opposite wedge. Hence (2) is fulfilled.

Conversely if (2) is satisfied then also the Requirement 4.1.13 is satisfied because the set $\mathcal{A}_r$ is Lorentz invariant and $*$–strong dense in $\mathcal{M}_0^r$ and hence also a core for $\Delta_r^{\frac{1}{2}}$. From this follows by Thm. 4.1.21 that the wedge–duality is fulfilled. □

The content of this subsection is taken from [Bch96].

4.2) *The reality condition and the Bisognano–Wichmann property*

In the discussion at the beginning of this section we saw that we must solve two problems before we can prove the PCT–theorem. The first was the wedge–duality, corresponding to the properties of the index space of Wightman fields. The second was the reality condition implying that every Wightman field has its conjugate complex partner. In analogy we pose:

4.2.1 Reality condition:

We say a Poincaré covariant theory of local observables in the vacuum sector, which satisfies the wedge duality, fulfils the reality condition if:

(i) Every $A(K,0)\in\mathcal{A}_r\cap\mathcal{A}_l$ and every x such that $K+x\subset W^r$ fulfils the relation

$$\widehat{A^*}(K,P_Wx)=\{\hat{A}(K,P_Wx)\}^*.$$

(ii) Ω is cyclic for the set

$$\{A(K,x);A(K,0)\in\mathcal{A}_r\cap\mathcal{A}_l,\ \text{and}\ K+x\subset W^r\}.$$

With this notation we obtain:

4.2.2 Theorem:

In a representation of a Poincaré covariant theory of local observables in the vacuum sector the modular group associated with the algebra of any wedge coincides with the corresponding Lorentz boosts iff the theory fulfils wedge duality and the above reality condition with respect to the Lorentz transformations.

Proof: If we know that $U(\Lambda(t))$ and $\Delta_W^{\mathrm{i}t}$ coincide then by Thm. 4.1.21 one has wedge duality. Moreover, the reality condition is fulfilled because for every $A(K,x)\in\mathcal{M}(W^r)$ one has

$$\widehat{A^*}(K,x)\Omega=U(\Lambda(-\frac{\mathrm{i}}{2}))A^*\Omega=\Delta_W^{\frac{1}{2}}A^*\Omega=JAJ\Omega,$$

and

$$\Delta_W^{\frac{1}{2}}\hat{A}(K,x)\Omega=JA^*J\Omega=\{JAJ\}^*\Omega.$$

Hence the reality condition is fulfilled.

Next assume wedge duality and the reality condition. Let $A(K,0) \in \mathcal{A}_r \cap \mathcal{A}_l$ and $A(K,x) \in \mathcal{M}(W^r)$. Take an element $B \in \mathcal{M}(W^l)$ and look at the matrix elements

$$F^+(s,t) = (\Omega, B\Delta_r^{\mathrm{i}s} U(\Lambda(t))A(K,x)\Omega)$$
$$F^-(s,t) = (\Omega, A(K,x)U(\Lambda(-t))\Delta_r^{-\mathrm{i}s} B\Omega)$$

Bringing B and $\Delta_r^{\mathrm{i}s}$ to the left side we see that $F^+(s,t)$ can be analytically continued into the tube domain $(s,t) \in S(0,\frac{1}{2}) \times S(-\frac{1}{2},0)$. Correspondingly $F^-(s,t)$ has an analytic continuation into the domain $(s,t) \in S(-\frac{1}{2},0) \times S(0,\frac{1}{2})$. Next we look at the coincidence domains. Since $A(K,x) \in \mathcal{M}(W^r)$ and $B \in \mathcal{M}(W^l)$ we have by wedge duality $F^+(s,t) = F^-(s,t)$ for all $(s,t) \in \mathbb{R}^2$. Next we look at $F^+(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2})$. The modular theory yields $\langle \Omega B\Delta_r^{\mathrm{i}s-\frac{1}{2}} = \langle \Omega J_r B^* J_r \Delta_r^{\mathrm{i}s}$. By the above result about the analytic continuation of $U(\Lambda(t))A(K,x)\Omega$ we know that there exists an element $\hat{A}(K,P_W x) \in \mathcal{M}(W^l)$ with $U(\Lambda(-\frac{\mathrm{i}}{2}))A(K,x)\Omega = \hat{A}(K,P_W x)\Omega$. Hence we find:

$$F^+(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) = (\Omega, J_r B^* J_r U(\Lambda(t))\hat{A}(K,P_W x)\Omega).$$

Next we want to compute $F^-(s-\frac{\mathrm{i}}{2}, t+\frac{\mathrm{i}}{2})$. We start with

$$F^-(s,t) = (U(\Lambda(t))A^*(K,x)\Omega, \Delta_r^{-\mathrm{i}s} B\Omega).$$

From this we obtain:

$$\begin{aligned} F^-(s-\frac{\mathrm{i}}{2}, t+\frac{\mathrm{i}}{2}) &= (U(\Lambda(t-\frac{\mathrm{i}}{2})A^*(K,x)\Omega, \Delta_r^{-\mathrm{i}s-\frac{1}{2}} B\Omega) \\ &= (U(\Lambda(t))\widehat{A^*}(K,P_W x)\Omega, \Delta^{-\mathrm{i}s} J_r B^* J_r \Omega). \end{aligned}$$

Because of the reality condition we find:

$$= (\Omega, \hat{A}(K,P_W x)U(\Lambda(-t))\Delta_r^{-\mathrm{i}s} J_r B^* J_r \Omega).$$

By the wedge duality we obtain $J_r B^* J_r \in \mathcal{M}(W^r)$. Since $\hat{A}(K,P_W x)$ belongs to $\mathcal{M}(W^l)$ we obtain

$$F^+(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) = F^-(s-\frac{\mathrm{i}}{2}, t+\frac{\mathrm{i}}{2}).$$

By both coincidences and the edge of the wedge theorem, Thm. 1.4.1, we obtain a bounded periodic function $F(s,t) = F(s-\mathrm{i}, t+\mathrm{i})$. Since bounded entire functions are constant we find

$$F(s,-s) = const = F(0,0),$$
$$(\Omega, B\Delta^{\mathrm{i}s} U(\Lambda(-s))A(K,x)\Omega) = (\Omega, BA(K,x)\Omega).$$

Since $\mathcal{M}(W^l)\Omega$ and $\{A(K,x)\Omega\}$ are dense in $\mathcal{H}$, where $A(K,0)$ fulfils the reality condition, we obtain $\Delta_r^{\mathrm{i}s} U(\Lambda(-s)) = \mathbb{1}$. □

4.3) *The PCT–theorem*

Now we are prepared for the proof of the PCT-theorem under the assumption that the wedge–duality and the reality condition are fulfilled. Starting from the Ansatz Eq. (4.0.1) one has to solve two problems:

1) Since Θ shall be a local transformation, also J_W must be local. Since the map $A\Omega \to A^*\Omega$ is local, and since by Thm. 4.2.2 $\Delta_W^{1/2}$ and $U(\Lambda_W(-\frac{\mathrm{i}}{2})$ coincide, we know that the product

$$S_W = J_W \Delta_W^{1/2} = J_W U(\Lambda_W(-\frac{\mathrm{i}}{2})) \tag{4.3.1}$$

acts local. Therefore, J_W and $U(\Lambda_W(-\frac{\mathrm{i}}{2}))$ must act local at the same time. The answer to this question is closely related to the next one.

2) The operator product $J_W U(R_W(\pi))$ shall be independent of the choice of the wedge W. Using Eq. (4.3.1) we obtain $J_W = U(\Lambda_W(-\frac{\mathrm{i}}{2}))S_W$ and consequently

$$J_W U(R_W(\pi)) = U(\Lambda_W(-\frac{\mathrm{i}}{2}))U(R_W(\pi))S_W, \tag{4.3.2}$$

where we have used the fact that $U(R_W(\pi))$ maps the algebra $\mathcal{M}(W)$ onto itself, which implies, that S_W and $U(R_W(\pi))$ commute. We will apply the expression (4.3.2) to vectors of the form $A(K,x)\Omega$ with $K+x \subset W$. Therefore, problem 2) is solved if $U(\Lambda_W(-\frac{\mathrm{i}}{2}))U(R_W(\pi))A^*(K,x)\Omega$ is independent of W. (As long as $K+x \subset W$.) The product $U(\Lambda_W(-\frac{\mathrm{i}}{2}))U(R_W(\pi))$ is nothing else but the element $-\mathbb{1}$. Since we get to $U(\Lambda_W(-\frac{\mathrm{i}}{2}))U(R_W(\pi))A(K,x)\Omega$ by analytic continuation, we have to make sure that for different W the continuation gives a unique answer.

We start with the uniqueness problem because its answer is needed for the solution of the locality–question. For simplicity of notation we restrict ourselves to the four–dimensional Minkowski space. In this case the Lorentz group is six–dimensional. First, with help of the Malgrange–Zerner theorem 1.4.2 we will construct a function on the complex Lorentz group. The points $U(\Lambda_W(-\frac{\mathrm{i}}{2}))$ will be points on the boundary of the domain which we construct. Therefore, we must convince ourselves that $U(\Lambda)$ is single valued on that domain.

Let D be a double cone such that its closure does not contain the origin. We choose a wedge with $D \subset W$. Let G be the (connected) Lorentz group and set

$$N(D) = \{g \in G;\ D \subset gW\}. \tag{4.3.3}$$

Since W is open, $N(D)$ is open and contains the identity of the group.

4.3.1 Lemma:

There exist $g_1 = 1, g_2, ..., g_6 \in N(D)$ and $T_1, ...T_6 > 0$ such that

$$D \subset \Lambda_{g_6 W}(t_6)...\Lambda_{g_1 W}(t_1)W$$

for $|t_i| < T_i,\ i = 1, ..., 6$. The elements $g_2, ..., g_6$ can be chosen in such a way that the generators of the groups $\Lambda_{g_i W}(t_i)$ are linearly independent.

Proof: Let the neigbourhood of the identity $N_1(D)$ be a subset of $N(D)$ such that $g_1, ..., g_6 \in N_1(D)$ implies $g_1...g_6 \in N(D)$. Since $N_1(D)$ is a neighbourhood of the identity exists $g_1 = 1, g_2, ..., g_6 \in N_1(D)$ such that the generators of $\Lambda_{g_i W}(t)$ are linearly independent. Choosing T_i such that $\Lambda_{g_i W}(t_i) \in N_1(D)$ for $|t_i| < T_i$ then the statements of the lemma are fulfilled. □

With help of the last lemma we can construct an analytic function on parts of the whole complex Lorentz group $\hat{G}$.

4.3.2 Proposition:

With the assumptions and notations of the last lemma, the function

$$U(\Lambda_{g_6 W}(t_6))...U(\Lambda_W(t_1))A\Omega, \quad A \in \mathcal{M}(D), \quad D \subset W \tag{4.3.4}$$

has an analytic continuation into all t–variables. The function Eq. (4.3.4) *is the boundary value of an analytic function holomorphic in some domain in $\hat{G}$.*

Proof: For $t_6, ..., t_{i+1}, t_{i-1}, ..., t_1$ real and in their proper domain the above function can in the variable t_i be analytically extended into the strip $S(-\frac{1}{2}, 0)$. Therefore, the Malgrange–Zerner theorem, Thm. 1.4.2, implies that the product Eq. (4.3.4) has an analytic continuation in all t variables into some domain which still has to be determined. It is clear from the construction that the real function is the boundary value of the analytic continuation. □

Next we want to determine the domain of holomorphy of this function. This calculation will be done by mapping the strip $S(-\frac{1}{2}, 0)$ bi–holomorphic onto itself in such a way that the interval $|x| < T$ is mapped onto $\mathbb{R}$ and the rest of the boundary onto $-\frac{i}{2} + \mathbb{R}$. This is achieved by the transformation

$$\zeta = \frac{1}{2\pi} \log \frac{1 - \mathrm{e}^{-2\pi T}}{\mathrm{e}^{2\pi T} - 1} \frac{\mathrm{e}^{2\pi T} - \mathrm{e}^{-2\pi z}}{\mathrm{e}^{-2\pi z} - \mathrm{e}^{-2\pi T}}. \tag{4.3.5}$$

In this new variables we obtain as domain of holomorphy

$$0 > \sum_{i=1}^{6} \Im m\, \zeta_i > -\frac{1}{2}. \tag{4.3.6}$$

If the elements $g_0, ..., g_6$ are properly chosen then an interior point of the ζ variables corresponds to an interior point in the $\hat{g}$ variables.

In the ζ–variable the domain (4.3.6) is convex and hence simply connected. Since the transformation (4.3.5) is bi–holomorphic, it follows that also the image in the t–variables is simply connected. Hence there are no monodromy problems in these variables. Therefore, we have to show that the inverse transformation of (4.3.5) sends the boundary points

$$\sum_{i=1}^{6} \Im m\, \zeta_i = -\frac{1}{2} \quad \text{and} \quad \Re e\, \zeta_i = 0 \tag{4.3.7}$$

to some set where the inverse map is unique. To this end we need the inverse transformation of (4.3.5), which is

$$z = \frac{1}{2\pi} \log \frac{(1 - \mathrm{e}^{-2\pi T}) + (\mathrm{e}^{2\pi T} - 1)\mathrm{e}^{2\pi\zeta}}{(\mathrm{e}^{2\pi T} - 1) + (1 - \mathrm{e}^{-2\pi T})\mathrm{e}^{2\pi\zeta}}. \tag{4.3.8}$$

For $t_i = \mathrm{i}\tau,\ -\frac{1}{2} \leq \tau \leq 0$ we obtain with $x^0, ..., x^3$ the basis in the W–frame

$$\begin{aligned}\Lambda_{gW}(\mathrm{i}\tau) = g_i x^0\rangle \cos 2\pi\tau \langle g_i x^0 - \mathrm{i} g_i x^0\rangle \sin 2\pi\tau \langle g_i x^1 \\ - \mathrm{i} g_i x^0\rangle \sin 2\pi\tau \langle g_i x^1 + g_i x^1\rangle \cos 2\pi\tau \langle g_i x^1 \\ + g_i x^2\rangle\langle g_i x^2 + g_i x^3\rangle\langle g_i x^3.\end{aligned} \tag{4.3.9}$$

As long as we restrict ourselves to the set $-\frac{1}{2} \leq \tau \leq 0$, the representation (4.3.9) is one to one. Because of the additivity theorems of the spherical functions this statement remains true for the domain (4.3.7). Hence we obtain $U(\Lambda_{gW}(-\mathrm{i}/2))A\Omega = U(-R_{gW}(\pi))A\Omega$ with a unique representation of the element -1. $\square$

Collecting the result of the discussion we obtain:

4.3.3 Proposition:

Let D be a double cone such that the closure of D does not contain the origin. Then for $A \in \mathcal{M}(D)$ and g such that $D \subset gW$

$$U(\Lambda_{gW}(-\frac{\mathrm{i}}{2}))U(R_{gW}(\pi))A\Omega$$

is independent of g.

Proof: From the above discussion we know that the statement is true for g in a sufficiently small neighbourhood of the identity in G. But this implies that it is true for all $g \in N(D)$. $\square$

Next we turn to the locality problem.

4.3.4 Proposition:

Let D be a double cone and let closure $D \subset W$*, then for $A \in \mathcal{M}(D)$ one finds*

$$J_W A J_W \in \mathcal{M}(P_W D),$$

where P_W denotes the reflection in the characteristic two–plane of W.

Proof: Let $K_W(D)$ be the cylindrical set generated from D by applying the translations in the directions perpendicular to the characteristic two–plane. Then Thm. 4.1.7 implies $U(\Lambda_W(-\frac{\mathrm{i}}{2}))A\Omega = \hat{A}\Omega$ with $\hat{A} \in \mathcal{M}(P_W K_W(D))$. Hence we obtain

$$U(R_W(\pi))U(\Lambda_W(-\frac{\mathrm{i}}{2}))A\Omega = U(R_W(\pi))\hat{A}\Omega.$$

Since by Prop. 4.3.3 the operator on the left side is independent of W we get

$$\mathrm{Ad}\, U(R_W(\pi))\hat{A} \in \bigcap_{g\in N(D)} \mathcal{M}(-K_{gW}(D)).$$

Using Lemma 4.1.9 we are allowed to shift D inside the wedge, then doing the reflection, and afterwards shift back without changing the result. So we get

$$\mathrm{Ad}\, U(R_W(\pi))\hat{A} \in \bigcap_{D+x\subset W} \bigcap_{g\in N(D+x)} \mathcal{M}(-K_{gW}(D) = \mathcal{M}(-D).$$

From this we obtain as mentioned before $J_W A\Omega = B\Omega$ with $B \in \mathcal{M}(P_W D)$. Since Ω is separating for $\mathcal{M}(W')$ we obtain $J_W A J_W \in \mathcal{M}(P_W D)$. This shows the proposition. □

From this we obtain with Eq. (4.0.1)

4.3.5 Theorem:
Every QFTLO which fulfils wedge duality and the reality condition is PCT convariant.

4.4) *The Bisognano–Wichmann property and the construction of the Poincaré group*

We saw that the PCT–theorem is closely connected with the Bisognano–Wichmann property (see Def. 3.1.8) i.e., the modular group of every wedge acts like the associated group of Lorentz boosts. If we assume that the theory fulfils the Bisognano–Wichmann property, then one can ask wether or not all these modular groups fit together and give rise to a representation of the Poincaré group. If the dimension of the Minkowski space is two then one has only the right and the left wedge and their translates. Since the Bisognano–Wichmann property implies that the translates of the wedge along the lightlike vectors fulfil the condition of half–sided modular inclusion, the translations are obtained by the construction of Wiesbrock [Wie93],[Wie97a] (see 2.6) which together with the modular group of the wedge give rise to a representation of the Poincaré group [Bch92]. Hence the construction procedure contains new aspects if the dimension of the Minkowski space is at least three.

A first treatment of this problem is due to Brunetti, Guido, and Longo [BGL94]. They used the first and the second cohomology of the Poincaré group and showed that the modular groups of all wedges give rise to a representation of the covering of the Poincaré group. In a second paper Guido and Longo [GL95] generalized their method to charged fields and showed that in this frame the connection between spin and statistics is fulfilled.

Here we will use a construction which is based entirely on the principle of half–sided modular inclusions. It has the advantage that it gives directly a representation of the Poincaré group and not of its covering [Bch98b]. In order to avoid index manipulation we represent the result for the four–dimensional Minkowski space. The construction is in three steps. First we construct the translations by using the half–sided modular inclusions of wedges and their translates. Then we show that the algebra of the intersection of two wedges with a common lightlike vector fulfil the condition of half–sided modular inclusion with respect to the algebras of the wedges. This will allow us to construct the translational part of the stabilizer group of the common lightlike vector. Since this group connects the modular groups of different wedges we can, in the third step, construct the whole Poincaré group.

First step: *Construction of the translations*

We start our investigation by looking at the family of wedges $W[\ell,\ell',a]$ where ℓ and ℓ' are fixed and a is of the form $a=\lambda\ell+\mu\ell'$. Therefore, we suppress in the first part the indices $[\ell,\ell']$ and write simply $W[a], \Delta[a]^{\mathrm{i}t}$, and so on.

Let $W[a]$ and $W[a+\lambda\ell]$ be two wedges and $\lambda>0$. Then by Bisognano–Wichmann property the algebra $\mathcal{M}(W[a+\lambda\ell])$ fulfils the condition of $-$ half–sided modular inclusion with respect to the algebra $\mathcal{M}(W[a])$. Hence by Thm. 2.6.2 a unitary group $U[a,\lambda\ell](t)$ exists with positive generator fulfilling

$$\operatorname{Ad} U[a,\lambda\ell](1)\mathcal{M}(W[a]) = \mathcal{M}(W[a+\lambda\ell]). \tag{4.4.1}$$

Furthermore, this group satisfies the following properties ($\mathrm{e}(t)=\mathrm{e}^{-2\pi t}$):

$$\begin{aligned} U[a,\lambda\ell](t)\Omega &= \Omega, \\ \operatorname{Ad}\Delta[a]^{\mathrm{i}t}U[a,\lambda\ell](s) &= U[a,\lambda\ell](\mathrm{e}(t)s), \\ \operatorname{Ad} U[a,\lambda\ell](s)\mathcal{M}(W[a]) &= \mathcal{M}(W[a+s\lambda\ell]), \\ U[a,\lambda\ell](1-\mathrm{e}(t)) &= \Delta[a+\lambda\ell]^{\mathrm{i}t}\Delta[a]^{-\mathrm{i}t}. \end{aligned} \tag{4.4.2}$$

These formulas follow from Thm. 2.6.2. together with Eq. (4.4.1). Because of Thm. 2.6.5 the group $U[a,\lambda\ell](s)$ is uniquely defined by the properties listed in the first and third line together with the positivity of the spectrum. From the last line of (4.4.2) we obtain

$$U[a,\lambda\ell](1) = \lim_{t\to\infty}\Delta[a+\lambda\ell]^{\mathrm{i}t}\Delta[a]^{-\mathrm{i}t}. \tag{4.4.3}$$

Notice that by the last line of (4.4.2) the limit converges in the weak and hence in the strong topology. Moreover, from representation (4.4.3) we see that $U[a,\lambda\ell](s)$ acts like the translation in the ℓ direction. Hence by the uniqueness Thm. 2.6.5 we find that this is independent of a, i.e.

$$U[a,\lambda\ell](t) = U[b,\lambda\ell](t). \tag{4.4.4}$$

The mentioned uniqueness of the groups $U[a,\lambda\ell](t)$ implies for $\lambda,\mu\neq 0$ the identity

$$U[a,\lambda\ell](s) = U[a,\mu\ell](\frac{\lambda}{\mu}s). \tag{4.4.4$'$}$$

Hence we only have to deal with groups $U[\ell](s)$.

Using the wedge $W[a]$ again we can construct a group $U[a,\ell'](s)$ in the same manner. By proper definition, this group satisfies again the spectrum condition and the relations similar to (4.4.2)-(4.4.4). The only change is

$$\operatorname{Ad}\Delta[a]^{\mathrm{i}t}U[a,\ell'](s) = U[a,\ell'](\mathrm{e}(-t)s). \tag{4.4.2$'$}$$

Also here we obtain a group $U[\ell'](s)$ which does not depend on the first parameter.

It remains to show that the groups $U[\ell](s)$ and $U[\ell'](s)$ commute. To this end we notice that we can map $\mathcal{M}(W[a])$ onto $\mathcal{M}(W[a+t\ell-s\ell'])$ in two different ways, namely using either $\mathcal{M}(W[a+t\ell])$ or $\mathcal{M}(W[a-s\ell'])$ as intermediate algebra. This yields

$$\mathcal{M}(W[a+t\ell-s\ell']) = \mathrm{Ad}\, U[\ell'](-s)U[\ell](t)\mathcal{M}(W[a]),$$
$$\mathcal{M}(W[a+t\ell-s\ell']) = \mathrm{Ad}\, U[\ell](t)U[\ell'](-s)\mathcal{M}(W[a]).$$

We want to show that the product of translation operators coincide. Therefore, we compute with help of (4.4.2) and (4.4.2') and obtain

$$\begin{aligned} U[a+t\ell,\ell'](-s(1-\mathrm{e}(-\mu)))U[a,\ell](t(1-\mathrm{e}(\mu))) &= \\ &\Delta[a+t\ell-s\ell']^{\mathrm{i}\mu}\Delta[a+t\ell]^{-\mathrm{i}\mu}\Delta[a+t\ell]^{\mathrm{i}\mu}\Delta[a]^{-\mathrm{i}\mu}, \\ U[a-s\ell',\ell](t(1-\mathrm{e}(\mu)))U[a,\ell'](-s(1-\mathrm{e}(-\mu))) &= \\ &\Delta[a+t\ell-s\ell']^{\mathrm{i}\mu}\Delta[a-s\ell']^{-\mathrm{i}\mu}\Delta[a-s\ell']^{\mathrm{i}\mu}\Delta[a]^{-\mathrm{i}\mu}. \end{aligned}$$

Using the independence of the first parameter we obtain

$$U[\ell'](a)U[\ell](b) = U[\ell](b)U[\ell'](a). \tag{4.4.5}$$

Having constructed the Poincaré group in two dimensions we have to go to higher dimensions. First we want to show that the translations defined in different two–planes also commute. To this end we fix a lightlike direction ℓ and look at the family of wedges defined by ℓ and another lightlike vector $\{W[\ell,\ell'];\ell'\neq\ell\}$. Using the $-$half–sided modular inclusions $\mathcal{M}(W[\ell,\ell_1,\lambda\ell])\subset\mathcal{M}(W[\ell,\ell_1,0])$ and $\mathcal{M}(W[\ell,\ell_2,\lambda\ell])\subset\mathcal{M}(W[\ell,\ell_2,0])$ we obtain two different translation groups $U[\ell,\ell_1,\ell](t)$ and $U[\ell,\ell_2,\ell](t)$ respectively. Both groups act like translations on every double cone and hence on every wedge. Therefore, by Thm. 2.6.5 they have to coincide. Hence the groups depend only on the direction of the translations and not on the two–plane which has been used for constructing them. Consequently we obtain groups $U[\ell](t)$. From this it follows that all these groups $U[\ell](s)$ commute for different ℓ, since for every two different ℓ's there exists a wedge which is defined by these two vectors. Since all these unitary groups fulfil the spectrum condition there exists a group $V(a), a\in\mathbb{R}^4$ such that $U[\ell](s)$ coincides with $V(s\ell)$. Hence we have constructed the translation group of $\mathbb{R}^4$ which transforms by the modular groups in the expected way.

We collect the results obtained so far:

4.4.1 Lemma:

Assume all modular groups of the wedge algebras act like their associated Lorentz groups. Then a unique continuous representation of the translation–group $V(a)$ exists which fulfils spectrum–condition and acts geometrically on the local algebras

$$\mathrm{Ad}\, V(a)\mathcal{M}(D) = \mathcal{M}(D+a),$$

where D denotes a double cone. (It is assumed, that $\mathcal{M}(D)$ coincides with the intersection of the wedge algebras of all wedges containing D.) This representation $V(a)$ is contained in the algebra generated by the modular groups.

Moreover, the modular groups of the wedges and the translations transform each other as if they were members of a unitary representation of the Poincaré group.

Proof. We know that $V(a)$ transforms the algebras of the wedges in the geometric manner. This implies the correct action on $\mathcal{M}(D)$ by passing to the intersection. The rest follows from (4.4.2) and (4.4.2') and the fact that every translation can be decomposed into translations in lightlike directions. □

From this result we obtain

4.4.2 Proposition:

Let a representation of a theory of local observables fulfil the above–mentioned conditions. Then this representation fulfils wedge duality, i.e.

$$\mathcal{M}(W[\ell,\ell'])' = \mathcal{M}(W[\ell',\ell]).$$

Proof. Since in every two–dimensional subspace associated with a wedge we have a representation of the Poincaré group which acts local and since the Lorentz boosts coincide with the modular group it follows that for every localized operator A belonging to the right wedge the expression $U(\Lambda(t))A\Omega$ has a bounded analytic continuation into the strip $-\frac{1}{2} < \Im m\, t < 0$ with continuous boundary values. This follows from the fact that $A\Omega$ is in the domain of $\Delta^{\frac{1}{2}}$. Hence the conditions of Thm. 4.1.22 are fulfilled and the theory obeys wedge duality. □

Since an algebra and its commutant has up to a sign the same modular group we obtain the following symmetry:

$$\Delta[\ell,\ell']^{\mathrm{i}t} = \Delta[\ell',\ell]^{-\mathrm{i}t}. \tag{4.4.6}$$

Second step: *The stabilizer group of a light ray*

Next we want to construct the translational part of the stabilizer group of any light ray $\ell \in \partial V^+$. To this end we look at the family of wedges having one light ray in common,

$$\{W[\ell,\ell_2];\ell \text{ fixed}\}. \tag{4.4.7}$$

It is well known that the stabilizer $S(\ell)$ of a lightlike vector is isomorphic to the euclidean transformation of $\mathbb{R}^2$. (See e.g. Gelfand, Minlos and Shapiro [GMS63].) The rotations are the transformations around the space–direction of the light ray. In order to understand the translations let us introduce a second lightlike vector ℓ_2 which we choose in such a way that ℓ, t, ℓ_2 lie in one two–plane. Let $T(\ell)$ be the tangent hyperplane at the forward lightcone V^+ containing the vector ℓ. Then the affine hyperplane $\ell_2 + T(\ell)$ intersects ∂V^+ in a two-dimensional set (parabola) homeomorphic to $\mathbb{R}^2$. The translations of $S(\ell)$ have this set as orbit.

In the concrete example $\ell = (1,1,0,0), \ell_2 = (1,-1,0,0)$, these translations become $(a = (a_1, a_2) \in \mathbb{R}^2)$

$$\Lambda^\ell(a) = \begin{pmatrix} 1+\frac{a^2}{2} & -\frac{a^2}{2} & a_1 & a_2 \\ \frac{a^2}{2} & 1-\frac{a^2}{2} & a_1 & a_2 \\ a_1 & -a_1 & 1 & 0 \\ a_2 & -a_2 & 0 & 1 \end{pmatrix}. \tag{4.4.8}$$

(See also R. Jost [Jo65] Appendix.) It is easy to check that this is a representation of the two–dimensional translation group,

$$\Lambda^\ell(a)\Lambda^\ell(b) = \Lambda^\ell(a+b).$$

Setting $\ell_2(a) = \Lambda^\ell(a)\ell_2$ then one finds

$$\Lambda[\ell, \ell_2(a)](t) = \Lambda^\ell(a)\Lambda[\ell,\ell_2](t)\Lambda^\ell(-a). \tag{4.4.9}$$

Using Eqs. (1.5.3) and (4.4.8) then simple calculations imply

$$\Lambda[\ell,\ell_2](t)\Lambda^\ell(a)\Lambda[\ell,\ell_2](-t) = \Lambda^\ell(e(t)a), \tag{4.4.9$'$}$$

or more general

$$\begin{aligned} \Lambda[\ell,\ell_2(a)](t)\Lambda[\ell,\ell_2(b)](-t) &= \Lambda[\ell,\ell_2(a)](t)\Lambda^\ell(b-a)\Lambda[\ell,\ell_2(a)](-t)\Lambda^\ell(a-b) \\ &= \Lambda^\ell((1-e(t))(a-b)). \end{aligned} \tag*{4.4.10}$$

From this we obtain:

$$\Lambda^\ell(a-b) = \lim_{t\to\infty} \Lambda[\ell,\ell_2(a)](t)\Lambda[\ell,\ell_2(b)](-t). \tag{4.4.11}$$

In order to show that the corresponding limits of products of modular operators exist and define a commutative group we need once more the principle of half–sided modular inclusion. The crucial result is:

4.4.3 Theorem:

Let the theory fulfil the Bisognano–Wichmann property. Then the algebra $\mathcal{M}\Big(W[\ell,\ell_1] \cap W[\ell,\ell_2]\Big)$ *fulfils the condition of* $-$*half–sided modular inclusion with respect to both algebras* $\mathcal{M}(W[\ell,\ell_1])$ *and* $\mathcal{M}(W[\ell,\ell_2])$.

Proof: By Def. 1.5.2 of the wedge one has

$$W[\ell_1,\ell_2] = \{x;\ (\ell_1,x) < 0, (\ell_2,x) > 0\}.$$

This implies

$$W[\ell,\ell_1] \cap W[\ell,\ell_2] = \{x;\ (l,x) < 0, (\ell_i,x) > 0, i = 1,2\}.$$

With $\ell_2 = \alpha\ell + \beta\ell_1 + \ell^-$, $\alpha, \beta > 0$ and $\Lambda[\ell,\ell_1](t)\ell = e(t)\ell, \Lambda[\ell,\ell_1](t)\ell_1 = e(-t)\ell_1$ we obtain:

$$\begin{aligned}\Lambda[\ell,\ell_1](t)\ell_2 &= e(t)\alpha\ell + e(-t)\beta\ell_1 + \ell^-,\\ (x, \Lambda[\ell,\ell_1](t)\ell_2) &= e(t)\alpha(x,\ell) + e(-t)\beta(x,\ell_1) + (x,\ell^-)\\ &= (x,\ell_2) + \alpha(e(t)-1)(x,\ell) + \beta(e(-t)-1)(x,\ell_1).\end{aligned}$$

This expression is positive for $t \leq 0$. This implies that the algebra of the intersection fulfils the condition of $-$half–sided modular inclusion with respect to the algebra of the wedge $W[\ell,\ell_1]$. By symmetry we obtain the statement of the theorem. □

4.4.4 Remark: If we look at three wedges with one common lightlike vector ℓ, i.e. $W[\ell,\ell_1], W[\ell,\ell_2], W[\ell,\ell_3]$ then the algebra of the intersection

$$\mathcal{M}\Big(W[\ell,\ell_1] \cap W[\ell,\ell_2] \cap W[\ell,\ell_3]\Big) \tag{4.4.12}$$

also fulfils the condition of $-$half–sided modular inclusion with respect to all three algebras $W[\ell,\ell_i],\ i = 1,2,3$. This is a consequence of the identity

$$\begin{aligned}&\mathcal{M}\Big(W[\ell,\ell_1] \cap W[\ell,\ell_2] \cap W[\ell,\ell_3]\Big)\\ &\qquad = \mathcal{M}\Big(W[\ell,\ell_1] \cap W[\ell,\ell_2[\Big) \cap \mathcal{M}\Big(W[\ell,\ell_1] \cap W[\ell,\ell_3]\Big)\end{aligned}$$

and the fact that both algebras on the right side fulfil the condition of $-$half–sided modular inclusion with respect to $\mathcal{M}(W[\ell,\ell_1])$ and hence also for the intersection. For the other two algebras the statement follows by symmetry.

Looking at Eq. (4.4.11) we see that we have to show that the corresponding product $\Delta[\ell,\ell_2(a)]^{\mathrm{i}t}\Delta[\ell,\ell_2(b)]^{-\mathrm{i}t}$ converges for $t \to \infty$ strongly to a unitary operator $U^\ell(a,b)$ and that this operator acts local on every double cone, i.e.,

$$\mathrm{Ad}\, U^\ell(a,0)\mathcal{M}(O) = \mathcal{M}(\Lambda^\ell(a)O).$$

4.4.5 Lemma:

The product

$$\Delta[\ell,\ell_2(a)]^{\mathrm{i}t}\Delta[\ell,\ell_2(b)]^{-\mathrm{i}t}$$

converges for $t \to \infty$ strongly to an operator $U^\ell(a,b)$. This operator acts geometrically on local algebras, i.e.

$$\mathrm{Ad}\, U^\ell(a,b)\mathcal{M}(D) = \mathcal{M}(\Lambda^\ell(a-b)D). \tag{4.4.13}$$

Proof. Since by Thm.4.4.3 $\mathcal{M}(W[\ell,\ell_2(a)] \cap W[\ell,\ell_2(b)])$ fulfils the condition of $-$ half–sided modular inclusion with respect to the algebras $\mathcal{M}(W[\ell,\ell_2(b)])$ and $\mathcal{M}(W[\ell,\ell_2(a)])$ there exist by Theorem 2.6.2 two one–parametric unitary groups $U^\ell[a,b;a](t), U^\ell[a,b;b](t)$ with the properties

$$\mathcal{M}\Big(W[\ell,\ell_2(a)] \cap W[\ell,\ell_2(b)]\Big) = \begin{cases}\mathrm{Ad}\, U^\ell[a,b;a](1)\mathcal{M}(W[\ell,\ell_2(a)]),\\ \mathrm{Ad}\, U^\ell[a,b;b](1)\mathcal{M}(W[\ell,\ell_2(b)]).\end{cases} \tag{4.4.14}$$

Both these groups fulfil similar properties as listed in (4.4.2). From this we derive

$$\mathrm{Ad}\, U^\ell[a,b;a](1)^{-1}U^\ell[a,b;b](1)\mathcal{M}(W[\ell,\ell_2(b)]) = \mathcal{M}(W[\ell,\ell_2(a)]). \tag{4.4.14'}$$

These operators are connected with the modular operators of the algebras and their intersections by the formulas

$$U^\ell[a,b;a](1-\mathrm{e}(t)) = \Delta[\cap]^{\mathrm{i}t}\Delta[\ell,\ell_2(a)]^{-\mathrm{i}t},$$
$$U^\ell[a,b;b](1-\mathrm{e}(t)) = \Delta[\cap]^{\mathrm{i}t}\Delta[\ell,\ell_2(b)]^{-\mathrm{i}t},$$

where $\Delta[\cap]$ denotes the modular operator of the intersection. We find

$$U^\ell[a,b;a](1-\mathrm{e}(t))^{-1}U^\ell[a,b;b](1-\mathrm{e}(t)) = \Delta[\ell,\ell_2(a)]^{\mathrm{i}t}\Delta[\ell,\ell_2(b)]^{-\mathrm{i}t}.$$

This shows that for $t \to \infty$ the product on the left converges weakly and hence also strongly. Therefore, also the right side converges strongly. Since the approximations $\Delta[\ell,\ell_2(a)]^{\mathrm{i}t}\Delta[\ell,\ell_2(b)]^{-\mathrm{i}t}$ act geometrically we see that this is also true for the limit

$$U^\ell(a,b) = \lim_{t\to\infty} U^\ell[a,b;a](1-\mathrm{e}(t))^{-1}U^\ell[a,b;b](1-\mathrm{e}(t)). \tag{4.4.15}$$

Equation (4.4.10) shows that the limit acts as stated in the lemma. □

Next we have to show that the operators $U(a,b)$ depend only on the difference $(a-b)$ and that the operators $V(a-b) = U(a,b)$ define a representation of the two dimensional translation group.

4.4.6 Lemma:

The operators $U^\ell(a,b)$ depend only on the difference of the arguments

$$U^\ell(a,b) = V^\ell(a-b).$$

These operators define a continuous representation of the two–dimensional abelian group $\Lambda^\ell(a)$

$$V^\ell(a)V^\ell(b) = V^\ell(a+b), \qquad a,b \in \mathbb{R}^2.$$

Proof: From the relation $U^\ell(a,b) = U^\ell[a,b;a](1)^{-1}U^\ell[a,b;b](1)$ we conclude

$$U^\ell(a,b)U^\ell(b,a) = 1. \tag{4.4.16}$$

Since the subalgebras

$$\mathcal{M}\Big(W[\ell,\ell_2(a)] \cap W[\ell,\ell_2(b)]\Big) \quad \text{and} \quad \mathcal{M}\Big(W[\ell,\ell_2(a)] \cap W[\ell,\ell_2(c)]\Big)$$

fulfil both the condition of $-$ half–sided modular inclusion with respect to the two algebras $\mathcal{M}(W[\ell,\ell_2(a)])$ and by Remark 4.4.4 the triple intersection $\mathcal{M}\Big(W[\ell,\ell_2(a)] \cap W[\ell,\ell_2(b)] \cap$

$W[\ell,\ell_2(c)]\Big)$ fulfils also the condition of $-$half–sided modular inclusion with respect to $\mathcal{M}(W[\ell,\ell_2(a)])$. By symmetry the same holds with respect to the algebras $\mathcal{M}(W[\ell,\ell_2(b)]), \mathcal{M}(W[\ell,\ell_2(c)])$. Hence we obtain, as in the last lemma, three groups

$$U^\ell[a,b,c;a](t), \quad U^\ell[a,b,c;b](t), \quad U^\ell[a,b,c;c](t).$$

Now we can represent $U^\ell(a,b)$ with help of these operators

$$U^\ell(a,b) = U^\ell[a,b,c;a](1)^{-1}U^\ell[a,b,c;b](1).$$

This leads to the relation

$$\begin{aligned} U^\ell(a,b)U^\ell(b,c)U^\ell(c,a) &= 1, \\ U^\ell(a,b)U^\ell(b,c) &= U^\ell(a,c). \end{aligned} \tag{4.4.16'}$$

Since Eq. (4.4.13) holds for every double cone D we conclude that $U^\ell(a,b)$ and $U^\ell(a+c,b+c)$ differ only by a phasefactor. If we set $U^\ell(a,b) = V^\ell(a-b)f(a,b)$ where $f(a,b)$ is the phasefactor, then the second equation of (4.4.16') implies that $V^\ell(a)$ is a representation of the central extension of the group $\Lambda^\ell(a)$. This implies by Eq. (4.4.9)

$$\begin{aligned} \mathrm{Ad}\, U^\ell(a,b)\Delta[\ell,\ell_2(c)]^{\mathrm{i}s} &= \Delta[\ell,\ell_2(c+a-b)]^{\mathrm{i}s}, \\ \mathrm{Ad}\, U^\ell(a,b)U^\ell(c,d) &= U^\ell(c+a-b,d+a-b). \end{aligned} \tag{4.4.17}$$

The second line follows from the first by inserting a product and taking the limit. Now we start from the last equation of (4.4.16') and use the last line of (4.4.17),

$$\begin{aligned} U^\ell(a,b)U^\ell(c,d) &= U^\ell(a,b)U^\ell(c,b)U^\ell(b,d) \\ &= U^\ell(c+a-b,a)U^\ell(a,b)U^\ell(b,d) = U^\ell(a+c-b,d). \end{aligned}$$

Taking the inverse of this relation we obtain with (4.4.16)

$$U^\ell(a,b)U^\ell(c,d) = U^\ell(a,d-(c-b)), \tag{$*$}$$

whereby the arguments have been renamed. Comparing the last two equations, we get with $f = c-d$ the equation

$$U^\ell(a+f,b) = U^\ell(a,b-f).$$

This shows that $U^\ell(a,b)$ depends only on the difference–variable. We set

$$V^\ell(a-b) = U^\ell(a,b) = U^\ell(a-b,0). \tag{4.4.18}$$

Inserting this into $(*)$ we find

$$V^\ell(a-b)V^\ell(c-d) = V^\ell(a-d+c-b).$$

Hence the $V^\ell(a)$ define an abelian representation of the two–dimensional translation group.

It remains to show that this is a continuous representation. Knowing that $V^\ell(a)$ is a representation of $\Lambda^\ell(a)$ we conclude from Eqs. (4.4.12),(4.4.9') and Lemma 4.4.5

$$\mathrm{Ad}\,\Delta[\ell,\ell_2(b)]^{\mathrm{i}t} V^\ell(a) = V^\ell(\mathrm{e}(t)a). \tag{4.4.19}$$

Since the modular group is continuous we see that $V^\ell(a)$ is continuous in radial direction. Multiplying this expression with $V(b)$ we see that $V(\mathrm{e}(t)a+b)$ is continuous in t for every value of a and b. Hence $V(a)$ is continuous. □

Third step: *construction of the rotations*

Our aim is to show that the operators $\Delta[\ell_1,\ell_2]^{\mathrm{i}t}$ generate a representation of the Lorentz group. Therefore, we have to show that

$$\prod \Delta[\ell_1^{(i)},\ell_2^{(i)}]^{\mathrm{i}t^{(i)}} = \mathbb{1} \tag{4.4.20}$$

holds in case the equation

$$\prod \Lambda[\ell_1^{(i)},\ell_2^{(i)}](t^{(i)}) = 1 \tag{4.4.21}$$

is fulfilled. To show this we are only allowed to make transformations which do not change the conclusion, i.e. the transformations implied by the Bisognano–Wichmann property and those derived from this. We will find other transformations by looking at half–sided modular inclusions.

We say two expressions containing elements of the Lorentz group are equivalent, if the corresponding products of operators $\Delta[\ell_1,\ell_2]^{\mathrm{i}t}$ and $V^\ell(a)$ fulfil the same equation.

4.4.7 Lemma:

Every element $\Lambda[\ell_1,\ell_2](t)$ is equivalent to a product of the form

$$\mathrm{Ad}\,\{\Lambda^\ell(a)\Lambda^{\ell'}(b)\}\Lambda[\ell,\ell'](t')$$

where t' is either t or $-t$.

Proof. We look at the transformation $\Lambda[\ell_1,\ell_2](t)$. If $\ell_1=\ell$ and $\ell_2=\ell'$ then we get the lemma with $a=b=0$. If $\ell_1=\ell'$ and $\ell_2=\ell$ then we use (4.4.6) for transforming the element to the previous situation. If one of the two vectors ℓ_1,ℓ_2 coincides with ℓ we can assume that this is ℓ_1. Then there is a transformation $\Lambda^\ell(a)$ mapping ℓ_2 onto a multiple of ℓ'. Therefore, $\Lambda[\ell,\ell_2](t)$ is equivalent to $\mathrm{Ad}\,\Lambda^\ell(a)\Lambda[\ell,\ell'](t)$. If one of the two vectors ℓ_1,ℓ_2 coincides with ℓ' we can assume that this is ℓ_2. By the same argument we find that $\Lambda[\ell,\ell_2](t)$ is equivalent to $\mathrm{Ad}\,\Lambda^{\ell'}(b)\Lambda[\ell,\ell'](t)$.

Assume next that ℓ_1 and ℓ_2 are not multiples of ℓ or ℓ'. Then there is a transformation $\Lambda^\ell(a)$ mapping ℓ_2 onto a multiple of ℓ'. By this transformation ℓ_1 is mapped onto ℓ_3. Hence we get a transformation $\Lambda^{\ell'}(b)$ which maps ℓ_3 onto a multiple of ℓ. Since this transformation does not change ℓ' the original transformation is mapped onto $\Lambda[\ell,\ell'](t)$. If one of the vectors is already in the right position then we need only one transformation.

If necessary we can change the order of vectors because of (4.4.6). Hence every element $\Lambda[\ell_1,\ell_2](t)$ is equivalent to an element of the form

$$\mathrm{Ad}\,\{\Lambda^{\ell}(a)\Lambda^{\ell'}(b)\}\Lambda[\ell,\ell'](t')$$

where t' is either t or $-t$. □

Using this lemma we show:

4.4.8 Lemma:

Every product

$$\prod_{i=1}^{n}\Lambda[\ell_1^{(i)},\ell_2^{(i)}](t^{(i)})$$

is equivalent to the product

$$\Lambda[\ell,\ell'](t^{(0)})\prod_{i=1}^{m}\Lambda^{\ell'}(b^{(i)})\Lambda^{\ell}(a^{(i)}). \tag{4.4.22}$$

Proof. Using the last lemma we replace every $\Lambda[\ell_1^{(i)},\ell_2^{(i)}](t^{(i)})$ by an element of the form
$\mathrm{Ad}\,\{\Lambda^{\ell}(a'(i))\Lambda^{\ell'}(b'(i))\}\Lambda[\ell,\ell'](t'(i))$. Using (4.4.19) in the form

$$\begin{aligned}\mathrm{Ad}\,\Lambda[\ell,\ell'](t)\Lambda^{\ell}(a) &= \Lambda^{\ell}(\mathrm{e}(t)a),\\ \mathrm{Ad}\,\Lambda[\ell,\ell'](t)\Lambda^{\ell'}(a) &= \Lambda^{\ell'}(\mathrm{e}(-t)a)\end{aligned} \tag{4.4.19$'$}$$

we can commute all $\Lambda[\ell,\ell'](t'(i))$ to the front and multiply them. Therefore, we end up with an expression listed in the lemma. Since the $\Lambda^{\ell}(a)$ and the $\Lambda^{\ell'}(a)$ are groups all arguments are unequal to zero except perhaps for the first $\Lambda^{\ell'}(b)$ or the last $\Lambda^{\ell}(a)$. □

Using this lemma we have to investigate expressions of the form (4.4.22). For further simplification of this expression we must investigate the rotations.

Let x_0 be a timelike vector in the two–plane spanned by ℓ and ℓ', and let $\Lambda^{\ell}(a)$ be an element in the stabilizer group of ℓ. Then $\Lambda^{\ell}(a)x_0$ is a vector on which we can apply $\Lambda^{\ell'}(b)$. There will be an element $b(a)$ such that $\Lambda^{\ell'}(b(a))\Lambda^{\ell}(a)x_0$ belongs to the two–plane containing ℓ,ℓ' and x_0. In this situation $s(a)$ exists such that $\Lambda[\ell,\ell'](s(a))$ maps this vector back to x_0. Therefore, the product represents a rotation

$$\Lambda[\ell,\ell'](s(a))\Lambda^{\ell'}(b(a))\Lambda^{\ell}(a)=R(\ell,a). \tag{4.4.23}$$

In the same manner we obtain a second rotation if we start with $\Lambda^{\ell'}(a)$,

$$\Lambda[\ell',\ell](s'(a))\Lambda^{\ell}(b'(a))\Lambda^{\ell'}(a)=R(\ell',a). \tag{4.4.23$'$}$$

First we need to determine $b(a), s(a)$ and the element $R(\ell, a)$ and $b'(a), s'(a)$ and $R(\ell', a)$ respectively. This we do in our standard coordinate system. $\Lambda^\ell(a)$ maps the vector x_0 onto $\Lambda^\ell(a)x_0 = (1 + \frac{a^2}{2}, \frac{a^2}{2}, a_1, a_2)$ and hence we get

$$\Lambda^{\ell'}(b)\Lambda^\ell(a)x_0 = \Big((1 + \frac{b^2}{2})(1 + \frac{a^2}{2}) + \frac{b^2}{2}\frac{a^2}{2} + (b, a),$$
$$-\frac{b^2}{2}(1 + \frac{a^2}{2}) + (1 - \frac{b^2}{2})\frac{a^2}{2} - (b, a), b_1(1 + a^2) + a_1, b_2(1 + a^2) + a_2\Big).$$

This vector belongs to the plane spanned by ℓ and ℓ' for

$$b(a) = -\frac{a}{1 + a^2}. \tag{4.4.24}$$

Inserting this we find

$$\Lambda^{\ell'}(b(a))\Lambda^\ell(a)x_0 = (\frac{2 + 2a^2 + a^4}{2(1 + a^2)}, \frac{2a^2 + a^4}{2(1 + a^2)}, 0, 0).$$

This implies

$$\Lambda[\ell, \ell'](s(a)) = \frac{1}{2(1 + a^2)}\begin{pmatrix} 2 + 2a^2 + a^4 & -(2a^2 + a^4) & 0 & 0 \\ -(2a^2 + a^4) & 2 + 2a^2 + a^4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4.24$'$}$$

from which follows

$$\mathrm{e}(s(a)) = \frac{1}{(1 + a^2)}. \tag{4.4.24$''$}$$

In order to compute the rotation $R(\ell, a)$ notice first that $\Lambda^\ell(a)$ leaves the vector $(1, 0, 0) \times (0, a_1, a_2)$ unchanged. The same holds for $\Lambda^{\ell'}(-\frac{a}{1+a^2})$ and $\Lambda[\ell, \ell'](s(a))$ so that $\frac{1}{\|a\|}(1, 0, 0) \times (0, a_1, a_2)$ is the axis of rotation. (The multiplication is the vector–product in $\mathbb{R}^3$.) The angle of rotation can be computed by applying $R(\ell, a)$ to the vector $(0, 1, 0, 0)$. One finds

$$\Lambda[\ell, \ell'](s(a))\Lambda^{\ell'}(b(a))\Lambda^\ell(a)(0, 1, 0, 0) =$$
$$\frac{1}{1 + a^2}(0, 1 - a^2, -2a_1, -2a_2) = \frac{1 - a^2}{1 + a^2}(0, 1, 0, 0) - \frac{2a^2}{1 + a^2}(0, 0, \frac{a_1}{\|a\|}, \frac{a_2}{\|a\|}).$$

This implies the following characterization of $R(\ell, a)$

$$R(\ell, a) : \begin{cases} \text{axis of rotation :} & \frac{1}{\|a\|}(1, 0, 0) \times (0, a_1, a_2), \\ \text{angle of rotation :} & \cos\varphi = \frac{1 - a^2}{1 + a^2}, \sin\varphi = -\frac{2\|a\|}{1 + a^2}. \end{cases} \tag{4.4.24$'''$}$$

By similar computation one finds

$$b'(a) = \frac{a}{1+a^2}, \tag{4.4.25}$$

$$\Lambda[\ell,\ell'](s'(a)) = \frac{1}{2(1+a^2)}\begin{pmatrix} 2+2a^2+a^4 & (2a^2+a^4) & 0 & 0 \\ (2a^2+a^4) & 2+2a^2+a^4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.4.25$'$}$$

$$\mathrm{e}(s'(a) = (1+a^2), \tag{4.4.25$''$}$$

$$R(\ell',a) : \begin{cases} \text{axis of rotation :} & \frac{1}{\|a\|}(1,0,0)\times(0,a_1,a_2), \\ \text{angle of rotation :} & \cos\varphi = \frac{1-a^2}{1+a^2}, \sin\varphi = \frac{2\|a\|}{1+a^2}. \end{cases} \tag{4.4.25$'''$}$$

Fixing the axis of rotation and replacing in (4.4.23) and (4.4.23') the Lorentz transformations by its representants then we obtain a family of representations

$$U(R(\ell,d,\varphi)) = \Delta[\ell,\ell']^{\mathrm{i}s(a)}V^{\ell'}(-\frac{a}{1+a^2})V^{\ell}(a). \tag{4.4.26}$$

In this formula d means the normalized rotation axis. The angle φ does not admit the value π. In the original definition φ was non–negative, but we can drop this restriction by identifying $R(\ell,d,-\varphi)$ with $R(\ell,-d,\varphi)$.

Next we investigate the rotations defined in (4.4.26). First we show:

4.4.9 Lemma:

The operators $U(R(\ell,d,\varphi))$ defined in Eq. (4.4.26) do not depend on the argument ℓ. They are continuous in the direction d and in the angle provided $-\pi<\varphi<\pi$.

Proof. First we show that $U(R(\ell,d,\psi))$ is continuous in ψ. Notice first that $V^{\ell'}(-\frac{a}{1+a^2})V^{\ell}(a)$ is weakly continuous in a and by the unitarity of the product also strongly continuous. Repeating this argument we find that the expression $\Delta[\ell,\ell']^{\mathrm{i}s(a)}$ $V^{\ell'}(-\frac{a}{1+a^2})V^{\ell}(a)$ is continuous in a. If we keep the direction of a fixed then we obtain that $U(R(\ell,d,\psi))$ is continuous in ψ.

Next we show that the expression $U(R(\ell,d,\varphi))$ depends continuously on ℓ. Notice first that the definition of $U(R(\ell,d,\psi))$ implies the relation

$$\mathrm{Ad}\,U(R(\ell,d,\psi))\Delta[\ell_1,\ell_2]^{\mathrm{i}t} = \Delta[R(\ell,d,\psi)\ell_1, R(\ell,d,\psi)\ell_2]^{\mathrm{i}t}.$$

Consequently (4.4.26) implies

$$\mathrm{Ad}\,U(R(\ell,d,\psi))U(R(\ell_1,d,\varphi) = U\Big(R(R(\ell,d,\psi)\ell_1,d,\varphi)\Big), \tag{4.4.26$'$}$$

in case ℓ_1 is perpendicular to d. From this we obtain continuity in ℓ since we know the continuity of $U(R(\ell,d,\psi))$ in ψ.

Let now φ be an irrational multiple of 2π. Then $\{n\varphi \mod 2\pi;\ n \in \mathbb{Z}\}$ is dense in the open interval $(-\pi, \pi)$. Choosing $\psi = \varphi$ in (4.4.26') and $\ell_1 = \ell$ then we obtain

$$U(R(\ell, d, \varphi)) = U(R(R(\ell, d, \varphi)\ell, d, \varphi)).$$

Iterating this equation we get:

$$\begin{aligned} U(R(\ell, d, \varphi)) &= U(R(R^n(\ell, d, \varphi)\ell, d, \varphi)) \\ &= U(R(R(\ell, d, n\varphi)\ell, d, \varphi)), \quad n \in \mathbb{Z}. \end{aligned}$$

Using the continuity in ℓ we find that $U(R(\ell, d, \varphi))$ is independent of ℓ, provided $\varphi/2\pi$ is irrational. Since $U(R(\ell, d, \varphi))$ is continuous in φ it follows the independence of ℓ for all φ. Since $U(R(d, \varphi))$ is continuous in φ we conclude from

$$\mathrm{Ad}\, U(R(d', \varphi))U(R(d, \psi)) = U(R(R(d', \varphi)d, \psi))$$

that $U(R(d, \varphi))$ is also continuous at d in any direction. Since this is true for any point d on the unit–sphere we obtain continuity in d. □

Knowing the identity of the different representations of the rotations we can make a further transformation of the expression (4.4.22).

4.4.10 Lemma:

The expression (4.4.22) *is equivalent to one of the expressions*

$$\begin{aligned} &\Lambda[\ell, \ell'](t_0)\Lambda^{\ell}(a_0)\prod_{i=1}^{m} R(\ell, d(i), \varphi(i)), \\ &\Lambda[\ell, \ell'](t_0)\Lambda^{\ell'}(a_0)\prod_{i=1}^{m} R(\ell, d(i), \varphi(i)). \end{aligned} \tag{4.4.27}$$

Proof. Assume that at the end of (4.4.22) there is an element $\Lambda^{\ell}(a)$. We can replace it by $\Lambda^{\ell'}(\frac{a}{1+a^2})\Lambda[\ell, \ell'](-s(a))\Lambda[\ell, \ell'](s(a))\Lambda^{\ell'}(-\frac{a}{1+a^2})\Lambda^{\ell}(a)$. The last three factors give rise to an element $R(\ell, d(a), \varphi(a))$. By using (4.4.19') the Λ–factor can be commuted to the left. The remaining $\Lambda^{\ell'}$–factor can be combined with the factor of the same kind which was to the left of $\Lambda^{\ell}(a)$. Therefore, at the end we find after these manipulations an expression of the form $\Lambda^{\ell'}(b)R(\ell, d(a), \varphi(a))$. Now we can perform with $\Lambda^{\ell'}(b)$ the similar manipulation and obtain a factor $R(\ell', d(b), \varphi(b))$. This can be replaced by $R(\ell, d(b), -\varphi(b))$. So we obtained for the last two factors of (4.4.22) the factors $R(\ell, d(b), -\varphi(b))R(\ell, d(a), \varphi(a))$. Repeating this procedure we end up with one of the expressions (4.4.27). If there is an element $\Lambda^{\ell'}(b)$ at the end of (4.4.22) the procedure is the same. □

We are interested in the situation where the expression (4.4.27) is of the value 1. In this situation (4.4.27) can be simplified.

4.4.11 Lemma:

Assume (4.4.27) *has the value* 1. *Then one finds* $\Lambda[\ell,\ell'](t_0) = 1$ *and* $\Lambda^{\ell}(a_0) = 1$ *and* $\Lambda^{\ell'}(a_0) = 1$.

Proof. We consider the first line of (4.4.27). Since the product has the value 1 it follows that ℓ is mapped onto itself. Since the first two factors leave the direction of ℓ unchanged the same must be true for the product of the rotations. But this implies that the product of the rotations, which does not change t, maps ℓ onto itself. Hence we get $\Lambda[\ell,\ell'](t_0)\ell = \ell$ which implies $t_0 = 0$. Since the product of the rotations maps ℓ onto itself it also keeps ℓ' fixed, which must be true also for $\Lambda^{\ell}(a_0) = 1$. This implies $a_0 = 0$. The second line of (4.4.27) can be handled in the same manner. □

Knowing that $U(R(d,\varphi))$ depends only on the direction of the axis of rotation and the rotation angle we have to show that these operators form for fixed axis of rotation a representation of the circle group.

4.4.12 Proposition:

For fixed axis of rotation the operators $U(R(d,\varphi))$ *give rise to a representation of the rotation group. This implies in particular that*

$$U(R(d,\pi)) = \lim_{\varphi\nearrow\pi} U(R(d,\varphi))$$

exists and $U(R(d,\varphi))$ *is continuous in* φ *on the whole circle.*

Since the proof of this proposition is straight forward but lengthy we will present it in the appendix.

Now we are prepared for the main result.

4.4.13 Theorem:

Assume the modular group of every wedge algebra $\mathcal{M}(W[\ell_1,\ell_2,a])$ *acts on every algebra of a double cone like the associated group of Lorentz boosts. Then the modular groups* $\Delta^{\mathrm{i}t}[\ell_1,\ell_2,a]$ *define a representation of the Poincaré group.*

Proof. In the beginning we have constructed the translation so that it remains to construct the Lorentz transformations. To this end we have to show that the equation $\prod \Lambda[\ell_1^{(i)},\ell_2^{(i)}](t^{(i)}) = 1$ implies the relation $\prod \Delta[\ell_1^{(i)},\ell_2^{(i)}]^{\mathrm{i}t^{(i)}} = \mathbb{1}$. We saw in (4.4.22) that the product can be transformed into

$$\Lambda[\ell,\ell'](t^{(0)}) \prod_{i=1}^{m} \Lambda^{\ell'}(b^{(i)})\Lambda^{\ell}(a^{(i)}) = 1.$$

To show this the principle of half–sided modular inclusion was needed. Using Lemma 4.4.10 and Lemma 4.4.11 the product in question can be transformed into $\prod R(\ell,d(i),\varphi(i)) = 1$. So it remains to show that this implies $\prod U(R(\ell,d(i),\varphi(i)) = \mathbb{1}$. From the relation

$$\mathrm{Ad}\, U(R(d_1,\varphi_1))U(R(d_2,\varphi_2) = U(R(R(d_1,\varphi_1)d_2,\varphi_2)$$

it follows that the operators $U(R(d,\varphi))$ give rise at most to a central extension of the rotation group. Since we know that the representations are unique for the rotations around a fixed axis we conclude by Mackey's method of induced representations [Mac68] that the $U(R(d,\varphi))$ form a single valued representation of the whole rotation group. Hence follows $\prod U(R(\ell,d(i),\varphi(i)) = \mathbb{1}$. □

Appendix:

Proof of Proposition 4.4.12: Due to the independence of $U(R(\ell,d,\varphi))$ from ℓ we obtain with $a^2=1$ the relation

$$\Delta[\ell,\ell']^{\mathrm{i}s(\alpha a)}V^{\ell'}(-\frac{\alpha a}{1+\alpha^2})V^{\ell}(\alpha a) = \Delta[\ell,\ell']^{-\mathrm{i}s(\alpha a)}V^{\ell}(\frac{\alpha a}{1+\alpha^2})V^{\ell'}(-\alpha a). \tag{4.4.28}$$

Applying $\mathrm{Ad}\,\Delta[\ell,\ell']^{\mathrm{i}t}$ to this relation we find by (4.4.19')

$$\Delta[\ell,\ell']^{2\mathrm{i}s(s)}V^{\ell'}(-\frac{\mathrm{e}(-t)\alpha a}{1+\alpha^2})V^{\ell}(\mathrm{e}(t)\alpha a) = V^{\ell}(\frac{\mathrm{e}(t)\alpha a}{1+\alpha^2})V^{\ell'}(-\mathrm{e}(-t)\alpha a). \tag{4.4.28$'$}$$

Notice: If we fix the vector ℓ and the axis of rotation d then we have also fixed a. Therefore, we obtain

$$\Delta[\ell,\ell']^{\mathrm{i}s(\lambda a)}V^{\ell'}(-\frac{\lambda a}{1+\lambda^2})V^{\ell}(\lambda a)\Delta[\ell,\ell']^{\mathrm{i}s(\mu a)}V^{\ell'}(-\frac{\mu a}{1+\mu^2})V^{\ell}(\mu a) \tag{4.4.29}$$

for the product of two rotations around the same axis. Using (4.4.19') this expression becomes

$$\Delta[\ell,\ell']^{\mathrm{i}(s(\lambda a)+s(\mu a))}V^{\ell'}(-\frac{\mathrm{e}(s(\mu a))\lambda a}{1+\lambda^2})V^{\ell}(\mathrm{e}(-s(\mu a))\lambda a)V^{\ell'}(-\frac{\mu a}{1+\mu^2})V^{\ell}(\mu a). \tag{4.4.29$'$}$$

We want to apply formula (4.4.28') to the third and fourth factor of the expression (4.4.29'). This implies the following identifications:

$$\begin{aligned}\mathrm{e}(-t)\alpha &= \frac{\mu}{1+\mu^2},\\ \frac{\mathrm{e}(t)\alpha}{1+\alpha^2} &= \mathrm{e}(-s(\mu a))\lambda = \lambda(1+\mu^2).\end{aligned} \tag{4.4.30}$$

For the last transformation we have used (4.4.24"). Since the left sides have the same sign, this must also hold for the right sides. Hence we get the restriction $\lambda\mu>0$. We can solve (4.4.30) and obtain

$$\begin{aligned}1+\alpha^2 &= \frac{1}{1-\lambda\mu},\\ \mathrm{e}(t)\alpha &= \frac{\lambda(1+\mu^2)}{1-\lambda\mu}, \quad \lambda\mu\neq 1, \lambda\mu>0,\\ \frac{\mathrm{e}(-t)\alpha}{1+\alpha^2} &= \frac{\mu(1-\lambda\mu)}{1+\mu^2}.\end{aligned} \tag{4.4.30$'$}$$

Inserting (4.4.30') into (4.4.29') then the expression (4.4.29) obtains the form

$$\begin{aligned}\Delta[\ell,\ell']^{\mathrm{i}(s(\lambda a)+s(\mu a))}V^{\ell'}(-\frac{\lambda a}{(1+\lambda^2)(1+\mu^2)})\Delta[\ell,\ell']^{2\mathrm{i}(s(\alpha a)}\\ \times V^{\ell'}(-\frac{\mu(1-\lambda\mu)}{1+\mu^2}a)V^{\ell}(\frac{\lambda(1+\mu^2)}{1-\lambda\mu}a)V^{\ell}(\mu a)\\ =\Delta[\ell,\ell']^{2\mathrm{i}(s(\alpha a)+2s(\alpha a)}V^{\ell'}(-\Big[\frac{\mathrm{e}(2s(\alpha a))\lambda a}{(1+\lambda^2)(1+\mu^2)})+\frac{\mu(1-\lambda\mu)}{1+\mu^2}a)\Big]a)\\ \times V^{\ell}(\Big[\frac{\lambda(1+\mu^2)}{1-\lambda\mu}+\mu\Big]a).\end{aligned}\tag{4.4.30''}$$

The argument of the operator V^ℓ becomes $\frac{\lambda+\mu}{1-\lambda\mu}$. For computing the argument of $V^{\ell'}$ notice first the relation

$$\mathrm{e}(2s(\alpha a))=\mathrm{e}(s(\alpha a))^2=\frac{1}{(1+\alpha^2)^2}=(1-\lambda\mu)^2.$$

Inserting this we find

$$\begin{aligned}\mathrm{e}(2s(\alpha a))\frac{\lambda}{(1+\lambda^2)(1+\mu^2)}+\frac{\mu(1-\lambda\mu)}{1+\mu^2})&=\frac{\lambda\mu}{1-\lambda\mu}\frac{(1-\lambda\mu)^2}{(1+\lambda^2)(1+\mu^2)}\\ &=\frac{\frac{\lambda+\mu}{1-\lambda\mu}}{1+\left(\frac{\lambda+\mu}{1-\lambda\mu}\right)^2}.\end{aligned}$$

If we set

$$\frac{\lambda+\mu}{1-\lambda\mu}=\rho\tag{4.4.31}$$

then the product (4.4.29) becomes

$$\Delta[\ell,\ell']^{\mathrm{i}(s(\lambda a)+s(\mu a)+2s(\alpha a))}V^{\ell'}(-\frac{\rho}{1+\rho^2})V^{\ell}(\rho).\tag{4.4.29'''}$$

Finally it remains to look at the exponent of the modular operator. We know $\mathrm{e}(s(\lambda a))=\frac{1}{1+\lambda^2}$, which implies $s(\lambda a)=-\log(1+\lambda^2)$. Hence we obtain

$$\begin{aligned}s(\lambda a)+s(\mu a)+2s(\alpha a)&=-\log(1+\lambda^2)(1+\mu^2)(1+\alpha^2)^2\\ &=-\log\frac{(1+\lambda^2)(1+\mu^2)}{(1-\lambda\mu)^2}=-\log\Big(1+\Big(\frac{\lambda+\mu}{1-\lambda\mu}\Big)^2\Big)=-\log(1+\rho^2).\end{aligned}$$

Since ρ is symmetric in λ and μ it follows that the rotations around a fixed axis commute and give rise to a rotation (provided $\lambda\mu\neq 1$). It remains to show that the relation

$\varphi(\lambda a) + \varphi(\mu a) = \varphi(\rho a)$ is fulfilled. From (4.4.24‴) we obtain $\mathrm{e}^{\mathrm{i}\varphi(\lambda a)} = \frac{(1-\mathrm{i}\lambda)^2}{1+\lambda^2}$ from which we get

$$\mathrm{e}^{\mathrm{i}\varphi(\lambda a)}\mathrm{e}^{\mathrm{i}\varphi(\mu a)} = \frac{(1-\mathrm{i}\lambda)^2}{1+\lambda^2}\frac{(1-\mathrm{i}\mu)^2}{1+\mu^2} = \frac{\left(1-\mathrm{i}\frac{\lambda+\mu}{1-\lambda\mu}\right)^2}{1+\left(\frac{\lambda+\mu}{1-\lambda\mu}\right)^2}.$$

This shows that the group–relations are fulfilled.

The restriction for the calculation was $\lambda\mu > 0$ and $\lambda\mu \neq 1$. Therefore, we have to look at the angle π and at the product with different signs of the angle. Let us regard the second problem first. We find with (4.4.24") and (4.4.28)

$$\begin{aligned}
\left\{\Delta[\ell,\ell']^{\mathrm{i}s(\alpha a)}V^{\ell'}(-\frac{\alpha a}{1+\alpha^2})V^{\ell}(\alpha a)\right\}^{-1} &= V^{\ell}(-\alpha a)V^{\ell'}(-\frac{\alpha a}{1+\alpha^2})\Delta[\ell,\ell']^{-\mathrm{i}s(\alpha a)}\\
&= \Delta[\ell,\ell']^{-\mathrm{i}s(\alpha a)}V^{\ell}(-\mathrm{e}(s(\alpha a))\alpha a)V^{\ell'}(\mathrm{e}(-s(\alpha a))\frac{\alpha a}{1+\alpha^2})\\
&= \Delta[\ell,\ell']^{-\mathrm{i}s(\alpha a)}V^{\ell}(-\frac{\alpha a}{1+\alpha^2})V^{\ell'}(\alpha a)\\
&= \Delta[\ell,\ell']^{-\mathrm{i}s(\alpha a)}\Delta[\ell,\ell']^{2\mathrm{i}s(\alpha a)}V^{\ell'}(\frac{\alpha a}{1+\alpha^2})V^{\ell}(-\alpha a).
\end{aligned}$$

This implies

$$R(d,\varphi)^{-1} = R(d,-\varphi), \quad -\pi < \varphi < \pi.$$

From this we obtain the multiplication rule ($\varphi > \psi$):

$$R(d,\varphi)R(d,-\psi) = R(d,\varphi-\psi)R(d,\psi)R(d,-\psi) = R(d,\varphi-\psi).$$

A similar calculation is valid for $\varphi < \psi$. We have to discuss the point $\varphi = \pi$.

We define $R(d,\pi) := R(d,\pi/2)^2$. Since $R(d,\varphi)$ is continuous in φ we see by the multiplication rule and the continuity of the square that $R(d,\varphi)^2$ is defined for all values of $\varphi \neq \pi$. It remains to show that the product rule is fulfilled also for $\varphi = \pi$. Notice first that $R(d,\pi/2)^2 = R(d,\varphi)R(d,\pi-\varphi) = R(d,\pi)$ holds for $0 < \varphi < \pi$. From this one obtains

$$\begin{aligned}
R(d,\pi)R(d,\varphi) &= R(d,\frac{\pi+\varphi}{2})R(d,\frac{\pi-\varphi}{2})R(d,\varphi) = R(d,\frac{\pi+\varphi}{2})^2\\
&= R(d,\varphi-\pi).
\end{aligned}$$

Moreover, we get with $0 < \varphi < \pi/2$,

$$\begin{aligned}
R(d,\pi)^2 &= R(d,\pi-\varphi)R(d,\varphi)R(d,\pi-\varphi)R(d,\varphi) = R(d,\pi-\varphi)^2R(d,\varphi)^2 =\\
&R(d,-2\varphi)R(d,2\varphi) = 1.
\end{aligned}$$

This implies $R(d,\pi) = R(d,-\pi)$ and the proposition is proved. □

4.5) *The approach of Buchholz and Summers*

We saw that the Bisognano–Wichmann property for the modular groups implies Lorentz covariance, wedge duality and the PCT–theorem, provided the algebras of the double cones are the intersection of the wedge algebras. This implies in particular, that the modular conjugations of the wedge algebras act as reflection, i.e.

$$J_W \mathcal{M}(D) J_W = \mathcal{M}(P_W D). \tag{4.5.1}$$

Here P_W is the reflection in the characteristic two–plane of the wedge W, which leaves the apex of the wedge unchanged. If a is in the characteristic two–plane of W and $W = W(\ell_1, \ell_2, a)$ then with $x = \lambda\ell_1 + \mu\ell_2 + x^-$ one obtains

$$P_W x = -\lambda\ell_1 - \mu\ell_2 + x^- + 2a. \tag{4.5.2}$$

If the theory fulfils Eq.(4.5.1) for every double cone then we say it fulfils the Bisognano–Wichmann property for the modular conjugations. Since the Poincaré group is generated by the reflections (if the dimension of the Minkowski space is larger than two), it is natural to ask whether or not one can derive the Poincaré covariance also from the Bisognano–Wichmann property for modular conjugations. Using some additional assumptions this question has been answered for the translation positively by Buchholz and Summers [BS93].

Since every double cone is the intersection of wedges, it is no restriction if one requires Eq. (4.5.1) only for wedges. In a recent paper Buchholz, Dreyer, Florig and Summers [BDFS98] have generalized this setting by requiring that the modular conjugation of every wedge algebra maps only the family of all wedge algebras onto itself. This contains a hidden version of the wedge duality. Adding to this the assumptions that the modular conjugations preserve (I) isotony and (II) stability of non–intersection, they were able to show the following: Every transformation T of the set of wedges onto itself, and which together with its inverse fulfils (I) and (II), is a Poincaré transformation. If, in addition, the considered set of transformations T is a group, which acts transitively on the set of wedges and if the Minkowski space is four dimensional, then this group contains the identity component of the Poincaré group. In a very recent paper Buchholz, Florig and Summers [BFS99] showed that the adjoint representation of the translations of this group, acting on the wedge algebras, is necessarily continuous.

The group representation obtained from the modular conjugations must not fulfil the spectrum condition. In order to obtain this condition one has to add additional assumptions. The authors of [BDFS99] called one of the possibilities the modular stability condition.

It is interesting to notice that the method of Buchholz, Summers and co–workers can be transcribed to quantum field theories on de Sitter space. Whether or not this method can be generalized to other manifolds can only be answered by future calculations.

Here we will present the construction of the Poincaré group and show the continuity property of the translations. The continuity of the Lorentz transformations will only be discussed. Our construction of the Poincaré transformation differs in some points from that of [BDFS99].

In this section we define wedges slightly different from the notation in Sect. 1.5. Here $W(\ell_1, \ell_2, a)$ means that the lightlike vectors ℓ_1, ℓ_2 belong either to $\partial V^+, \partial V^-$ or to $\partial V^-, \partial V^+$, i.e. $(\ell_1, \ell_2) < 0$, and that the wedge $W(\ell_1, \ell_2, a + \rho_1 \ell_1 + \rho_2 \ell_2) \subset W(\ell_1, \ell_2, a)$ for $\rho_1, \rho_2 \geq 0$. This description is symmetric in both lightlike vectors and is better suited for dealing with time– or space reflections.

4.5.1 Definition:

Let $\mathcal{W}$ denote the set of all wedges. By $\mathcal{T}$ we denote the set of all transformations T,

$$T : \mathcal{W} \longrightarrow \mathcal{W},$$

such that T^{-1} exists and T as well as T^{-1} fulfil:
(I) Isotony, i.e. $W_1 \subset W_2$ implies $T(W_1) \subset T(W_2)$ and $T^{-1}(W_1) \subset T^{-1}(W_2)$.
(II) Stability of non–intersection, i.e. $W_1 \cap W_2 = \emptyset$ implies $T(W_1) \cap T(W_2) = \emptyset$ and $T^{-1}(W_1) \cap T^{-1}(W_2) = \emptyset$.

With these assumptions we will show:

4.5.2 Theorem:

Let the dimension of the Minkowski space be larger than 2. Then every transformation $T \in \mathcal{T}$ is an element of the full Poincaré group enlarged by the dilatations.

Before we come to the proof we introduce some

4.5.3 Notation:

(i) Let ℓ be a lightlike vector, then $H(\ell, a)$ denotes the set of vectors x such that $(x - a, \ell) = 0$. This is an affine hyperplane of dimension $d - 1$.
(ii) Recall that the characteristic two–plane of a wedge $W(\ell_1, \ell_2, a)$ is the plane generated by ℓ_1 and ℓ_2.
(iii) The supporting plane of the wedge $W(\ell_1, \ell_2, a)$ is the intersection of the two affine hypersurfaces $H(\ell_1, a)$ and $H(\ell_2, a)$.
(iv) Let ℓ be a lightlike vector. By $\mathcal{X}(\ell)$ we denote the set of wedges such that one of its vectors coincides (up to a positive factor) with ℓ. This means that the vector ℓ belongs always to V^+ or always to V^-.
(v) Let ℓ be a lightlike vector. By $X(\ell, a)$ we denote the set of wedges in $\mathcal{X}(\ell)$ such that their supporting planes belong to $H(\ell, a)$.
(vi) Let ℓ_1, ℓ_2 belong to different light cones. Then $\mathcal{Y}(\ell_1, \ell_2)$ denotes the set of all translates of $W(\ell_1, \ell_2, 0)$.
(vii) $Y(\ell_1, \ell_2; a, \ell_1)$ denotes the set of wedges in $\mathcal{Y}(\ell_1, \ell_2)$, such that their supporting planes belong to $H(\ell_1, a)$. $Y(\ell_1, \ell_2; a, \ell_2)$ is defined similarly.
(viii) $F(a)$ denotes the set of all wedges such that a is contained in their supporting planes.

Using only the isotony property of the elements in $\mathcal{T}$ we show:

4.5.4 Lemma:

Every $T \in \mathcal{T}$ maps the sets $Y(\ell_1, \ell_2; a, \ell_1)$ onto the sets $Y(\ell_1', \ell_2'; a', \ell_1')$ and also classes $\mathcal{Y}(\ell_1, \ell_2)$ onto classes $\mathcal{Y}(\ell_1', \ell_2')$.

Proof: Since all elements in $\mathcal{Y}(\ell_1, \ell_2)$ have the same lightlike vectors it follows that to two of the wedges exists a third containing both. Hence by isotony T maps $\mathcal{Y}(\ell_1, \ell_2)$ onto a class of the same kind. Since the intersection of the characteristic two–plane with the supporting plane is one point, the elements in $\mathcal{Y}(\ell_1, \ell_2)$ can be uniquely characterized by a point in the characteristic two–plane. This two–plane is isomorphic to the two dimensional Minkowski space. Defining a map such that the multiples of the vector ℓ belonging to V^+ are mapped onto the x^+ axis and the negative of the multiples of the vector ℓ belonging to V^- are mapped onto x^-, then the order of inclusion in $\mathcal{Y}(\ell_1, \ell_2)$ becomes the order by V^+ in the two–dimensional Minkowski space. By this transformation T induces an order preserving map $\gamma(T)$ on the two–dimensional Minkowski space. Hence by a result of Zeeman [Ze64] $\gamma(T)$ sends light rays onto light rays. This is equivalent to the statement that T sends $Y(\ell_1, \ell_2, a)$ onto sets of the same kind. □

Next we want to show that T maps families $\mathcal{X}(\ell)$ onto families of the same kind. For this result also the non–intersection property is needed.

4.5.5 Lemma:

Every $T \in \mathcal{T}$ sends families of the form $X(\ell, a)$ onto families of the same form. In particular classes $\mathcal{X}(\ell)$ are sent onto classes $\mathcal{X}(\ell')$.

Proof: Choose a family $Y(\ell_1, \ell_2; a, \ell_1)$ and look at the elements in $\mathcal{W}$ which have an empty intersection with every element in $Y(\ell_1, \ell_2; a, \ell_1)$. These consist of the union of all $Y(-\ell_1, \ell_3; b, -\ell_1)$ such that $b = a + \rho(-\ell_2)$ with $\rho \geq 0$. Notice that the family $Y(-\ell_1, \ell_3; b, -\ell_1)$ is ordered. We say $Y(-\ell_1, \ell_3; b_1, -\ell_1) \succ Y(-\ell_1, \ell_3; b_2, -\ell_1)$ if every element in $Y(-\ell_1, \ell_3; b_2, -\ell_1)$ is contained in one element belonging to $Y(-\ell_1, \ell_3; b_1, -\ell_1)$. Thus the maximal element is of the form $Y(-\ell_1, \ell_3; a, -\ell_1)$, and hence the union of the maximal elements is just $X(-\ell_1, a)$. Since T preserves order of inclusion it also preserves the order $\succ$. Hence T maps $X(-\ell_1, a)$ into sets of the same kind. But since T is a bijection it follows that the map is surjective. Using isotony again we find that the family of sets $\mathcal{X}(\ell)$ is mapped by T onto itself. □

Next we want to look at the families $F(a)$ and want to show that they are mapped onto families of the same kind. For this we need several preparations.

4.5.6 Corollary:

Every element $T \in \mathcal{T}$ maps opposite wedges onto opposite wedges.

Proof: A wedge $W(\ell_1, \ell_2, a)$ is the unique element belonging to $X(\ell_1, a) \cap X(\ell_2, a)$. The opposite wedge is the intersection of $X(-\ell_1, a)$ with $X(-\ell_2, a)$. Since $X(-\ell_i, a)$ is the maximal element in the complement of $X(\ell_i, a)$, $i = 1, 2$ we see that T maps $X(-\ell_i, a)$ onto the maximal element in the complement of $T(X(\ell_i, a))$. This implies the statement of the corollary. □

Let us take d linear independent lightlike vectors in V^+. Then the point $\{0\}$ can be characterized in d different ways, namely by the intersection of the supporting planes of the wedges $W(-\ell_i, \ell_j, 0)$, $i \neq j$. If we apply T to this situation then every of these families define a point a'_i. We want to show that all these points coincide.

4.5.7 Lemma:

Let $\ell_i,\ i=1,...d$ be lightlike vectors belonging to ∂V^-, and $T\in\mathcal{T}$. Let the vectors ℓ_i be such that their images ℓ'_i are linearly independent. (Such families exist because of the property of T^{-1}.) Let a be fixed. Consider the d families of $d-1$ wedges $\{W(-\ell_i,\ell_j,a),\ i\neq j\}$. For $T\in\mathcal{T}$ let $T(W(-\ell_i,\ell_j,a_{i,j}))=W(-\ell'_i,\ell'_j,a'_{i,j})$. The intersection of the supporting planes of $W(-\ell'_i,\ell'_j,a'_{i,j})$ for fixed i defines a point a'_i. Then all the a'_i coincide.

Proof: From Lemma 4.5.5 and Cor. 4.5.6 we know that $T(W(-\ell_i,\ell_j,a))$ is of the form $W(-\ell'_i,\ell'_j,a'_{i,j})$, where ℓ'_i is independent of the other arguments. Moreover, Cor. 4.5.6 implies that we can choose $a'_{i,j}=a'_{j,i}$. Since $a'_{i,j}$ and a'_i belong both to the supporting space of $T(W(-\ell_i,\ell_j,a))$ we can write $a'_{i,j}=a'_i+\xi_{i,j}$, where the vector $\xi_{i,j}$ is perpendicular to ℓ'_i and ℓ'_j. From this we obtain by taking the difference of $a'_{i,j}$ and $a'_{j,i}$ that $a'_i-a'_j$ is perpendicular to ℓ'_i and ℓ'_j. Because of $a'_i-a'_j=(a'_i-a'_k)-(a'_j-a'_k),\ k\neq i,j$ we obtain that $a'_i-a'_j$ is also perpendicular to ℓ'_k. Since these vectors are linear independent we get $a'_i=a'_j$. □

Next we generalize this result.

4.5.8 Lemma:

Let $T\in\mathcal{T}$. Let $\ell_i,\ i=1...d+1$ be vectors belonging to V^-. Assume that the vectors ℓ_i are such that the d vectors $\ell'_j,\ i\neq j,\ i$ fixed are linear independent. Assume all wedges $W(-\ell_i,\ell_j,a_{i,j})$ are such that their supporting planes contain the point a. Then the supporting planes of the images $W(-\ell'_i,\ell'_j,a'_{i,j})$ contain a unique point a'.

Proof: Let $a'i$ be the unique point of the family $W(-\ell'_i,\ell'_j,a'_{i,j})$ described in the last lemma. Then one has the relation $a'_{i,j}=a'_i+\xi_{i,j}$, where the vector $\xi_{i,j}$ is perpendicular to ℓ'_i and ℓ'_j. From $a'_{i,j}=a'_{j,i}$ one obtains $a'_i-a'_j$ is perpendicular to ℓ'_i and ℓ'_j. Writing again $a'_i-a'_j=(a'_i-a'_k)-(a'_j-a'_k)$ we obtain that $a'_i-a'_j$ is also perpendicular to $\ell'_k,\ k\neq i,j$. Hence these differences vanish. □

Combining the last two lemmata we obtain

4.5.9 Corollary:

Let $T\in\mathcal{T}$. Then T maps a family $F(a)$ onto a family of the same type, i.e. onto $F(a')$. The induced map $\tau: a\to a'$ is a bijection of the Minkowski space.

Proof: We start with d vectors $\ell_i\in\partial V^-$ such that their images ℓ'_i are linear independent. Then the images of the wedges $W(-\ell_1,\ell_i,a)$ define the point a'. It remains to show that the image of $W(\ell_{d+1},\ell_{d+2},a)$ contains the point a' in its supporting plane. Assume the additional vector belonging to ∂V^+ is ℓ_{d+1}. If now $\ell'_i,\ i=1...d$ and $-\ell'_{d+1}$ are such that every d–tuple of these vectors are linear independent, then we can replace ℓ_1 by $-\ell_{d+1}$ without changing the point a'. If ℓ_{d+1} is not in such situation, then we may succesively vary the vectors ℓ_i without changing the point a' such that the new vectors ℓ_i and $-\ell_{d+1}$ are in the situation described above. Since $-\ell_{d+1}$ and ℓ_{d+2} and hence also $-\ell'_{d+1}$ and ℓ'_{d+2} are different we can repeat this procedure without changing the first vector, which is $-\ell_{d+1}$. Replacing now ℓ_2 by ℓ_{d+2} we find that the supporting plane of $W(\ell_{d+1},\ell_{d+2},a)$ contains the point a'. So T maps $F(a)$ onto $F(a')$. From the uniqueness of the action on classes

$Y(\ell_1, \ell_2; a, \ell_1)$ it follows that the associated map τ on the Minkowski space is injective. Since T^{-1} has the same property it follows that the associated map τ is a bijection. □

Now we are in the position to show Thm. 4.5.2.

Proof of Thm. 4.5.2: From the investigation of $\mathcal{X}(\ell)$ we know that the map $\ell \to \ell'$ sends ∂V^+ either to ∂V^+ or to ∂V^-. The second situation can be reduced to the first by multiplying T with the time inversion. Let $a_2 = a_1 + \rho\ell$ with $\ell \in \partial V^+$ and $\rho > 0$. We look at the sub–family of $F(a_1)$ respectively of $F(a_2)$ consisting of $F(a_i) \cap X(\ell, a_1)$. We denote these sets by $F(a_i, \ell)$. The situation $a_2 = a_1 + \rho\ell$ with the above conditions can be characterized by the condition: To every W_2 in $F(a_2, \ell)$ exists an element W_1 in $F(a_2, \ell)$ such that $W_2 \subset W_1$ holds. Since a wedge $W(\ell', \ell_2', a)$ is contained in a wedge belonging to $F(a_1', \ell')$ only if $a = \rho_1\ell' + \rho_2\ell_2'$, $\rho_i \geq 0$, and since the inclusion holds for all possible ℓ_2', we conclude by the isotony condition $a_2' = \rho'\ell'$. This implies that τ maps light rays onto lightrays. Hence by the result of Zeeman [Zee64] or Borchers and Hegerfeldt [BH72] we conclude that τ is from the Poincaré group enlarged by the dilatations. From the relation $T(F(a)) = F(\tau a)$ we conclude that $\mathcal{T}$ defines the same group as the group of the τ's. □

Now we restrict our attention to the modular conjugations of the wedges $W \in \mathcal{W}$. Let T_W be the corresponding map in $\mathcal{T}$. Before we introduce the requirements for the maps T_W, we add one remark. If G is a subgroup of $\mathcal{T}$ which acts transitive on $\mathcal{W}$, and if the Minkowski space has dimension 4, then it is claimed in [BDFS98] that G contains the identity component of the Poincaré group. We will use this result which is plausible for arbitrary dymension. For the following treatment of T_W we need some

4.5.10 Notations:

(1) Let W be a wedge, then we denote by G_W the subgroup of the Poincaré group which maps W onto itself.
(2) G_W^1 denotes the intersection of G_W with the identity component of the Poincaré group.
(3) Let ℓ be a lightlike vector, then we denote by G^ℓ the translational part of the stabilizer group of ℓ (see Sect. 4.4).

Because of the special structure of wedges, D_W consists of two parts, the euclidean group of the supporting plane of W, and that part of the invariance group of the characteristic two–plane, which maps the wedge onto itself. The latter part consists of the Lorentz boosts associated with the wedge, and the time reflection (wich is different for different wedges),

For the transformations T_W associated with the modular conjugation of the algebra belonging to W we impose the following

4.5.11 Requirements:

(i) The group generated by all the T_W acts transitive on $\mathcal{W}$.
(ii) If two wedges have a non–trivial intersection, then the intersection of the wedge–algebras is large. This implies

$$T_W(W) \cap W = \emptyset.$$

(iii) Since T_W is the image of the conjugation associated with $\mathcal{M}(W)$ we require that T_W commutes with every element in G_W^1, i.e.

$$T_W T_g = T_g T_W, \qquad T_g \in G_W^1.$$

(iv) Since T_W represents a conjugation we must have

$$T_W^2 = 1.$$

With these requirements one obtains

4.5.12 Theorem:

Let T_W fulfil the requirements 4.5.1, then one has $T_W = P_W$, where P_W is the total reflection in the characteristic two–plane. This implies in particular the wedge duality

$$T_W(W) = W'.$$

Proof: We first draw a consequence of condition (ii), and assume that 0 is contained in the supporting plane of the wedge. Let $W = W(\ell_1, \ell_2, 0)$ then we know that $T_W(W)$ must belong to the union of $X(-\ell_1, a),\ a = -\rho\ell_2,\ \rho \geq 0$ and $X(-\ell_2, a),\ a = -\rho\ell_1,\ \rho \geq 0$. In order to obtain a wedge in that set we put $T_W^0 = (\Lambda^\ell P_W, a)$ with $\Lambda^\ell \in G^{\ell_1}$ or $\Lambda^\ell \in G^{\ell_2}$. The vector a belongs to the characteristic two–plane of the wedge and one has to distinguish three possibilities:
(i) $\Lambda^\ell \in G^{\ell_1}, \Lambda^\ell \neq 1$ implies $a = -\rho\ell_2 + \lambda\ell_1, \rho > 0, \lambda \in \mathbb{R}$.
(ii) $\Lambda^\ell \in G^{\ell_2}, \Lambda^\ell \neq 1$ implies $a = -\rho\ell_1 + \lambda\ell_2, \rho > 0, \lambda \in \mathbb{R}$.
(iii) $\Lambda^\ell = 1$ implies that a belongs to the complement of W.

Since T_W^0 and T_W map W onto the same wedge, they differ only by an element $T_g \in G_{T_W(W)}$. For simpler writing we commute T_g with T_W^0 and obtain an element in G_W. Therefore, T_W has the form

$$T_W = (\Lambda^\ell P_W T_g, a),$$

with $\Lambda^\ell, P_W, T_g, a$ as described above.

Next we turn to the requirement (iii). From the above mentioned structure of G_W we know that T_g maps the supporting plane of the wedge onto itself. The same is true for P_W. If Λ^ℓ is not the identity then the image of the supporting plane is no longer the original supporting plane. Since T_W has to commute with all translations in the supporting plane we conclude $\Lambda^\ell = 1$. Since $(P_W, 0)$ and $(1, a)$ commute both with the transformation group of the supporting plane, also T_g must commute with these transformations. Hence T_g must have the form $T_g = T_g^c T_g^s$, where T_g^s is the identity on the characteristic plane and a multiple of the identity in the supporting plane, i.e. either the identity or the total reflection on this plane T_g^c lies in the group generated by the Lorentz boosts and the time reflection. Since P_W commutes with the Lorentz boosts also T_g^c has to commute with the Lorentz boosts. This implies that T_g^c contains no time reflection, and hence it can only be a Lorentz boost of the wedge. It remains lo look at the translations $(1, a)$. Since the only vector in the characteristic two–plane invariant under the Lorentz boosts is $\{0\}$, also a must vanish. Therefore, T_W has the form

$$T_W = P_W \Lambda_W(t) T_g^s.$$

$\Lambda_W(t)$ is a Lorentz boost of the wedge W and T_g^s is either the identity or the total reflection of the supporting plane. Notice that these transformations commute. Therefore condition (iv) implies $\Lambda_W(t) = 1$. So it remains

$$T_W = \begin{cases} P_W, \\ -1. \end{cases}$$

Finally condition (i) implies

$$T_W = P_W.$$

For shifted wedges we obtain T_W by translations. Assume for instance $W = W_0 + a$ with a in the characteristic two–plane of W. Then one obtains

$$P_W^a = P_W^0 + 2a.$$

This implies wedge duality. □

Let $\mathcal{T}_j$ be the subgroup generated by the modular conjugations of all the wedges in $\mathcal{W}$. Assume one is dealing with a QFT on a Hilbert space $\mathcal{H}$ and that there exists a vector $\Omega \in \mathcal{H}$, which is cyclic and separating for all wedge algebras $\mathcal{M}(W)$. Assume, moreover, that the modular conjugation J_W fulfils the relation

$$J_W \mathcal{M}(W_1) J_W = \mathcal{M}(T_W(W_1)),$$
$$J_W J_{W_1} J_W = J_{T_W W_1}, \quad J_W = J_W.$$

Then the J_W generate an adjoint representation of the determinant $+1$ part of the Poincaré group.

Next we want to show that the representation generated by the J_W is a true representation. Let $W_1, ...W_n$ be wedges such that

$$\prod_{i=1}^{n} T_{W_i} = 1 \tag{4.5.3}$$

holds, then one has to show

$$\prod_{i=1}^{n} J_{W_i} = \mathbb{1}. \tag{4.5.4}$$

To this end we choose an arbitrary W and look at the expression $\prod_{i=1}^{n} J_{W_i}.J_W$. Using the above relation one obtains

$$\prod_{i=1}^{n} J_{W_i} J_W = \prod_{i=1}^{n} J_{W_i} J_{T_{W_n} W} J_{W_n} = ...$$
$$= J_{T_{W_1} ... T_{W_n} W} \prod_{i=1}^{n} J_{W_i} = J_W \prod_{i=1}^{n} J_{W_i}.$$

Therefore, $\prod_{i=1}^{n} J_{W_i}$ belongs to the center of the group generated by the J_W's. We now restrict to the four–dimensional situation. Later we will see that the group representation is continuous.

It remains to show that we are dealing with a true representation of tha Poincaré group. We know from section 4.4 how tedious such calculations are. Therefore, we skip this calculation and refere to the original paper [BDFS98]. Collecting the results we obtain

4.5.13 Theorem:

Let the dimension of the Minkowski space be 4, the representation of the "+"part of the Poincaré group induced by the J_W's is a true representation.

Next we are coming to the continuity problem and its solution described in [BFS99].

4.5.14 Proposition:

Let $U(\Lambda, a)$ be the representation of the Poincaré group obtained by the products of the J_W's. Then $U(1, a)$ is strongly continuous.

Proof: Let W be a wedge such that $\{0\}$ belongs to the supporting plane of W. Choose $a \in W$ (in the characteristic two–plane of W) and define $\{\bigcup_{t>0} \mathcal{M}(W + ta)\}'' = \mathcal{M}$. By construction one has $\mathcal{M} \subset \mathcal{M}(W)$.Let J_t be the modular conjugation of $\mathcal{M}(W + ta)$. We want to show that J_t converges strongly to J_0. We know from Thm. 2.1.1 that J_t converges strongly to J, where J denotes the modular conjugation of $\mathcal{M}$. Moreover, one has for sufficiently small t_1

$$\mathrm{Ad}\,(J_0 J_{t_1})\mathcal{M}(W + t_2 a) = \mathcal{M}(W + (t_2 - t_1)a) \subset \mathcal{M}.$$

Consequently

$$\mathrm{Ad}\,(J_0 J \mathcal{M}(W + t_2 a) \subset \mathcal{M}(W + t_2) \subset \mathcal{M}.$$

From this we obtain

$$\mathcal{M} \subset \mathcal{M}(W) = \mathrm{Ad}\,(J_0)\mathcal{M}(W') \subset \mathrm{Ad}\,(J_0)\mathcal{M}' = \mathrm{Ad}\,(J_0 J)\mathcal{M} \subset \mathcal{M},$$

this means $\mathcal{M} = \mathcal{M}(W)$ and hence J_t converges strongly to J_0. This implies that the representation of the translations in the a–direction is weakly– and by unitarity also strongly continuous. Hence the translations in the characteristic two–plane of W are strongly continuous. Changing W we obtain that the translations are continuously represented. □

The proof of the continuity of the Lorentz transformations will not be presented here. However, one can imagine how the above proof can be adapted to the situation where one looks at one–parametric subgroups $\Lambda(t)$ of the Lorentz group. One wants to compare the algebra $\mathcal{M}(\Lambda(t)W)$ with $\mathcal{M}(W)$. In order to do this one must assume, that Ω is also cyclic for the algebras $\mathcal{M}(\Lambda(t)W \cap W)$, provided t is sufficiently small. If this is the case one can look at the limit $t \searrow 0$ and argue as above.

Finally we come to the spectrum condition. As mentioned before, the representation of the translations induced by the J_W's does not have to fulfil the spectrum condition. In order to obtain the spectrum condition, Buchholz, Dreyer, Florig and Summers introduced a new assumption, which they called

4.5.15 Modular stability condition:

The modular group of every wedge is contained in the group generated by the modular conjugations.

Since the group generated by the J_W's is the + part of the Poincaré group, it is easy to see that the modular group of the wedge coincides (up to a scale factor) with the group of the Lorentz boosts associated with the wedge. Since Ω is also cyclic for the shifted wedges one can conclude, as in Sect. 4.4, that the spectrum of the translations is contained in the closure of either V^+ or V^-. In order to obtain this result one can also use the method of Wiesbrock [Wie92] which leads to the same conclusion.

We end this section with some

4.5.16 Remarks:

(1) If one knows that the operators J_W fulfil all the conditions we have used in this section, and if one knows from other sources that the theory enjoys the spectrum condition, then the group generated by the J_W's must not necessarily contain the modular groups of the wedge algebras. Even in the situation where one knows that the J_W are modular conjugations and that the spectrum condition is fulfilled, a proof is missing that $\mathcal{T}_j$ contains the modular groups of the wedges.
(ii) There exist QFTLO's which do not fulfil wedge duality, or others where the Lorentz covariance is missing (also for the wedge algebras). Such theories do not fulfil the Bisognano–Wichmann property neither for the modular groups nor for the modular conjugations. Hence these criteria are a selection criterium for both, the field theory and the vacuum state. The criterium in [BDFS98] has the advantage that it also applies to certain theories without spectrum condition. If these methods apply to QFT's on curved manifolds this might be an advantage. Whether or not it is an advantage for theories on Minkowski space is a question of taste, in particular since the so–called modular stability requirement is a sufficient but not a necessary condition implying that the spectrum is contained in the forward or backward light cone.

4.6) *Remarks, additions and problems*

(I) If the local algebras are generated by Wightman fields with finite components then the result of Bisognano and Wichmann Thm. 3.1.5 shows that the modular groups of the wedges coincide with the associated Lorentz boosts. On the other hand if we know the Bisognano–Wichmann property then we can derive Poincaré– and PCT–covariance for the local net. (Section 4.4 and 4.3.) But it is still an open problem whether or not the Bisognano–Wichmann property for a local net implies that this net is generated by Wightman fields. The existing attempts of constructing Wightman fields from local nets try to relate the field operator to the Hamilton operator (generator of the time translations, H–bounds methods) Fredenhagen and Hertel [FH81]. It might be useful to try to find relations with respect to the modular operator of the algebra of the wedge.

(II) The construction of the Poincaré group from the modular groups of the wedges is possible if the Bisognano–Wichmann property holds. The first construction under this condition has been given by Brunetti, Guido and Longo [BGL94]. Their method is based on group cohomology and therefore more elegant than the method presented here. However, their method has the disadvantage that it leads to a representation of the covering group. In order to obtain a true group representation Guido and Longo [GL95] enlarged the group by the modular conjugations. In addition they incorporated charged fields. In this frame they proved the PCT– and the spin and statistics theorem. This result implies that in the vacuum sector one has a true representation of the Poincaré group.

(III) In Tomita's modular theory one makes statements about the action of the modular group only on the algebra and its commutant. Therefore, it is unnatural to formulate the Bisognano–Wichmann property for all local algebras $\mathcal{M}(D)$. It should only be formulated for such D which belong to W or to W'. If one does this, one does not loose any information. This is a consequence of the following reason: The knowledge about the action inside W suffices to conclude that the algebras associated with the translates of a wedge along one of its defining lightlike vectors fulfil the condition of $\mp$half–sided modular inclusion with respect to $\mathcal{M}(W)$. With help of Thm. 2.6.2 one obtains the translations in the characteristic two–plane of W. Since by Thm. 2.5.2 one knows the commutation between these translations and the modular group one can determine the action of this group on arbitrary $\mathcal{M}(D)$. One finds the full Bisognano–Wichmann property for the modular groups. This procedure has been worked out by D. Guido [Gui95].

Unfortunately the Bisognano–Wichmann property for the modular conjugations can not be replaced by a local version. If we only know the action inside the wedge then we cannot compute the action of J_W on $J_{W'}$. Therefore, we are not able to conclude that the products $J_W J_{W'}$ give rise to a representation of a central extension of the Poincaré group. Hence if we assume that the modular group of the wedge algebra is contained in the group generated by the J_W's, we are not able to conclude that the modular groups fulfil the Bisognano–Wichmann property.

(IV) The Bisognano–Wichmann property for the modular groups is essential for the derivation of the CPT–theorem. Since this condition is probably hard to verify in concrete examples, one has to look for conditions which imply this property. The whole Buchholz Summers program, if restricted to the Minkowski space, is of this nature. If we start from a Poincaré covariant theory, then the wedge duality and the reality condition also implies the Bisognano–Wichmann property for the modular groups. One should add other assumptions implying this property.

(V) If a Poincaré covariant QFTLO fulfils the Bisognano–Wichmann property for the modular groups then it can happen that the theory is covariant under two different representations of the Poincaré group. In this case holds [Bch98b]:

4.6.1 Theorem:

Assume we are dealing with a local quantum field theory in the vacuum sector, which is covariant under two different vacuum representations of the Poincaré group. Let $U_0(\Lambda, a)$ be the representation generated by the modular groups of the wedge algebras and $U_1(\Lambda, a)$ the second representation. Then there exists a local gauge transformation of the Lorentz group $G(\Lambda)$ with

$$U_1(\Lambda, a) = U_0(\Lambda, a)G(\Lambda).$$

Moreover, $G(\Lambda)$ commutes with $U_0(\Lambda', a)$ for all a, Λ, Λ'. In addition $G(\Lambda)$ is a gauge transformation, i.e. it maps every local algebra onto itself.

That this situation occurs shows the following example: Take an infinite number of copies of a finite component Wightman field. Let $U(\Lambda, a)$ be the representation of the Poincaré group transforming the Wightman field. Let $G(\Lambda)$ be a representation of the Lorentz group which acts on the indices numbering the copies. Then $U(\Lambda, a) \otimes \mathbb{1}$ is the group generated by the modular groups and $U(\Lambda, a) \otimes G(\Lambda)$ is the second representation.

(VI) The reality condition together with the wedge duality implies the Bisognano–Wichmann property. Recently Guido and Wiesbrock (see Schroer and Wiesbrock [SW98]) have given a different condition which replaces the reality condition 4.2.1.

4.6.2 Theorem:

Assume we are dealing with a QFTLO on the vacuum sector. Assume that for every wedge the map

$$A\Omega \longrightarrow U(\Lambda_W(-\frac{\mathrm{i}}{2}))A^*\Omega$$

is bounded for $A \in \mathcal{M}(W)$. Here $U(\Lambda_W(t))$ denotes the group of boosts associated with W. Then the theory fulfils the Bisognano–Wichmann property.

(VII) Inspired by the result that $\mathcal{M}(W[\ell, \ell_1]) \cap W[\ell, \ell_2])$, $\ell_1 \neq \ell_2$ fulfils the condition of $-$half–sided modular inclusion with respect to both algebras $\mathcal{M}(W[\ell, \ell_1])$ and $\mathcal{M}(W[\ell, \ell_2])$ (see Thm. 4.4.1) H.-W. Wiesbrock has introduced the concept of "modular intersection".

4.6.3 Definition:

Let $\mathcal{M}, \mathcal{N}$ be two von Neumann algebras with a common cyclic and separating vector Ω. One says that $(\mathcal{M}, \mathcal{N}, \Omega)$ have the $\mp$modular intersection property if:
I. $\mathcal{M} \cap \mathcal{N}$ fulfils the condition of $\mp$half–sided modular inclusion with respect to both algebras $\mathcal{M}$ and $\mathcal{N}$.
II. There holds

$$J_{\mathcal{N}}(s - \lim_{t\to\pm\infty} \Delta_{\mathcal{N}}^{\mathrm{i}t}\Delta_{\mathcal{M}}^{-\mathrm{i}t})J_{\mathcal{N}} = (s - \lim_{t\to\pm\infty} \Delta_{\mathcal{M}}^{\mathrm{i}t}\Delta_{\mathcal{N}}^{-\mathrm{i}t}).$$

In a QFTLO which fulfils the Bisognano–Wichmann property the modular intersection condition is fulfilled for the algebras of two wedges which have the first– or the second light ray in common. The condition II is a consequence of Lemma 4.4.5. In particular the existence of the strong limit is guaranteed by the first condition. If we set $(s - \lim_{t\to\pm\infty} \Delta_{\mathcal{N}}^{\mathrm{i}t}\Delta_{\mathcal{M}}^{-\mathrm{i}t}) = U$ then condition II reads $J_{\mathcal{N}}UJ_{\mathcal{N}} = U^*$.

Using a finite number of pairs fulfilling the condition of modular intersection one is able to reconstruct the algebras of all non–translated wedges. This program has been taken up by H.-W. Wiesbrock [Wie97b],[Wie98], where he solved the problem for $\mathbb{R}^3$. Here he needs three wedges which are localized in such a way that the algebras of every pair fulfils the condition of $-$ or $+$modular intersection. Adding one shifted wedge which fulfils the condition of half–sided modular inclusion, he was able to construct the algebras of all wedges (including the translated ones) and a continuous representation of the Poincaré group which fulfils the spectrum condition.

Taking the intersection of wedge algebras on can construct the algebras for the double cones. Unfortunately one is not able to conclude that Ω is also cyclic for these algebras except one starts from a QFTLO.

5. Properties of local algebras

For several applications one wants to know the structure of the local algebras. The questions of interest are usually the factor property, the type of the algebra, and the action of symmetry groups. Before entering into the subject we have to collect some results of the Tomita–Takesaki theory.

5.1) *Some mathematical consequences of the modular theory*

The first concept is the generalization of the center of a von Neumann algebra.

5.1.1 Definition:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Set $\omega(A) = (\Omega, A\Omega)$, $A \in \mathcal{M}$. The centralizer of ω consists of all elements $Z \in \mathcal{M}$ for which

$$\omega(ZA) = \omega(AZ), \quad \forall \quad A \in \mathcal{M}$$

holds.

If Z belongs to the centralizer, then the KMS–condition implies

$$\sigma^t(Z) = Z, \qquad t \in \mathbb{R}$$

and viceversa. In particular the center of $\mathcal{M}$ belongs to the centralizer.

It might happen that a von Neumann algebra is too large in order to possess separating states. In this case one has to generalize the concept of states. They are called weights.

5.1.2 Definition

(a) Let $\mathcal{M}$ be a von Neumann algebra. A weight is a mapping

$$\omega : \ \mathcal{M}^+ \longrightarrow [0, \infty]$$

with the properties:
(α) $\omega(\rho A) = \rho\omega(A), \quad \rho \in \mathbb{R}^+,\ A \in \mathcal{M}^+$
with the multiplication rule $0.\infty = 0$.
(β) $\omega(A + B) = \omega(A) + \omega(B), \quad A, B \in \mathcal{M}^+$
(b) A weight ω is called semi–finite if

$$n_\omega := \{A \in \mathcal{M}; \omega(A^*A) < \infty\}$$

is strongly dense in $\mathcal{M}$.

(c) ω is called faithful if $A \in \mathcal{M}^+$ and $\omega(A) = 0$ implies $A = 0$.

(d) A weight is called normal if for every increasing net $A_\alpha \in \mathcal{M}^+$ there holds

$$\omega(\lim_\alpha A_\alpha) = \lim_\alpha \omega(A_\alpha).$$

The set n_ω is a linear space and by the linear extension of ω this becomes a pre–Hilbert space. Moreover, n_ω is a left–ideal so that one gets a representation of $\mathcal{M}$ by

$$\pi_\omega(B)A\rangle = BA\rangle.$$

If ω is a normal, faithful, semi–finite weight, then one can handle the Tomita–Takesaki theory in almost the same manner as with normal faithful states. (See U. Haagerup [Hgr75].) The advantage of this concept is the existence of normal, faithful, semi–finite weights for every von Neumann algebra. We need weights only for the discussion of symmetries in section 5.4. Otherwise we use only von Neumann algebras which have normal, faithful states.

Another important aspect of the Tomita–Takesaki theory is the natural cone associated with a von Neumann algebra. It is often denoted by $\mathcal{P}^\natural$. Here we will use the notation $\mathcal{H}^+$.

5.1.3 Lemma:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector Ω. Let (Δ, J) be the modular operator and conjugation of $(\mathcal{M}, \Omega)$. Then the following sets coincide and are called the natural cone of $(\mathcal{M}, \Omega)$.

(i) Closure of $\Delta^{1/4}\mathcal{M}^+\Omega$.
(ii) Closure of $\Delta^{-1/4}\mathcal{M}'^+\Omega$.
(iii) Closure of $\{Aj(A)\Omega;\ A \in \mathcal{M}\}$.

For the proof see [BR79] Prop. 2.5.26. Some of the properties of $\mathcal{H}^+$ are listed in the following

5.1.4 Proposition:

Let $\mathcal{H}^+$ be the natural cone of $(\mathcal{M}, \Omega)$. Then holds:

(i) $\mathcal{H}^+$ is a proper cone, i.e. $\mathcal{H}^+ \cap (-\mathcal{H}^+) = \{0\}$.
(ii) With $\mathcal{H}_r = \{\psi \in \mathcal{H}; J\psi = \psi\}$ one gets $\mathcal{H}_r = \mathcal{H}^+ - \mathcal{H}^+$.
(iii) $\mathcal{H}^+$ is a self–dual cone in $\mathcal{H}_r$, i.e. $\psi \in \mathcal{H}_r$ and $(\psi, \varphi) \geq 0\ \forall\ \varphi \in \mathcal{H}^+$ implies $\psi \in \mathcal{H}^+$.
(iv) For every $\psi \in \mathcal{H}^+$ and $A \in \mathcal{M}$ one has $Aj(A)\psi \in \mathcal{H}^+$.
(v) $\Delta^{\mathrm{i}t}\mathcal{H}^+ = \mathcal{H}^+$ for all $t \in \mathbb{R}$.

For the proof see [BR79] Props. 2.5.26, 2.5.27, 2.5.28. The natural cone has some universality properties listed in the following

5.1.5 Theorem:

Let $\mathcal{H}^+$ be the natural cone of $(\mathcal{M}, \Omega)$. Then:

(*i*) *To every normal, positive linear functional ω on $\mathcal{M}$ exists a unique vector $\psi_\omega \in \mathcal{H}^+$ with*

$$\omega(A) = (\psi_\omega, A\psi_\omega), \quad A \in \mathcal{M}.$$

(*ii*) *The mapping $\omega \leftrightarrow \psi_\omega$ is continuous in both directions. The following estimate holds:*

$$\|\psi_\omega - \psi_\rho\|^2 \leq \|\omega - \rho\| \leq \|\psi_\omega - \psi_\rho\| \|\psi_\omega + \psi_\rho\|.$$

(*iii*) *Assume the vector $\psi \in \mathcal{H}^+$ is cyclic and separating for $\mathcal{M}$ then the natural cones*

$$\mathcal{H}^+(\mathcal{M}, \Omega) \quad \text{and} \quad \mathcal{H}^+(\mathcal{M}, \psi)$$

coincide.

(*iv*) *Let $\alpha \in \mathrm{Aut}\mathcal{M}$ and define*

$$U(\alpha)\psi_\omega = \psi_{(\alpha^{-1*}\omega)}$$

then by linearity this map can be extended to all of $\mathcal{H}$. This extension is a unitary operator. The set

$$\{U(\alpha);\ \alpha \in \mathrm{Aut}\mathcal{M}\}$$

defines a unitary representation of $\mathrm{Aut}\mathcal{M}$, *the adjoint action of which implements the automorphisms.*

For the proof see [BR79] Thm. 2.5.31, Prop. 2.5.30, Cor. 2.5.32. Another important result is due to A. Connes [Co74] which says that the algebras $\mathcal{M}$ and $\mathcal{M}'$ are uniquely characterized by the natural cone. First some notations:

5.1.6 Definition:

(i) A face of a cone C is a subcone $F \subset C$ with $a, b \in C$, $a < b$ in the order of the cone C and $b \in F$ implies $a \in F$.
(ii) The set $\mathcal{D}(\mathcal{H}^+) := \{\delta \in \mathcal{B}(\mathcal{H});\ \mathrm{e}^{t\delta}\mathcal{H}^+ = \mathcal{H}^+ \ \forall\, t \in \mathbb{R}\}$ is a Lie algebra.
(iii) A map $I : \mathcal{D}(\mathcal{H}^+) \to \mathcal{D}(\mathcal{H}^+)$ is called an orientation of $\mathcal{H}^+$ if it fulfils:
$I^2 = -1$, $[I\delta_1, \delta_2] = [\delta_1, I\delta_2] = I[\delta_1, \delta_2]$ and $I(\delta^*) = -I(\delta)^*$. (To be precise, for this definition one first has to devide $\mathcal{D}(\mathcal{H}^+)$ by its center.)
(iv) Let F be a face of $\mathcal{H}^+$, then F^- denotes the face of $\mathcal{H}^+$ which is perpendicular to F. By a result of Connes one has closure $F = F^{--}$. P_F denotes the projection onto the Hilbert subspace generated by F. $\mathcal{H}^+$ is called facially homogenious if $\mathrm{e}^{t(P_F - P_{F^-})}\mathcal{H}^+ = \mathcal{H}^+$, $t \in \mathbb{R}$ and this for all faces F of $\mathcal{H}^+$.

The concept of orientation and homogeneity can also be formulated for arbitrary cones. The result of Connes is the following:

5.1.7 Theorem:

There is a one to one correspondence between von Neumann algebras $\mathcal{M}$ acting on $\mathcal{H}$ and selfdual, orientable, and facially homogenious cones of $\mathcal{H}$.

Von Neumann has classified the factors by three types denoted by I,II, and III. For a long time there were only very few different type III factors known. Using canonical anti–commutation relations, R. Powers [Pow67] was able to construct a continuous family of different type III factors. An attempt to classify these factors were made by Araki and Woods [AW68]. The question of the classification has finally be settled by A. Connes [Co73a]. This classification is based on the invariant S which is defined as follows:

5.1.8 Definition:

Let $\mathcal{M}$ be a von Neumann algebra and ω be a normal weight on $\mathcal{M}$. Let $E \in \mathcal{M}$ be the support of ω. Then ω is faithful on $E\mathcal{M}E$. Hence there exists a modular operator Δ_ω for this algebra. One defines:

$$S(\mathcal{M}) = \cap\{\text{spectrum } \Delta_\omega; \omega \text{ is a normal, semi} - \text{finite weight on } \mathcal{M}\}.$$

If $\mathcal{M}$ is of type III, then there are the following possibilities:

5.1.9 Theorem:

Let $\mathcal{M}$ be a type III factor, then for the Connes invariant exist the following possibilities:

(1) $S(\mathcal{M}) = \{0, 1\}$,
(2) $S(\mathcal{M}) = \underline{\{0\}} \cup \{\lambda^n; n \in \mathbb{Z}, 0 < \lambda < 1\}$,
(3) $S(\mathcal{M}) = \overline{\mathbb{R}^+}$.

If $S(\mathcal{M})$ is $\{1\}$ then $\mathcal{M}$ is not of type III.

5.1.10 Notation:

A factor with $S(\mathcal{M}) = \{0, 1\}$ is called a III_0–factor. The factors with the set (2) are called III_λ, and those with $S(\mathcal{M}) = \overline{\mathbb{R}^+}$ are named III_1–factors.

Let $\mathcal{M}$ be a von Neumann algebra and ω be a normal faithful state on $\mathcal{M}$. Then it can happen, that for some $t \in \mathbb{R}$ the modular transformation σ_ω^t is inner, i.e. there exists a unitary $U \in \mathcal{M}$ with $\sigma_\omega^t(A) = UAU^*$, $A \in \mathcal{M}$. In this case one shows

$$\Delta_\omega^{it} = UJ_\omega UJ_\omega. \tag{5.1.1}$$

If σ_ω^t is inner for one normal faithful state then this is true for every such state.

A. Connes [Co73a] has introduced the invariant $T(\mathcal{M})$, consisting of all $t \in \mathbb{R}$ such that σ^t is inner. It is clear that $T(\mathcal{M})$ is a subgroup of $\mathbb{R}$. For instance an algebra $\mathcal{M}$ is semi–finite iff $T(\mathcal{M}) = \mathbb{R}$. We do not need the full relation between $T(\mathcal{M})$ and $S(\mathcal{M})$. We are only interested in the type III_1 case. The result is the following:

5.1.11 Theorem:

A von Neumann factor is of Type III_1 iff $T(\mathcal{M}) = \{0\}$. This means that all σ^t, $t \neq 0$ are outer automorphisms of $\mathcal{M}$.

In every class III_λ, $0 \leq \lambda \leq 1$ no classification is known except for one algebra. These are the hyperfinite factors.

5.1.12 Definition:

A factor $\mathcal{M}$ is called hyperfinite if there exists an increasing net $\mathcal{N}_\alpha \subset \mathcal{M}$ of type I algebras with

$$\mathcal{M} = \{\bigcup_\alpha \mathcal{N}_\alpha\}''.$$

The importance of this concept is the following result [Co76], [Hgr87]:

5.1.13 Proposition:

Every of the classes III_λ contains exactly one element which is hyperfinite.

5.2) *The factor problem*

The locality and the spectrum conditions together with the existence of a vacuum–vector imply that the global algebra is of type I. One finds that the commutant of the algebra $\mathcal{M}(\mathbb{R}^d)$ is abelian, and that the projection E_0 onto all translational invariant vectors is an abelian projection in $\mathcal{M}$ with central support $\mathbb{1}$. In this case the center is pointwise invariant under the translations. This has first been observed by Araki [Ara64]. The properties of the projection E_0 is a consequence of the cluster property.

The first proof of the cluster property is due to the author [Bch62]. A systematic study of this property was started by Doplicher, Kadison, Kastler, and Robinson [DKKR67] using the notation of asymptotic abelian systems introduced by Doplicher, Kastler, and Robinson in [DKR66] and independently by Ruelle [Ru66]. This notation has been weakend by Lanford and Ruelle [LR67] introducing the concept of G-abelian systems. The most general concept leading to the cluster property has been introduced by Størmer [Stø67]. He called it large groups of automorphisms. One important consequence of the cluster property of the vacuum state is the additivity of the spectrum. The result is due to Wightman [Wi64].

Next we are looking at the algebra of the wedge. Here the following result is known:

5.2.1 Theorem:

Assume we are dealing with a QFTLO on the vacuum sector. Let $\mathcal{M}(W)$ be the algebra of the wedge domain. Then

$$\mathcal{Z}(\mathcal{M}(W)) \subset \mathcal{Z}(\mathcal{M}(\mathbb{R}^d)),$$

where $\mathcal{Z}(\mathcal{M})$ denotes the center of $\mathcal{M}$.

This result has first been obtained by Driessler [Dri75]. Our demonstration is taken from [Bch98a]. First we show a result which has its interest of its own, and from which Thm. 5.2.1 follows easily.

5.2.2 Lemma:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Assume $U(s) \in \mathcal{H}str(\mathcal{M})^+$ or $U(s) \in \mathcal{H}str(\mathcal{M})^-$. Then:

a. If we write $U(s) = \mathrm{e}^{\mathrm{i}Ht}$ and denote by $\mathcal{D}(H)$ the domain of definition for H then

$$\Delta^{\mathrm{i}t}\mathcal{D}(H) \subset \mathcal{D}(H).$$

b. If E_0 denotes the projection onto the eigenspace to the value 0 of H then E_0 commutes with Δ^{it}.

c. If F_1 denotes the projection onto the eigenspace to the value 1 of Δ, then one has

$$F_1 \leq E_0.$$

Proof: We show the lemma for $U(s) \in \mathcal{H}str(\mathcal{M})^+$. For $U(s) \in \mathcal{H}str(\mathcal{M})^-$ the arguments are essentially the same.

a. Let $\varphi, \psi \in \mathcal{D}(H)$ then we obtain from Thm. 2.5.2 $(\varphi, \Delta^{it} H\psi) = \mathrm{e}^{-2\pi t}(H\varphi, \Delta^{it}\psi)$. Since the left side is continuous in φ it follows that $\Delta^{it}\psi \in \mathcal{D}(H)$.

b. Let $H\psi = 0$ then we obtain $0 = \Delta^{it}H\psi = H\mathrm{e}^{-2\pi t}\Delta^{it}\psi$. From this we conclude $\Delta^{it}E_0\mathcal{H} \subset E_0\mathcal{H}$. Because of the group property of Δ^{it} we get $\Delta^{it}E_0\mathcal{H} = E_0\mathcal{H}$.

c. Keep s real and $s \geq 0$. From the assumption $\mathrm{Ad}\, U(s)\mathcal{M} \subset \mathcal{M}$ for $s \geq 0$ and from $\mathcal{D}(\Delta^{\frac{1}{2}}) = \{X\Omega; X\eta\mathcal{M},\ \Omega \in \mathcal{D}(X) \cap \mathcal{D}(X^*)\}$ we conclude that on $\mathcal{D}(\Delta^{\frac{1}{2}})$ the relation $\Delta^{it}U(s) = U(\mathrm{e}^{-2\pi t}s)\Delta^{it}$ can be analytically continued in t as long as $-\frac{1}{2} \leq \Im m\, t \leq 0$. If we choose $t = -\mathrm{i}\frac{1}{4}$ then we find

$$\Delta^{1/4}U(s) = \mathrm{e}^{-Hs}\Delta^{1/4}, \quad s \geq 0.$$

Multiplying this equation from both sides with F_1 we find $F_1U(s)F_1 = F_1\mathrm{e}^{-Hs}F_1$.
Since the right side is positive we obtain

$$F_1U(-s)F_1 = (F_1U(s)F_1)^* = F_1U(s)F_1 \geq 0.$$

Hence by the spectrum condition and by Schwarz reflection principle the function $F_1U(s)F_1$ is bounded and entire analytic which must be constant. This implies $F_1\mathrm{e}^{-Hs}F_1 = F_1$ which is only possible for $E_0 \geq F_1$. □

Next we have to show that the elements in $\mathcal{Z}(\mathcal{M})$ commute with the half–sided translations.

5.2.3 Lemma:

Let $U(t) \in \mathcal{H}str(\mathcal{M})^+$, then

$$[U(t), Z] = 0 \qquad \forall\, Z \in \mathcal{Z}(\mathcal{M}) \quad \text{and} \quad \forall\, t \in \mathbb{R}.$$

This result can also be found in [Dri75].

Proof: Let $Z = Z^* \in \mathcal{Z}(\mathcal{M})$ and set $Z_t = \mathrm{Ad}\, U(t)Z$. For $t \geq 0$ the element Z_t belongs to $\mathcal{M}$ and for $t \leq 0$ to $\mathcal{M}'$. This implies that Z commutes with Z_t for all $t \in \mathbb{R}$. Applying $\mathrm{Ad}\, U(s)$ to the commutator we obtain $[Z_{t_1}, Z_{t_2}] = 0$. Hence $\{Z_t\}$ generates an abelian von Neumann algebra invariant under $U(t)$. Since $U(t)$ has a positive generator it follows that $\mathrm{Ad}\, U(t)$ is inner in $\{Z_t\}$" [Bch66]. This implies $Z_t = Z$. □

*Proof of Thm.*5.2.1: Let $W = W(\ell_1, \ell_2)$, then the translations along the ℓ_1– and ℓ_2–direction belong to $\mathcal{H}str(\mathcal{M})^+$ and $\mathcal{H}str(\mathcal{M})^-$ respectively. Let E_0 be the projection onto the vectors invariant under both of these translations, then Lemma 5.2.2 implies $F_1 \le E_0$. The translation group of the characteristic two–plane contains the time translations. This implies that E_0 is the projection onto vectors invariant under all translations. Since Ω is also separating for $\mathcal{M}(W) \vee \mathcal{Z}(\mathcal{M}(\mathbb{R}^d))$ we conclude that the centralizer of $\mathcal{M}(W)$ is a subset of the global center. $\square$

For the algebras of the double cones no similar result can be obtained. Even in the case where $\mathcal{M}(\mathbb{R}^d)$ is a factor, one can easily construct examples where $\mathcal{M}(D)$ has a non–trivial center. (See 5.5.(II).) Up to now there are no conditions known, implying, that $\mathcal{M}(D)$ is a factor.

5.3) *The type question*

From the investigations of Kadison [Ka63] and from Guenin and Misra [GM63] it is known that the local algebras can not be of finite type. In 1967 Borchers [Bch67] showed the following result:

5.3.1 Theorem:

(1) *Let $O_1 \subset O_2$ such that there exists $O_3 \subset (O_2 \cap O_1')$. Assume E is a projection in $\mathcal{M}(O_1)$, then E is equivalent to its central support in $\mathcal{M}(O_2)$,* mod $\mathcal{M}(O_2)$.
(2) *If $O_1 + x \subset O_2$ for x in some open neighbourhood of $\mathbb{R}^d$, then the central support of E in $\mathcal{M}(O_2)$ belongs to the center of the global algebra.*

there is not known more under the general assumptions. If one wants to obtain better results, one has to impose additional requirements.

The situation is much better for the algebra of the wedge. This is due to the existence of half–sided translations. The first result in this direction is due to Driessler [Dri75]. But he uses the additional assumption that the spectrum has a mass gap. Here we follow the method of Longo [Lo79], with a slight variation, applying Thm. 5.1.11. There exists also a proof which uses the invariant $S(\mathcal{M})$ and Prop. 5.1.9. (See [Bch98a].)

5.3.2 Theorem:

In a QFTLO on the vacuum Hilbert space with one vacuum vector the algebra $\mathcal{M}(W)$ is of type III_1.

Proof: Since the center of $\mathcal{M}(W)$ is contained in the center of the global algebra, the statement is true in case that it is true when $\mathcal{M}(\mathbb{R}^d)$, and hence by Thm. 5.2.1 $\mathcal{M}(W)$ is a factor. We will use Thm. 5.1.11 for the proof of the type question. Assume σ^t is inner for one fixed $t \neq 0$. We want to leed this to a contradiction. In that case there exists a unitary $U \in \mathcal{M}(W)$ with $\sigma^t(A) = UAU^*$, $A \in \mathcal{M}(W)$. From this we obtain $\mathrm{Ad}\,\{\sigma^t(U^*)U\}A = A$. This implies $\sigma^t(U^*)U = \lambda \mathbb{1}$ with $|\lambda| = 1$. But $\sigma^t(U^*) = UU^*U^* = \lambda U^*$ implies $\lambda = 1$. Hence U^* and U belong to the centralizer of $\mathcal{M}(W)$. (See Def. 5.1.1.) From Lemma 5.2.2 (c) we get $U\Omega = \mu\Omega$, $|\mu| = 1$, and since Ω is separating we find $U = \mu\mathbb{1}$ and by Eq. (5.1.1) $\Delta^{it} = \mathbb{1}$. This contradicts the existence of half–sided translations. (See Thm. 2.5.2.) $\square$.

This result has used only the existence of half–sided translations. Therefore, the theorem remains true for arbitrary algebras with half–sided translation. In conformal field theory these are the algebra of the forward light–cone and the algebras of the double cones.

The determination of the type of local algebras $\mathcal{M}(D)$ is burdened with some difficulties. It is known from examples, as the free massive field, that local algebras fulfil the split property [DL84] if specific conditions are fulfilled. This property is the following: Let $D_1 \subset D$ be such that $D_1 + x \subset D$ for x in some open neighbourhood of the origin. In that case one can find a type I algebra $\mathcal{N}$ with $\mathcal{M}(D_1) \subset \mathcal{N} \subset \mathcal{M}(D)$. This implies that one cannot expect any statement about the type from purely local considerations. Some more information about the structure of $\mathcal{M}(D)$ has to be used.

This difficulty has been circumvented by Fredenhagen [Fre85] by observing that there exists no intermediate type I algebra if the domains D_1 and D have boundary points in common. Therefore, he puts the double cone D into the corner of the wedge and tries to compare the Connes invariant S of $\mathcal{M}(D)$ and $\mathcal{M}(W)$. To do this he needs the assumption that the local algebras are generated by Wightman fields wich have the Haag–Narnhofer–Stein property [HNS84].

Let us first explain this concept. Let $\Phi(x)$ be a Wightman field, then we say for $\Phi(x)$ exists a scaling limit if there exists a non–negative function $N(\lambda)$ defined for $\lambda > 0$ such that for all n

$$N(\lambda)^n(\Omega, \Phi(\lambda x_1)...\Phi(\lambda x_n)\Omega)$$

converges for $\lambda \to 0$ to some non–trivial Wightman functional. With this concept we introduce the following

5.3.3 Requirement:

There exists a Wightman field $\Phi(x)$ such that:
(i) For every $f \in \mathcal{D}$ with supp. $f \in \mathcal{D}$ the operator $\Phi(f)$ is affiliated with $\mathcal{M}(D)$.
(ii) $\Phi(x)$ fulfils the Haag–Narnhofer–Stein scaling property.
(iii) The theory fulfils the Bisognano–Wichmann property. (If the set of Wightman fields, which fulfil (i), generate $\mathcal{M}(D)$ then (iii) is implied by the result of Bisognano and Wichmann Thm. 3.1.5.)

With this requirement Fredenhagen has shown the following result:

5.3.4 Theorem:

We are dealing with a QFTLO in the vacuum sector, such that the global algebra is a factor, and which fulfils the Requirement 5.3.3. *Let W be a wedge such that zero belongs to its edge. Let $D \subset W$ be a double cone such that zero belongs to the boundary of D. Let $\mathcal{N}$ be a von Neumann algebra with*

$$\mathcal{M}(D) \subset \mathcal{N} \subset \mathcal{M}(W).$$

Then $\mathcal{N}$ is of type III_1.

For the proof of this result we need some preparations. The first is concerned with the characterization of points in the spectrum of the modular operator. This will be given without proof. The second deals with consequences of the Haag–Narnhofer–Stein scaling property.

5.3.5 Proposition:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Let Δ, J be the modular operator and conjugation of the pair $(\mathcal{M}, \Omega)$ and let $j(A)$ stand for JAJ. Then the following statements are equivalent:

1. $\lambda \in$ spec Δ.
2. *For every $\epsilon > 0$ exists an operator $A \in \mathcal{M}$ with $\|A\Omega\| \geq 1$ and*
$$\|(\lambda^{1/2}A - j(A^*))\Omega\| + \|(A^* - \lambda^{1/2}j(A))\Omega\| \leq \epsilon.$$
3. *For every $\epsilon > 0$ exists an operator $A \in \mathcal{M}$ with $\|A\Omega\| \geq 1$, such that*
$$|(\Omega, AB\Omega) - \lambda(\Omega, BA\Omega)| \leq \epsilon\{(\Omega, B^*B\Omega) + \lambda(\Omega, BB^*\Omega)\}^{1/2}$$

holds for every $B \in \mathcal{M}$.

For the first equivalence see [Ped79] Lemma 8.15.8, and for the second [Fre85] Prop. 4.1. The next result is concerned wit consequences of the Haag–Narnhofer–Stein scaling property.

5.3.6 Lemma:

Make the assumptions of Thm. 5.3.4 and let J be the modular conjugation of $\mathcal{M}(W)$. Then for every $\lambda > 0$ and $\epsilon > 0$ exists a uniformly bounded sequence $A_n \in \mathcal{M}(\frac{1}{n}D)$ such that

$$\|A_n\Omega\| \geq 1 \quad \text{and} \quad \|(j(A_n) - \lambda^{1/2}A_n)\Omega\| \leq \epsilon \tag{5.3.1}$$

holds.

Proof: Let us choose coordinates in such a way that the characteristic two–plane of W is the (0,1)–plane and that the center of D is the point $(0, 1, 0, ..., 0) =: x_0$. Then one finds

$$\mathrm{Ad}\,\Delta_W^{it}\Phi(\rho x_0) = \Phi(-\rho\sinh 2\pi t, \rho\cosh 2\pi t, 0, ...).$$

We set

$$\Phi_\rho(t) = N(\rho)\Phi(-\rho\sinh 2\pi t, \rho\cosh 2\pi t, 0, ...).$$

It is known, that it is sufficient to integrate $\Phi(x)$ only in the time coordinate in order to obtain a well defined operator [Bch64]. Hence it is sufficient to test $\Phi_\rho(t)$ with functions $f(t) \in \mathcal{D}$. Moreover we have $\Phi_\rho(f)\ \eta\ \mathcal{M}(\rho' D)$ for supp $f(t) \subset \{t; |t| < \frac{1}{2\pi}\ln 2\frac{\rho'}{\rho}\}$. This implies for $p > 0$:

$$\Phi_{\frac{1}{n(1+p)}}(f)\ \eta\ \mathcal{M}(\frac{1}{n}D), \quad \text{if} \quad \text{supp } f(t) \subset \{t; |t| < \frac{1}{2\pi}\ln 2(1+p)\}.$$

Choose p and $f(t)$ with supp $f \subset \{t; |t| < \frac{1}{2\pi}\ln 2(1+p)\}$ (depending on ϵ) such that the Fourier transform of $f(t)$ is centered around $\log\lambda$. More precisely, since $\Phi_\rho(x)$ converges for $\rho \to 0$ we can choose p and f such that

$$\|\Phi_{\frac{1}{n(1+p)}}(f)\Omega\| \geq 1 + \frac{\epsilon}{3}, \quad \text{and} \quad \|(\Delta_W^{\frac{1}{2}} - \lambda^{\frac{1}{2}})\Phi_{\frac{1}{n(1+p)}}(f)\Omega\| \leq \frac{\epsilon}{3} \tag{5.3.2}$$

for sufficiently large n. Now we set

$$A_n = \{1 + c\Phi_{\frac{1}{n(1+p)}}(f)\Phi_{\frac{1}{n(1+p)}}(f)^*\}^{-1}\Phi_{\frac{1}{n(1+p)}}(f)$$

with a constant c which we have to determine. (Here Φ means the closure of the tested operator.) Writing Φ for $\Phi_{\frac{1}{n(1+p)}}(f)$ we obtain the estimate

$$\begin{aligned}\|(A_n - \Phi)\Omega\|^2 &= (\Omega, \Phi^* \frac{c^2(\Phi\Phi^*)^2}{(1 + c\Phi\Phi^*)^2}\Phi\Omega) \\ &= c^2(\Omega, (\Phi^*\Phi)^*\Phi^* \frac{1}{(1 + \Phi\Phi^*)^2}\Phi(\Phi^*\Phi)\Omega \\ &\leq \frac{c}{4}\|\Phi^*\Phi\Omega\|^2\end{aligned} \tag{5.3.3}$$

The last estimate is obtained by inserting $\Phi^* = U|\Phi^*|$ and observing that the unitary U drops out in the estimate of the norm. From this we obtain

$$\|\Delta_W^{1/2}(A_n - \Phi)\Omega\|\|J(A_n - \Phi)\Omega\| = \|(A_n - \Phi)\Omega\| \leq \frac{\sqrt{c}}{2}\|\Phi^*\Phi\Omega\|.$$

Since $\Phi_{\frac{1}{n(1+p)}}(x)$ converges for $n \to \infty$ we can choose c such that $\frac{\sqrt{c}}{2}\|\Phi^*\Phi\Omega\| \leq \frac{\epsilon}{3}$ uniformly in $n > n_0$. Combining Eqs. (5.3.2) and (5.3.3) with this estimate we obtain the lemma. □

Proof of Thm 5.3.4: For the proof we make use of the fact, that the algebra of a point consists only of multiples of the identity. (See e.g. [Bch96] Thm IV.6.3.) Since the sequence A_n is bounded in norm and since $A_n \subset \mathcal{M}(\frac{1}{n}D)$ it follows that every weak limit point of $A_n^*A_n, (j(A_n^*) - \lambda^{1/2}A_n)^*(j(A_n^*) - \lambda^{1/2}A_n), (A_n^* - \lambda^{1/2}j(A_n))^*(A_n^* - \lambda^{1/2}j(A_n))$ is a multiple of the identity. This implies that for $\psi \in \mathcal{H}$ and sufficient large n one has

$$\|A_n\psi\| \leq 1 - \epsilon', \quad \|(j(A_n^*) - \lambda^{1/2}A_n)\psi\| \leq \epsilon, \quad \|(A_n^* - \lambda^{1/2}j(A_n))\psi\| \leq \epsilon.$$

From this we obtain for $B \in \mathcal{N}$ the estimate

$$\begin{aligned}(\psi, (A_nB - BA_n)\psi) &= ((A_n^* - \lambda^{1/2}j(A_n))\psi, B\psi) + \lambda^{1/2}(j(A_n)\psi, B\psi) \\ &+ \lambda^{1/2}(B^*\psi, (j(A_n^*) - \lambda^{1/2}A_n)\psi) - \lambda^{1/2}(B^*\psi, j(A_n^*)\psi) \\ &= \epsilon\{\|B\psi\| + \lambda^{1/2}\|B^*\psi\| \leq \epsilon\sqrt{2}\{(\psi, B^*B\psi) + \lambda(\psi, BB^*\psi)\}^{1/2}.\end{aligned}$$

Hence every $\lambda > 0$ belongs to $S(\mathcal{N})$. Since $S(\mathcal{N})$ is closed it follows that $S(\mathcal{N})$ is the closed positive real axis. This implies that the central decomposition of $\mathcal{N}$ contains only factors of type III_1. □

More about the structure of the local algebras can be said, if in addition, one makes more assumptions, in particular the nuclearity condition introduced by Buchholz and Wichmann [BW86].

First we must explain this concept. Let H be the generator of the time translation and Ω the vacuum vector. The map $\Theta_\beta : \mathcal{M} \to \mathcal{H}$ defined by

$$\Theta_\beta(A) = \mathrm{e}^{-\beta H} A\Omega$$

is called nuclear if one can write it

$$\Theta_\beta(A) = \sum_n \varphi(A)\psi_n, \quad \varphi \in \mathcal{M}^*,\ \psi_n \in \mathcal{H} \tag{5.3.4}$$

with $\sum_n \|\varphi_n\|\|\psi_n\| < \infty$.

The expression

$$\mathcal{N}(\Theta_\beta) := \inf\{\sum_n \|\varphi_n\|\|\psi_n\|\}$$

where the infimum is taken over all possible representations Eq. (5.3.4). Buchholz and Wichmann suggested the nuclearity condition by comparing the situation in a bounded region with that of a thermodynamical system in a box. If one does so, one obtains some suggestion about the behaviour of the norm $\mathcal{N}(\Theta_\beta)$ as function of β, the dimension of the Minkowski space and the diameter of the double cone D, when Θ_β is applied to $\mathcal{M}(D)$. In the coming investigation we only need the behaviour in β. This we formulate as an assumption.

5.3.7 Condition:

We say a QFTLO fulfils the Buchholz–Wichmann property if the map $\mathcal{M}(D) \to \mathcal{H}$ defined by

$$\Theta_\beta(A) = \mathrm{e}^{-\beta H} A\Omega, \quad A \in \mathcal{M}(D)$$

is nuclear and the nuclear norm fulfils the estimate

$$\mathcal{N}(\Theta_\beta) \leq M \mathrm{e}^{(\frac{\beta_0}{\beta})^n},$$

where M, β_0, n are constants which may depend on the dimension of the space and the diameter of the double cone D.

With help of this condition Buchholz, D'Antoni and Fredenhagen [BDF87] showed the following result:

5.3.8 Theorem:

Assume a QFTLO fulfils the Buchholz–Wichmann property, Condition 5.3.7*. Let* $D_1 \subset D$ *such that the closure of* D_1 *is contained in the interior of* D*. Then there exists a type I factor* $\mathcal{P}$ *with*

$$\mathcal{M}(D_1) \subset \mathcal{P} \subset \mathcal{M}(D).$$

For the proof of this theorem we refer to the orginal paper [BDF87]. We want to combine this result with Thm. 5.3.4 and obtain:

5.3.9 Theorem:

Assume we are dealing with a QFTLO in the vacuum sector. Assume that the theory fulfils the Haag–Narnhofer–Stein assumption, Requirement 5.3.3, *and the Buchholz–Wichmann property, Condition* 3.5.7. *Assume in addition that $\mathcal{M}(D)$ is continuous from inside or from outside.* (The first statement means $\mathcal{M}(D) = \{\cup \mathcal{M}(D_i)\}''$ with closure $D_i \subset$ interior D_{i+1} and $\cup D_i = D$.) *Then every local algebra is isomorphic to:*

$$\mathcal{M}(D) \cong \mathcal{R}\overline{\otimes}\mathcal{Z},$$

where $\mathcal{R}$ is the unique hyperfinite type III_1 factor and $\mathcal{Z}$ is the center of $\mathcal{M}(D)$.

Proof: From Thm. 5.3.8 we know that $\mathcal{M}(D)$ can be approximated from inside (or outside) by type I factors. Hence $\mathcal{M}(D)$ is hyperfinite. Since in the central decomposition of a hyperfinite von Neumann algebra there appear only hyperfinite factors, it follows that $\mathcal{M}(D)$ can be expressed as an integral $\mathcal{M}(D) = \int d\mu(z)\mathcal{R}(z)$ of hyperfinite III_1 factors. Because of the uniqueness of this factor, Prop. 5.1.13, we obtain the statement of the theorem. (The integral decomposition causes no problem, since $\mathcal{M}(D)$ is countably decomposable.) □

5.4) *On the implementation of symmetry groups*

Assume we are describing a physical theory in terms of a C*-algebra $\mathcal{A}$ and a symmetry group G, i.e. we have a representation of G by automorphisms of $\mathcal{A}$

$$\alpha : \; G \longrightarrow \text{Aut}\,(\mathcal{A}).$$

This situation is usually called a C*-dynamical system and denoted by the triple $\{\mathcal{A}, G, \alpha\}$. For applications it is of interest to characterize those representations π of $\mathcal{A}$, for which there exists in $\mathcal{H}_\pi$ a continuous unitary representation $U(g)$ of the symmetry group which implements the automorphism:

$$U(g)\pi(x)U^*(g) = \pi(\alpha_g x). \tag{5.4.1}$$

Let α_g act strongly continuous, which means that the function $g \to \alpha_g(A)$ is a continuous function on G with values in the normed space $\mathcal{A}$. If in addition the group is locally compact, then one can integrate over the group. This led Doplicher, Kastler and Robinson [DKR66] to introduce the C^*-completion of the algebra of continuous $\mathcal{L}^1$ functions on G with values in $\mathcal{A}$. They called it the covariance algebra. Nowadays it is called the crossed product of $\mathcal{A}$ with G. The importance of the covariance algebra stems from the fact that there is a one to one correspondence of covariant representations of $\mathcal{A}$ and representations of the covariance algebra. For details see the book of G.K.Pedersen [Ped79].

If one is dealing with a C*-dynamical system and a representation $\{\pi, \mathcal{H}\}$ of $\mathcal{A}$, then it is usually hard to decide whether or not this representation can be extended to a representation of the covariance algebra. The difficulties are twofold: If $\pi(\mathcal{A})$ has a center then the multiplicity problem may appear. Moreover, by passing to the adjoint representation of the group, one has to be aware of central extensions of the group. Both problems can be circumvented by passing to quasi–equivalent representations. The reason for the first problem is clear. The reason for the second problem is the following: If $U(g)$ is a ray–representation of G on $\mathcal{H}$, then there exists a second representation $\hat{U}(g)$ which is also a ray–representation, but with the complex conjugate phase–factor. Therefore $U(g) \otimes \hat{U}(g)$ is a representation of the group on $\mathcal{H}\overline{\otimes}\mathcal{H}$. Replacing π by $\pi \otimes \mathbb{1}$ we obtain a covariant representation. This leads to the following notation:

5.4.1 Definition:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system and $\{\pi, \mathcal{H}\}$ be a representation of $\mathcal{A}$ then $\{\pi, \mathcal{H}\}$ is called quasi-covariant, if there exists a covariant representation $\{\pi_1, U, \mathcal{H}_1\}$ such that $\{\pi, \mathcal{H}\}$ and $\{\pi_1, \mathcal{H}_1\}$ are quasi-equivalent.

Quasi–covariant representations are much easier to characterize than covariant representations. The first result was obtained in [Bch69] which was based on the assumptions of strong continuity and the locally compactness of the group. Some time later Borchers [Bch83] observed, that it is neither necessary to assume that α_g acts strongly continuous nor that G is locally compact. To prove this the natural cone will be used, in particular Thm. 5.1.5.(iv).

5.4.2 Theorem:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system. Let π be a representation of $\mathcal{A}$. Then this representation is quasi–covariant iff:

(α) *The dual action α_g^* maps the folium of $\pi(\mathcal{A})$ onto itself.*
(β) *α_g^* acts strongly continuous on the folium of π. This means the function*

$$g \longrightarrow \alpha_g^*(\omega)$$

is a continuous function on G with values in the folium of π, furnished with the norm topology.

The folium of a representation is the set of states, which extend to normal states of $\pi(\mathcal{A})$".

Proof: Condition (α) is clearly necessary. If $U(g)$ is a continuous representation, then one has for $\psi \in \mathcal{H}$

$$\begin{aligned} |(\psi, U(g)AU^*(g)\psi) - (\psi, U(g_0)AU^*(g_0)\psi)| &= |((U^*(g) - U^*(g_0))\psi, AU^*(g)\psi) \\ &+ (U^*(g_0)\psi, (U^*(g) - U^*(g_0))\psi) \leq 2\|A\|\|\psi\|\|(U^*(g) - U^*(g_0))\psi\|. \end{aligned}$$

Since $U(g)$ is strongly continuous it follows that α_g^* acts strongly continuous on the vector states. Since the foliun consists of the norm closure of the linear span of the vector states, we see that α_g^* acts strongly on the folium. If conversely α_g^* acts strongly continuous on the folium, then we use the standard representation of $\pi(\mathcal{A})$" and we obtain by Thm. 5.1.5.(iv) a representation of the group which is strongly continuous because of Thm. 5.1.5.(ii). □

This result suggests to investigate closer that part of $\mathcal{A}^*$ on which α_g^* acts strongly continuous. We introduce:

5.4.3 Definition:

By $\mathcal{A}_c^*$ we denote the set of $\phi \in \mathcal{A}^*$, ($\mathcal{A}^*$ denotes the topological dual of $\mathcal{A}$), such that for every $\epsilon > 0$ exists a neighbourhood $\mathcal{U}$ of the identity of G such that

$$\|\phi \circ \alpha_g - \phi\| \leq \epsilon$$

holds for $g \in \mathcal{U}$.

Some properties of this set are described in the following

5.4.4 Proposition:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system and assume $G(\tau)$ is a topological group, then the space $\mathcal{A}_c^*$ has the following properties:

(i) *$\mathcal{A}_c^*$ is a linear norm–closed space.*
(ii) *$\mathcal{A}_c^*$ is invariant under the action of the group i.e. $\phi \in \mathcal{A}_c^*$ implies $\phi \circ \alpha_g \in \mathcal{A}_c^*$ for every $g \in G$.*
(iii) *With $\phi \in \mathcal{A}_c^*$ one finds also that ϕ^* and $|\phi|$ belong to $\mathcal{A}_c^*$. $\mathcal{A}_c^*$ is generated by its positive elements.*

Since this result has no connection with the Tomita–Takesaki theory, we refer for the proof to the original paper [Bch83].

Recall that for every positive linear functional $\omega \in \mathcal{A}^+$ exists a vector $\xi_\omega \in \mathcal{H}^+$, ($\mathcal{H}^+$ denotes the natural cone of $\mathcal{A}^{**}$) with $\omega(A) = (\xi_\omega, A\xi_\omega)$. Next we introduce some concepts:

5.4.5 Notation:

Let $\{\mathcal{A}, G, \alpha\}$ be a C^*-dynamical system with G being a topological group. Let $\mathcal{H}$ be the Hilbert-space of the standard representation of $\mathcal{A}^{**}$ and let $\mathcal{H}^+$ be the natural cone associated with this representation then we denote

(i) $\mathcal{H}_c^+ = \{\psi_\omega; \omega \in (\mathcal{A}_c^*)^+\}$.
(ii) $\mathcal{H}_c =$ *smallest sub-Hilbert-space of $\mathcal{H}$ containing $\mathcal{H}_c^+$.*
(iii) *Denote the canonical involution associated with the standard representation of $\mathcal{A}^{**}$ by J.*
(iv) *The algebra $\mathcal{A}^{**}$ will usually be denoted by $\mathcal{M}$. Then $\mathcal{A}^*$ and $\mathcal{M}_*$ are the same space.*

About this set we know:

5.4.6 Proposition:

With the assumptions and notations of 5.4.5 one obtains

(i) *$\mathcal{H}_c^+$ is a closed cone.*
(ii) *The space $\mathcal{H}_c$ is invariant under the canonical involution J.*
(iii) *If $\mathcal{H}_c^r$ denotes the vectors $\psi \in \mathcal{H}_c$ with $J\psi = \psi$ then $\mathcal{H}_c^+$ is a self-dual cone in $\mathcal{H}_c^r$ and $\mathcal{H}_c$ is algebraically generated by $\mathcal{H}_c^+$.*
(iv) *If P_c denotes the projection onto $\mathcal{H}_c$ then for every $\psi \in \mathcal{H}^+$ one has $P_c\psi \in \mathcal{H}_c^+$.*

Proof: (i) Let $\xi_1, \xi_2 \in \mathcal{H}_c^+$ then it follows from Prop. 5.4.4 that the functional $A \to (\xi_1, A\xi_2)$ belongs to $\mathcal{M}_{*,c}$. Hence the functional generated by $\xi_1 + \xi_2$ is in $\mathcal{M}_{*,c}$ which implies that $\mathcal{H}_c^+$ is a cone.
(ii) This follows from the fact that $\mathcal{H}_c^+$ is pointwise invariant under the involution J.
(iii) Assume $\xi_i, \eta_i \in \mathcal{H}_c^+, \quad i = 0, ...3.$ then the functional $A \to \left(\sum(\mathrm{i})^k \xi_k, A\sum(\mathrm{i})^l \eta_l\right)$ belongs to $\mathcal{M}_{*,c}$. Since $\mathcal{M}_{*,c}$ is norm closed it follows that $A \to (\xi, A\eta) \in \mathcal{M}_{*,c}$ for all $\xi, \eta \in \mathcal{H}_c$. This implies that $\mathcal{H}^+ \cap \mathcal{H}_c$ is a closed cone. Now let $\eta \in \mathcal{H}_c$ with $J\eta = \eta$, then $(\eta, .\eta) \in \mathcal{M}_{*,c}^+$ and hence exists a vector $\xi \in \mathcal{H}_c^+$ with $(\eta, .\eta) = (\xi, .\xi)$ and consequently a partial isometry $W' \in \mathcal{M}'$ with $\eta = W'\xi$. From $J\eta = \eta$ and $J\xi = \xi$ it follows with $W = JW'J$ that also $\eta = W\xi$ holds. Without loss of generality we may assume that W^*W is the support of ξ. Now from $W\xi = JWJ\xi$ we obtain $W^2\xi = WJWJ\xi \in \mathcal{H}^+$. This implies for $A \in \mathcal{M}$

$$\begin{aligned}(W^2\xi, AW^2\xi) &= (WJWJ\xi, AWJWJ\xi) = (W\xi, AWJW^*W\xi) \\ &= (W\xi, AW\xi) = (JWJ\xi, AJWJ\xi) = (\xi, A\xi).\end{aligned}$$

By the uniqueness of the representing vectors we obtain $W^2\xi = (JWJ)^2\xi = \xi$. From the minimality of W we obtain that W^2 is the support projection of ξ. This implies $W = W^*$. Since $(\eta, .\xi) = (W\xi, .\xi)$ it follows that $(\eta, .\xi)$ is selfadjoint and this formula gives the polar decomposition. From this we see $\xi^+ = W^+\xi \in \mathcal{H}_c^+$ and $\xi^- = W^-\xi \in \mathcal{H}_c^+$. Hence $\mathcal{H}_c^+ - \mathcal{H}_c^+ = \mathcal{H}_c^r$. Finally $\xi_1, \xi_2 \in \mathcal{H}_c^+$ implies $(\xi_1, \xi_2) \geq 0$ since $\mathcal{H}_c^+ \subset \mathcal{H}^+$. If $\eta \in \mathcal{H}_c^r$ then from the previous calculation $\eta = \xi_1 - \xi_2$ with $\xi_1, \xi_2 \in \mathcal{H}_c^+$ and $(\xi_1, \xi_2) = 0$. Hence $(\eta, \xi) \geq 0$ for all $\xi \in \mathcal{H}_c^+$ implies $(\eta, \xi^-) = -\|\xi^-\|^2 \geq 0$. From this we obtain $\eta \in \mathcal{H}_c^+$.
(iv) Let $\xi \in \mathcal{H}^+$ then we obtain for all $\xi_c \in \mathcal{H}_c^+$ the estimate $(P_c\xi, \xi_c) = (\xi, \xi_c) \geq 0$. This implies $P_c\xi \in \mathcal{H}_c^+$. □

Next we want to look at the facial structure of $\mathcal{H}_c^+$. In the following investigations we will use only the properties (1)...(4) of $\mathcal{H}_c^+$, so that the results can be used for arbitrary sub-cones of $\mathcal{H}^+$ with these properties. For these investigations we need some

5.4.7 Notations

Let f be a face of $\mathcal{H}_c^+$, then we denote by f^c the complementary face in $\mathcal{H}_c^+$, it is:

$$f^c = \{\xi \in \mathcal{H}_c^+; \xi - f\}.$$

By f^- we denote the complementary face of f in $\mathcal{H}^+$:

$$f^- = \{\xi \in \mathcal{H}^+; \xi - f\}.$$

We remark that the map $f \to f^c$ is only defined for faces of $\mathcal{H}_c^+$, while $f \to f^-$ is also defined for faces of $\mathcal{H}^+$.

For every face f of $\mathcal{H}_c^+$ we associate two faces of $\mathcal{H}^+$, namely, $F^+(f) = (f^c)^-$ and $F^-(f) = (f^-)^-$.

Note that $f^c, f^-, F^-(f)$, and $F^+(f)$ are closed faces. If possible we will denote the faces of $\mathcal{H}_c^+$ by small and those of $\mathcal{H}^+$ by capital letters.

The elementary properties of these faces are described in the following

5.4.8 Lemma

For any face f of $\mathcal{H}_c^+$ we obtain:

(a) $f \subset F^-(f) \subset F^+(f)$.
(b) $\xi \in \mathcal{H}^+$ then $P_c\xi \in (f^c)^c$ iff $\xi \in F^+(f)$
(c) $P_cF^-(f) = P_cF^+(f) = (f^c)^c$.
(d) $(F^+(f))^- = F^-(f^c)$ *and* $(F^-(f))^- = F^+(f^c)$.
(e) *The set* $\{\eta \in \mathcal{H}^+;\ \text{such that there exists }\ \xi \in f\ \text{ with } \eta \leq \xi\}$ *is dense in* $F^-(f)$.
(f) f *is dense in* $(f^c)^c$.

Proof: (a) Note first that the operations $f \to f^c$ and $f \to f^-$ reverse the order of inclusion. Moreover, the relation $f \subset (f^c)^c$ follows directly from the definition. From $f^c \subset f^-$ we conclude therefore, $f \subset (f^-)^- \subset (f^c)^-$, which is the first statement by the definition of F^- and F^+.
(b) We know that $\xi \in \mathcal{H}^+$ implies $P_c\xi \in \mathcal{H}_c^+$. Hence $P_c\xi \in (f^c)^c$ iff $(P_c\xi, \eta) = 0$ for all $\eta \in f^c \subset \mathcal{H}_c^+$. Consequently $(P_c\xi, \eta) = (\xi, \eta) = 0$ and therefore $\xi \in (f^c)^- = F^+(f)$ and vice versa.
(c) The last result implies $P_cF^+(f) = (f^c)^c$. On the other hand $f^c \subset f^-$ and $P_cf^- = f^c$, since $P_cf = f$. Therefore, $\eta \in (f^c)^c$ and $\xi \in f^-$ implies $(\eta, \xi) = (\eta, P_c\xi) = 0$, from which we conclude $\eta \in F^-(f)$. Hence $(f^c)^c \subset F^-(f) \subset F^+(f)$ which yields (c).
(d) We have by definition $F^+(f) = (f^c)^-$ and hence $(F^+(f))^- = \left((f^c)^-\right)^- = F^-(f^c)$. Inserting now f^c for f we obtain $(F^+(f^c))^- = F^-((f^c)^c)$. Now remark, if $\xi \in f^-$ then $(\xi, \eta) = 0$ for all $\eta \in f$ and hence $P_c\xi \in f^c$. But this implies for $\eta \in (f^c)^c$ the equation $(\xi, \eta) = (P_c\xi, \eta) = 0$ and hence $f^- = \left((f^c)^c\right)^-$ or $F^-(f) = F^-((f^c)^c)$. This leads to $F^-(f) = (F^+(f^c))^-$ or $(F^-(f))^- = (F^+(f^c))^{-,-} = (F^+(f^c))$ since $(F^+(f^c))$ is a closed face of $\mathcal{H}^+$.
(e) Let F be the face $\{\eta \in \mathcal{H}^+; \eta \leq \xi \text{ for some } \xi \in f\}$. Then by the above mentioned result of A.Connes (5.1.6.(iv)) one has $\bar{F} = (F^-)^-$. Let $\eta \in f^-$, then $(\eta, \xi) = 0$ for all $\xi \in f$ and hence for all $\xi \in F$, since the scalar product preserves order. This shows $f^- \subset F^-$. On the other hand by definition of F we have $f \subset F$ and hence $F^- = f^-$. But this yields $\bar{F} = (F^-)^- = (f^-)^- = F^-(f)$.
(f) By the proof of (c) we know $(f^c)^c \subset F^-(f)$. Hence by (e) we find for a given $\xi \in (f^c)^c$ and $\epsilon > 0$ elements $\eta \in F^-(f)$ and $\zeta \in f$ such that $\eta \leq \zeta$ and $\|\xi - \eta\| < \epsilon$. Now $P_c\eta \leq P_c\xi = \zeta$ and hence $P_c\eta \in f$. On the other hand $\|\xi - P_c\eta\| = \|P_c(\xi - \eta)\| \leq \|\xi - \eta\| < \epsilon$. This shows that f is dense in $(f^c)^c$. □

In order to formulate the next results we need some more

5.4.9 Notations:

(a) Let F be a face in $\mathcal{H}^+$ then $\sum_0^3 (\mathrm{i})^n F$ is a linear sub-space of $\mathcal{H}$. The projection onto its closure will be denoted by P_F. (See 5.1.6.(iv).)

(b) If we perform the same construction in $\mathcal{H}_c$ with a face f then we will denote the corresponding projection by p_f.

5.4.10 Remark: If F is a face of $\mathcal{H}^+$ then P_F is of the form $P_F = EJEJ$, where E is a projection in $\mathcal{M}$. (See e.g. [Co74].) In particular, $P_F\mathcal{H}$ is the standard representation-space for the algebra $\mathcal{M}_E$. In particular, $\bar{F}$ is the natural cone of $\mathcal{M}_E$. This implies that $\sum_0^3 (\mathrm{i})^n \bar{F}$ is closed. At the moment we do not know whether or not $\sum_0^3 (\mathrm{i})^n \bar{f}$ is a closed sub-space of $\mathcal{H}_c$. That it is, indeed, closed is a consequence of the following

5.4.11 Lemma:

For any face f of $\mathcal{H}_c^+$ follows:

(a) $[P_{F^+(f)}, P_c] = [P_{F^-(f)}, P_c] = 0.$
(b) $p_f = P_c P_{F^-(f)} = P_c P_{F^+(f)}.$
(c) $[P_{F^+(f)}, P_{F^-(f)}] = 0.$
(d) $P_{F^-(f)} = P_{F^+(f)} P_{F^-(\mathcal{H}_c^+)}.$

Proof: (a) From Lemma 5.4.8 we know $P_c P_{F^+(f)} \mathcal{H}^+ = \bar{f} \subset P_{F^+(f)} \mathcal{H}^+$. Hence we get $P_c P_{F^+(f)} \mathcal{H} = P_c \sum (\mathrm{i})^n F^+(f) = \sum (\mathrm{i})^n \bar{f} \subset P_{F^+(f)} \mathcal{H}$. This implies $P_c P_{F^+(f)} \mathcal{H} = P_{F^+(f)} P_c P_{F^+(f)} \mathcal{H}$ and hence $P_{F^+(f)} P_c P_{F^+(f)} = P_c P_{F^+(f)}$, which shows that the two projections commute. The same argument holds for $P_{F^-(f)}$.
(b) Due to the commuting of P_c and $P_{F^-(f)}$ the product is a projection. Since $P_c F^-(f) \mathcal{H}^+ = \bar{f}$ we learn that $P_c P_{F^-(f)} \mathcal{H} = \sum (\mathrm{i})^n \bar{f}$ is closed and equal to $p_f \mathcal{H}$. The same argument is again true for $P_{F^+(f)}$.
(c) First note $f^c \subset \mathcal{H}_c^+$, and hence $F^+(f) \supset \mathcal{H}_c^{+,-}$. Let now $\xi \in \mathcal{H}^+$ then by Remark 5.4.10 one has $P_{F^-(\mathcal{H}_c^+)}\xi \in F^-(\mathcal{H}_c^+)$ and $P_{F^+(f)} P_{F^-(\mathcal{H}_c^+)} \xi \in F^+(f)$. Take $\eta \in (\mathcal{H}_c^+)^-$ then both equations lead to $(\eta, P_{F^+(f)} P_{F^-(\mathcal{H}_c^+)} \xi) = (\eta, P_{F^-(\mathcal{H}_c^+)} \xi) = 0$, hence we have $P_{F^+(f)} P_{F^-(\mathcal{H}_c^+)} \xi \in F^-(\mathcal{H}_\xi^+)$ for every $\xi \in \mathcal{H}^+$. Since such vectors generate $\mathcal{H}$ the equation $P_{F^-(\mathcal{H}_c^+)} P_{F^-(\mathcal{H}_c^+)} = P_{F^-(\mathcal{H}_c^+)}$ follows, which is equivalent to (c).
(d) Since for $\xi \in \mathcal{H}^+$ the vector $P_{F^-(\mathcal{H}_c^+)} \xi$ belongs to $F^-(\mathcal{H}_c^+)$ there exist for every $\epsilon > 0$ vectors $\eta \in F^-(\mathcal{H}_c^+)$ and $\zeta \in \mathcal{H}_c^+$ with $\eta \leq \zeta$ and $\|P_{F^-(\mathcal{H}_c^+)} \xi - \eta\| \leq \epsilon$. So we obtain $\|P_{F^+(f)} P_{F^-(\mathcal{H}_c^+)} \xi - P_{F^+(f)} \eta\| \leq \epsilon$ and $P_{F^+(f)} \eta \leq P_{F^+(f)} \zeta \in \bar{f}$. This shows $P_{F^+(f)} P_{F^-(\mathcal{H}_c^+)} \mathcal{H}^+ \subset F^-(f)$. Since the inverse inclusion $F^-(f) \subset F^-(\mathcal{H}_c^+)$ holds both sides coincide. □

As a consequence of the last lemma we obtain:

5.4.12 Corollary:

$\mathcal{H}_c^+$ is a homogenious cone in the sense of A.Connes [Co74]. (See also Def. 5.1.6.)

Proof: If f is any face of $\mathcal{H}_c^+$ then we have to verify the equation

$$\exp\{t(p_f - p_{f^c})\} \mathcal{H}_c^+ = \mathcal{H}_c^+$$

for every $t \in \mathbf{R}$. Since $\mathcal{H}^+$ is a homogenious cone it follows for every face F of $\mathcal{H}^+$

$$\exp\{t(P_F - P_{F^-})\} \mathcal{H}^+ = \mathcal{H}^+.$$

Choosing $F = F^+(f)$ then by Lemma 5.4.8 $(F^+(f))^- = F^-(f^c)$. Multiplying the above equation by P_c we obtain

$$\begin{aligned}\mathcal{H}_c^+ &= P_c\mathcal{H}^+ = P_c \exp\{t(P_{F^+(f)} - P_{F^-(f^c)})\}\mathcal{H}^+ \\ &= \exp\{t(p_f - p_{f^c})\}P_c\mathcal{H}^+ = \exp\{t(p_f - p_{f^c})\}\mathcal{H}_c^+.\end{aligned}$$

This gives the desired result. □

The aim is to show that the cone $\mathcal{H}_c^+$ is the natural cone of a von Neuman algebra. First we introduce some candidates.

5.4.13 Definition:

(1) We define

$$\mathcal{M}_c^0 = \{A \in \mathcal{M}; [A, P_c] = 0\}.$$

(2) Let $A\omega(.) := \omega(A.)$ and $\omega A(.) := \omega(.A)$. Then we put

$$\mathcal{M}_m^0 = \{A \in \mathcal{M}; A\omega \in \mathcal{M}_{*,c},\ \omega A \in \mathcal{M}_{*,c},\ \forall \omega \in \mathcal{M}_{*,c}\}.$$

(3) Let E_c be the smallest projection in $\mathcal{M}$ with $E_cP_c = P_c$.

All these objects are invariant under α_g. First note that both sets are von Neumann algebras. The two algebras are not different. We have

5.4.14 Lemma:

(1) *The two algebras $\mathcal{M}_c^0$ and $\mathcal{M}_m^0$ coincide.*
(2) *Every element in $\mathcal{M}_c^0$ commutes with E_c.*

Proof: (1) If $A \in \mathcal{M}_c^0$ then it commutes with P_c which implies that with $\xi \in \mathcal{H}_c$ also $A\xi$ belongs to $\mathcal{H}_c$, hence $\mathcal{M}_c^0 \subset \mathcal{M}_m^0$. Conversely let $U = e^{\mathrm{i}Ht} \in \mathcal{M}_m^0$ be unitary then $UJUJ\mathcal{H}_c^+ = \mathcal{H}_c^+$ which implies that $UJUJ$ commutes with P_c. Since P_c and J commute we obtain $[P_c, H] + J[P_c, H]J = 0$. Replacing H by $\mathrm{i}H$ we see that H belongs to $\mathcal{M}_c^0$. Since the von Neumann algebra is generated by its unitaries follows $\mathcal{M}_m^0 \subset \mathcal{M}_c^+$.
(2) $A \in \mathcal{M}_c^0$ implies that it maps the Hilbert space $\mathcal{H}_c$ into itself. Consequently it maps also $closure\{\mathcal{M}'\mathcal{H}_c\} = E_c\mathcal{H}$ into itself. □

For the coming investigation we introduce with A. Connes [Co74] the sets

5.4.15 Definition:

We denote by

(a) $\mathcal{D}(\mathcal{H}^+) = \{\delta \in \mathcal{B}(\mathcal{H}); \mathrm{e}^{t\delta}\mathcal{H}^+ \subset \mathcal{H}^+ \quad \forall t \in \mathbf{R}\}$.
(b) $\mathcal{D}(\mathcal{H}_c^+) = \{\delta \in \mathcal{B}(\mathcal{H}_c); \mathrm{e}^{t\delta}\mathcal{H}_c \subset \mathcal{H}_c \quad \forall t \in \mathbf{R}\}$

where $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded linear operators on $\mathcal{H}$.

For facially homogenious self-dual cones the following characterization of $\mathcal{D}(\mathcal{H}^+)$ has been given by A.Connes [Co74]:

(a) Let $\mathcal{H}^+$ be a facially homogenious self-dual cone and $\delta \in \mathcal{B}(\mathcal{H})$ then $\delta \in \mathcal{D}(\mathcal{H}^+)$ iff

(i) $J\delta = \delta J$ *and*
(ii) *For* $\xi, \eta \in \mathcal{H}^+$ *with* $(\xi, \eta) = 0$ *it follows* $(\delta\xi, \eta) = 0$.
(b) If $\mathcal{H}^+$ *is the natural cone of* $\mathcal{M}$ *then* $\delta \in \mathcal{D}(\mathcal{H}^+)$ *iff* $\delta = x + JxJ$ *for a suitable* $x \in \mathcal{M}$.

We start the investigation from Remark 5.4.10 showing that to every face $f \subset \mathcal{H}_c^+$ is associated a minimal face $F^-(f)$ and hence a projection $E(f) \in \mathcal{M}$ with $P(F^-(f)) = E(f)JE(f)J$. First we want to investigate these special projections:

5.4.16 Lemma:

Let f *be a closed face of* $\mathcal{H}_c^+$ *and let* $F^-(f)$ *be the minimal closed face of* $\mathcal{H}^+$ *containing* f. *Let* $E(f) \in \mathcal{M}$ *be the unique projection such that* $P(F^-(f)) = E(f)JE(f)J$ *holds then* $E(f) \in \mathcal{M}_c := \mathcal{M}_c^0 E_c$.

Proof: We know from Lemma 5.4.11 that $P(F^-(f)) = E(f)JE(f)J$ commutes with P_c. Then δ_{F^-} commutes with P_c. On the other hand, we know that δ_{F^-} is of the form $\delta_{F^-} = 1/2(E(f) + JE(f)J)$.

$e^{t/2E(f)}Je^{t/2E(f)}J$ maps $\mathcal{H}_c^+$ onto $\mathcal{H}_c^+$, consequently for $\omega \in \mathcal{M}_{*,c}^+$ we have $e^{t/2E(f)}Je^{t/2E(f)}J\omega e^{t/2E(f)}Je^{t/2E(f)}J \in \mathcal{M}_{*,c}^+$. The left side of this expression has some analyticity property namely the first factors are entire analytic in t and the last are entire anti–analytic. Therefore the inclusion also holds for complex t. (The second t has to be replaced by $\bar{t}$.) But this can only be true if $e^{t/2E(f)}\omega e^{t/2E(f)} \in \mathcal{M}_{*,c}^+$. Hence $e^{t/2E(f)} \in \mathcal{M}_m^0$ and by differentiation $E(f) \in \mathcal{M}_m^0$. But since $E(f)$ is smaller than E_c we obtain $E(f) \in \mathcal{M}_c$. □

5.4.17 Definition:

We put

$$\mathcal{D}_c(\mathcal{H}^+) = \{\delta \in \mathcal{D}(\mathcal{H}^+) \text{ with } \delta = A + JAJ,\ A \in \mathcal{M}_c\}$$

Using the last result we obtain:

5.4.18 Proposition:

(1) *For every* $\delta \in \mathcal{D}(\mathcal{H}^+)$ *one has* $\pi_0(\delta) \in \mathcal{D}(\mathcal{H}_c^+)$ *with*

$$\pi_0(\delta) = P_c \delta P_c$$

where P_c *denotes the projection onto* $\mathcal{H}_c$.
(2) *If we denote by* π_c *the restriction of* π_0 *to* $\mathcal{D}_c(\mathcal{H}^+)$ *then* π_c *defines a bijection between the self-adjoint parts of* $\mathcal{D}_c(\mathcal{H}^+)$ *and* $\mathcal{D}(\mathcal{H}_c^+)$.

Proof: (1) Since P_c and δ both commute with J, also $P_c\delta P_c$ commutes with J. Assume next $\xi, \eta \in \mathcal{H}_c^+$ with $(\xi, \eta) = 0$ then follows from $(\xi, P_c\delta P_c\eta) = (\xi, \delta\eta) = 0$ that $P_c\delta P_c$ belongs to $\mathcal{D}(\mathcal{H}_c^+)$.
(2) The spaces $\mathcal{D}(\mathcal{H}^+)$ and $\mathcal{D}(\mathcal{H}_c^+)$ are both invariant under involution and weakly closed. Therefore, one can pass to the self-adjoint and positive part. The sets $\mathcal{D}_{sa}(\mathcal{H}^+)_1^+$ and $\mathcal{D}_{sa}(\mathcal{H}_c^+)_1^+$ are both weakly compact and convex. The extremal elements are of the form

$$\delta_F = \frac{1}{2}(1_{\mathcal{H}} + P(F) - P(F^-))$$

or

$$\delta_f = \frac{1}{2}(1_{\mathcal{H}_c} + p(f) - p(f^c))$$

respectively. Bellissard and Iochum [BI79] have shown that for every facially homogenious self-dual cone $\mathcal{H}^+$ the elements in $\mathcal{D}_{sa}(\mathcal{H}^+)$ permit an integral representation

$$\delta = \int_{-\|\delta\|}^{+\|\delta\|} \lambda d\delta_{F(\lambda)}.$$

This holds in particular for our cone $\mathcal{H}_c^+$. Since we know that every δ_f is of the form $\pi_0(\delta_{F(f)})$ with $\delta_{F(f)} = \frac{1}{2}(1_{\mathcal{H}} + P(F^+(f)) - P(F^-(f^c)))$ we see that every self-adjoint element in $\mathcal{D}(\mathcal{H}_c^+)$ is of the form $\pi_0(\delta)$.

Moreover, we know by Lemma 5.4.15

$$P(F^+(f)) = (E(f) + 1 - E_c)J(E(f) + 1 - E_c)J$$
$$P(F^-(f^c))) = E(f^c)JE(f^c)J$$

which implies $\delta_{F-(f)} \in \mathcal{D}_c(\mathcal{H}^+)$. Consequently π_c is surjective.

Since π_c is the multiplication by the projector P_c one has $\|\pi_c\delta\| \leq \|\delta\|$. On the other hand, the carrier of P_c in $\mathcal{M}_c$ is 1. Therefore the reconstruction of δ from $\pi_c\delta$ by means of the integral representation shows that (for self-adjoint) δ and $\pi_c\delta$ have the same spectrum. This shows $\|\delta\| = \|\pi_c\delta\|$ and hence ker $\pi_c = 0$. Consequently π_c is an isomorphism. □

Combining all results we obtain:

5.4.19 Theorem:

(1) *The cone $\mathcal{H}_c^+$ is facial homogenious and oriented and is, therefore, the natural cone of a von Neumann-algebra $\mathcal{N}_c$.*

(2) *The von Neumann-algebra $\mathcal{N}_c$ is isomorphic to the sub-von Neumann-algebra $\mathcal{M}_c \subset \mathcal{M}_{E_c}$ where*

(α) E_c is the smallest projection in $\mathcal{M}$ which is larger than the support projections of all states belonging to $\mathcal{M}_{*,c}$.

(β) $\mathcal{M}_c$ is the set of operators in $\mathcal{M}_{E_c}$ which are right and left multipliers of $\mathcal{M}_{*,c}$.

(γ) The automorphisms α_g are automorphisms of $\mathcal{M}_c$.

(3) *$\mathcal{M}_{*,c}$ is the pre-dual of $\mathcal{M}_c$.*

Proof: We know from Proposition 5.4.17 that the self-adjoint elements of $\mathcal{D}_c(\mathcal{H}^+)$ and $\mathcal{D}(\mathcal{H}_c^+)$ coincide. Therefore, we have to look only at the skew-symmetric elements in $\mathcal{D}(\mathcal{H}_c^+)$. Let $\delta = -\delta^* \in \mathcal{D}(\mathcal{H}_c^+)$, then $\mathrm{e}^{t\delta}$ defines a unitary group. Set $\alpha_t(\delta_1) = \mathrm{e}^{t\delta}\delta_1\mathrm{e}^{-t\delta}$ for $\delta_1^* = \delta_1 \in \mathcal{D}(\mathcal{H}_c^+)$. Since $\delta_1 = A + JAJ$ with $A = A^* \in \mathcal{M}_c$ we know that we can write $\alpha_t(\delta_1) = \hat{\alpha}_t(A) + J\hat{\alpha}_t(A)J$, where $\hat{\alpha}_t$ defines a linear mapping of $(\mathcal{M}_c)_h$ into itself. By linear extension we obtain a linear mapping of $\mathcal{M}_c$ onto itself. Since this mapping is given by a unitary group it must be an automorphism. Using now the theorem of Kadison [Ka66] and Sakai [Sa66] we obtain by standard arguments that δ belongs to $\mathcal{M}_c$. This shows that the map π_c, defined in Proposition 5.4.17, is a bijection between $\mathcal{D}_c(\mathcal{H}^+)$ and $\mathcal{D}(\mathcal{H}_c^+)$. Therefore, $\mathcal{H}_c^+$ is oriented by the orientation induced by $\mathcal{M}_c$. The isomorphism property of π_c shows the second statement of the theorem. The statement (γ) is a consequence of the invariance of $\mathcal{M}_{*,c}$. The third statement is due to the fact that $\mathcal{H}_c^+$ generates all functionals in $\mathcal{M}_{*,c}$ and that $\mathcal{H}_c^+$ is the natural cone of $\mathcal{M}_c$. □

The construction of the algebra $\mathcal{M}_c$ is taken from [Bch93a].

5.5) *Remarks, additions and problems*

(I) Since physical observables should be real, i.e. represented by selfadjoint operators, some physicists like to start with Jordan algebras instead of C^*– or von Neumann algebras. In this connection it is worthwhile to mention that Connes' theory of the equivalence of von Neumann algebras with cones, fulfilling some properties, extends to certain Jordan algebras, which are the analogue of von Neumann algebras. This has been worked out by B. Iochum [Io83] in his thesis.

(II) It is easy to construct examples of QFTLO, where $\mathcal{M}(D)$ is not a factor. Let $\{\mathcal{M}(O), \mathcal{H}, \mathbb{R}^{d+1}\}$ be a QFTLO on the (d+1)–dimensional Minkowski space. Define a theory on the d–dimensional space as follows. Let $\hat{D}$ be a double cone in $\mathbb{R}^d$ and D its extension to $\mathbb{R}^{d+1}$. Let $K(\hat{D})$ be the cylindrical set in $\mathbb{R}^{d+1}$. i.e. $(x^0, ..., x^{d-1}) \in \hat{D}$ and x^d arbitrary. Then $D' \cap K(\hat{D})$ contains interior points. Choose an abelian algebra $\mathcal{A}(\hat{D}) \subset \mathcal{N}(D' \cap K(\hat{D}))$ and define $\mathcal{M}(\hat{D}) = \mathcal{M}(D) \vee \mathcal{A}(\hat{D})$. This algebra has at least $\mathcal{A}$ as center. It is clear that one can choose $\mathcal{A}(\hat{D})$ in an $\mathbb{R}^d$ invariant manner. Notice that we obtain for the wedge

$$\mathcal{M}(\hat{W}) = \vee\{\mathcal{M}(D), D \subset W\}$$

because of the double cone theorem 1.4.4.

Problem: Do there exist conditions implying that $\mathcal{M}(D)$ is a factorΓ

(III) Also for the algebras of spacelike cones one knows their type. Driessler [Dri77] showed that the algebra of a spacelike cone $\mathcal{M}(C)$ is of type III. Borchers and Wollenberg [BW91] showed the following result:

5.5.1 Theorem:

Let C be a spacelike cone and e be a direction inside C. Let W be a wedge which is invariant in the e–direction. Then $\mathcal{M}(C \cap W)$ is of type III_1.

Notice if C is a cone which is causally stable, i.e. $C = C$" then exists a larger cone $C' \supset C$ such that $C = C' \cap W$. Therefore, the algebras of such cones are of type III_1.

(IV) If one deals with special assumptions then the result of section 5.4 can sometimes be strengthened. If the group is the translation group of $\mathbb{R}^d$ and one is interested in those representations where the spectrum of $U(a)$ is contained in some proper cone C then one obtains a stronger result. But first we need some notation.

5.5.2 Definition:

Let $\{\mathcal{A}, \mathrm{I\!R}^d, \alpha\}$ be a C*–dynamical system and $C \subset \mathrm{I\!R}^d$ be a closed, convex, proper cone with interior points. Let $\hat{C}$ denote the dual cone of C. Then we denote by

(1) $\mathcal{A}_0^*(C)$ the set of elements $\varphi \in \mathcal{A}^*$ with the properties:

(α) $a \to \varphi(x\alpha_a y)$ is a continuous function on $\mathrm{I\!R}^d$, $\quad x, y \in \mathcal{A}$.

(β) $\varphi(x\alpha_a y)$ is the boundary value of an analytic function $W(z)$ holomorphic in the tube

$$T(\hat{C}) = \{z \in \mathbb{C}^d;\ \Im m\, z \in \text{interior of} C\}.$$

(γ) There exists a constant m such that

$$|W(z)| \leq \|\varphi\|\|x\|\|y\|\mathrm{e}^{m\|\Im m\, z\|}$$

holds for $z \in T(C)$.

(δ) φ^* fulfils the same conditions as φ.

(2) $\mathcal{A}^*(C)$ is the norm–closure of $\mathcal{A}_0^*(C)$.

With this notation one obtains:

5.5.3 Theorem:

Let $\{\mathcal{A}, \mathrm{I\!R}^d, \alpha\}$ be a C*–dynamical system and $C \subset \mathrm{I\!R}^d$ be a closed, convex, proper cone with interior points. Then there exists a projection $E(C)$ in the center of $\mathcal{A}^{**}$ with

(1) *$\varphi \in \mathcal{A}^*(C)$ iff there holds*

$$\varphi(E(C)A) = \varphi(A), \qquad \forall\ A \in \mathcal{A}.$$

(2) *Let $\{\mathcal{H}, \pi\}$ be a representation of $\mathcal{A}$. Then one can find a continuous unitary representation $V(a)$ acting on $\mathcal{H}$, which implements α_a with spectrum $V(a) \subset C$ if and only if every vector state ω_ψ belongs to $\mathcal{A}^*(C)$.*

(3) *The representation $V(a)$ can be chosen to be in $\pi(\mathcal{A})$".*

For details see [Bch96].

(V) Part 5.4 has some interest in connection with broken symmetries. If $\{\mathcal{A}, G, \alpha\}$ is a C^*–dynamical system with G a topological group, then one is not only interested in representations where the symmetry is implemented by a continuous unitary representation of the group G, but also in representations with broken symmetries. By this we mean representations where the symmetry is no longer exact, but where there is enough symmetry left in order that it can be observed as symmetry on some observables. One possibility is to assume that there is an exact symmetry on some subalgebra. Adapting this point of view one should look for some algebra which is isomorphic to a subalgebra of $\mathcal{M}_c$, introduced in the last section. (Lagrangean field theory suggests to look at some deformed algebra. But, in the general theory it is not clear what deformation means.)

6. Tensor product decomposition of quantum field theories

The axioms of quantum field theory are such that they allow to describe two or more independent theories in one object. There are several mathematical procedures which permit to construct a new theory out of two or more independent theories. In all the known examples the new theory does not describe new physics. The simplest example is the direct sum, or more generally, the direct integral of theories. The inverse operation is the integral decomposition with respect to the center of the global algebra. There are effective criteria implementing that a theory is indecomposable with respect to the direct sum operation. This is the cluster decomposition property or equivalently the uniqueness of the vacuum vector [Bch62],[DKKR67].

More complicated is the direct product of theories. Starting with two theories $\{\mathcal{M}_i(O), U_i(\Lambda,x), \mathcal{H}_i, \Omega_i\}$, $i = 1,2$ one can define a new theory on $\mathcal{H}_1 \overline{\otimes} \mathcal{H}_2$ by $\mathcal{M}(O) = \mathcal{M}_1(O) \overline{\otimes} \mathcal{M}_2(O)$, $U(\Lambda,x) = U_1(\Lambda,x) \otimes U_2(\Lambda,x)$ and $\Omega = \Omega_1 \otimes \Omega_2$. The new theory $\{\mathcal{M}(O), U(\Lambda,x), \mathcal{H}, \Omega\}$ fulfils again all axioms of local quantum field theory. In order to discover the direct product structure one has to look at the sub–theory $\{\mathcal{M}_1(O) \otimes \mathbb{1}, U(\Lambda,x), \mathcal{H}, \Omega\}$ which fulfils the assumptions of the theory of local observables except the cyclicity assumption for the vacuum vector. In this section we want to develop the theory for the converse operation, i.e. decomposition of tensor products. Besides the usual assumptions we require that the global algebra is a factor, and that the theory satisfies the Bisognano–Wichmann property.

6.0.1 Remark:

(1) As a consequence of the Bisognano–Wichmann property one concludes that the theory fulfils the wedge duality, i.e, for every wedge the relation

$$\mathcal{M}(W)' = \mathcal{M}(W')$$

holds, where W' denotes the opposite wedge of W. For the proof see Prop. 4.4.2.

(2) If one identifies the algebra of the double cone D with

$$\mathcal{M}(D) = \cap\{\mathcal{M}(W);\ D \subset W\}. \tag{6.0.1}$$

then the general duality property

$$\mathcal{M}(D)' = \mathcal{M}(D')$$

holds, where D' denotes the (interior) of the spacelike complement of D.

6.1) *On modular covariant subalgebras*

In order to understand the problem let us start with the assumption that our theory is a tensor product.

$$\{\mathcal{M}_1(O) \overline{\otimes} \mathcal{M}_2(O), U_1(x) \otimes U_2(x), \mathcal{H}_1 \overline{\otimes} \mathcal{H}_2, \Omega_1 \otimes \Omega_2\}.$$

First we look at one algebra $\mathcal{M}$ for a suitable chosen domain. Then we have $\mathcal{M} = \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$. Since Ω is a product state we know that also the modular group splits, i.e.

$$\Delta^{\mathrm{i}t} = \Delta_1^{\mathrm{i}t} \otimes \Delta_2^{\mathrm{i}t}.$$

If this is the case then $\mathcal{M}_1 \otimes \mathbb{1}$ is a subalgebra of $\mathcal{M}$ which is mapped by σ^t onto itself

$$\sigma^t(\mathcal{M}_1 \otimes \mathbb{1}) = \mathcal{M}_1 \otimes \mathbb{1}.$$

Subalgebras which are mapped by σ^t onto itself are "modular covariant subalgebras".

We start our investigation by introducing modular covariant subalgebras and describing their relations to normal and faithful conditional expectations In addition we describe Takesaki's result on the structure of modular covariant subalgebras [Tak72].

Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and let the vector $\Omega \in \mathcal{H}$ be cyclic and separating for $\mathcal{M}$. Then we denote by Δ, J the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \Omega)$.

6.1.1 Definition:

A von Neuman subalgebra $\mathcal{N} \subset \mathcal{M}$ ($\mathbb{1} \in \mathcal{N}$) is called modular covariant if it fulfils the equation

$$\Delta^{\mathrm{i}t} \mathcal{N} \Delta^{-\mathrm{i}t} = \mathcal{N}, \qquad \forall\, t \in \mathbb{R}.$$

The set of modular covariant subalgebras of $\mathcal{M}$ will be denoted by $\mathcal{M}cs(\mathcal{M})$

Notice that the vector Ω is separating for $\mathcal{N}$ but not cyclic, because cyclicity implies $\mathcal{N} = \mathcal{M}$. (See e.g. Kadison and Ringrose [KR86] Thm. 9.2.36.)
The symbol $[\mathcal{N}\Omega]$ denotes the projection onto the Hilbert subspace generated by $\mathcal{N}\Omega$.

Modular covariant subalgebras have the following well known and easy to verify properties. (See [Tak72],[Ko86],[KK92] and [Bch98b].)

6.1.2 Lemma:

Let $\mathcal{N} \in \mathcal{M}cs(\mathcal{M})$. Let $\mathcal{H}_\mathcal{N}$ be the closure of $\mathcal{N}\Omega$ and denote by $E_\mathcal{N}$ the projection onto $\mathcal{H}_\mathcal{N}$. By $\widehat{\mathcal{N}}$ we denote the restriction of $\mathcal{N}$ to $\mathcal{H}_\mathcal{N}$. Then:

1. *$E_\mathcal{N}$ commutes with $\Delta^{\mathrm{i}t}$ and J. The restriction of Δ and J to $\mathcal{H}_\mathcal{N}$ will be denoted by $\widehat{\Delta}$ and $\widehat{J}$.*
2. *$\widehat{\Delta}$ and $\widehat{J}$ are the modular group and modular conjugation of $(\widehat{\mathcal{N}}, \Omega)$.*
3. *The commutant of $\widehat{\mathcal{N}}$ in $\mathcal{H}_\mathcal{N}$ coincides with $\widehat{J}\widehat{\mathcal{N}}\widehat{J}$.*
4. *The map $\mathcal{N} \longrightarrow \widehat{\mathcal{N}}$ is an isomorphism of von Neumann algebras.*
5. *$A \in \mathcal{M}$ and $[A, E_\mathcal{N}] = 0$ implies $A \in \mathcal{N}$.*
6. *$A \in \mathcal{M}$ and $A\Omega \in \mathcal{H}_\mathcal{N}$ implies $A \in \mathcal{N}$.*

Proof: 1. Since $\mathcal{N}$ is invariant under the action of the modular group we get $\Delta^{\mathrm{i}t}\mathcal{N}\Omega = \mathcal{N}\Omega$. Hence $\Delta^{\mathrm{i}t}$ maps $\mathcal{H}_\mathcal{N}$ onto itself. Therefore, it commutes with $E_\mathcal{N}$. For $A \in \mathcal{N}$ we get the identity $JA\Omega = \Delta^{1/2}J\Delta^{1/2}A\Omega = \Delta^{1/2}A^*\Omega$. This implies that J maps $\mathcal{H}_\mathcal{N}$ onto itself. Hence J commutes with $E_\mathcal{N}$.

2. For $A, B \in \mathcal{N}$ we find: $(\Omega, \hat{B}\widehat{\Delta}^{\mathrm{i}(t-\mathrm{i})}\hat{A}) = (\Omega, B\Delta^{\mathrm{i}(t-\mathrm{i})}A\Omega) = (\Omega, A\Delta^{-\mathrm{i}t}B\Omega) = (\Omega, \hat{A}\widehat{\Delta}^{-\mathrm{i}t}\hat{B}\Omega)$. This implies that $\widehat{\Delta}^{\mathrm{i}t}$ fulfils the KMS–condition with respect to the algebra $\widehat{\mathcal{N}}$. Hence $\widehat{\Delta}^{\mathrm{i}t}$ is the modular group of $\widehat{\mathcal{N}}$. The equation $(A \in \mathcal{N})$ $\widehat{J}\widehat{\Delta}^{1/2}\hat{A}\Omega = \hat{A}^*\Omega$ implies that $\widehat{J}$ is the modular conjugation of $\widehat{\mathcal{N}}$.
3.$J\mathcal{N}J$ is a von Neumann subalgebra of $\mathcal{M}'$. Since J and $\mathcal{N}$ commute with $E_{\mathcal{N}}$ we obtain $E_{\mathcal{N}}J\mathcal{N}J = \widehat{J}\widehat{\mathcal{N}}\widehat{J} = \widehat{\mathcal{N}}'$.
4. The algebra $\mathcal{N}'$ contains $\mathcal{M}'$ which implies that Ω is cyclic for this algebra. Hence we find for $A \in \mathcal{N}$:

$$\begin{aligned}\|A\|^2 &= \sup_{B\in\mathcal{M}'} (BE_{\mathcal{N}}\Omega, A^*ABE_{\mathcal{N}}\Omega)/(\Omega, E_{\mathcal{N}}B^*BE_{\mathcal{N}}\Omega)\\ &= \sup_{B\in\mathcal{M}'} (\sqrt{E_{\mathcal{N}}B^*BE_{\mathcal{N}}}\Omega, A^*A\sqrt{E_{\mathcal{N}}B^*BE_{\mathcal{N}}}\Omega)/(\Omega, E_{\mathcal{N}}B^*BE_{\mathcal{N}}\Omega) = \|\hat{A}\|^2.\end{aligned}$$

5,6. Let $\tilde{\mathcal{N}}$ be the subalgebra of those elements in $\mathcal{M}$ which commute with $E_{\mathcal{N}}$. Then this is again a modular covariant subalgebra which contains $\mathcal{N}$. The restriction of $\tilde{\mathcal{N}}$ to $\mathcal{H}_{\mathcal{N}}$ has again $\widehat{\Delta}$ as modular operator. But this implies that this restriction and $\widehat{\mathcal{N}}$ coincide. Since Ω is separating for $\mathcal{M}$ it follows that $\tilde{\mathcal{N}}$ and $\mathcal{N}$ coincide. □

The results of the last lemma have been strengthened.

6.1.3 Theorem: (Takesaki [Tak72])

With the assumptions and notations of Lemma 6.1.2 *we obtain:*
1) *For $A \in \mathcal{M}$ one has $EAE \in \widehat{\mathcal{N}}$.*
2) *There exists a normal faithful conditional expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{N}$.*
3) *$\mathcal{E}$ commutes with the modular action:*

$$\mathcal{E}(\mathrm{Ad}\,\Delta^{\mathrm{i}t}A) = \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{E}(A),\ A \subset \mathcal{M}.$$

4) *There exists also a conditional expectation $\mathcal{E}'$ from $\mathcal{M}'$ to $J\mathcal{E}(\mathcal{M})J$ defined by*

$$\mathcal{E}'(A') = J\mathcal{E}(JA'J)J, \qquad A' \in \mathcal{M}'.$$

5) *Let E be a projection with $E\Omega = \Omega$. If there is a von Neumann algebra $\mathcal{N} \subset \mathcal{M}$ with $E \in \mathcal{N}'$ and the central support of E in $\mathcal{N}'$ is $\mathbb{1}$ and if addition one has $E\mathcal{M}E = \mathcal{N}E$ then $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$.*

Proof: 1) Let $A \in J\mathcal{N}J$ then A commutes with $E_{\mathcal{N}}$ and with $\mathcal{M}$. Hence A commutes with $E_{\mathcal{N}}BE_{\mathcal{N}}$ for $B \in \mathcal{M}$. Since $\widehat{\mathcal{N}}$ is the commutant of $\widehat{J\mathcal{N}J}$ in $\mathcal{B}(\mathcal{H}_{\mathcal{N}})$ it follows that $E_{\mathcal{N}}BE_{\mathcal{N}} \in \widehat{\mathcal{N}}$.
2) Let ρ be the map $\mathcal{N} \longrightarrow \widehat{\mathcal{N}}$ which is a normal isomorphism. For $B \in \mathcal{M}$ define

$$\mathcal{E}(B) = \rho^{-1}(E_{\mathcal{N}}BE_{\mathcal{N}}).$$

Since $E_{\mathcal{N}}BE_{\mathcal{N}} \in \widehat{\mathcal{N}}$ it follows that $\mathcal{E}$ is a normal map. For $N_i \in \mathcal{N},\ i = 1, 2$ we obtain with $B \in \mathcal{M}$

$$\begin{aligned}\mathcal{E}(N_1BN_2) &= \rho^{-1}\big(\rho(N_1)E_{\mathcal{N}}BE_{\mathcal{N}}\rho(N_2)\big)\\ &= N_1\rho^{-1}(E_{\mathcal{N}}BE_{\mathcal{N}})N_2 = N_1\mathcal{E}(B)N_2.\end{aligned}$$

Hence $\mathcal{E}$ is a normal conditional expectation.
3) Since $E_{\mathcal{N}}$ commutes with Δ it follows that for $N \in \mathcal{N}$ the equation $\rho(\mathrm{Ad}\,\Delta^{\mathrm{i}t}N) = \mathrm{Ad}\,\Delta^{\mathrm{i}t}\rho(N)$ holds. This implies for $A \in \mathcal{M}$:

$$\begin{aligned}\mathcal{E}(\mathrm{Ad}\,\Delta^{\mathrm{i}t}A) &= \rho^{-1}(E_{\mathcal{N}}\mathrm{Ad}\,\Delta^{\mathrm{i}t}AE_{\mathcal{N}}) = \rho^{-1}(\mathrm{Ad}\,\Delta^{\mathrm{i}t}E_{\mathcal{N}}AE_{\mathcal{N}})\\ &= \mathrm{Ad}\,\Delta^{\mathrm{i}t}\rho^{-1}(E_{\mathcal{N}}AE_{\mathcal{N}}) = \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{E}(A).\end{aligned}$$

4) This is a trivial consequence of 2).
5) Let S be the Tomita conjugation of $\mathcal{M}$, then the assumptions imply that S commutes with E. Next we show that S^*S also commutes with E. Let $B \in \mathcal{M}'$ then EBE commutes with $\widehat{\mathcal{N}}$, since E commutes with $\mathcal{N}$. Let now $C \in \widehat{\mathcal{N}}'$, then we want to show that it is of the form $E\tilde{C}E$ with $\tilde{C} \in \mathcal{M}'$. Define $\tilde{C}$ by the equation $\tilde{C}A\Omega = AC\Omega$, $A \in \mathcal{M}$. If $\tilde{C}$ is bounded , then it belongs to $\mathcal{M}$ and has the properties we need. We get the following estimate:

$$\begin{aligned}\|\tilde{C}A\Omega\| &= \|AC\Omega\| = (\Omega, C^*A^*AC\Omega)^{1/2}\\ &= (\Omega, C^*EA^*AEC\Omega)^{1/2} = (\Omega, [EA^*AE]^{1/2}C^*C[EA^*AE]^{1/2}\Omega)^{1/2}\\ &\le \|C\|\|[EA^*AE]^{1/2}\Omega\| \le \|C\|\|\,|A|\,\Omega\|.\end{aligned}$$

Hence we get $\widehat{\mathcal{N}}' = E\mathcal{M}'E$ which implies that S^*S commutes with E: Hence $\mathcal{N}$ is a modular covariant subalgebra. This proves the theorem. □

6.2) *Conditional expectations and half–sided translations*

If $\mathcal{M}$ is a von Neumann algebra with cyclic and separating vector then we call the anti–linear operator $S_{\mathcal{M}} := J_{\mathcal{M}}\Delta_{\mathcal{M}}^{1/2}$ the Tomita conjugation of $(\mathcal{M}, \Omega)$. In this section we will deal with operators of the same kind, i.e. operators S fulfilling:

(*i*) S is a densely defined closed anti–linear operator with domain of definition $\mathcal{D}(S)$.
(*ii*) $S^2 = \mathbb{1}$ on $\mathcal{D}(S)$.
(*iii*) $\Omega \in \mathcal{D}(S)$ and $S\Omega = \Omega$.

We will call such operators generalized Tomita conjugations.

Since S is closed it has a polar decomposition $S = J\Delta^{1/2}$. Then Δ is invertible and J is a conjugation, i.e.

$$J\Delta J = \Delta^{-1}, \qquad J = J^* = J^{-1}. \tag{6.2.1}$$

These properties follow from the condition $S^2 = \mathbb{1}$. (See e.g, Bratteli and Robinson [BR79] Prop.2.5.11.)

We often deal with the situation that we have a generalized Tomita conjugation S and a Tomita conjugation $S_{\mathcal{M}}$ which is an extension of S. From Eq. (2.1.3) we know $(1+\Delta_{\mathcal{M}})^{-1} \ge (1+\Delta)^{-1}$. This implies that the operator–valued function $C(t) := \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$ has a bounded analytic extension into the strip $S(0, \frac{1}{2})$. We are interested in determining the value of this function at the upper boundary. We obtain:

6.2.1 Lemma:

Let S be a generalized Tomita conjugation and $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$ such that the latter is an extension of S. Define $C(t) := \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$. Then $C(t)$ has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ and at the upper boundary one has

$$C(t + \frac{\mathrm{i}}{2}) = J_{\mathcal{M}}C(t)J. \tag{6.2.2}$$

Moreover, the following estimate holds:

$$\|C(\tau)\| \leq 1.$$

Proof: Since $\Delta_{\mathcal{M}} \leq \Delta$ it follows by standard arguments that $C(t)$ has a bounded extension into the strip $S(0, \frac{1}{2})$. This extension is bounded in norm by 1. Choose $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S_{\mathcal{M}})$ then we have

$$\begin{aligned}(\varphi, C(t + \frac{\mathrm{i}}{2})\psi) &= (\Delta_{\mathcal{M}}^{\frac{1}{2}}\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}\Delta^{-\frac{1}{2}}\psi) \\ &= (J_{\mathcal{M}}S_{\mathcal{M}}\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi) = (J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi, S_{\mathcal{M}}\varphi).\end{aligned}$$

Since $S^*\psi \in \mathcal{D}(S^*)$ we find $J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi \in \mathcal{D}(S^*_{\mathcal{M}})$. Hence we obtain

$$= (\varphi, S^*_{\mathcal{M}}J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}JS^*\psi).$$

With $S^*_{\mathcal{M}}J_{\mathcal{M}} = J_{\mathcal{M}}S_{\mathcal{M}}$ and the commutation of $S_{\mathcal{M}}$ with $\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ we find

$$= (\varphi, J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}S_{\mathcal{M}}\Delta^{\mathrm{i}t}JS^*\psi).$$

Because $S_{\mathcal{M}}$ is an extension of S, we can replace $S_{\mathcal{M}}$ by S which commutes with $\Delta^{\mathrm{i}t}$. Hence we obtain

$$= (\varphi, J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}SJS^*\psi).$$

With $SJS^* = J$ we get

$$(\varphi, C(t + \frac{\mathrm{i}}{2})\psi) = (\varphi, J_{\mathcal{M}}C(t)J\psi).$$

Since $\mathcal{D}(S_{\mathcal{M}})$ and $\mathcal{D}(S^*)$ are both dense in $\mathcal{H}$ the lemma follows. □

We saw in Sect.2.3 that the elements in $Char(\mathcal{M})$ are in one to one correspondence with the von Neumann subalgebras belonging to $\mathcal{S}ub(\mathcal{M})$. Therefore, it is interesting to know which condition of Lemma 2.3.2 is the crucial one. It turns out that the conditions (1)–(6) can easily be satisfied, but that condition (7) is the essential one. In order to overcome the lack of condition 7 of Lemma 2.3.2 we will use a property similar to that of half–sided modular inclusions.

6.2.2 Theorem:

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω and let $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$. Let S be a generalized Tomita conjugation and assume $S_{\mathcal{M}}$ is an extension of S. Assume in addition that S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \leq 0$. Then:

1. *There exists a unitary group $U(t)$ with*
   α *$U(t)\Omega = \Omega$ for all $t \in \mathbb{R}$.*
   β *$U(t)$ has a non–negative generator.*
2. *Between the modular group of $\mathcal{M}$ and $U(t)$ exist the relations*

$$\Delta_{\mathcal{M}}^{\mathrm{i}t} U(s) \Delta_{\mathcal{M}}^{-\mathrm{i}t} = U(\mathrm{e}^{-2\pi t} s), \qquad J_{\mathcal{M}} U(t) J_{\mathcal{M}} = U(-t).$$

3. *Define*

$$S_t = \Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$$

which is monotonously increasing with t and set

$$S_\infty = \lim_{t \to \infty} S_t.$$

Then there holds for $s > 0$

$$U(s) S_\infty U(-s) = S_{-\frac{1}{2\pi} \log s}.$$

Notice: There exists a variant of this theorem which is obtained by replacing everywhere t by $-t$.

The statement of the theorem needs some explanation. By assumption the family $\Delta_{\mathcal{M}}^{\mathrm{i}t} S \Delta_{\mathcal{M}}^{-\mathrm{i}t}$ is increasing with t. Hence the projections onto the graphs are an increasing family of projections which converges strongly. Since all these projections are majorized by the projection onto the graph of $S_{\mathcal{M}}$ the limit is smaller or equal to the majorant.

The proof of this theorem is a variation of the proof of Wiesbrock's theorem on half–sided modular inclusions presented in section 2.4.

6.2.3 Lemma:

Let $S = J\Delta^{1/2}$ be a generalized Tomita conjugation. In addition let V be a unitary operator with
a. $V\mathcal{D}(S) \subset \mathcal{D}(S)$.
b. $V\Omega = \Omega$.
c. For $\psi \in \mathcal{D}(S)$ one has $SV\psi = VS\psi$.
Then:

The operator–valued function

$$\Delta^{-\mathrm{i}t} V \Delta^{\mathrm{i}t} =: V(t)$$

has a bounded analytic continuation into the strip $S(0, \frac{1}{2})$ which fulfils the estimate

$$\|V(t + \mathrm{i}\tau)\| \leq 1. \quad 0 \leq \tau \leq \frac{1}{2}.$$

At the upper boundary $V(z)$ obeys the equation

$$V(t+\frac{\mathrm{i}}{2}) = JV(t)J.$$

Proof: Since S commutes with $\Delta^{\mathrm{i}t}$ it follows that S commutes with $V(t)$. Moreover, since $V\mathcal{D}(S) \subset \mathcal{D}(S)$ it follows by the usual argument that $\Delta^{-\mathrm{i}t}V\Delta^{\mathrm{i}t}$ has a bounded analytic continuation into $S(0,\frac{1}{2})$. Choose $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S)$. Then one has

$$\begin{aligned}(\varphi, V(t+\frac{\mathrm{i}}{2})\psi) &= (\Delta^{1/2}\varphi, \Delta^{-\mathrm{i}t}V\Delta^{\mathrm{i}t}\Delta^{-1/2}\psi) = (JS\varphi, V(t)JS^*\psi)\\ &= (S^*J\varphi, V(t)JS^*\psi) = (SV(t)JS^*\psi, J\varphi) = (V(t)J\psi, J\varphi) = (\varphi, JV(t)J\psi).\end{aligned}$$

This shows the lemma. □

Next we have a look at the expression $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t}$ under the assumption of the theorem.

6.2.4 Lemma:

Assume S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \leq 0$. Then for the operator–valued function $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\Delta^{\mathrm{i}t} =: C(t)$ the following holds:

- *(i) The inclusion properties:*
  - *α. $C(t)\mathcal{D}(S) \subset \mathcal{D}(S)$ for $t \geq 0$.*
  - *β. $C(t)\mathcal{D}(S^*) \subset \mathcal{D}(S^*)$ for $t \leq 0$.*
  - *γ. $C(t+\frac{\mathrm{i}}{2})\mathcal{D}(S^*) \subset \mathcal{D}(S^*)$ for $t \in \mathbb{R}$.*
- *(ii) This implies:*
  - *α. For $\psi \in \mathcal{D}(S)$ one has $SC(t)\psi = C(t)S\psi$ provided $t \geq 0$.*
  - *β. For $\varphi \in \mathcal{D}(S^*)$ one has $S^*C(t)\varphi = C(t)S^*\varphi$ if $t \leq 0$.*
  - *γ. For $\varphi \in \mathcal{D}(S^*)$ one has $S^*C(t+\frac{\mathrm{i}}{2})\varphi = C(t+\frac{\mathrm{i}}{2})S^*\varphi$ for all $t \in \mathbb{R}$.*

Proof: S is for $t \leq 0$ an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$. This implies $\Delta_{\mathcal{M}}^{\mathrm{i}t}\mathcal{D}(S) \subset \mathcal{D}(S) \subset \mathcal{D}(S_{\mathcal{M}})$. Hence we obtain $C(t)\mathcal{D}(S) \subset \mathcal{D}(S)$ for $t \geq 0$. Next choose $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S)$ then we obtain for $t \leq 0$:

$$\begin{aligned}(\psi, S\Delta_{\mathcal{M}}^{\mathrm{i}t}S\varphi) &= (\psi, S_{\mathcal{M}}\Delta_{\mathcal{M}}^{\mathrm{i}t}S\varphi)\\ &= (\psi, \Delta_{\mathcal{M}}^{\mathrm{i}t}S_{\mathcal{M}}S\varphi) = (\psi, \Delta_{\mathcal{M}}^{\mathrm{i}t}\varphi) = (\Delta_{\mathcal{M}}^{-\mathrm{i}t}\psi, \varphi).\end{aligned}$$

On the other hand we get

$$(\psi, S\Delta_{\mathcal{M}}^{\mathrm{i}t}S\varphi) = (S\varphi, \Delta_{\mathcal{M}}^{-\mathrm{i}t}S^*\psi).$$

Since the expression is continuous in φ we conclude $\Delta_{\mathcal{M}}^{-\mathrm{i}t}S^*\psi \in \mathcal{D}(S^*)$ and from $S^*\mathcal{D}(S^*) = \mathcal{D}(S^*)$ we get for $t \leq 0$ $\Delta_{\mathcal{M}}^{-\mathrm{i}t}\mathcal{D}(S^*) \subset \mathcal{D}(S^*)$. This implies (i),$\beta$. Using Lemma 6.2.1 we obtain

$$C(t+\frac{\mathrm{i}}{2})\mathcal{D}(S^*) = J_{\mathcal{M}}C(t)J\mathcal{D}(S^*) = J_{\mathcal{M}}C(t)\mathcal{D}(S).$$

Because of $\mathcal{D}(S) \subset \mathcal{D}(S_{\mathcal{M}})$ we obtain by the definition of $C(t)$ the inclusion $C(t+\frac{i}{2})\mathcal{D}(S^*) \subset J_{\mathcal{M}}\mathcal{D}(S_{\mathcal{M}}) = \mathcal{D}(S^*_{\mathcal{M}}) \subset \mathcal{D}(S^*)$. This shows (i),$\gamma$. For $t \geq 0$ we obtain from $\Delta_{\mathcal{M}}^{-it}\mathcal{D}(S) \subset \mathcal{D}(S) \subset \mathcal{D}(S_{\mathcal{M}})$

$$\begin{aligned} SC(t)\mathcal{D}(S) &= S\Delta_{\mathcal{M}}^{-it}\Delta^{it}\mathcal{D}(S) = S_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}\Delta^{it}\mathcal{D}(S) = \Delta_{\mathcal{M}}^{-it}S_{\mathcal{M}}\Delta^{it}\mathcal{D}(S) \\ &= \Delta_{\mathcal{M}}^{-it}S\Delta^{it}\mathcal{D}(S) = \Delta_{\mathcal{M}}^{-it}\Delta^{it}S\mathcal{D}(S) = C(t)S\mathcal{D}(S). \end{aligned}$$

Next we calculate for $\psi \in \mathcal{D}(S^*)$ and $\varphi \in \mathcal{D}(S)$ and $t \leq 0$

$$(\varphi, S^*C(t)\psi) = (\Delta_{\mathcal{M}}^{-it}\Delta^{it}\psi, S\varphi) = (\Delta^{it}\psi, \Delta_{\mathcal{M}}^{it}S\Delta_{\mathcal{M}}^{-it}\Delta_{\mathcal{M}}^{it}\varphi).$$

As $\Delta_{\mathcal{M}}^{it}S\Delta_{\mathcal{M}}^{-it}$ is the generalized Tomita conjugation with domain $\Delta_{\mathcal{M}}^{it}\mathcal{D}(S) \subset \mathcal{D}(S)$ it follows that $(\Delta_{\mathcal{M}}^{it}S\Delta_{\mathcal{M}}^{-it})^*$ is an extension of S^*. This implies

$$= (\Delta_{\mathcal{M}}^{it}\varphi, (\Delta_{\mathcal{M}}^{it}S\Delta_{\mathcal{M}}^{-it})^*\Delta^{it}\psi) = (\Delta_{\mathcal{M}}^{it}\varphi, S^*\Delta^{it}\psi) = (\varphi, \Delta_{\mathcal{M}}^{-it}\Delta^{it}S^*\psi).$$

This shows (ii),β. Finally

$$S^*C(t+\frac{i}{2})\mathcal{D}(S^*) = S^*J_{\mathcal{M}}C(t)J\mathcal{D}(S^*).$$

As in the proof of (i),γ we have $J_{\mathcal{M}}C(t)J\mathcal{D}(S^*) \subset \mathcal{D}(S^*_{\mathcal{M}}) \subset \mathcal{D}(S^*)$. Hence we obtain

$$= S^*_{\mathcal{M}}J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}\Delta^{it}J\mathcal{D}(S^*) = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}S_{\mathcal{M}}\Delta^{it}J\mathcal{D}(S^*).$$

Since $S_{\mathcal{M}}$ is an extension of S we get

$$= J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}S\Delta^{it}J\mathcal{D}(S^*) = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-it}\Delta^{it}JS^*\mathcal{D}(S^*) = C(t+\frac{i}{2})S^*\mathcal{D}(S^*).$$

This shows the lemma. □

$C(t)$ has an analytic extension into $S(0,\frac{1}{2})$. For $t \geq 0$ it maps $\mathcal{D}(S)$ into $\mathcal{D}(S)$ and for the rest of the boundary it maps $\mathcal{D}(S^*)$ into $\mathcal{D}(S^*)$. Therefore, we will map $S(0,\frac{1}{2})$ bi–holomorphic onto $S(0,\frac{1}{2})$ in such a way that $\mathbb{R}^+$ ist mapped onto $\mathbb{R}$ and the rest of the boundary is mapped onto $\frac{i}{2}+\mathbb{R}$. This is achieved by the transformation

$$\zeta = \frac{1}{2\pi}\log(e^{2\pi z}-1), \qquad z = \frac{1}{2\pi}\log(e^{2\pi\zeta}+1).$$

We introduce

$$B(t) := C(\frac{1}{2\pi}\log(e^{2\pi t}+1)), \tag{6.2.3}$$

then together with Lemma 6.2.4 holds

$$\begin{aligned} B(t)\mathcal{D}(S) \subset \mathcal{D}(S), \text{ for } t \in \mathbb{R} \quad &\text{and} \quad SB(t)\mathcal{D}(S) = B(t)S\mathcal{D}(S), \\ B(t+\frac{i}{2})\mathcal{D}(S^*) \subset \mathcal{D}(S^*), \text{ for } t \in \mathbb{R} \quad &\text{and} \quad S^*B(t+\frac{i}{2})\mathcal{D}(S^*) = B(t+\frac{i}{2})S^*\mathcal{D}(S^*). \end{aligned} \tag{6.2.4}$$

The last inclusion is valid with the possible exception of the point $\frac{\mathrm{i}}{2}$. Next we show:

6.2.5 Lemma:

Define $B(s,t) = \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}$ *with* $B(t)$ *from Eq.* (6.2.3). $B(s,t)$ *has an analytic extension into the tube based on the quadrangle with the corners*

$$(\Im m\, s, \Im m\, t) = (0,0),\ (\frac{1}{2}, -\frac{1}{2}),\ (\frac{1}{2}, 0),\ (0, \frac{1}{2}). \tag{6.2.5}$$

In the domain of holomorphy one has

$$\|B(\sigma,\tau)\| \leq 1.$$

In the four corners $B(\sigma,\tau)$ *takes the values*

$$\begin{aligned} B(s,t) &= \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}, \\ B(s+\frac{\mathrm{i}}{2}, t) &= \Delta^{-\mathrm{i}s}JB(t)J\Delta^{\mathrm{i}s}, \\ B(s, t+\frac{\mathrm{i}}{2}) &= \Delta^{-\mathrm{i}s}B(t+\frac{\mathrm{i}}{2})\Delta^{\mathrm{i}s}, \\ B(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) &= \Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}. \end{aligned}$$

Proof: For t real we get by Lemma 6.2.3 in s an analytic extension into $S(0,\frac{1}{2})$ which is bounded in norm by 1. Moreover, we have $B(s+\frac{\mathrm{i}}{2}, t) = JB(s,t)J = \Delta^{-\mathrm{i}s}JB(t)J\Delta^{\mathrm{i}s}$. For s real Lemma 6.2.1 yields an analytic extension in t into $S(0,\frac{1}{2})$ which is also bounded in norm by 1. Moreover, we have $B(s,t+\frac{\mathrm{i}}{2}) = \Delta^{-\mathrm{i}s}B(t+\frac{\mathrm{i}}{2})\Delta^{\mathrm{i}s}$. Since J is anti–linear the expression $JB(t)J$ can be analytically continued into $S(-\frac{1}{2},0)$ wich is norm–bounded by 1. At the lower boundary one finds $B(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) = \Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}$. Using the Malgrange-Zerner theorem Thm. 1.4.2 we obtain the statement of the lemma. □

Now we are prepared for the first crucial step:

6.2.6 Proposition:

Between the group $\Delta^{\mathrm{i}s}$ *and the operator–valued function* $B(t)$ *exist the relations*

$$\Delta^{\mathrm{i}s}B(t)\Delta^{-\mathrm{i}s} = B(t-s) \quad \text{and} \quad JB(t)J = B(t+\frac{\mathrm{i}}{2}).$$

Proof: Choose $\psi \in \mathcal{D}(S)$ and $\varphi \in \mathcal{D}(S^*)$ and define the two functions

$$\begin{aligned} F^+(s,t) &= (\varphi, B(s,t)\psi) = (\varphi, \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi), \\ F^-(s,t) &= (S\psi, B(s,t)^*S^*\varphi) = (S\psi, \Delta^{-\mathrm{i}s}B(t)^*\Delta^{\mathrm{i}s}S^*\varphi). \end{aligned}$$

By Lemma 6.2.5 $F^+(s,t)$ has a bounded analytic extension into the tube given by Eq. (6.2.5) and $F^-(s,t)$ into the conjugate complex of that domain, which is also the negative of the domain given by Eq. (6.2.5). By Eq. (6.2.4) we obtain for real s, t

$$\begin{aligned} F^+(s,t) &= (S^*S^*\varphi, \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi) = (S\Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi, S^*\varphi) \\ &= (\Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}S\psi, S^*\varphi) = F^-(s,t). \end{aligned}$$

Moreover, one obtains with Eq. (6.2.4) and Lemma 6.2.5

$$\begin{aligned} F^+(s+\frac{\mathrm{i}}{2}, t-\frac{\mathrm{i}}{2}) &= (S^*S^*\varphi, \Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}\psi) = (S\Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}\psi, S^*\varphi) \\ &= (\Delta^{-\mathrm{i}s}JS^*B(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}\psi, S^*\varphi) = (\Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})S^*J\Delta^{\mathrm{i}s}\psi, S^*\varphi) \\ &= (\Delta^{-\mathrm{i}s}JB(t+\frac{\mathrm{i}}{2})J\Delta^{\mathrm{i}s}S\psi, S^*\varphi) = F^-(s-\frac{\mathrm{i}}{2}, t+\frac{\mathrm{i}}{2}). \end{aligned}$$

Using the edge of the wedge theorem Thm. 1.4.1 we obtain a function which is periodic, i.e.

$$F(s,t) = F(s+n\mathrm{i}, t-n\mathrm{i}), \qquad n \in \mathbb{Z}.$$

(As mentiond in Sect. 2.6 the discontinuity which might exist at $\frac{\mathrm{i}}{2}$ is harmless.) Since $F(\sigma,\tau)$ is bounded by $\max\{\|\psi\|\|\varphi\|, \|S\psi\|\|S^*\varphi\|\}$ the function must be constant in the direction of periodicity, i,e.

$$F(s,t) = F(s+z, t-z), \qquad z \in \mathbb{C}.$$

Choosing $z = -s$ and inserting the expression for F we obtain:

$$(\varphi, \Delta^{-\mathrm{i}s}B(t)\Delta^{\mathrm{i}s}\psi) = (\varphi, B(t+s)\psi).$$

For $s = \frac{\mathrm{i}}{2}$ and $z = -\frac{\mathrm{i}}{2}$ one finds

$$(\varphi, JB(t)J\psi) = (\varphi, B(t+\frac{\mathrm{i}}{2})\psi).$$

Since $\mathcal{D}(S)$ and $\mathcal{D}(S^*)$ are both dense in $\mathcal{H}$ we obtain the statement of the proposition. □

The last result is the basis of the following

6.2.7 Proposition:

The operator–valued function $C(t)$ is a commutative family of unitary operators. Moreover, there exists a continuous unitary group $U(s)$ with non–negative generator such that

$$C(t) = U(\mathrm{e}^{2\pi t} - 1) \tag{6.2.6}$$

holds.

The proof of this statement is based on the last proposition and it is an exact copy of the corresponding part of the proof of Thm. 2.6.2. Therefore, it does not need to be repeated here.

Proof of Theorem 6.2.2: The first statement of the theorem is the content of Proposition 6.2.7. We know that $C(t)$ fulfils the cocycle relation, which we use in the form $\Delta^{-\mathrm{i}s}C(t)\Delta^{\mathrm{i}s} = C(s+t)C(s)^*$. Inserting Eq.(6.2.6) we find

$$\Delta_{\mathcal{M}}^{-\mathrm{i}s}U(\mathrm{e}^{2\pi t}-1)\Delta_{\mathcal{M}}^{\mathrm{i}s} = U(\mathrm{e}^{2\pi(s+t)}-1)U(-\mathrm{e}^{2\pi s}+1) = U(\mathrm{e}^{2\pi s}(\mathrm{e}^{2\pi t}-1)).$$

Since $U(t)$ fulfils the spectrum condition the last equation can analytically be continued to arbitrary arguments. This shows the first part of statement 2. From (6.2.6) we obtain $C(\frac{\mathrm{i}}{2}) = U(-2)$. Hence we obtain $J_{\mathcal{M}} = C(\frac{\mathrm{i}}{2})J = U(-2)J$. If we insert Eq. (6.2.3) into the second expression of Proposition 6.2.6 we get

$$\mathrm{Ad}\, JC(\frac{1}{2\pi}\log(\mathrm{e}^{2\pi t}+1)) = C(\frac{1}{2\pi}\log(-\mathrm{e}^{2\pi t}+1)).$$

Using Eq. (6.2.6) this reads $\mathrm{Ad}\, JU(\mathrm{e}^{2\pi t}) = U(-\mathrm{e}^{2\pi t})$. With the above expression for $J_{\mathcal{M}}$ we obtain

$$\mathrm{Ad}\, J_{\mathcal{M}}U(\mathrm{e}^{2\pi t}) = \mathrm{Ad}\,\{U(-2)J\}U(\mathrm{e}^{2\pi t}) = U(-\mathrm{e}^{2\pi t}).$$

By analytic continuation we obtain the second relation of statement 2. Finally with $\mathrm{Ad}\,\Delta_{\mathcal{M}}^{\mathrm{i}t}S = S_t$ and $\mathrm{Ad}\,\Delta^{\mathrm{i}t}S = S$ we obtain $\mathrm{Ad}\, C(-t)S = S_t$. Inserting Eq. (6.2.6) we find $\mathrm{Ad}\, U(\mathrm{e}^{-2\pi t}-1)S = S_t$. With $S_\infty = \lim_{t\to\infty} S_t = \lim_{t\to\infty}\mathrm{Ad}\, U(\mathrm{e}^{-2\pi t}-1)S$ we get $S_t = \mathrm{Ad}\, U(\mathrm{e}^{-2\pi t})S_\infty$ or $\mathrm{Ad}\, U(s)S_\infty = S_{-\frac{1}{2\pi}\log s}$, $s>0$. This proves the theorem. □

From Thm. 6.2.2 one can draw several conclusions. We start with the following result:

6.2.8 Corollary:

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω and let $S_{\mathcal{M}}$ be the Tomita conjugation of $\mathcal{M}$. Let S be a generalized Tomita conjugation and assume $S_{\mathcal{M}}$ is an extension of S. Assume also that S is an extension of $\Delta_{\mathcal{M}}^{\mathrm{i}t}S\Delta_{\mathcal{M}}^{-\mathrm{i}t}$ for $t \le 0$. If we have in addition

$$S_{\mathcal{M}} = \lim_{t\to\infty} S_t,$$

then S is the Tomita conjugation of a von Neumann algebra $\mathcal{N}$ which has Ω as cyclic and separating vector. Moreover, on has

$$\mathcal{N} = U(1)\mathcal{M}U(-1).$$

6.2.9 Remark:

Unfortunately I could not show that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, although it is suggested by the fact that $S_{\mathcal{M}}$ is an extension of $S_{\mathcal{N}}$. Up to now one needs additional information in order to conclude that $\mathcal{N}$ is a subalgebra of $\mathcal{M}$.

Proof of the Corollary: With $S_\infty = \lim_{t\to\infty} S_t$ we know from Thm. 6.2.2 the relation $S = U(1)S_\infty U(-1)$. With $S_\infty = S_{\mathcal{M}}$ it follows $S = U(1)S_{\mathcal{M}}U(-1)$. Since $\mathcal{M}\Omega$ is a core for $S_{\mathcal{M}}$ it follows with $\mathcal{N} = U(1)\mathcal{M}U(-1)$ that $\mathcal{N}\Omega$ is a core for S. Hence the corollary is proved. □

In connection with conditional expectations one can conclude that the algebra $\mathcal{N}$, described in Corollary 6.2.8, is a subalgebra of $\mathcal{M}$.

A half–sided translation associated with $\mathcal{M}$ is a one–parametric unitary group $V(t)$ fulfilling:
(i) $V(t)\Omega = \Omega$ for all $t \in \mathbb{R}$.
(ii) $V(t)$ has a non–negative generator.
(iii) $V(t)\mathcal{M}V(-t) \subset \mathcal{M}$ for $t \geq 0$ (or for $t \leq 0$).
With these concepts we show:

6.2.10 Theorem:

Let $\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω. Assume $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$ and $\mathcal{E}$ the associated conditional expectation. (See Thm. 6.1.3.) Denote by $\widehat{\mathcal{N}}$ resp. $\widehat{\mathcal{E}}$ the restriction of $\mathcal{N}$ resp. $\mathcal{E}$ to the cyclic subspace of $\mathcal{N}$. Assume $V(t)$ is a +half–sided translation for $\mathcal{M}$. Then:
(i) $\mathcal{E}(V(t)\mathcal{M}V(-t))$ is dense in the von Neumann algebra $\{\mathcal{E}(V(t)\mathcal{M}V(-t))\}''$.
(ii) There exists a +half–sided translation for $\widehat{\mathcal{N}} = \widehat{\mathcal{E}}(\mathcal{M})$ with

$$U(t)\widehat{\mathcal{N}}U(-t) = \{\widehat{\mathcal{E}}(V(t))\mathcal{M}V(-t))\}''.$$

Proof: From Thm. 6.1.3 and from $E = [\mathcal{N}\Omega]$ we get the relation $\widehat{\mathcal{E}}(V(t)\mathcal{M}V(-t))\Omega = EV(t)\mathcal{M}\Omega$. Since $V(t)$ has a non–negative generator we conclude that $EV(t)\mathcal{M}\Omega$ is dense in $E\mathcal{H}$. Let $S_{-\frac{1}{2\pi}\log t}$ be the map $EV(t)AV(-t)\Omega \longrightarrow EV(t)A^*V(-t)\Omega$. Since $J_{\mathcal{M}}\widehat{\mathcal{N}}J_{\mathcal{M}}$ is the commutant of $\widehat{\mathcal{N}}$ in $E\mathcal{H}$ it follows that $S_{-\frac{1}{2\pi}\log t}$ is pre–closed. Denote the closure again by $S_{-\frac{1}{2\pi}\log t}$. Since $V(t)\mathcal{M}V(-t) \subset V(t_0)\mathcal{M}V(-t_0)$ for $t \geq t_0$ we obtain with $\Delta^{\mathrm{i}t}_{\mathcal{M}}V(s)\Delta^{-\mathrm{i}t}_{\mathcal{M}} = V(\mathrm{e}^{-2\pi t}s)$ and with $\Delta^{\mathrm{i}t}_{\mathcal{N}} = \Delta^{\mathrm{i}t}_{\mathcal{M}}|E\mathcal{H}$ that $S_{\mathcal{N}}$ is an extension of S_0 which is an extension of $\Delta^{\mathrm{i}t}_{\mathcal{N}}S_0\Delta^{-\mathrm{i}t}_{\mathcal{N}}$ for $t \leq 0$. Hence the family $\{S_t\}$ fulfils the conditions of Thm. 6.2.2. Consequently exists a +half–sided translation $U(t)$ of $\widehat{\mathcal{N}}$ with

$$S_t = U(\mathrm{e}^{2\pi t})S_{\mathcal{N}}U(-\mathrm{e}^{2\pi t}).$$

Since $\{EV(\mathrm{e}^{2\pi t})A\Omega; A \in \mathcal{M}\}$ is a core for S_t there exists an operator B affiliated with $\mathcal{N}$ such that $U(\mathrm{e}^{2\pi t})BU(-\mathrm{e}^{2\pi t})\Omega = EV(\mathrm{e}^{2\pi t})AV(-\mathrm{e}^{2\pi t})\Omega$ holds. (See [BR79] Prop. 2.9.5.) Since Ω is separating for $\widehat{\mathcal{N}}$ we obtain $U(\mathrm{e}^{2\pi t})\hat{B}U(-\mathrm{e}^{2\pi t}) = EV(\mathrm{e}^{2\pi t})AV(-\mathrm{e}^{2\pi t})E$ which implies $\|B\| \leq \|A\|$. Hence we get $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E \subset U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}U(-\mathrm{e}^{2\pi t})$. The sets $EV(\mathrm{e}^{2\pi t})\mathcal{M}\Omega$ and $U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}\Omega$ are both a core for S_t which implies that $EV(\mathrm{e}^{2\pi t})\mathcal{M}\Omega$ is dense in $U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}\Omega$ in the graph topology of S_t. Since the graph topology of S_t is stronger than the Hilbert space topology we get the density in the Hilbert space topology. Since Ω is separating and since $EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E$ is convex we conclude that

$EV(\mathrm{e}^{2\pi t})\mathcal{M}V(-\mathrm{e}^{2\pi t})E$ is strongly dense in $U(\mathrm{e}^{2\pi t})\widehat{\mathcal{N}}U(-\mathrm{e}^{2\pi t})$. Hence the theorem is proved. □

6.3) *Construction of sub–theories*

If we start with a wedge W and assume the algebra $\mathcal{M}(W)$ has a modular covariant subalgebra $\mathcal{N}(W)$. Let $\mathcal{E}_W$ be the associated conditional expectation and E_W the projection onto $[\mathcal{N}(W)\Omega]$. If we now change the wedge to $\Lambda W + x$ then of course $U(\Lambda, x)\mathcal{N}(W)U(\Lambda, x)^*$ is a modular covariant subalgebra of $\mathcal{M}(\Lambda W + x)$. But in order to obtain a decomposition of the global field theory the projections E_W and $E_{\Lambda W+x}$ have to coincide. If this is the case then we also need conditional expectations for the algebras $\mathcal{M}(D)$ associated with double cones. In order to be able to construct such conditional expectations the algebras must be closely related to the algebras of wedges. Therefore, we set

$$\mathcal{M}(D) = \cap\{\mathcal{M}_{\Lambda W+x};\ D \subset \Lambda W + x\}.$$

Now we can define what we mean by the coherence property.

6.3.1 Definition:

Assume we deal with a quantum field theory in the vacuum sector. Assume with every double cone D and every wedge W is associated a modular covariant subalgebra $\mathcal{N}(D) \subset \mathcal{M}(D)$ and $\mathcal{N}(W) \subset \mathcal{M}(W)$. Then we call this family coherent if the projections E_D and E_W coincide for all double cones D and for all wedges W.

Unfortunately it is not always possible to transport the conditional expectation from one wedge to all others in a coherent way. Half–sided translations can only be used if the positive linear maps $L_t(A) : \mathcal{M} \to \widehat{\mathcal{N}}$ defined by

$$L_t(A) = U(-t)EV(t)AV(-t)EU(t)$$

are trivial. These half–sided translations of $\mathcal{M}(W)$ would be necessary in order to transport the conditional expectation to the shifted wedges or to pass to other wedges with one light ray in common. (See Sect. 4.4.)

In case one knows that the translations in the characteristic two–plane of the wedge W commute with E_W one can conclude more:

6.3.2 Lemma:

Let the dimension of the Minkowski space be larger than 2. Let $\mathcal{N}(W)$ be a modular covariant subalgebra of $\mathcal{M}(W)$. Assume E_W commutes with the translations in the characteristic two–plane of W. Then E_W commutes with all translations.

Proof: From the projection E_W we define the projection $E_W(a)$ by $E_W(a) = \mathrm{Ad}\,U(a)E_W$. For x in the characteristic two–plane we get $\mathrm{Ad}\,U(x)E_W(a) = E_W(a)$. Let $\mathcal{P}$ be the von Neumann algebra generated by all $E_W(a)$. This algebra is invariant under the group $U(a)$. Since this group fulfils the spectrum condition it is inner, i.e there exists a unitary group $V(a) \in \mathcal{P}$ with $V(a)AV(-a) = U(a)AU(-a)$ for all $A \in \mathcal{P}$. The group

$V(a)$ fulfils also the spectrum condition and maps Ω onto Ω [Bch96]. Moreover, we get $V(x) = 1$ for x in the characteristic two–plane. Denote the spectral family of $V(a)$ by $F(\Delta)$. Let $F(\omega)$ be the spectral projection associated with the set $\{p; p_0 < \omega\}$ then the constance of $V(x)$ implies $F(\omega) = F(\{0\})$. Hence for every compact set Δ not containing the origin we get $F(\Delta) = 0$. This implies $V(a) = \mathbb{1}$ and hence the lemma. □

Assume we have a coherent family of modular covariant subalgebras for all wedges. It remains to construct a modular covariant subalgebra for every double cone.

6.3.3 Lemma:

Let $\mathcal{N}(W)$ be a coherent family of modular covariant subalgebras of $\mathcal{M}(W)$. Define for any double cone

$$\mathcal{N}(D) = \cap\{\mathcal{N}(W);\ D \subset W\}.$$

Then $\mathcal{N}(D)$ is a modular covariant subalgebra of

$$\mathcal{M}(D) = \cap\{\mathcal{M}(W);\ D \subset W\}.$$

Moreover, one has

$$[\mathcal{N}(D)\Omega] = [\mathcal{N}(W)\Omega].$$

Proof: Because of the coherence we know that E_W is independent of W. Therefore, we call it E. For every W we know by Lemma 6.1.2 $\mathcal{N}(W) = \mathcal{M}(W) \cap \{E, \mathbb{1}\}'$. Hence we obtain by definition of $\mathcal{N}(W)$ the relation

$$\mathcal{N}(D) = \mathcal{M}(D) \cap \{E, \mathbb{1}\}'. \tag{6.3.1}$$

This shows that $\mathcal{N}(D)$ is a subalgebra of $\mathcal{M}(D)$. For $A \in \mathcal{M}(D)$ and a wedge $W \supset D$ one has $\mathcal{E}_W(A)\Omega = EA\Omega$. Since the right side is independent of W we obtain $\mathcal{N}(D)\Omega = E\mathcal{M}(D)\Omega$. Hence Ω is cyclic for $\widehat{\mathcal{N}}(D)$ in $E\mathcal{H}$. Since this vector is also separating for $\mathcal{N}$ and since E commutes with $\mathcal{N}(D)$ it follows that the central carrier of E in $\mathcal{N}'$ is $\mathbb{1}$, i.e. the map

$$\alpha : \mathcal{N}(D) \longrightarrow \widehat{\mathcal{N}}(D)$$

is an isomorphism of von Neumann algebras. Define for $A \in \mathcal{M}(D)$

$$\mathcal{E}_D(A) = \alpha^{-1}(EAE)$$

then Eq. (6.3.1) implies that $\mathcal{E}_D$ is a conditional expectation. This implies by Thm. 6.1.3,4 that $E\mathcal{H}$ is invariant under the modular group of $\mathcal{M}(D)$. Hence $\mathcal{N}(D)$ is a modular covariant subalgebra of $\mathcal{M}(D)$. This shows the lemma. □

We saw that the coherence property is not automatic. Therefore we have to assume this in the future. Next we show:

6.3.4 Lemma:

Let $\{\mathcal{M}(D), U(\Lambda, x), \Omega\}$ be a theory of local observables fulfilling the Bisognano–Wichmann property. Let $\{\mathcal{N}(W), \mathcal{N}(D)\}$ be a coherent family of modular covariant subalgebras and $E = E_W$ be the associated projection. Then $E\mathcal{H}$ is invariant under the Poincaré transformations $U(\Lambda, x)$. Moreover, for every wedge the restrictions $\widehat{\Delta}_W^{it}$ and $\widehat{U}(\Lambda_W(t), 0)$ coincide. Here $\Lambda_W(t)$ denotes the Lorentz boosts which map W onto itself.

Proof: We know that E commutes with the modular group of every wedge. Since the theory has the Bisognano–Wichmann property it follows that the modular group coincides with the corresponding Lorentz boosts and hence E commutes with these boosts. Since the Lorentz boosts and the translations generate the whole (connected part of the identity) Poincaré group (see Sect. 4.4), the projection E commutes with all $U(\Lambda, x)$. Since $U(\Lambda_W(t), 0)$ and Δ_W^{it} coincide it follows that also their restrictions to $E\mathcal{H}$ coincide. □

We collect the main results of this section in the following

6.3.5 Theorem:

Let $\{\mathcal{M}(D), U(\Lambda, x), \mathcal{H}, \Omega\}$ be a theory of local observables fulfilling the assumptions of the introduction. Assume there exists a coherent family of modular covariant subalgebras $\mathcal{N}(W)$ of $\mathcal{M}(W)$. Then a local quantum field theory $\{\widehat{\mathcal{N}}(D), \widehat{U}(\Lambda, x), E\mathcal{H}, \Omega\}$ exists which fulfils the axioms listed in the introduction. In particular one has for every wedge

$$\widehat{\mathcal{N}}(W) = \vee\{\widehat{\mathcal{N}}(D); D \subset W\}$$

Proof: Let $\{\mathcal{N}(D), \mathcal{N}(W)\}$ be the coherent family of modular covariant subalgebras where $\mathcal{N}(D)$ is constructed as in Lemma 6.3.3. Let $\{\widehat{\mathcal{N}}(D), \widehat{\mathcal{N}}(W)\}$ be the restriction of this family to the Hilbert space $E\mathcal{H}$ and let $\widehat{U}(\Lambda, x)$ be the restriction of the unitary representation of the Poincaré group described in Lemma 6.3.4. From $\mathrm{Ad}\, U(\Lambda, x)\mathcal{M}(W) = \mathcal{M}(\Lambda W + x)$ and $\mathcal{N}(W) = \mathcal{M}(W) \cap \{\mathbb{1}, E\}'$ we obtain $\mathrm{Ad}\, U(\Lambda, x)\mathcal{N}(W) = \mathcal{N}(\Lambda W + x)$. Since $\mathcal{N}(D)$ is the intersection of $\{\mathcal{N}(W); D \subset W\}$ it follows $\mathrm{Ad}\, U(\Lambda, x)\mathcal{N}(D) = \mathcal{N}(\Lambda D + x)$. Finally from $\mathcal{M}(W) = \vee\{\mathcal{M}(D); D \subset W\}$ we get $\mathcal{M}(W)' = \cap\{\mathcal{M}(D)'; D \subset W\}$. Hence

$$\begin{aligned}\cap\{\mathcal{N}(D)'; D \subset W\} &= \cap\{\mathcal{M}(D)' \vee \{\mathbb{1}, E\}"; D \subset W\} \\ &=\subset \{\cap\{\mathcal{M}(D)' : D \subset W\} \vee \{\mathbb{1}, E\}"\} = \mathcal{M}(W) \vee \{\mathbb{1}, E\}" = \mathcal{N}(W)'\end{aligned}$$

implies

$$\mathcal{N}(W) \subset \vee\{\mathcal{M}(D) \cap \{\mathbb{1}, E\}'; D \subset W\} = \vee\{\mathcal{N}(D); D \subset W\} \subset \mathcal{N}(W).$$

Since all $\mathcal{N}(D)$, and $U(\Lambda, x)$ commute with E and Ω is cyclic for $\widehat{\mathcal{N}}(D)$ in $E\mathcal{H}$ the set $\{\widehat{\mathcal{N}}(D), \{\widehat{U}(\Lambda, x), E\mathcal{H}, \Omega\}$ defines a theory of local observables as described in the introduction.
□

6.4) *Decomposition of the global algebra*

The investigations of this subsection are based on a result of Takesaki [Tak72]. Notice if $\mathcal{N}$ is a modular covariant subalgebra of $\mathcal{M}$, then this is also true for $\mathcal{N}^c := \mathcal{N}' \cap \mathcal{M}$.

The existence of the two conditional expectations $\mathcal{E}$ and $\mathcal{E}^c$ has some important consequences.

6.4.1 Theorem:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω. Assume $\mathcal{N} \in \mathcal{M}cs(\mathcal{M})$ is a von Neumann subfactor. Let $\mathcal{N}^c$ be the relative commutant of $\mathcal{N}$ in $\mathcal{M}$ and let $\mathcal{R} = \mathcal{N} \vee \mathcal{N}^c$ be the von Neumann algebra generated by $\mathcal{N}$ and $\mathcal{N}^c$. Then the map

$$\pi : \sum A_i \otimes B_i \in \mathcal{N} \otimes \mathcal{N}^c \longrightarrow \sum A_i B_i \in \mathcal{R} \subset \mathcal{M}$$

extends to an isomorphism of $\mathcal{N} \overline{\otimes} \mathcal{N}^c$ onto $\mathcal{R} = \mathcal{N} \vee \mathcal{N}^c$. Moreover the vacuumstate $(\Omega, .\Omega)$ is a product–state on $\mathcal{R}$, i.e. $A \in \mathcal{N}$ and $B \in \mathcal{N}^c$ implies

$$(\Omega, AB\Omega) = (\Omega, A\Omega)(\Omega, B\Omega).$$

Proof: Let $A \in \mathcal{N}$ and $B \in \mathcal{N}^c$ and let $\mathcal{E}$ be the conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. Then one finds

$$A\mathcal{E}(B) = \mathcal{E}(AB) = \mathcal{E}(BA) = \mathcal{E}(B)A.$$

Since $\mathcal{N}$ is a factor and $\mathcal{E}(B) \in \mathcal{N}$ we see that $\mathcal{E}(B)$ is a scalar. This implies in particular

$$\mathcal{E}(B) = (\Omega, B\Omega).$$

Hence we obtain by Lemma 6.2.2

$$(\Omega, AB\Omega) = (\Omega\mathcal{E}(AB)\Omega) = (\Omega, A\mathcal{E}(B)\Omega) = (\Omega, A\Omega)(\Omega, B\Omega)$$

implying that $(\Omega, .\Omega)$ is a product–state on $\mathcal{R}$. Let $A_i \in \mathcal{N}$ be such that $\{A_i\Omega\}$ forms an orthonormal basis of $\mathcal{H}_{\mathcal{N}}$ and $B_j \in \mathcal{N}^c$ be such that $\{B_j\Omega\}$ forms an orthonormal basis of $\mathcal{H}_{\mathcal{N}^c}$. Then one finds

$$(A_i B_j \Omega, A_k B_l \Omega) = (\Omega, A_i^* A_k \Omega)(\Omega, B_j^* B_l \Omega) = \delta_{i,k}\delta_{j,l}.$$

Because of the separating property of Ω the set $\{A_i\}$ is weakly total in $\mathcal{N}$ and $\{B_j\}$ is weakly total in $\mathcal{N}^c$. This implies that $\{A_i B_j\}$ is weakly total in $\mathcal{R}$. This shows that

$$U : A_i\Omega \otimes B_j\Omega \longrightarrow A_i B_j \Omega$$

defines a unitary map from $\mathcal{H}_{\mathcal{N}} \overline{\otimes} \mathcal{H}_{\mathcal{N}^c}$ onto $\mathcal{H}_{\mathcal{R}}$. Hence

$$U \sum A_i \otimes B_i U^* = \sum A_i B_i$$

extends to a normal isomorphism of von Neumann algebras. $\square$

In order to apply Takesaki's result on tensor products we have to know that the modular covariant subalgebra $\mathcal{N}(W)$ of $\mathcal{M}(W)$ is a factor, which will be show under the assumption that $\mathcal{M}(W)$ itself is a factor. This is known to be the case if the global algebra is a factor. Since the factor property for $\mathcal{M}(D)$ is not known we are not able to show that $\mathcal{N}(D)$ is a factor. Hence we can not use Takesaki's result. Here we will use a characterization of tensor products due to Ge and Kadison [GK66].

For the factor property of $\mathcal{N}(W)$ we use Lemma 5.2.2: Let $U(t)$ be a half–sided translation of the von Neumann algebra $\mathcal{M}$. Denote by E_0 the projection onto the $U(t)$ invariant vectors and by F_1 the projection onto the eigenvectors of $\Delta_{\mathcal{M}}$ to the eigenvalue 1. Then one has

$$F_1 \leq E_0.$$

From this we conclude:

6.4.2 Proposition:

Let $\{\mathcal{M}(D), U(\Lambda, x), \mathcal{H}, \Omega\}$ be a theory of local observables. Assume the global algebra is a factor and hence $\mathcal{M}(W)$ is a factor. Then every modular covariant subalgebra of $\mathcal{M}(W)$ is a factor.

Proof: Let $\mathcal{N}(W)$ be a modular covariant subalgebra of $\mathcal{M}(W)$ and let Z be in the center of $\mathcal{N}(W)$. Then $\widehat{Z}$ is in the center of $\widehat{\mathcal{N}}(W)$ and hence it commutes with $\widehat{\Delta}_W^{it}$. Since the map $\mathcal{N}(W) \to \widehat{\mathcal{N}}(W)$ is an isomorphism we find that Z commutes with Δ_W^{it}. This implies $Z\Omega \in F_1\mathcal{H} \subset E_0\mathcal{H}$. As the group generated by half–sided tranlations for $\mathcal{M}(W)$ contains the time translation it follows $E_0\mathcal{H} = \mathbb{C}\Omega$. Hence $Z\Omega = z\Omega$, $z \in \mathbb{C}$ and the separability of Ω implies $Z = z\mathbb{1}$. This shows the proposition. $\square$

Knowing that $\mathcal{N}(W)$ is a factor, we can use Takesaki's result for the construction of tensor products. But first we have to look at the relative commutants.

6.4.3 Lemma:

Assume $\{\mathcal{N}(W)\}$ is a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$. Let $\mathcal{N}^c(W)$ be the relative commutant of $\mathcal{N}(W)$ in $\mathcal{M}(W)$. Define $\mathcal{N}^p(W) = \mathcal{N}(W) \vee \mathcal{N}^c(W)$. Then $\{\mathcal{N}^c(W)\}$ and $\{\mathcal{N}^p(W)\}$ are both coherent families of subalgebras of $\{\mathcal{M}(W)\}$.

Proof: We know that the two families $\{\mathcal{M}(W)\}$ and $\{\mathcal{N}(W)\}$ are covariant under the Poincaré group. Hence the family $\{\mathcal{N}^c(W)\}$ is covariant under the Poincaré group. Since $\mathcal{N}^c(W + a) \subset \mathcal{N}^c(W)$ for $a \in W$ we obtain by a Reeh–Schlieder type argument that the projection $[\mathcal{N}^c(W)\Omega]$ commutes with the translations. In order to show that it commutes also with the Lorentz transformations we use the half–sided translations of $\mathcal{M}(W)$ which are explained in Sect. 4.4 and which are connected with Lorentz transformations.

Let $W(\ell, \ell_1), W(\ell, \ell_2)$ be two wedges with the same first vector then the algebra $\mathcal{M}(W(\ell, \ell_1) \cap W(\ell, \ell_2))$ fulfils the condition of half–sided modular inclusion with respect to both algebras $\mathcal{M}(W(\ell, \ell_1))$ and $\mathcal{M}(W(\ell, \ell_2))$ (See Thm. 4.4.3). Since the same arguments are true for the algebras $\{\mathcal{N}(W)\}$ and since $[\mathcal{N}\Omega]$ commutes with Poincaré transformations we conclude that the unitary groups $V_i(t)$ which map $\mathcal{M}(W(\ell, \ell_i))$ onto $\mathcal{M}(W(\ell, \ell_1) \cap W(\ell, \ell_2))$ commute with $[\mathcal{N}(W)\Omega]$. Hence V_i maps $\mathcal{N}^c(W(\ell, \ell_i))$ onto $\mathcal{N}^c(W(\ell, \ell_1) \cap$

$W(\ell, \ell_2)$). Since the groups $V_i(t)$ fulfil the spectrum condition we get by a Reeh–Schlieder type argument that $[\mathcal{N}^c(W(\ell, \ell_i))\Omega]$ and $[\mathcal{N}^c(W(\ell, \ell_1) \cap W(\ell, \ell_2))\Omega]$ are the same projections. Since one can repeat these arguments we obtain that $[\mathcal{N}^c(W)\Omega]$ commutes also with all Lorentz transformations. Hence $\{\mathcal{N}^c(W)\}$ is a coherent family. The same method can be used the corresponding result for $\{\mathcal{N}^p(W)\}$. □

6.4.4 Remark:

The relative commutant of $\mathcal{N}^p(W)$ is trivial, because $(\mathcal{N}^p(W))^c$ belongs to the center of the factor $\mathcal{N}^p(W)$ (see Prop. 6.4.1).

Since we do not know whether or not $\mathcal{M}(D)$ and $\mathcal{N}(D)$ are factors, we will define $\mathcal{N}^c(D)$ and $\mathcal{N}^p(D)$ differently.

6.4.5 Definition:

With the assumptions as before we set for double cones

$$\begin{aligned} \mathcal{N}^c(D) &= \cap\{\mathcal{N}^c(W);\ D \subset W\}, \\ \mathcal{N}^p(D) &= \cap\{\mathcal{N}^p(W);\ D \subset W\}. \end{aligned} \tag{6.4.1}$$

Since these definitions are similar to those in Lemma 6.3.3, the conclusion of that lemma holds for $\mathcal{N}^c(D)$ and $\mathcal{N}^p(D)$ with the obvious changes.

Next we have to look at conditions which imply that $\mathcal{M}(D)$ is isomorphic to a tensor product. For the proof of such condition we need a result of Ge and Kadison which is based on the tensor slice mapping introduced by Tomijama [Tmj57]. First we have to explain this concept.

Let $\mathcal{R}$ and $\mathcal{S}$ be von Neumann algebras acting on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Let ω and ρ be normal linear functionals on $\mathcal{R}$ and $\mathcal{S}$ respectively. Then their product $\omega \otimes \rho$ defines a linear functional on $\mathcal{R}\overline{\otimes}\mathcal{S}$ which is defined on $\mathcal{H}\overline{\otimes}\mathcal{K}$. Keeping ω fixed and taking $\psi, \chi \in \mathcal{K}$ and choosing $T \in \mathcal{R}\overline{\otimes}\mathcal{S}$ then the expression $\omega \otimes \rho_{\psi,\chi}(T)$ defines a sesquilinear form on $\mathcal{K}$. This form is continuous and defines by the Riesz representation theorem a linear operator $\Psi_\omega(T)$. Since the commutant of $\mathcal{R}\overline{\otimes}\mathcal{S}$ is $\mathcal{R}'\overline{\otimes}\mathcal{S}'$ it is easy to see that $\Psi_\omega(T)$ belongs to $\mathcal{S}$. This is the tensor slice mapping introduced by Tomijama. In the same manner there exists a mapping $\Psi_\rho : \mathcal{R}\overline{\otimes}\mathcal{S} \to \mathcal{R}$.

With this concept the following result of L. Ge and R. Kadison [GK66] holds, which we quote without proof:

6.4.6 Proposition:

Let $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{R}\overline{\otimes}\mathcal{S}$, then $\mathcal{M}$ splits, i.e. $\mathcal{M} = \mathcal{R}_1\overline{\otimes}\mathcal{S}_1$ with $\mathcal{R}_1 \subset \mathcal{R}$ and $\mathcal{S}_1 \subset \mathcal{S}$ exactly if every tensor slice mapping sends $\mathcal{M}$ into $\mathcal{M}$.

Using this result we obtain:

6.4.7 Proposition:

Let $\mathcal{N}(D)$ be defined as in Lemma 6.3.3 and $\mathcal{N}^c(D), \mathcal{N}^p(D)$ as in Eq. (6.4.1) then one has

$$\mathcal{N}^p(D) \cong \mathcal{N}(D)\overline{\otimes}\mathcal{N}^c(D)$$

Proof: Assume $D \subset W$ then $\mathcal{N}^p(D)$ is a subalgebra of $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$. Since Ω is cyclic and separating for $\mathcal{N}^c$ restricted to $\mathcal{H}_{\mathcal{N}^c}$, every normal functional ρ of $\mathcal{N}^c$ is of the form $\rho = (\varphi, .\,\psi)$ with $\varphi, \psi \in \mathcal{H}_{\mathcal{N}^c}$. Looking at $\Psi_\rho(A)$ is equivalent to looking at

$$(E_{\mathcal{N}} \otimes P_\varphi)A(E_{\mathcal{N}} \otimes P_\psi)$$

where P_φ and P_ψ are the projections onto φ and ψ respectively. This shows by Eq. (6.3.1) that $\Psi_\rho(A)$ is proportional to $\mathcal{E}_D(A)$, and hence $\Psi_\rho(A) \in \mathcal{N}^p(D)$. By symmetry we get also $\Psi_\omega(A) \in \mathcal{N}^p(D)$. So we find $\mathcal{N}^p(D) \cong \mathcal{N}(D)\overline{\otimes}\mathcal{N}^c(D)$. □

Collecting the results of this section we obtain:

6.4.8 Theorem:

Let $\{\mathcal{M}(O), U(\Lambda, x), \mathcal{H}, \Omega\}$ be a theory of local observables fulfilling the assumptions listed in the introduction. Assume that $\{\mathcal{N}(W)\}$ is a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$. Let $\mathcal{N}^c(W)$ be the relative commutant of $\mathcal{N}(W)$ in $\mathcal{M}(W)$ and $\mathcal{N}^p(W) = \mathcal{N}(W) \vee \mathcal{N}^c(W)$. Then:

(1) *There exists on $\mathcal{H}$ a sub–theory of local observables*

$$\{\mathcal{N}^p(D), \mathcal{N}^p(W), U(\Lambda, x)\}$$

covariant under the existing unitary group $U(\Lambda, x)$. Moreover, $\{\mathcal{N}^p(D), \mathcal{N}^p(W)\}$ are modular covariant subalgebras of $\{\mathcal{M}(D), \mathcal{M}(W)\}$ such that $\mathcal{N}^p(W)$ has a trivial relative commutant in $\mathcal{M}(W)$. If E^p denotes the projection onto $[\mathcal{N}^p(W)\Omega]$ then E^p commutes with $\mathcal{N}^p(D), \mathcal{N}^p(W)$ and the group–representation $U(\Lambda, x)$. Moreover, Ω is cyclic for $\mathcal{N}^p(D)$ in $E^p\mathcal{H}$. If we denote the restriction of $\mathcal{N}^p(D)$ and $U(\Lambda, x)$ by $\widehat{\mathcal{N}}^p(D)$ and $\widehat{U}(\Lambda, x)$ respectively then

$$\{\widehat{\mathcal{N}}^p(D), \widehat{U}(\Lambda, x), E^p\mathcal{H}, \Omega\}$$

defines a theory of local observables satisfying the axioms listed in the introduction.

(2) *There exists two coherent families $\{\mathcal{N}(D), \mathcal{N}(W)\}$ and $\{\mathcal{N}^c(D), \mathcal{N}^c(W)\}$ of modular covariant subalgebras of $\{\mathcal{M}(D), \mathcal{M}(W)\}$. If E and E^c are the projections onto $[\mathcal{N}(W)]$ and $[\mathcal{N}^c(W)]$ respectively then these projections commute with $U(\Lambda, x)$ and E with $\mathcal{N}(D)$ and E^c with $\mathcal{N}^c(D)$. With this we obtain:*

$$\{\widehat{\mathcal{N}}^p(D), \widehat{U}(\Lambda, x), E^p\mathcal{H}, \Omega\} \cong \{\widehat{\mathcal{N}}^0(D)\overline{\otimes}\widehat{\mathcal{N}}^c(D), \widehat{U}^0(\Lambda, x) \otimes \widehat{U}^c(\Lambda, x), E\mathcal{H}\overline{\otimes}E^c\mathcal{H}, \Omega^0 \otimes \Omega^c\}.$$

In this formula $\widehat{X}^0$ denotes the restriction to $E\mathcal{H}$ and $\widehat{X}^c$ the restriction to $E^c\mathcal{H}$.

Proof: The existence of the local field theory $\{\mathcal{N}^p(D), \mathcal{N}^p(W), U(\Lambda, x)\}$ such that its restriction to $E^p\mathcal{H}$ is a theory of local observables with cyclic vector has been shown in Lemma 6.4.3. That the relative commutant of $\mathcal{N}^p(W)$ in $\mathcal{M}(W)$ is trivial follows from Remark 6.4.4. That the restrictions $\widehat{\mathcal{N}}^p(W)$ and $\widehat{\mathcal{N}}^p(D)$ split into a tensor product has been shown in Lemma 6.4.3 and Prop. 6.4.7. From the coherence property shown in Thm.

6.3.2 we conclude that also the Hilbert space $E^p\mathcal{H}$ splits into a tensor product $E\mathcal{H}\overline{\otimes}E^c\mathcal{H}$. Since this splitting is independent of the domains W and D the theorem is proved. □

6.5) *The hidden charge problem*

If we look at the modular covariant subalgebras $\mathcal{N}(W)$ of $\mathcal{M}(W)$, then it can happen that the relative commutant $\mathcal{N}^c(W)$ of $\mathcal{N}(W)$ in $\mathcal{M}(W)$ is trivial, i.e. $\mathcal{N}^c(W) = \mathbb{C}\mathbb{1}$. This is called the hidden charge problem because of the following reason: If we start with a theory of local observables $\{\mathcal{N}(O), U(\Lambda, x), \mathcal{H}, \Omega\}$ such that the theory has charged sectors which are connected by localized Bose fields, then we can add these Bose fields and obtain a field algebra $\{\mathcal{F}(O), \widehat{U}(\Lambda, x), \widehat{\mathcal{H}}, \Omega\}$ which also fulfils the assumptions of the theory of local observables. Knowing only the latter theory one would like to discover the local net $\{\mathcal{N}(O), U(\Lambda, x), \mathcal{H}, \Omega\}$ and the structure of the charged fields. The simplest case has been discussed in [Bch65] namely that the charged fields are covariant under the action of a compact abelian group. In this case one has unitary operators in $\mathcal{M}(W)$ which define automorphisms of $\mathcal{N}(W)$. This is no longer true in the general situation. The next, more complicated case is described by Doplicher, Haag and Roberts [DHR69]. Here, or more general in the situation described by Buchholz and Fredenhagen [BF82], the commutant of $\mathcal{N}(W) \vee \mathcal{N}(W')$ is generated by minimal projections. In general one has to cope with the situation where the commutant of $\mathcal{N}(W) \vee \mathcal{N}(W')$ is not generated by minimal projections. In both cases, the tensor product decomposition and the hidden charge situation, one has to look at sub–theories. Therefore, both problems are mingled and one has to disentangle and to solve them.

Let $\{\mathcal{N}(W)\}$ be a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$ and assume that the relative commutant $\mathcal{N}^c(W)$ of $\mathcal{N}(W)$ in $\mathcal{M}(W)$ is trivial. Let E be the projection onto $[\mathcal{N}(W)\Omega]$. We introduce:

6.5.1 Definition:

(1) $\mathcal{G}$ denotes the set of wedges, double cones, and spacelike complements of double cones.
(2) For $G \in \mathcal{G}$ we define

$$\mathcal{M}_1(G) = \mathcal{M}(G) \vee \{\mathbb{1}, E\}''.$$

(3) $\mathcal{N}_1^c(G)$ denotes the relative commutant of $\mathcal{N}(G)$ with respect to $\mathcal{M}_1(G)$. Since by Remark 6.0.1 duality holds inside $\mathcal{G}$ one has

$$\mathcal{M}_1(G) = \mathcal{N}(G').$$

(4) $\mathcal{N}_\infty$ denotes the von Neumann algebra generated by all $\mathcal{N}(G)$.

The following properties of $\mathcal{M}_1(G)$ are easy to derive.

6.5.2 Lemma:

Let $\mathcal{M}_1(G)$ be the algebra defined in 6.5.1. *Then:*
(1) *For every wedge the algebra $\mathcal{M}_1(W)$ is a factor.*
(2) *For the relative commutant of $\mathcal{M}(G)$ in $\mathcal{M}_1(G)$ one has*

$$\mathcal{M}_1(G) \cap \mathcal{M}(G)' = \mathcal{M}(G') \cap \mathcal{N}(G')' = \mathcal{N}^c(G').$$

Hence for every wedge $\mathcal{M}_1(W) \cap \mathcal{M}(W)'$ *is trivial.*
(3) *For the relative commutant* $\mathcal{N}_1^c(G)$ *one has*

$$\mathcal{N}_1^c(G) = \mathcal{M}_1(G) \cap \mathcal{M}_1(G') = \mathcal{N}_1^c(G').$$

(4) $\mathrm{Ad}\,\Delta_G^{\mathrm{i}t}\mathcal{M}_1(G) = \mathcal{M}_1(G)$ *and hence*
$\mathrm{Ad}\,\Delta_G^{\mathrm{i}t}\mathcal{N}_1^c(G) = \mathcal{N}_1^c(G)$.

Proof: (1) Prop. 6.4.2 implies that $\mathcal{N}(W') = \mathcal{M}_1(W)'$ is a factor. (2) The relative commutant of $\mathcal{M}(G)$ in $\mathcal{M}_1(G)$ is trivial if $\mathcal{N}^c(G')$ is trivial. This is the case for all wedges by assumption and Lemma 6.4.3.(3). The definition of $\mathcal{N}_1^c(G)$ implies together with Def. 6.5.1.(3) $\mathcal{N}_1^c(G) = \mathcal{M}_1(G) \cap \mathcal{N}'(G)) = \mathcal{M}_1(G) \cap \mathcal{M}_1(G')$. Since this is symmetric in G and G' we get the statement. (4) E_G commutes with $\Delta_G^{\mathrm{i}t}$ (Lemma 6.1.2). □

Our first goal is to look at partial isometries in $\mathcal{M}(W)$.

6.5.3 Definition:

Let $\mathcal{N}(W)$ be a modular covariant subalgebra of $\mathcal{M}(W)$. We set:
(i) $\mathcal{J}(W) = \{V \in \mathcal{M}(W);\ V$ partial isometry with $V^*V = \mathbb{1},\ VV^* = R(V)\}$.
(ii) $\mathcal{P}(W) = \{VEV^* =: F;\ V \in \mathcal{J}(W)\}$, where $E = [\mathcal{N}(W)\Omega] = [\mathcal{N}(W')\Omega]$.
(iii) By $\mathcal{U}(W)$ we denote the set of unitaries in $\mathcal{M}(W)$.

With this notation we show:

6.5.4 Lemma:

1) *Let* $F \in \mathcal{P}(W)$ *and* P *be a projection in* $\mathcal{M}_1(W)$ *with* $P \leq F$. *Then:*

- α $P \in \mathcal{P}(W)$, *i.e. there exists an element* $V_1 \in \mathcal{J}(W)$ *with* $P = V_1EV_1^*$.
- β *There exists an element* $W \in \mathcal{J}(W) \cap \mathcal{N}(W)$ *with* $V_1 = VW$ *where* V *is defined by* $F = VEV^*$.
- γ *If* $F = P$ *then* W *is unitary.*

2) *Let* $F_1 = V_1EV_1^*, F_2 = V_2EV_2^*$ *be in* $\mathcal{P}(W)$. *Assume* $(V_1V_1^*)(V_2V_2^*) = 0$. *Then exists an element* $V \in \mathcal{J}(W)$ *with* $VEV^* = F_1 + F_2$.
3) *Let* $F \in \mathcal{P}(W)$ *then exists a unitary element* $U \in \mathcal{U}(W)$ *with* $F \leq UEU^*$.

Proof: 1.α) By assumption one has $V_1^*PV_1 \leq E$. Since $V_1^*PV_1$ commutes with $\mathcal{N}(W')$ there exists a projection $H \in \mathcal{N}(W)$ with $V_1^*PV_1 = HE$. Since $\mathcal{N}(W)$ is a factor of type III exists a partial isometry $W \in \mathcal{N}(W)$ with $W^*W = \mathbb{1}$ and $WW^* = H$. The operator V_1W belongs to $\mathcal{J}(W)$ and one finds $V_1WEW^*V_1^* = V_1EHV_1^* = V_1V_1^*PV_1V_1^* = P$. This implies also β.
1.γ) From $V_1EV_1^* = F_1 = F_2 = V_2EV_2^*$ we obtain $V_2^*V_1EV_1^*V_2 = E$. Hence $V_2^*V_1$ commutes with E which implies $V_2^*V_1 = W \in \mathcal{N}$, Since V_1 and V_2 have the same range it follows that W is unitary.
2) Choose a projection $H \in \mathcal{N}(W)$ with $H \neq 0 \neq (\mathbb{1} - H)$. Choose $W_1 \in \mathcal{N}(W)$ with $W_1W_1^* = \mathbb{1},\ W_1^*W_1 = H$ and $W_2 \in \mathcal{N}(W)$ with $W_2W_2^* = \mathbb{1},\ W_2^*W_2 = (\mathbb{1} - H)$. Define

$$V = V_1W_1 + V_2W_2$$

then we find with $V_2^* V_1 = 0$ and with $W_1 W_2^* = 0$

$$\begin{aligned} V^*V = (V_1W_1 + V_2W_2)^*(V_1W_1 + V_2W_2) =& W_1^*V_1^*V_1W_1 + W_2^*V_2^*V_2W_2 = \\ & H + (\mathbb{1} - H) = \mathbb{1}, \\ VV^* = (V_1W_1 + V_2W_2)(V_1W_1 + V_2W_2)^* =& V_1W_1W_1^*V_1^* + V_2W_2W_2^*V_2^* = \\ & R(V_1) + R(V_2). \end{aligned}$$

Moreover, since W_i commutes with E we obtain

$$VEV^* = (V_1W_1 + V_2W_2)E(V_1W_1 + V_2W_2)^* = V_1EV_1^* + V_2EV_2^* = F_1 + F_1.$$

3) Let $F = V_1EV_1^*$ with $R(V_1) \neq \mathbb{1}$. Since $\mathcal{M}(W)$ is of type III exists an element $V_2 \in \mathcal{J}(W)$ with $R(V_2) = (\mathbb{1} - R(V_1))$. From this follows 3) by statement 2). □

By the result of the last lemma it is sufficient to look at unitary elements in $\mathcal{J}(W)$, i.e. at elements of $\mathcal{U}(W)$. Now we introduce the sectors associated with elements $V \in \mathcal{J}(W)$.

6.5.5 Definition:

Let $\{\mathcal{N}(W)\}$ be a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$.
1) For $V \in \mathcal{J}(W)$ we set

$$S(V) = [\mathcal{N}(W)VE\mathcal{H}].$$

2) $\mathcal{N}'_\infty = \bigcap_D \mathcal{N}(D)' \bigcap_W \mathcal{N}(W)'$

Notice that the projection $S(V)$ does not only belong to $\mathcal{N}(W)'$ but also to $\mathcal{N}(W')'$ Since the Hilbert space $E\mathcal{H}$ is invariant under $\mathcal{N}(W')$. We observe

6.5.6 Theorem:

Let $\{\mathcal{N}(W)\}$ be a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$. Then for every $V \in \mathcal{J}(W)$ the projection $S(V)$ belongs to $\mathcal{N}'_\infty$.

Proof: The proof of this theorem consists of three parts. First assume V belongs to $\mathcal{J}(W + a)$ where a belongs to the interior of the wedge W, then the statement is true. Next we have to show that $S(\mathrm{Ad}\, U(\lambda a)V)$ depends weakly continuous on λ. The third part consists of showing that the statement remains true if one takes limits of elements desribed in the first part.

The first part follows from the fact that $D_1 = W \cap (W' + a)$ is not empty. Let D be contained in D_1 such that $D + x$ is contained in D_1 for some open set N. Hence $\cup\{\mathcal{N}(D+x);\ D+x \subset W \cup (W'+a)\}$ commutes with $S(V)$. Next we look at the commutator between $S(V)$ and $\mathrm{Ad}\, U(x)A$ with $A \in \mathcal{N}(D)$. Taking matrix elements of this commutator between vectors which are entire analytic for the translations then one obtains with help of the Jost–Lehmann–Dyson representation (Thm. 1.4.6.) that these vanish for all $x \in \mathbb{R}^d$. Since the analytic vectors are dense in $\mathcal{H}$, the commutator vanishes everywhere. Hence $S(V)$ belongs to $\mathcal{N}'_\infty$.

For the second part we write $S_W(V)$ in order to indicate that S depends on W. Let $a \in W$ then the coherence implies the commutativity of E with $U(\lambda a)$. The relation $S_W(\mathrm{Ad}\, U(\lambda a)V)\mathcal{H}$ = closure $\mathcal{N}(W)U(\lambda a)VU(-\lambda a)E\mathcal{H} = U(\lambda a)$ closure $\mathcal{N}(W -$

$\lambda a)VE\mathcal{H} = \mathrm{Ad}\,U(\lambda a)S_{W-\lambda a}(V)\mathcal{H}$. Since $\mathcal{N}(W-\lambda a)$ is a continuous monotonous increasing family of von Neumann algebras it follows that $[\mathcal{N}(W-\lambda a)VE\mathcal{H}]$ is a strongly continuous monotonous increasing family of projections. Also $U(\lambda a)$ depends strongly continuous on λ. Since $\mathcal{H}$ is invariant we conclude that $S_W(\mathrm{Ad}\,U(\lambda a)V)$ is weakly continuous in λ. Notice: In order that $S_W(\mathrm{Ad}\,U(\lambda a)V)$ belongs also to $\mathcal{N}(W')'$ one has to keep $\lambda \geq 0$.

For the third part choose $a \in W$. Then for $\lambda > 0$ the projection $S(\mathrm{Ad}\,U(\lambda a)V)$ belongs to $\mathcal{N}'_\infty$. Hence for $A \in \mathcal{N}_\infty$ we obtain $[A, S(\mathrm{Ad}\,U(\lambda a)V)] = 0$. Since $S(\mathrm{Ad}\,U(\lambda a)V)$ depends weakly continuous on λ for $\lambda \geq 0$ we get $[A, S(V)] = 0$. This implies $S(V) \in \mathcal{N}'_\infty$.□

Before we continue the investigation of $\mathcal{N}_1^c(W)$ we need the following preparation:

6.5.7 Lemma:

Assume $V \in \mathcal{U}(W)$ and set $P = VEV^*$. Then:
(i) The vector $V\Omega$ is cyclic and separating for $\mathcal{M}(W)$.
(ii) An element $A \in \mathcal{M}(W)$ belongs to $V\mathcal{N}V^*$, iff $[P, A] = 0$.
(iii) Let γ be the isomorphism $\gamma : V\mathcal{N}V^* \to V\mathcal{N}V^*P$ then

$$\mathcal{E}_V(A) := \gamma^{-1}(PAP), \tag{6.5.1}$$

defines a normal faithful conditional expectation from $\mathcal{M}(W)$ onto $V\mathcal{N}(W)V^*$.

Proof: (i) Since $V \in \mathcal{M}(W)$ is unitary we get $\mathcal{M}(W)V\Omega = \mathcal{M}(W)\Omega$. Next assume $A \in \mathcal{M}(W)$ and $AV\Omega = 0$ then we get $AV = 0$ and hence $A = 0$.
(ii) The equation $[A, P] = 0$ implies $AVEV^* - VEV^*A = 0$ and hence $[V^*AV, E] = 0$. This holds only if $V^*AV \in \mathcal{N}(W)$. This implies $[A, P] = 0$ iff $A \in V\mathcal{N}(W)V^*$.
(iii) Since the vector $V\Omega$ is separating for $\mathcal{M}(W)$ it follows that the map $\gamma(VNV^*) = VNV^*P$, $N \in \mathcal{N}(W)$ is an isomorphism. This implies that $\mathcal{E}_V(A) := \gamma^{-1}(PAP)$ is a normal, faithful, positive linear map from $\mathcal{M}(W)$ onto $V\mathcal{N}(W)V^*$. Since the elements in $V\mathcal{N}(W)V^*$ commute with P we see by the definition of γ, that $\mathcal{E}_V$ is a conditional expectation. □

Little is known about the structure of $\mathcal{N}_1^c$. A special situation appears if one has $S(V) = VEV^*$. In this case we obtain

6.5.8 Proposition:

Assume $V \in \mathcal{J}(W)$ is such that $S(V) = VEV^*$. Then it fulfils the following properties:
(i) V is unitary.
(ii) $S(V)$ is a minimal in $\mathcal{N}_1^c(W)$.
(iii) V^* induces an isomorphism of $\mathcal{N}$, i.e.

$$V^*\mathcal{N}(W)V = \mathcal{N}(W).$$

Proof: (i) Since $E\mathcal{H}$ contains the cyclic vector one obtains together with the previous lemma $R(V) = [V\mathcal{H}] = [V\mathcal{M}(W')E\mathcal{H}] = [\mathcal{M}(W')VE\mathcal{H}] = \mathbb{1}$.
(ii) Assume $P_1 \in \mathcal{N}_1^c(W)$ is such that $P_1 \leq P = S(V)$ then by Lemma 6.5.4.(1.β) exists a partial isometry $W \in \mathcal{N}(W)$ with $W^*W = \mathbb{1}$ and $P_1 = VWEW^*V^*$. From the first

statement we get VW is unitary. Since V is unitary we get W is unitary. Hence we get $P_1 = P$ and P is minimal.
(iii) For $N \in \mathcal{N}(W)$ one has $V^*NV \in \mathcal{M}(W)$. Moreover, one finds $EV^*NV = VS(V)NV$ $= V^*NS(V)V = V^*NVE$. Hence V^*NV commutes with E which implies $V^*NV \in \mathcal{N}$. Hence $\gamma(N) := V^*NV$ is an endomorphism of $\mathcal{N}(W)$. Since P is minimal in $\mathcal{N}_1^c(W) = \mathcal{N}(W)' \cap \mathcal{N}(W')'$ we get $P\mathcal{N}(W) \vee \mathcal{N}(W') = \mathcal{B}(P\mathcal{H})$. The relation $[V^*PV, \mathcal{N}(W)] = 0$ implies $[P, V\mathcal{N}(W)V^*] = 0$. Since $V\mathcal{N}(W)V^*$ commutes with $\mathcal{N}(W')$ we conclude $PV\mathcal{N}(W)V^* \subset P\mathcal{N}(W)$. Let β be the isomorphism $\mathcal{N}(W) \to P\mathcal{N}(W)$, then $\delta(N) := \beta^{-1}(PVNV^*)$ defines a second endomorphism of $\mathcal{N}(W)$. We get $\delta \circ \gamma(N) =$
$\beta^{-1}(PVV^*NVV^*) = N$. Moreover, we find $E\gamma \circ \beta(N) = EV^*\beta^{-1}(PVNV^*)V =$
$EV^*P\beta^{-1}(PVNV^*)V = EV^*P(VNV^*)V = EV^*VNV^*V = EN$. This inplies $\gamma \circ \beta(N) = N$. Hence γ is an isomorphism. □

Finally we are interested in the structure of the set of V's such that $S(V_1) = S(V_2)$ holds. We obtain a result only if $S(V_1)$ is a minimal projection in $\mathcal{N}_1^c(W)$.

6.5.9 Theorem:

Assume $V_1, V_2 \in \mathcal{U}(W)$ such that $S(V_1) = S(V_2) \neq E$ holds. If in addition $S(V_1)$ is a minimal projection in $\mathcal{N}_1^c(W)$ then there exist two unitary operators $W_1, W_2 \in \mathcal{M}(W)$ with

$$V_2 = W_1 V_1 W_2.$$

Proof: If $V_1EV_1^* = S(V_1)$. Assume $V_2EV_2 < S(V_1)$. Then one has $V_1^*V_2EV_2^*V_1 < E$. Since this operator commutes with $\mathcal{N}(W')$ we obtain $V_1^*V_2EV_2^*V_1 \in \mathcal{N}(W)E$. Hence exists a partial isometry $W \in \mathcal{N}(W)$ with domain $\mathbb{1}$ and range such that E.range W coincides with $V_1^*V_2EV_2^*V_1$. But this implies $W^*V_2 \in \mathcal{N}(W)$. This is only possible for $S(V_1) = E$. If this is not the case then Lemma 6.6.4,γ implies $V_2 = V_1W$ with a unitary W in $\mathcal{N}(W)$. If $S(V_1) \neq E$ and $V_1EV_1^* \neq S(V_1)$ then one has also $V_2EV_2 \neq S(V_1)$. Next notice that the minimality implies $S(V_1)\mathcal{N}_1^c(W)S(V_1) = \mathbb{C}S(V_1)$. Hence we find $S(V_1)\mathcal{N}_1^c(W)S(V_1)V_iE\mathcal{H} = V_iE\mathcal{H}$, $i = 1, 2$. Therefore, $V_iE\mathcal{H}$, $i = 1, 2$ are invariant under $S(V_1)\mathcal{N}(W') \vee \mathcal{N}_1^c(W)S(V_1)$ which implies $V_iEV_i^* \in S(V_1)\mathcal{N}(W)S(V_1)$. Since $S(V_1)\mathcal{N}(W)S(V_1)$ is of type III there exists a unitary $\hat{W} \in \mathcal{N}(W)S(V_1)$ with $\hat{W}V_1E\mathcal{H} = V_2E\mathcal{H}$. Since the map $\mathcal{M}(W) \to \mathcal{N}(W)S(V_1)$ is an isomorphism we can replace $\hat{W}$ by its inverse image in $\mathcal{N}$. The relation $WV_1E\mathcal{H} = V_2E\mathcal{H}$ implies by Lemma 6.6.4,γ $V_2 = WV_1W_1$.
□

From this result we learn that the "minimal sectors" $S(V)$ are characterized by the left–right co–sets $\mathcal{U}(\mathcal{N}(W))V\mathcal{U}(\mathcal{N}(W))$. Hence one can multiply minimal sectors and decompose the product into sectors. Unfortunately it is not known whether or not the algebra $\mathcal{N}_1^c(W)$ is of type I.

6.6) *Structure of decomposable theories*

In this section it will always be assumed that $\{\mathcal{N}(W)\}$ is a coherent family of modular covariant subalgebras of $\{\mathcal{M}(W)\}$.

Having solved the decomposition problem for tensor products and the hidden charge problem we shall have a look at the situations which might occur.

1. The simplest case is that, where $\mathcal{N}(W)$ and $\mathcal{N}^c(W)$ together generate $\mathcal{M}(W)$. In this situation the theory is the tensor product of two "simpler" theories.

2. The other extreme is the case where $\mathcal{N}^c(W)$ consists of multiples of the identity. This is the pure hidden charge situation.

3. If $\mathcal{N}^c(W)$ is not trivial then $\mathcal{N}(W)$ and $\mathcal{N}^{cc}(W)$ are not necessarily the same. Since the relative commutant of $\mathcal{N}(W)$ in $\mathcal{N}^{cc}(W)$ is trivial, the passage from $\mathcal{N}(W)$ to $\mathcal{N}^{cc}(W)$ is again a hidden charge problem. If we have solved this problem, then there are again two possibilities:

3.a. $\mathcal{N}^c(W)$ and $\mathcal{N}^{cc}(W)$ generate the whole algebra $\mathcal{M}(W)$. This is the same as situation 1.

3.b. $\mathcal{N}^c(W)$ and $\mathcal{N}^{cc}(W)$ generate only a subalgebra $\mathcal{N}^p(W) = \mathcal{N}^c(W)\overline{\otimes}\mathcal{N}^{cc}(W)$. In order to get to $\mathcal{M}(W)$ one has to solve the hidden charge problem for the algebra $\mathcal{N}^p(W)$.

4. Starting from $\mathcal{N}(W)$ and $\mathcal{N}^c(W)$ then it can happen that $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W) = \mathcal{N}^p(W)$ is not the whole algebra $\mathcal{M}(W)$. In this situation one has to solve the hidden charge problem for $\mathcal{N}^p(W)$.

The discussion of the cases 1—4 can be summarized in the following diagram:

$$\mathcal{N}^c = \mathcal{M} \cap \mathcal{N}'.$$

$$\begin{tikzcd}
\{\mathcal{N},\mathcal{N}^c\} \arrow[r, "\text{B.f.}"] \arrow[dd, "\text{t.p.}"] & \{\mathcal{N}^{cc},\mathcal{N}^c\} \arrow[d, "\text{t.p.}"] & \\
& \mathcal{N}^{cc}\overline{\otimes}\mathcal{N}^c \arrow[dr, "\text{B.f}"] & \\
\mathcal{N}\overline{\otimes}\mathcal{N}^c \arrow[ur, "\text{B.f}"] \arrow[rr, "\text{B.f.}"] & & \mathcal{M}
\end{tikzcd}$$

t.p. stands for the construction of the tensor product.

B.f. stands for the construction of the Bose field.

If we have reached the algebra $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ then one has to solve a hidden charge problem in order to get to $\mathcal{M}(W)$. But the algebra $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ is a subalgebra of $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$. If these algebras are different then the relative commutant of $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ in $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$ consists again of the multiples of the identity. Hence the passage from $\mathcal{N}(W)\overline{\otimes}\mathcal{N}^c(W)$ to $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$ is a hidden charge problem.

It remains to explain why the algebra $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^c(W)$ does not need to coincide with $\mathcal{M}(W)$, although we have solved a hidden charge problem in order to pass from $\mathcal{N}(W)$

to $\mathcal{N}^{cc}(W)$. It might happen that both theories constructed from $\mathcal{N}^{cc}(W)$ and $\mathcal{N}^{c}(W)$ have sectors associated with Fermi fields. Let us denote these theories by $\{\mathcal{F}^{cc}(O)\}$ and $\{\mathcal{F}^{c}(O)\}$. Now let us take the tensor product $\{\mathcal{F}^{cc}(O)\overline{\otimes}\mathcal{F}^{c}(O)\}$. In this situation the theory $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^{c}(W)$ has as well Bose– as Fermi sectors because the tensor product of two Fermi fields is a Bose field. If we restrict the theory to all Bose sectors, then there are sectors which are Bose sectors but not tensor products of Bose sectors. Therefore, $\mathcal{N}^{cc}(W)\overline{\otimes}\mathcal{N}^{c}(W)$ do not need to coincide with $\mathcal{M}(W)$.

6.7) *Remarks, additions and problems*

(i) The decomposition theory is based on the existence of modular covariant subalgebras $\mathcal{N}(W) \in \mathcal{M}(W)$. Therefore, the structure of this set $\mathcal{M}cs(\mathcal{M})$ defined in 6.1.2 is of interest. In particular one would like to know whether or not two different modular covariant subalgebras must have a non–trivial intersection.

(ii) The main problem of the decomposition theory is the construction of coherent families of modular covariant subalgebras. In Sect. 6.2 we have investigated the relation of half–sided translations to modular covariant subalgebras. Thm. 6.2.10 indicates that the family of modular covariant subalgebras obtained from one such subalgebra by means of Poincaré transformations is often coherent. But conditions are missing implementing that this is the case.

(iii) If $\mathcal{N}^{c}(W)$ is trivial then only little is known about the algebra $\mathcal{N}_1^{c}(W)$. In the usual theory of superselection sectors (d=4) one finds that $S(V)\mathcal{N}_1^{c}(W)$ is of type I. Is this true in the general case of hidden chargesΓ If this holds then with help of the method of Doplicher and Roberts [DR89] one should be able to construct the compact gauge group. However, if $S(V)\mathcal{N}_1^{c}(W)$ is of type II or III then this implies that the gauge group cannot be compact.

(IV) Nothing has been said about the statistics of sectors. It would be nice if one could repeat the arguments of Doplicher, Haag and Roberts in the scheme presented here.

(V) During the investigation of the hidden charge problem we have envisaged the possibility of a continuous family of charged sectors. Can one construct such an example, eventually with help of Guichardet's continuous tensor product [Gch69]Γ During the construction one has to face the problem that the field algebra shall be countably decomposable. The opposite possibility is the case where the center of $\mathcal{N}_1^{c}(W)$ is purely atomic. To answer these questions further investigations are needed.

(VI) Although we derived the structure of the superselection sectors only for Bose fields, it should be possible to do the same also for Bose– and Fermi Fields. In this case $\mathcal{F}(O)$ is a graded algebra which can be handled with small modifications as the pure Bose case.

(VII) The content of Sect. 6 has partly be explained in [Bch99]. The structure of sub–theories of QFTLO has also been investigated by D.R. Davidson in his thesis [Dav88].

7. Problems for the future

At the end of every section we have mentioned some problems. Nevertheless, there are some questions which should be discussed because they are, in my opinion, of importance for the future development of QFTLO.

7.1) *About the restriction to lower dimensions*

Axiomatic approach to QFTLO has, compared to the Lagrangean setting, the disadvantage, that there exist mathematical operations, which allow to construct new theories out of two or more given ones. These new theories do not contain any new physics. Examples of such operations are the direct sum, direct product, and additions of charged Bose fields to the observables. Therefore, one is interested in characterizing theories which are indecomposable with respect to such operations. However, there is one operation which is of different nature. This is the restriction to lower dimensions. For Wightman fields it is known [Bch64] that the field operators are C^∞–functions in spacelike directions with values in the space of operator valued distributions (in the time direction). Hence one can restrict Wightman fields to lower dimensions, as long as the lower dimensional space contains the time direction. The restriction in x–space corresponds to integration in momentum space. Therefore, if the orginal theory has an isolated mass, then such information gets lost by this operation. Hence also this operation is unwanted.

In QFTLO exists a similar operation. Assume $\{\mathcal{M}(O), \mathbb{R}^{d+1}, \alpha\}$ is a given theory, then one can construct a theory on $\mathbb{R}^d$ as follows: Let $\hat{D}$ be a double cone in $\mathbb{R}^d$, then this is the intersection of a double cone $D(\hat{D})$ in $\mathbb{R}^{d+1}$ with $\mathbb{R}^d$. On the other hand denote by $K(\hat{D})$ the cylindrical set obtained by choosing the first d variables in $\hat{D}$ and the last variable arbitrary. $\hat{D}$ is again the intersection of $K(\hat{D})$ with $\mathbb{R}^d$. Now we choose $\mathcal{N}(\hat{D})$ such that

$$\mathcal{M}(D(\hat{D})) \subset \mathcal{N}(\hat{D}) \subset \mathcal{M}(K(\hat{D}))$$

holds. Then $\{\mathcal{N}(\hat{D}), \mathbb{R}^d, \alpha\}$ defines a QFTLO provided we choose that $\mathcal{N}(\hat{D})$ fulfils covariance (in $\mathbb{R}^d$) and isotony. But these conditions are easily fulfilled. Therefore, there exist many different restrictions. Notice that for the wedge–algebras all these different restrictions coincide and are equal to $\mathcal{M}(W)$. This follows from the double cone theorem. Thm. 1.4.4.

Since the restriction leads to unwanted effects one would like to reconstruct the original theory. I hope, that with help of Tomitas modular theory this will be possible one day. Let us look at examples, in order to see, that my hope is not completely unjustified.

7.1.1 Example: Take a conformal QFT in two dimensions. Choose a fixed timelike direction and restrict the theory to this line. As algebra of an interval take the algebra of the associated double cone, i.e. if $(a, b), a < b$ is the interval then we associate to it the algebra of the double cone $(a + V^+) \cap (b - V^+)$ where V^+ denotes the forward light–cone. By this we obtain a theory on the line.

The algebra $\mathcal{M}(V^+ + a)$ with <u>a not on the line</u> fulfils the condition of half–sided modular inclusion with respect to the algebra of $\mathbb{R}^+$. This algebra is not associated with any set of $\mathbb{R}^1$. Moreover, the associated translation commutes with the translation along the time–axis. From the two–dimensional group of translations it should be possible to reconstruct the orginal theory on $\mathbb{R}^2$.

7.1.2 Example: Take a standard QFTLO in three dimensions and restrict it to two dimensions. Then one should be able to recover the original theory since the algebra $\mathcal{M}\Big(W(\ell_1,\ell_2)\cap W(\ell_1,\ell_3)\Big)$ fulfils the condition of half–sided modular inclusion with respect to the wedge algebra. This algebra is not associated to a subset of $\mathbb{R}^2$. But the corresponding half–sided translations allow to reconstruct the translational part of the stabilizer group of ℓ_1. Also here one should be able to reconstruct the orginal theory on $\mathbb{R}^3$.

In order to be able to reconstruct the original theory one has to understand the spaces of $\pm$half–sided translations (and the spaces of half–sided modular inclusions) for the algebras of the wedge domains. In conformal field theories one has to look also at the algebra of the forward light–cone.

When we constructed the Poincaré group from the modular groups of the wedges (section 4.4) we were able to show that certain half–sided translations commute. One has to understand better the principle behind this phenomenon.

Looking at the example of the forward lightcone in conformal field theory one sees, that the algebras of any subdomain S fulfilling $S+V^+=S$ belong to $\mathcal{H}smi(\mathcal{M}(V^+))^-$. Hence there exists a half–sided translation associated with it. For $a\in S$ one has the half–sided translation of $\mathcal{M}(V^++a)$ with its generator denoted by H_a. It should be possible to express the generator of the group associated with $\mathcal{M}(S)$ in terms of the family $\{H_a\}$.

The spaces $\mathcal{H}smi(\mathcal{M})^-$ and $\mathcal{H}smi(\mathcal{M})^+$ have certain order and convexity properties. These are explaind in [Bch96a]. Moreover, one can introduce an equivalence relation in $\mathcal{H}smi(\mathcal{M})^-$ (and also in $\mathcal{H}smi(\mathcal{M})^+$) as follows:

7.1.3 Definition:

Let $\mathcal{N}_1,\mathcal{N}_2\in\mathcal{H}smi(\mathcal{M})^-$ and $U_i(t),\ i=1,2$ their associated translations. Then $\mathrm{Ad}\,U_i(t-1)\mathcal{N}_i$ will be denoted by $\mathcal{N}_i(t)$. We call $\mathcal{N}_1$ and $\mathcal{N}_2$ equivalent

$$\mathcal{N}_1\sim\mathcal{N}_2$$

if there exist two non–zero positive numbers λ_1,λ_2 with

$$\mathcal{N}_1(\lambda_1)\subset\mathcal{N}_2\subset\mathcal{N}_1(\lambda_2).$$

Because of the decreasing monotony of $\mathcal{N}_1(\lambda)$ one must have $\lambda_2\le\lambda_1$.

It is interesting to notice that this order structure survives if one passes to the space of equivalence classes. This discussion shows that $\mathcal{H}smi(\mathcal{M})^-$ has a rich structure, but up to now it is not clear how to get to the geometric structure on which the algebra $\mathcal{M}$ is based.

In the example of the wedge one has to construct the algebra $\mathcal{M}(W(\ell_1,\ell_3))$ from the knowledge of the algebra $\mathcal{M}\Big(W(\ell_1,\ell_2)\cap W(\ell_1,\ell_3)\Big)$. This is possible since the half–sided translation connecting $\mathcal{M}(W(\ell_1,\ell_3))$ with $\mathcal{M}\Big(W(\ell_1,\ell_2)\cap W(\ell_1,\ell_3)\Big)$ is also a half–sided translation of the latter algebra. Knowing this translation one can reconstruct

$\mathcal{M}(W(\ell_1, \ell_3))$. The only problem here is the normalization of the group. If $U(t) \in \mathcal{H}str(\mathcal{M})^+$ and $\lambda > 0$, then $U(\lambda t) \in \mathcal{H}str(\mathcal{M})^+$. Therefore, λ has to be fixed for the correct application.

7.2) *Vacuum states on the hyperfinite* III_1 *algebra*

As discussed in Thm. 5.3.9 the Buchholz–Wichmann nuclearity property Cond. 5.3.7 implies that the local algebras are hyperfinite III_1 algebras. Therefore, the algebras belonging to wedges are also hyperfinite and of type III_1. By a result of Haagerup [Hgr87] there exists (up to unitary equivalence) only one hyperfinite III_1 factor. Therefore, it is tempting to ask wether or not the vacuum state of a QFTLO can be characterized by algebraic means. What I have in mind is the structure of the set of half–sided translations, or equivalently half–sided modular inclusions connected with the vacuum state of the given theory. The situation shall be explained by examples.

7.2.1 Example: The QFTLO on the line.

Here the wedge algebra is associated with the half–line $\mathbb{R}^+ = \{(0,\infty)\}$. If we look at the algebra associated with the set $(1,\infty)$, then this fulfils the condition of −half–sided modular inclusion and the algebra belonging to $(0,1)$ fulfils the condition of +half–sided modular inclusion. In this situation $\mathcal{M}((0,1))$ is the relative commutant of $\mathcal{M}((1,\infty))$ in $\mathcal{M}(\mathbb{R}^+)$ and the corresponding half–sided translations together with the modular group of $\mathcal{M}(\mathbb{R}^+)$ generate the Möbius group.

7.2.2 Example: QFTLO on the d–dimensional Minkowski space, $d > 1$.

For $d = 2$ one has for the algebra of the wedge two half–sided translations with opposite sign. These are the translations along the two lightlike directions. In this case the two translations commute and the two translations together with the modular group of the wedge–algebra generate the two–dimensional Poincaré group. In higher dimension we will restrict to theories fulfilling the Bisognano–Wichmann property. In this situation we know from Thm. 4.4.3 that the algebra $\mathcal{M}(W[\ell,\ell_1]) \cap W[\ell,\ell_2])$ fulfils the condition of −half–sided modular inclusion with respect to the algebras $\mathcal{M}(W[\ell,\ell_1])$ and $\mathcal{M}(W[\ell,\ell_2])$. In this situation we obtain for $\mathcal{M}(W[\ell,\ell_1])$ a family of half–sided modular inclusions labeled by the direction of ℓ_2. A precise characterization of this situation is still missing. This is due to the fact that one is looking for Loerentz transformations and not for the group generated by the half–sided translations.

7.2.3 Example: Conformal field theories in higher dimension.

In this situation the set of half–sided modular inclusions is much larger. This is due to the fact that one has timelike commutativity. Let G be a set with $G + V^+ = G$ then it is easy to see that $\mathcal{M}(G \cap W)$ fulfils the condition of −half–sided modular inclusion with respect to the algebra $\mathcal{M}(W)$. But the importance of the associated half–sided translations is not known.

7.2.4 Example: QFT on the two dimensional de Sitter space.

The two–dimensional de Sitter space is isomorphic to the one–sheeted hyperboloid in the three–dimensional Minkowski space. A wedge in this space is the intersection of the wedge in the ambient space with the hyperboloid. It turns out, that also in this situation the translations along the the lightlike directions are half–sided translations. But the situation is different as well from the field theory on the two–dimensional Minkowski space as from the field theory on the line. Since the "shifted wedges" of the de Sitter space can have an empty intersection it follows that the vacuum vector is not cyclic for the corresponding algebras. This implies that the two translations do not commute. Hence the situation is different from the Minkowski space theory. The situation is probably different from that of the line, because it is unlikely, that the different subalgebras fulfilling the condition of ±half–sided modular inclusion are relative commutants of eachother. (For details on QFT on de Sitter see e.g. [BB98].)

7.2.5 Problems:

1) Can one characterize those states on a hyperfinite III_1 factor which permit one or more ±half–sided modular inclusionsΓ
2) If a state permits at least one half–sided modular inclusion, what are the different families of such inclusions which can appearΓ
3) Can one discriminate different theories of local observables by means of the set of half–sided modular inclusionsΓ

7.3) *Can one interprete the local modular groups as local dynamics?*

For many questions in quantum physics it is advantageous to have a local dynamics. This is in particular the case if one is interested in defining Gibbs states of a system. If one starts from the usual quantum theory one chooses as subsystems the particle in a box with reflecting walls or periodic boundary conditions. This defines a quantum system and the corresponding Hamiltonian is considered as the local one. In Lagrangean quantum field theory the energy is usually given as an integral over a Hamiltonian density. In this situation one takes as local energy the integral of the energy density over the region one is interested in. Sometimes one has to take for the integration a smooth testfunction which is one in the domain of interest and which tends to zero in a small neighbourhood of that region. In the theory of local observables a definition of a local dynamics or an energy density is up to now only possible if the theory fulfils the nuclearity condition of Buchholz and Wichmann [BW86]. For the construction of a local dynamics see e.g. Buchholz and Junglas [BJ89] and for the energy density see Buchholz, Doplicher and Longo [BDL86]. Since for a general QFTLO there exists no concept which could be used as local dynamics, it is tempting to interprete the properly scaled modular groups of local regions as local dynamics.

First we have to explain what we want to understand by a local dynamics. Let us fix a vector x_0 in the forward light cone V^+ with $x_0^2 = 1$. The double cones $D_R^{x_0}$ are defined by

$$D_R^{x_0} = \{Rx_0 - V^+\} \cap \{-Rx_0 + V^+\}. \tag{7.3.1}$$

Let $U_R(t)$ be a family of unitary groups depending continuously on R such that the group $\mathrm{Ad}\, U_R(t)$ belongs to the automorphisms of $\mathcal{M}(D_R^{x_0})$. Then we say that these groups define a local dynamics if for every bounded set O the expression

$$U_R(t)A\Omega, \qquad A \in \mathcal{M}(O)$$

converges for $R \to \infty$ to $T(tx_0)A\Omega$ in the topology of the Hilbert space and this uniformly on every compact of the t–axis.

That the modular groups might be a good candidate is indicated by the following two examples.

7.3.1 Example: For a fixed double cone we choose $D = \{x; |x^0| + \|\vec{x}\| < 1\}$ and the running double cone will be replaced by a running family of wedges $W_R := W - Rx^1$ with $R > 1$ and x^1 is a fixed vector perpendicular to the time direction x^0 with $(x^1)^2 = -1$. If we denote the modular group of W_R by $\Delta_R^{\mathrm{i}t}$ then we choose as local dynamics

$$U_R(t) = \Delta_R^{-\mathrm{i}\frac{t}{2\pi R}}.$$

Because of $\Delta_R^{-\mathrm{i}\frac{t}{2\pi R}} = T(-Rx^1)\Delta_0^{-\mathrm{i}\frac{t}{2\pi R}}T(Rx^1)$ this becomes with Remark 2.5.3 $= T((\Lambda_W(-\frac{t}{2\pi R}) - 1)Rx^1)\Delta_0^{-\mathrm{i}\frac{t}{2\pi R}}$, where $T(x)$ denotes the representation of the translations. With Eq. (1.5.3) we find:

$$(\Lambda_W(-\frac{t}{2\pi R}) - 1)Rx :^1= x^0 R \sinh\frac{t}{R} + x^1 R(\cosh(-\frac{t}{R}) - 1) = x^0 t + O(\frac{1}{R}).$$

This implies

$$U_R(t)A\Omega = T(tx^0 + O(\frac{1}{R}))\Delta_0^{-\mathrm{i}\frac{t}{2\pi R}}A\Omega.$$

Since $\Delta_0^{\mathrm{i}t}$ is strongly continuous we obtain by the unitarity of the operators

$$s - \lim_{R\to\infty} U_R(t)A\Omega = T(t)A\Omega, \quad A \in \mathcal{M}(D).$$

7.3.2 Example: As a second example we look at conformal field theory, where the modular groups of the double cones are known (Thm. 3.2.2). We choose as running domains the double cones of radius R and choose

$$U_R(t) = \Delta_R^{-\mathrm{i}\frac{t}{\pi R}}.$$

With the notation of Thm. 3.2.2 this corresponds to the transformation

$$x^{\pm}(-\frac{t}{\pi R}) = R\frac{-(1 - x^{\pm}/R) + \mathrm{e}^{2t/R}(1 + x^{\pm}/R)}{(1 - x^{\pm}/R) + \mathrm{e}^{2t/R}(1 + x^{\pm}/R)}.$$

For small $x^{\pm}$ and large R we obtain

$$x^{\pm}(-\frac{t}{\pi R}) = x^{\pm} + t + O(\frac{1}{R}).$$

Since the representation of the conformal group is continuous it follows, also in this example, that $U(t)$ converges for large R to the time translation.

There is one essential difference between the two examples, namely the scaling of the corresponding modular groups differs by the facter 2. I think that one has to understand the origin of the difference in the scaling factors before one is able to prove that $\Delta_R^{-\mathrm{i}\frac{t}{\pi R}}$ converges to the time translation also in the general case.

7.4) *Modular theory in charged sectors*

Almost all the results described in this review are based on the fact that cyclic and separating vector Ω for the local algebras is at the same time the only vector which is invariant under the representation of the Poincaré group. We do not have this situation in the charged sectors. But if we take a vector ψ which has compact energy contribution and if ℓ is one of the lightlike vectors defining the wedge $W(\ell,\ell')$, then $U(\lambda\ell)$, $\lambda\in\mathrm{I\!R}$ is again a group with positive generator which maps $\mathcal{M}(W(\ell,\ell')$ into itself. Moreover the vector $U(\lambda\ell)\psi$ is again a vector which is cyclic and separating for $\mathcal{M}(W)$. In addition the modular group of $U(\lambda\ell)\psi$ can be computed from that of ψ with help of the cocycle Radon Nikodym derivative $[DU(\lambda\ell)\psi : D\psi]_t$, [Co73b], [CT77]. If we denote the Radon Nikodym derivative for a moment by u_t, then the cocycle relation means

$$u_{s+t}=u_s\sigma_\psi^s(u_t). \tag{7.4.1}$$

The action of the modular group belonging to $U(\lambda\ell)\psi$ can be computed with help of the formula

$$\sigma^t_{U(\lambda\ell)\psi}(A)=[DU(\lambda\ell)\psi : D\psi]_t\sigma^t_\psi(A)[DU(\lambda\ell)\psi : D\psi]^*_t,\quad A\in\mathcal{M}(W). \tag{7.4.2}$$

7.4.1 Problems:

(i) We know that the group $U(\lambda\ell)$ has an analytic continuation into the upper complex half–plane. What does this imply for the Radon Nikodym derivative $[DU(\lambda\ell)\psi : D\psi]_t\Gamma$ Note that for complex λ the vector $U(\lambda\ell)\psi$ is again cyclic and separating for $\mathcal{M}(O)$, which implies that the Radon Nikodym derivative is also defined for those values of λ.
(ii) Does there exist any relation between $\Delta^{\mathrm{i}t}_\psi, [DU(\lambda\ell)\psi : D\psi]_t$ and $U(\lambda\ell)$ besides the known standard onesΓ

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Appendix 1 (K.-H. Rehren)

Bibliography on the algebraic theory of superselection sectors in low dimensions

In the last decade, the algebraic theory of superselection sectors was supplemented by a vast reservoir of examples originating in two-dimensional conformal quantum field theory. As is well known, in low dimensions the possibility of braid group statistics is a new feature beyond the original DHR analysis, which is however easily incorporated into the original framework.

The following is a list of prominent references in the algebraic theory of superselection sectors in low dimensions.

The DHR theory was adapted to the case of braid group statistics in [FRS89], [FG91]. The local von Neumann algebras for specific models based on non–abelian current algebras were constructed and analysed in [Wa95], [Wa98], [Lo94]. Modular theory was applied to a general study of global properties of chiral nets concerning Haag duality, conformal covariance, spin-statistics theorem and CPT theorem in [BGL93], [GL96]. Models with a breakdown of Haag duality and the construction of the associated dual net were discussed in [BSM90], [Mü98a], [GLW98]. Sufficient conditions to reconstruct, using modular theory [Bch92], a chiral net with conformal symmetry and spectrum condition from a single half-sided modular inclusion of von Neumann algebras were formulated by [Wie94]. For models with Haag duality in two dimensions it was shown that the split property for wedges (presumably related to a mass gap) excludes the existence of localized superselection sectors at all [Mü98b], while solitonic sectors will generically emerge. Properties of the latter were studied in [Sch98], [Mü98c], [Re98].

The issue of charged fields which create superselection sectors from the vacuum, and of an underlying symmetry principle, was addressed from various sides. A reconstruction theorem comparable to the result by Doplicher and Roberts [DR90] cannot be achieved since non–abelian braid group statistics poses an obvious obstruction. In the abelian case, an anyonic field algebra was constructed in [BMT88]. The reduced field bundle (RFB) of intertwining non–local fields was introduced as a general construction in [FRS89], and conformal covariance properties of these algebras were analysed in [FRS92]. Pointlike exchange fields associated with the RFB were constructed in [FJ96], and the weak C* Hopf symmetry of the RFB was discovered in [Ni94], [Re97]. Other, ultimately unsatisfactory, symmetry concepts were discussed in [MS90], [Re90]. A theory of sector induction and restriction between a theory and a subtheory equipped with a global conditional expectation was initiated in [LR95] and was further elaborated with a view on specific chiral models in [BE98].

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Appendix 2 (R. Verch)

References for applications of Tomita-Takesaki theory in quantum field theory on curved spacetime

Listed below are references containing applications of Tomita-Takesaki theory to quantum field theory on curved spacetime.

On a generic curved spacetime, there are in general no symmetries (spacetime isometries) present, and hence there is no natural candidate for a vacuum state. Likewise, in a generic curved spacetime, it is in general not clear which spacetime regions, if any, play a similar role as the wedge regions in Minkowski spacetime in the sense that the modular objects corresponding to von Neumann algebras associated with these regions and preferred vacuum-like vectors act in a suitable sense geometrical. Therefore, most applications of Tomita-Takesaki theory to quantum field theory in curved spacetime so far have been restricted to a class of spacetimes possessing a structure which to certain extent mimics the geometrical features underlying the Bisognano-Wichmann situation, i.e. there are natural wedge-regions and Killing flows leaving these wedge regions invariant. In this case, a variety of versions of a geometric action of modular objects associated with wedge-regions and certain preferred states has been investigated in the works [BB99],[BEM98],[BDFS98],[GLRV99],[Kay85],[KW91],[Sew82],[SV96]. The pioneering work of this list is [Sew96], where a situation analogous to the Bisognano-Wichmann setting is modelled on Schwarzschild-Kruskal spacetime. An operator-algebraic version of it appears in [SV96]. The works [Kay85],[KW91] deal with an investigation of this Bisognano-Wichmann-like situation on black-hole spacetimes for free scalar field models. In [BB99],[BEM98], Bisognano-Wichmann-like scenarios are investigated on de Sitter spacetime.

An attractive line of thought is to try and characterize vacuum states on a generic spacetime by a suitable form of geometric modular action with respect to von Neumann algebras associated with a class of distinguished regions (e.g. wedge regions, cf. also [BB99]). On a generic spacetime without isometries such a geometric action of modular objects cannot be expected to be given by point-transformations on the underlying spacetime-manifold. A more general approach addressing this issue is developed in [BDFS98].

In [CR94] a somewhat different approach, compared to the works just cited, is taken towards the physical interpretation of modular objects in generally covariant quantum theories.

The type of the local von Neumann algebras of a quantum field theory is related to the spectra of their associated modular operators (Connes' invariant) and can, like on Minkowski spacetime, be fixed on curved spacetime via assumptions on the quantum field theory's short-distance scaling limits. This question is considered in [Ver97],[Wol92].

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