Introduction to representation theory of braid groups

Toshitake Kohno

The University of Tokyo

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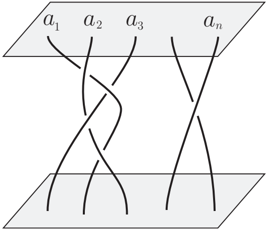
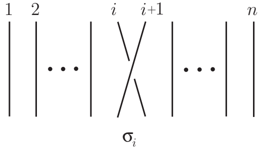
Braid groups were studied by E. Artin in the 1920's.



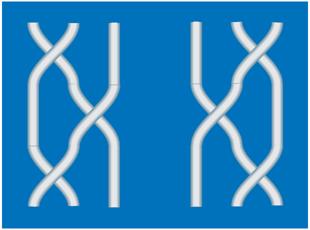
The isotopy classes of geometric braids as above form a group by composition. This is the braid group with n strands denoted by B_n.

Braid relations

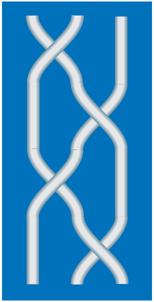


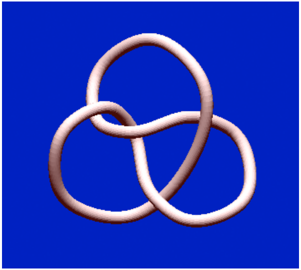
B_n is generated by σ_i, $1 \leq i \leq n-1$ with relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1$$

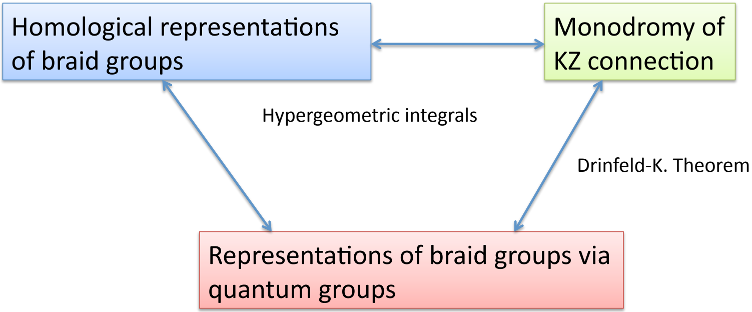


A braid and its closure (figure eight knot):





Quantum symmetry in representations of braid groups



Plan

- Monodromy representations of logarithmic connections

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- Knizhnik-Zamolodchikov (KZ) connection

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- Homological representations and KZ connections
- Quantum symmetry in homological representations

Configuration spaces

$\mathcal{F}_n(X)$: configuration space of ordered distinct n points in X.

$$\mathcal{F}_n(X) = \{(x_1, \cdots, x_n) \in X^n \ ; \ x_i \neq x_j \text{ if } i \neq j\},$$

$$\mathcal{C}_n(X) = \mathcal{F}_n(X)/\mathfrak{S}_n$$

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Suppose $X = \mathbf{C}$.

$$\pi_1(\mathcal{F}_n(\mathbf{C})) = P_n, \quad \pi_1(\mathcal{C}_n(\mathbf{C})) = B_n$$

We set $X_n = \mathcal{F}_n(\mathbf{C})$

Logarithmic forms

We set

$$\omega_{ij} = d\log(z_i - z_j), \quad 1 \leq i \neq j \leq n.$$

Consider a total differential equation of the form $d\phi = \omega\phi$ for a logarithmic form

$$\omega = \sum_{i<j} A_{ij}\omega_{ij}$$

with $A_{ij} \in M_m(\mathbf{C})$.

Infinitesimal pure braid relations

As the integrability condition we infinitesimal pure braid relations

$$[A_{ik}, A_{ij} + A_{jk}] = 0, \quad (i, j, k \text{ distinct}),$$
$$[A_{ij}, A_{k\ell}] = 0, \quad (i, j, k, \ell \text{ distinct})$$

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- As the holonomy of the flat connection ω we obtain linear representation of the pure braid group P_n.
- The horizontal section of ω is expressed as an infinite sum of iterated integrals of logarithmic forms (hyperlogarithms).
- Infinitesimal pure braid relations describe the nilpotent completion of the pure braid group P_n over $\mathbf{Q}$ (Malcev algebra).

KZ connections

$\mathfrak{g}$: complex semi-simple Lie algebra.
$\{I_\mu\}$: orthonormal basis of $\mathfrak{g}$ w.r.t. Killing form.
$\Omega = \sum_\mu I_\mu \otimes I_\mu$
$r_i : \mathfrak{g} \to End(V_i)$, $1 \le i \le n$ representations.

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Ω_{ij} : the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$.

$$\omega = \frac{1}{\kappa} \sum_{i<j} \Omega_{ij} d\log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}$$

ω defines a flat connection for a trivial vector bundle over the configuration space $X_n = \mathcal{F}_n(\mathbf{C})$ with fiber $V_1 \otimes \cdots \otimes V_n$ since we have

$$\omega \wedge \omega = 0$$

As the holonomy we have representations

$$\theta_\kappa : P_n \to GL(V_1 \otimes \cdots \otimes V_n).$$

In particular, if $V_1 = \cdots = V_n = V$, we have representations of braid groups

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We shall express the horizontal sections of the KZ connection : $d\varphi = \omega\varphi$ in terms of homology with coefficients in local system homology on the fiber of the projection map

$$\pi : X_{m+n} \longrightarrow X_n.$$

$X_{n,m}$: fiber of π, $\quad Y_{n,m} = X_{n,m}/\mathfrak{S}_m$

$\mathfrak{g} = sl_2(\mathbf{C})$ has a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$\lambda \in \mathbf{C}$
M_λ : Verma module of $sl_2(\mathbf{C})$ with highest weight vector v such that

$$Hv = \lambda v, \ \ Ev = 0$$

M_λ is spanned by

$$v, Fv, F^2 v, \cdots$$

For a non-negative integer λ we obtain an irreducible irreducible representation V_λ of dimension $\lambda + 1$ as a quotient of M_λ.

KZ equation for $sl_2(\mathbf{C})$

Consider the case $\lambda = 1$. Put $V = V_\lambda$.
The monodromy representations of braid groups

$$\theta_\kappa : B_n \to GL(V^{\otimes n}).$$

Set $q = e^{2\pi\sqrt{-1}/\kappa}$ and

$$g_i = q^{1/4}\theta_\kappa(\sigma_i)$$

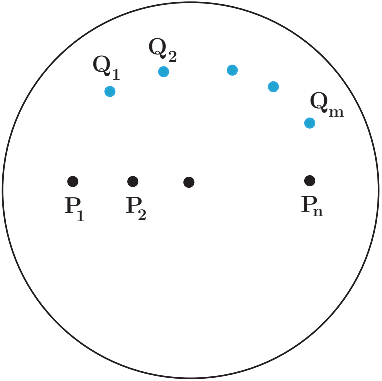
Then we have

$$(g_i - q^{1/2})(g_i + q^{-1/2}) = 0.$$

The monodromy representations factor through the Iwahori-Hecke algebra $\mathcal{H}(q)$. The above quadratic relation leads to the skein relation of the Jones polynomial.

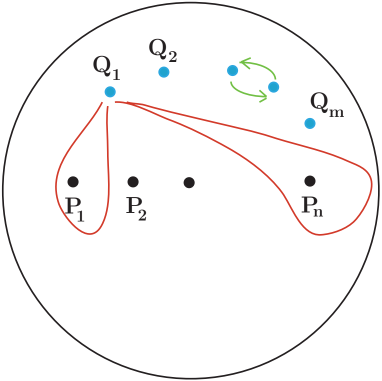
Fix $P = \{(1,0), \cdots, (n,0)\} \subset D$, where D is a 2 dimensional disc.
$\Sigma = D \setminus P$

$$\mathcal{F}_{n,m}(D) = \mathcal{F}_m(\Sigma), \quad \mathcal{C}_{n,m}(D) = \mathcal{F}_m(\Sigma)/\mathfrak{S}_m$$



Homology of relative configuration spaces

$$H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z}$$



Consider the homomorphism

$$\alpha : H_1(\mathcal{C}_{n,m}(D); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

defined by $\alpha(x_1, \cdots, x_n, y) = (x_1 + \cdots + x_n, y)$.

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Composing with the abelianization map
$\pi_1(\mathcal{C}_{n,m}(D), x_0) \to H_1(\mathcal{C}_{n,m}(D); \mathbf{Z})$, we obtain the homomorphism

$$\beta : \pi_1(\mathcal{C}_{n,m}(D), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}.$$

$\pi : \widetilde{\mathcal{C}}_{n,m}(D) \to \mathcal{C}_{n,m}(D)$: the covering corresponding to $\mathrm{Ker}\,\beta$.

Homological representations

$H_*(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$ considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module by deck transformations.

Express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$.

$$H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}(D); \mathbf{Z})$$

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$H_{n,m}$ is a free R-module of rank

$$d_{n,m}=\left(\begin{array}{c} m+n-2 \\ m \end{array}\right).$$

$\rho_{n,m}: B_n \longrightarrow \mathrm{Aut}_R\, H_{n,m}$: homological representations $(m>1)$ extensively studied by Bigelow and Krammer ; they are faithful representations.

Space of null vectors

$\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n, \quad |\Lambda| = \lambda_1 + \cdots + \lambda_n$
Consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$.

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m : non-negative integer

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \; ; \; Hx = (|\Lambda| - 2m)x\}$$

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The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$, hence it acts on the space of null vectors $N[|\Lambda| - 2m]$.
The monodromy of KZ connection

$$\theta_{\kappa,\lambda} : P_n \longrightarrow \mathrm{Aut}\, N[|\Lambda| - 2m]$$

Comparison theorem

We fix a complex number λ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$. We have

$$\theta_{\kappa,\lambda} : B_n \longrightarrow \mathrm{Aut}\, N[n\lambda - 2m].$$

Theorem

There exists an open dense subset U in $(\mathbf{C}^*)^2$ such that for $(\lambda, \kappa) \in U$ the homological representation $\rho_{n,m}$ with the specialization

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$$

is equivalent to the monodromy representation of the KZ connection $\theta_{\lambda,\kappa}$ with values in the space of null vectors

$$N[n\lambda - 2m] \subset M_\lambda^{\otimes n}.$$

Local system over the configuration space

$\pi : X_{n+m} \to X_n$: projection defined by
$(z_1, \cdots, z_n, t_1, \cdots, t_m) \mapsto (z_1, \cdots, z_n)$.
$X_{n,m}$: fiber of π.

$$\Phi = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{\kappa}} \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \times \prod_{1 \le i < j \le m} (t_i - t_j)^{\frac{2}{\kappa}}$$

(multi-valued function on X_{n+m}).
Consider the local system $\mathcal{L}$ associated with Φ.

Solutions to KZ equation

Notation:
$W[|\Lambda| - 2m]$ has a basis

$$F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$$

with $|J| = j_1 + \cdots + j_n = m$ and $v_{\lambda_j} \in M_{\lambda_j}$ the highest weight vector.

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Theorem (Schechtman-Varchenko, Date-Jimbo-Matsuo-Miwa, ...)

The hypergeometric integral

$$\sum_{|J|=m} \left(\int_\Delta \Phi R_J(z,t) dt_1 \wedge \cdots \wedge dt_m \right) F^J v$$

lies in $N[|\Lambda| - 2m]$ *and is a solution of the KZ equation, where* Δ *is a cycle in* $H_m(Y_{n,m}, \mathcal{L}^*)$.

Quantum symmetry

Theorem

There is an isomorphism

$$N_h[\lambda n - 2m] \cong H_m(Y_{n,m}, \mathcal{L}^*)$$

which is equivariant with respect to the action of the braid group B_n*, where* $N_h[\lambda n - 2m]$ *is the space of null vectors for the corresponding* $U_h(\mathfrak{g})$*-module with* $h = 1/\kappa$*.*

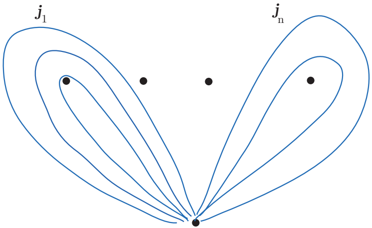
Quantum symmetry for twisted chains

There is the following correspondence:

twisted multi-chains $\Longleftrightarrow$ weight vectors $F^{j_1}v_1 \otimes \cdots \otimes F^{j_n}v_n$

twisted boundary operator $\Longleftrightarrow$ the action of $E \in U_h(\mathfrak{g})$

$$H_m(Y_{n,m}, \mathcal{L}^*) \Longleftrightarrow N_h[\lambda n - 2m]$$



twisted multi-chains